Grothendieck toposes
fibred over elementary toposes

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Grothendieck toposes as generalized spaces?
Classifying topos S[T] = "space of models of T"
- but depends on choice of elementary topos S.
For some T, can use any S with nno.
Show how construction S[T] can be fibred over 2-category of S's.

Vickers:
"Arithmetic universes and classifying toposes" (arXiv)

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Grothendieck topos = generalized point-free space

Ungeneralized: locale X

Frame = algebraic theory of opens

X -> Sierpinski $

Lattice, finite /\, arbitrary \/

Map = function (backwards) preserving those

Generalized: topos X

Grothendieck topos = algebraic theory of sheaves (local homeomorphisms)

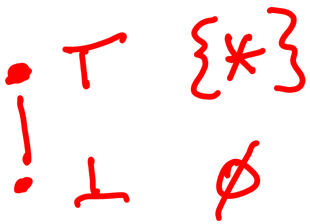
X -> {sets}

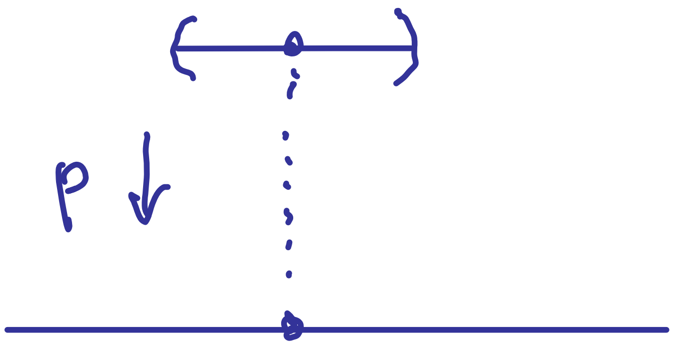
Category, finite limits, arbitrary colimits

Map = functor (backwards) preserving those

~ geometric morphism

X -> $

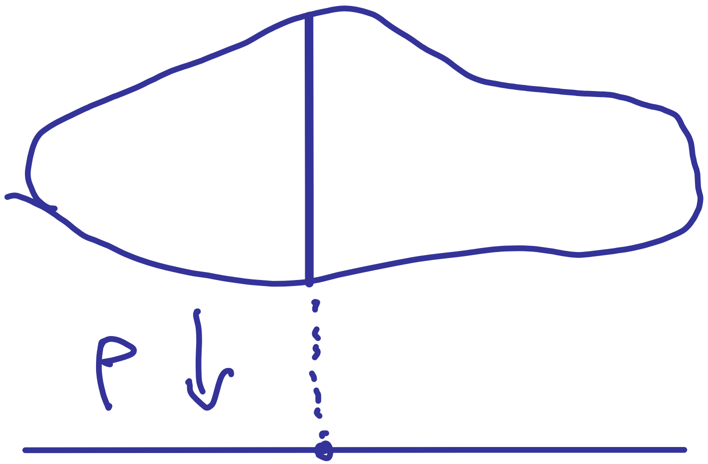




X

X -> {sets}

x |-> fibre



Presentations: Geometric theories $\mathbb{T}$

generators = signature: sorts, functions, predicates
relations = axioms

$\phi(x_1, \ldots x_n) \vdash \psi(x_1, \ldots x_n)$

formulae built with $\wedge \vee = \exists$

Ungeneralized: propositional
no sorts,
signature just propositional symbols

Present frame by generators and relations:
Lindenbaum algebra
= formulae modulo equivalence

Generalized: predicate

Grothendieck topos generated using finite limits, arbitrary colimits "making axioms hold"
= classifying topos $\mathbf{Set}[\mathbb{T}]$

Injection of generators gives generic model of theory.

Example: "space of sets" (object classifier)

Theory $\mathbb{O}$ one sort, nothing else.

Classifying topos $Set[\mathbb{O}] = [Fin, Set]$

Conceptually object = continuous map {sets} -> {sets}
Continuity is (at least) functorial + preserves filtered colimits
Hence functor {finite sets} -> {sets}

Generic model is the subcategory inclusion Inc: Fin -> Set

Example: "space of pointed sets"

Theory $\emptyset, pt$ one sort X, one constant x: 1 -> X.

Classifying topos $\mathbf{Set}[\emptyset, pt] \simeq [\mathrm{Fin}, \mathrm{Set}]/\mathrm{Inc}$

In slice category: 1 becomes Inc, Inc becomes Inc x Inc

Generic model is Inc with

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{id}"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

1 in slice

Inc in slice

Suppose you don't like Set?

the base topos

$\mathcal{S}$

Replace with your favourite elementary topos S.
Needs nno N.

