Work in progress

# Coherence for Geometricity

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# Dependent type theory of spaces

Doesn't exist yet

Compose substitutions in two orders

$$\vdash t : \mathbb{R}$$

$$X\ \mathrm{Space} \vdash H(X)\ \mathrm{Space}$$

$$x : \mathbb{R} \vdash Y(x)\ \mathrm{Space}$$

fibre of $Y$ over $x$

$$\begin{tikzcd}
 & x : \mathbb{R} \vdash Y(x)\ \mathrm{Space} \arrow[dl, "x := t"'] \arrow[dr, "X := Y(x)"] & \\
\vdash Y(t)\ \mathrm{Space} \arrow[dr, "X := Y(t)"'] & & x : \mathbb{R} \vdash H(Y(x))\ \mathrm{Space} \arrow[dl, "x := t"] \\
 & \vdash H(Y(t))\ \mathrm{Space} &
\end{tikzcd}$$

fibre of $Y$ over $t$

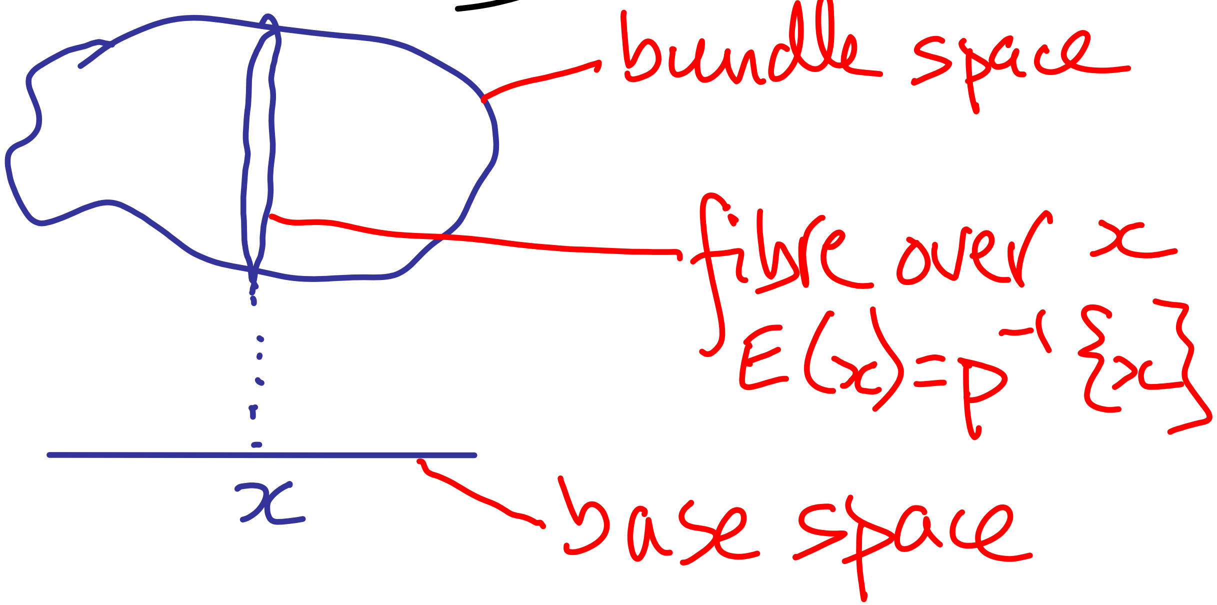
$H$ applied to $Y(t)$ = fibre of $H(Y(x))$ over $t$

$H$ (applied to $Y(x)$) "works fibrewise"

# Dependent type theory of spaces

## = fibrewise topology

Bundle $\begin{array}{c} E \\ p \big\downarrow \\ B \end{array}$




$$\Gamma \vdash \sim\sim\sim$$

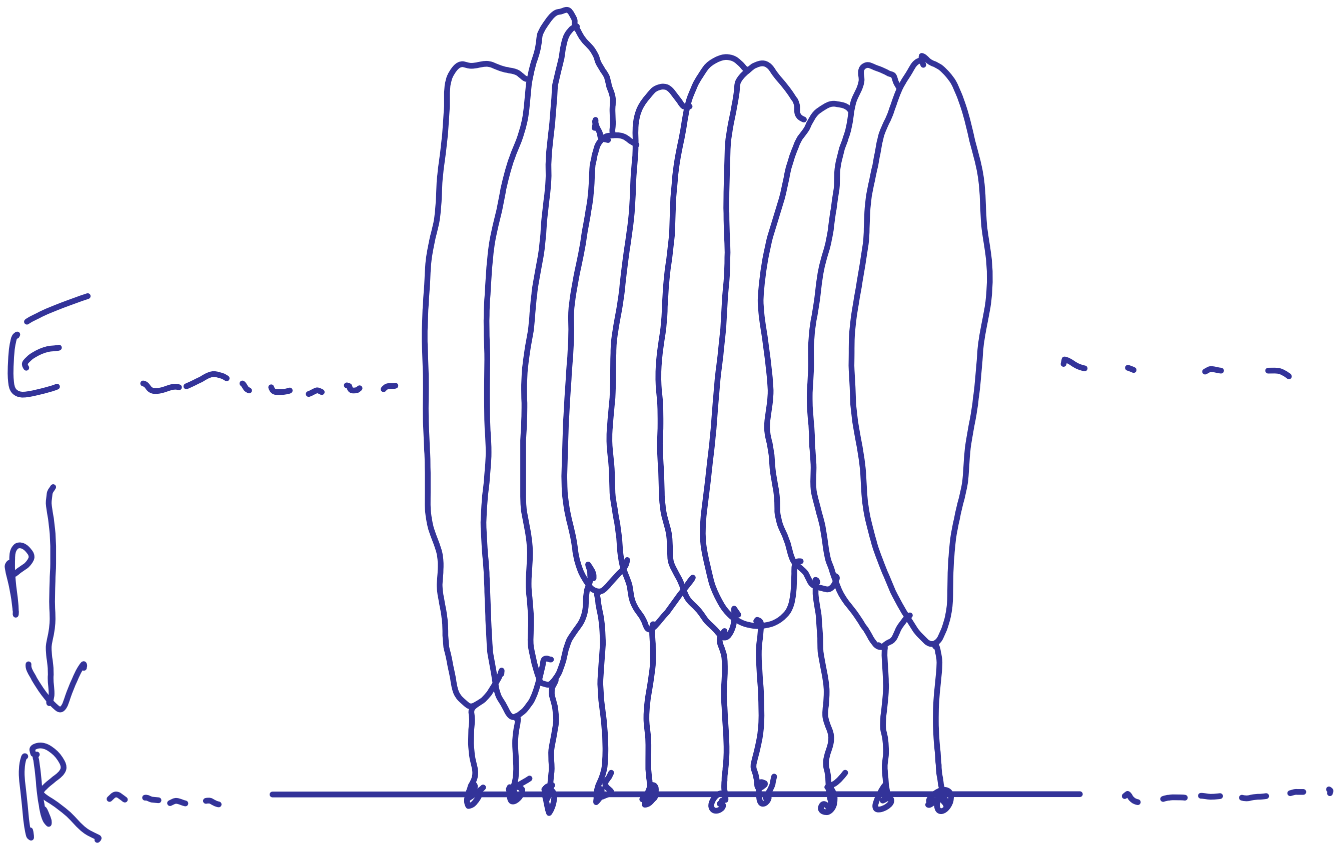
context declares generic base point $x : B$

construction in fibre over $x$

# Fibrewise topology – can it work?

e.g. $x : \mathbb{R} \mapsto E(x)$ Space

describes fibres




- Each fibre has topology
- Union of fibres = bundle space $E$
- $E$ needs a topology

– Where does it come from?

# Fibrewise topology in topos theory

All spaces now point-free

Topos of sheaves $\mathcal{S}B$ =

classifying topos for points of $B$:

$x : B \vdash$ internal maths of $\mathcal{S}B$

generic point of $B$

geometric

no exponentials!

Joyal Tierney

Localic bundle theorem

Internal locale $\simeq$ bundle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$

$x : B \vdash E(x)$ Space

Logical constraints

$\Downarrow$

topology on $E$

Question:

somehow

Suppose $X$ Space $\mapsto H(X)$ Space

What kind of constructions

- are preserved by substitution?
- are preserved by pullback of bundles?
- "work fibrewise"?

geometric

# Topos formulation of question

Take point-free space = frame (dually)