Fin becomes internal category in S.

Finite functions
f: m -> n

$$\begin{tikzcd} Fin_1 \arrow[r, shift left, "dom"] \arrow[r, shift right, "cod"'] & Fin_0 = N \end{tikzcd}$$

n = {0, ..., n-1}

Classifying topos becomes
- category of internal diagrams on Fin

$$\mathcal{S}[\mathbb{O}] = [Fin, \mathcal{S}]$$

(f: m -> n, x in X(m))

X(f)(x) in X(n)

$$\begin{tikzcd} dom^* X \arrow[r, shift left] \arrow[r, shift right, bend right] \arrow[d] & X \arrow[d] \\ Fin_1 \arrow[r, shift left, "dom"] \arrow[r, shift right, bend right, "cod"'] & N \end{tikzcd}$$

X(n) = fibre over n

Other classifier is slice, as before.

Roles of S

(1) Supply infinities for infinite disjunctions:
get theories T geometric over S.

(2) Classifying topos built over S: geometric morphism $\mathcal{S}[\mathbb{T}] \to \mathcal{S}$

Suppose T has disjunctions all countable

It's geometric over any S with nno.

But different choices of S give different classifying toposes.

Idea: describe construction fibred over 2-category of S's,
describe change of base along geometric morphisms.

$\mathrm{Top}_{\cong}$

0-cells: elementary toposes with nno
1-cells: geometric morphisms
2-cells: natural isomorphisms

Indexed categories

$$\mathcal{B}^{op} \xrightarrow{F} \mathrm{Cat}$$

F(X) is fibre over X

$f^* = F(f): F(Y) \to F(X)$ is
reindexing along $f: X \to Y$

In general, pseudofunctor -

F(f;g) only isomorphic to
F(f) o F(g).

Need coherence conditions.

Fibrations

- Grothendieck construction

Bundle fibres together
to make total category C.

object - (X in B, x in F(X))
morphism - (f, u) $x \xrightarrow{u} f^*y$

$$\begin{tikzcd}
x \arrow[r, "u"'] \arrow[d, dotted] & f^*y \arrow[dl, dotted] & y \arrow[d, dotted] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Composite of:
vertical (Id_X, u)
- f part is identity

cartesian (f, Id_{f*y})
- u part is iso

A cartesian lifting
of f to y.

Abstract essence of cartesianness of f

For all g, for all h_, there is unique h:

$$\begin{tikzcd}
z \arrow[r, dashed, "h"] \arrow[rr, bend left, "g"] \arrow[d, dotted] & x \arrow[r, "f"'] \arrow[d, dotted] & y \arrow[d, dotted] \\
Z \arrow[r, "\underline{h}"'] \arrow[rr, bend left, "\underline{g} = p(g)"] & X \arrow[r, "\underline{f} = p(f)"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "p"] \\
\mathcal{B}
\end{tikzcd}$$

If $x \cong \underline{f}^* y$

$z \to \underline{g}^* y \cong \underline{h}^* \underline{f}^* y \cong \underline{h}^* x$

defines h

Indexed 2-categories?

$\mathcal{B}^{op} \xrightarrow{F} 2\text{-}\mathcal{C}at$

F(X) is fibre over X

f* = F(f): F(Y) -> F(X) is
reindexing along f: X -> Y?

In general, pseudo2-functor -?

F(f;g) only isomorphic to
F(f) o F(g).

Need coherence conditions.??

2-Fibrations

- Grothendieck construction

Won't try to explain coherence conditions for indexed 2-categories.

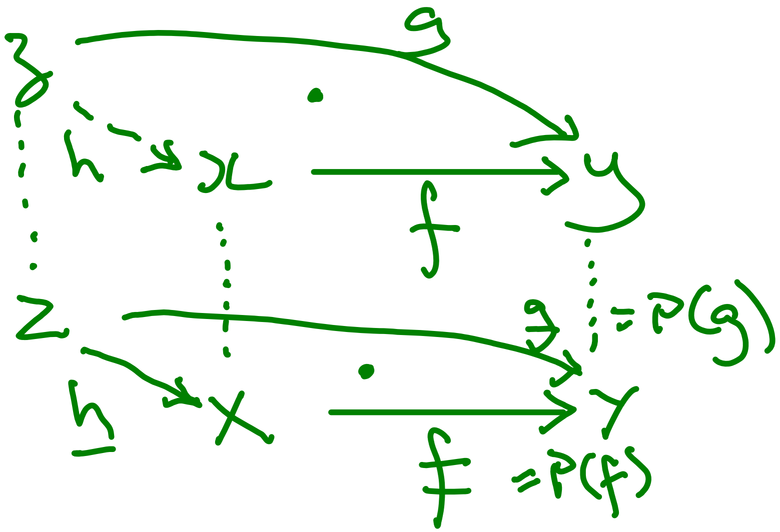
Instead go straight to 2-fibrations.