Topos $\mathcal{E}$ has $Fr_{\mathcal{E}}$ = cat. of internal frames

elementary, nno

Geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$

$$\begin{tikzcd} \cdot \arrow[r, shift left, "f_*"] \arrow[r, phantom, "\perp"] & \cdot \arrow[l, shift left, "f^*"] \end{tikzcd}$$

$$\begin{tikzcd} Fr_{\mathcal{E}} \arrow[r, shift left, "f_*"] \arrow[r, phantom, "\perp"] & Fr_{\mathcal{F}} \arrow[l, shift left, "f^{\#}"] \end{tikzcd}$$

reindexing along $f$

pullback of bundles

$$\begin{tikzcd} f^{\#}\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d] \\ \mathcal{E} \arrow[r, "f"'] & \mathcal{F} \end{tikzcd}$$

# Fr indexed over Top

Toposes
geometric morphisms

$$\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \qquad \mathrm{Fr}_{\mathcal{E}} \xleftarrow{f^\#} \mathrm{Fr}_{\mathcal{F}} \xleftarrow{g^\#} \mathrm{Fr}_{\mathcal{G}}$$

$$(gf)^\# \cong f^\# g^\#$$ coherently

Suppose $H_{\mathcal{E}} : \mathrm{Fr}_{\mathcal{E}} \to \mathrm{Fr}_{\mathcal{E}}$ for all $\mathcal{E}$

When is $H$ an indexed endofunctor of Fr?

geometricity of $H$

$$f^\# H_{\mathcal{F}} \cong H_{\mathcal{E}} f^\#$$ coherently

substitution along $f$

$X$ space $\mapsto$ $H(X)$ space

# Sufficient conditions for indexed endofunctor H

Hybrid: Half predicative, half impredicative

Predicative half

- Work geometrically on frame presentations

AUs

- Replace toposes by arithmetic universes

  ..... geometric maths .... arithmetic maths

- Use universal algebra of AUs to describe H as a single generic construction

- Can then be specialized to any topos & space, automatically indexed for presentations

Substitution

# Impredicative half

- Relate frames (impredicative) to presentations (predicative)
- Show automatic indexing on presentations
  $\to$ indexing on frames
- Need condition that:
  if $Q, Q'$ present isomorphic frames
  so do $H(Q), H(Q')$

# Frame presentations

finite powerset

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ \mathcal{P}_f L & R \arrow[l, "\lambda"] \end{tikzcd} \qquad D = \sum_r D_r$$

$$\mathrm{Fr}\left\langle L \;\middle|\; \bigwedge \lambda(r) \leq \bigvee_{d \in D_r} \bigwedge \rho(d) \quad (r \in R) \right\rangle$$

# DL-site

geometric theory

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ L & R \arrow[l, "\lambda"] \end{tikzcd} \qquad D = \sum_r D_r$$

$$\mathrm{Fr}\langle L \text{ (qua DL)} \mid \lambda(r) \leq \bigvee_{d\in D_r} \rho(d) \quad (r \in R)\rangle$$

- $L$ a DL
- $D$ a poset, $\rho, \pi$ monotone — wrt $(R, =)$

  each fibre of $\pi$ directed
- $\wedge : R \times L \to R$, $\wedge : D \times L \to D$

  $\pi(d \wedge x) = \pi(d) \wedge x$, $\rho(d \wedge x) = \rho(d) \wedge x$

  $\lambda(d \wedge x) = \lambda(d) \wedge x$

  meet stability
- Similar "join stability"

NB Frame presented $\cong \mathrm{dcpo}\langle L \text{ (qua poset)} \mid \text{same relations}\rangle$

dcpo coverage theorem

DLS

**Morphisms** $(L, R, D) \to (L', R', D')$

$\theta_L : L \to L'$

$\theta_R : R \to R'$

$\theta_D : D \to D'$

- preserve all structure (e.g. $\theta_L$ a DL-homomorphism)

Another geometric theory: 2 DL-sites + morphism

$+ \quad \theta_D$ fibrewise surjective

$$\pi'(d') = \theta_R(r) \to \exists d.\,(\pi(d) = r \wedge \theta_D(d) = d')$$

Category $DLS_{\mathcal{E}}$ for any topos $\mathcal{E}$.

# Adjunction

$$\mathfrak{K}_{32}(L,R,D) = \mathrm{Fr}\langle L \text{ (qua DL)} \mid \lambda(r) \le \bigvee^{\uparrow}_{\pi d = r} \rho(d) \ (r \in R) \rangle$$

$$\begin{tikzcd} \mathrm{Fr}_{\mathcal{E}} \arrow[d, "\mathfrak{K}_1", shift left=2] \arrow[d, phantom, "\dashv"] \\ \mathrm{DLS}_{\mathcal{E}} \arrow[u, "\mathfrak{K}_{32}", shift left=2] \end{tikzcd}$$

$\mathfrak{K}_1(A) =$ canonical presentation

$L = A$

$R = \{(a,S) \mid a \in A,\ S \subseteq A \text{ directed},\ a \le \bigvee^{\uparrow} S\}$

Counit $\varepsilon$ an iso ($\mathfrak{K}_1(A)$ presents $A$)

Unit $\eta$ not an iso – but $\mathfrak{K}_{32}\eta$ is

$$\begin{tikzcd}
\mathrm{Fr}_{\mathcal{E}} \arrow[rr, shift left, "f_*"] \arrow[d, shift left, "K_1"] & & \mathrm{Fr}_{\mathcal{F}} \arrow[ll, shift left, "f^\#", "\top"'] \arrow[d, shift left, "K_1"] \\
\mathrm{DLS}_{\mathcal{E}} \arrow[u, shift left, "K_{32}"] \arrow[u, phantom, "\dashv"] & & \mathrm{DLS}_{\mathcal{F}} \arrow[ll, "f^*"] \arrow[u, shift left, "K_{32}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

DLS strictly indexed using $f^*$

① $K_{32} f^* K_1 \dashv f_*$

$\therefore$ take $f^\# = K_{32} f^* K_1$

$$
\begin{tikzcd}
\mathrm{Fr}_{\mathcal{E}} \arrow[rr, "f_*"] \arrow[d, shift left=2, "K_1"] & & \mathrm{Fr}_{\mathcal{F}} \arrow[d, shift left=2, "K_1"] \\
\mathrm{DLS}_{\mathcal{E}} \arrow[u, shift left=2, "K_{32}"] \arrow[u, phantom, "\dashv"] & & \mathrm{DLS}_{\mathcal{F}} \arrow[ll, "f^*"] \arrow[u, shift left=2, "K_{32}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

② $f^*$ has Kleisli lifting

$\therefore$ $f^*$ preserves Kleisli isos (e.g. $\eta$)

# The predicative part: AUs

Joyal
Maietti
Maietti-Vickers

AU =

pretopos – finite limits
– stable disjoint finite coproducts
– stable effective quotients of equivalence relations

in fact all stable finite colimits in AU case

+ parametrized list objects

$$\begin{tikzcd} 1 \arrow[r, "\varepsilon"] & \mathrm{List}A & \mathrm{List}A \times A \arrow[l, "\mathrm{snoc}"'] \end{tikzcd}$$

$$\begin{tikzcd} & B \times \mathrm{List}A \arrow[d, "\exists!"', "\mathrm{rec}(b,g)"] & B \times \mathrm{List}A \times A \arrow[l, "B \times \mathrm{snoc}"'] \arrow[d, "\mathrm{rec}(b,g) \times A"] \\ B \arrow[ur, "\langle B, \varepsilon \rangle"] \arrow[r, "b"'] & Y & Y \times A \arrow[l, "g"] \end{tikzcd}$$

## Example

Any elementary topos $\mathcal{E}$ with nno is an AU

If $\mathcal{E} \xrightarrow{f} \mathcal{F}$ a geometric morphism then $f^*: \mathcal{F} \to \mathcal{E}$ is a (non-strict) AU-functor

# Theory of AUs is cartesian essentially algebraic

- Two sorts (object, morphism)
- Operations for domain, codomain, identities, composition, terminal, initial, pullback, pushout, list object, fillins, ... some partial
- Various equations

Hence can present by generators & relations.

# AU-sketches

- Graph of nodes & edges — modeled as objects & morphisms
- Diagrams — specify equations between paths
- Limit cones / Colimit cocones — specify objects as limits/colimits, with specified projections/injections
- List universals — specify objects as list objects

# Sketches present AUs

- by universal algebra of cartesian theories

$$
\begin{tikzcd}
\mathbb{T} \arrow[r, dashed] \arrow[dr, dashed, "\text{model of } \mathbb{T} \text{ in } \mathcal{A}"'] & AU\langle\mathbb{T}\rangle \arrow[d, "\exists!"', "\text{AU functor}"] \\
 & \mathcal{A}
\end{tikzcd}
$$

AU freely generated by generic model of $\mathbb{T}$

BUT algebra relates to strict AU functors

– preserve limits etc on the nose

Models of sketches normally understood non-strictly

# AU-contexts

Want each non-strict model
$\cong$ unique strict one.

Key coherence result

Context = sketch built by following steps.

1. Add fresh node
2. Add fresh edge
3. Add new diagram
4. Add new universal, in which the subjects are fresh

between old nodes

using old nodes & edges

Ensures equations of objects are definitional in nature.

## Universals - for limits, colimits, list objects

e.g. pullback :

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

subjects $P, p, q$

Specifies $P$ pullback, $p, q$ projections

e.g. list universal : subjects $T, L, P, \varepsilon$, snoc, $p, q$

$$\begin{tikzcd} T \arrow[r, "\varepsilon"] & L & P \arrow[l, shift right, "\mathrm{snoc}"'] \arrow[l, shift left, "p"] \arrow[r, "q"] & A \end{tikzcd}$$

Specifies
$T$ terminal,
$L$ List$(A)$,
$P$ $L \times A$,
+ relevant structure maps

e.g. $\exists$ AU-context $DLS_0$

---

model = DL-site

For simplicity overlook morphisms here

Topos $\mathcal{E}$ (elementary + nno) is an AU

$DLS_{\mathcal{E}}$ = category of strict models of $DLS_0$ in $\mathcal{E}$

$\simeq AU_s[AU\langle DLS_0\rangle, \mathcal{E}]$ - strict AU-functors

# Reindexing along $\mathcal{E} \xrightarrow{f} \mathcal{F}$

$f^*$ is non-strict AU-functor

$\therefore$ converts strict $DLS_0$ models to non-strict

$$
\begin{tikzcd}
\mathcal{E} & \mathcal{F} \arrow[l, "f^*"'] & AU\langle DLS_0 \rangle \arrow[l, "Q"'] \arrow[ll, bend left=20, "f^*Q"'] \arrow[ll, bend left=40, phantom, "\cong"] \arrow[ll, bend left=50]
\end{tikzcd}
$$

by property of contexts

$$
\therefore \quad DLS_{\mathcal{E}} \xleftarrow{\;f^*\;} DLS_{\mathcal{F}}
$$

Strict indexing : $(gf)^* = f^* g^*$

# Generic constructions

- $AU\langle DLS_0\rangle$ constructed out of generic DL-site $Q_g$
- Suppose $H$ a model of $DLS_0$ in $AU\langle DLS_0\rangle$
- hence strict AU-endofunctor of $AU\langle DLS_0\rangle$
- $H$ is a single generic construction, but can be specialized to any specific DL-site

$$
\begin{tikzcd}
\mathcal{E} & & AU\langle DLS_0\rangle \arrow[ll, "Q"'] \\
& & AU\langle DLS_0\rangle \arrow[u, "H"'] \arrow[llu, "H(Q)"]
\end{tikzcd}
$$

all strict

# H is strictly indexed endofunctor

made strict

$$\mathcal{Y} \xleftarrow{Q} Au\langle DLS_0\rangle \xleftarrow{H} Au\langle DLS_0\rangle$$

$$\begin{tikzcd}
f^*Q \arrow[dddd, mapsto] & & & Q \arrow[lll, mapsto] \arrow[dddd, mapsto] \\
& DLS_{\mathcal{E}} \arrow[d, "H_{\mathcal{E}}"'] & & DLS_{\mathcal{Y}} \arrow[ll, "f^*"'] \arrow[d, "H_{\mathcal{Y}}"] & \\
& DLS_{\mathcal{E}} & & DLS_{\mathcal{Y}} \arrow[ll, "f^*"] & \\
& & & & \\
f^*QH & & & QH \arrow[lll, mapsto]
\end{tikzcd}$$

made strict

Example Double powerlocale

$\mathbb{P} = P_U P_L$ localic hyperspaces

On frames: $A \mapsto \mathrm{Fr}\langle A \text{ (qua dcpo)}\rangle$

On presentations:

First, from $(L, R, D)$ get

$$\mathrm{Fr}\langle L \text{ (qua poset)} \mid \lambda(r) \leq \bigvee^{\uparrow}_{\pi d = r} \rho(d) \quad (r \in R)\rangle$$

instead of DL

Next, complete to DL-site $(L', R', D')$

$L' = DL\langle L \text{ (qua poset)}\rangle$ etc.

# Conclusions

For construction $X : \mathrm{Space} \vdash H(X)\ \mathrm{Space}$

Aim: a single generic construction,
specialized by substitution

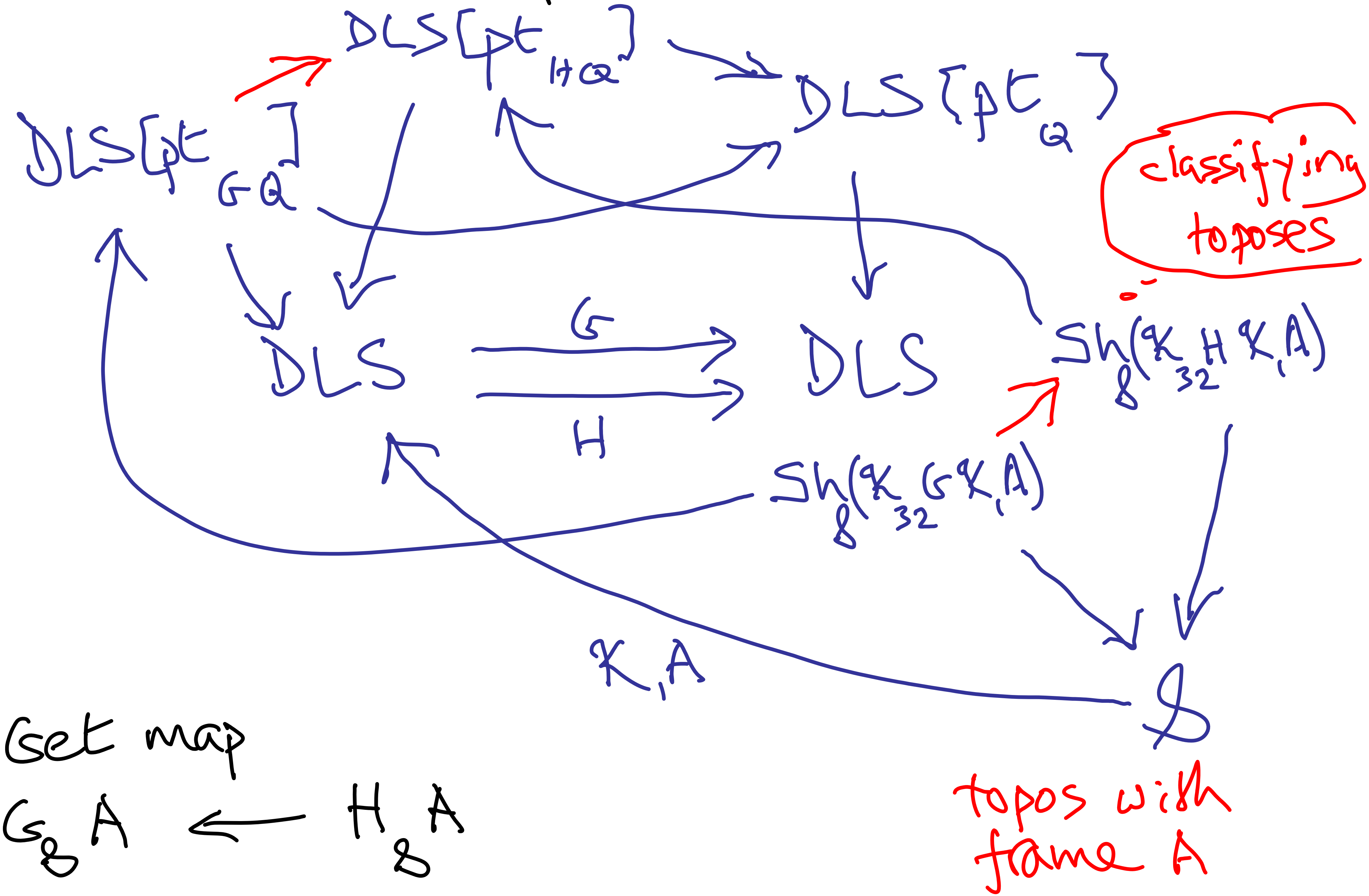
Algebra of AUs allows this predicatively.

Impredicative arguments transfer to toposes,
using frames.

# Post conclusion: points and maps

$$\begin{tikzcd}
 & \mathrm{DLS}[pt_{HQ}] \arrow[r] \arrow[d] & \mathrm{DLS}\{pt_{Q}\} \arrow[dd] \\
\mathrm{DLS}[pt_{GQ}] \arrow[ur] \arrow[dr] \arrow[urr, bend right] & \\
 & \mathrm{DLS} \arrow[r, shift left, "G"] \arrow[r, shift right, "H"'] & \mathrm{DLS}
\end{tikzcd}$$

# Post conclusion: points and maps




Get map

$G_{\otimes} A \longleftarrow H_{\otimes} A$