Buckley "Fibred 2-categories and bicategories"
- following Hermida, Bakovic

Cartesian 1-cells

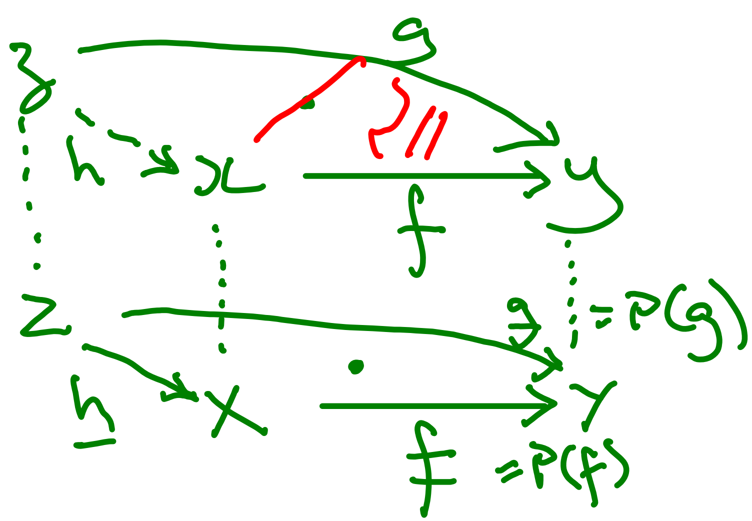
Strong version for 2-categories

For all g, for all h_, there is unique h:



Weak version for bicategories

For all g, for all h_, there is ~~unique~~ h:



I need both, even though I only have 2-functors between 2-categories.

For 2-cells: basically look at P_xy: C(x,y) -> B(P(x), P(y))

Grothendieck toposes over S

$$\begin{array}{c}\mathcal{E}\\ {\scriptstyle p}\downarrow\\ \mathcal{S}\end{array}$$ classifies some T (over S) iff p is **bounded** geometric morphism

$$\exists A \text{ in } \mathcal{E}\ \forall B \text{ in } \mathcal{E}\ \exists C \text{ in } \mathcal{S}$$

$$\begin{tikzcd} A \times p^* C & & B \\ & \cdot \arrow[ul] \arrow[ur] & \end{tikzcd}$$

GTop(S): 0-cell = bounded g.m. to S.
1-cell = triangle with iso in it.
2-cell = n.t. pasting correctly with isos.

$$\begin{tikzcd} \mathcal{E}' \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Uparrow"] \arrow[dr, "p'"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{S} \arrow[u, phantom, "\cong\ \cong"] & \end{tikzcd}$$

Reindexing along f_: S' -> S:
  pseudopullback

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{S}' \arrow[r, "f"] & \mathcal{S} \end{tikzcd}$$

In fibred form GTop

$$
\begin{tikzcd}
\mathcal{E}' \arrow[rr, bend left, "\bar{g}"] \arrow[rr, bend right, "\bar{f}"'] \arrow[rr, phantom, "\Uparrow"] \arrow[d, "p'"'] \arrow[drr, phantom, "\cong"] & & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}' \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\wr\Vert"] & & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
GTop^{co} \arrow[d, "Q"] \\
Top^{co}_{\cong}
\end{tikzcd}
$$

2-fibration in bicategorical (weak) sense

1-cell cartesian iff it's a pseudoullback square
2-cell cartesian if upstairs n.t. is iso

Models of T $\quad \mathcal{E} \longmapsto \mathcal{E}\text{-Mod-}\mathbb{T}$

category of models of T in E

Reindexing $\quad f : \mathcal{E}' \longrightarrow \mathcal{E}$ $\qquad$ $\begin{tikzcd} \mathcal{E}' \arrow[r, shift left, "f_*"] \arrow[r, phantom, "\top"] & \mathcal{E} \arrow[l, shift left, "f^*"] \end{tikzcd}$

geometric morphism

$$\mathcal{E}'\text{-Mod-}\mathbb{T} \xleftarrow[\text{apply } f^*]{} \mathcal{E}\text{-Mod-}\mathbb{T}$$

2-functor $\quad \mathrm{Top}^{op} \longrightarrow \mathrm{Cat}$

$$\mathrm{Top}^{op} = (\mathrm{Top}^{co})^{coop}$$

Minor annoyance:
Standard treatment expects contravariance on both 1-cells (op) and 2-cells (co).

Strictness

Vickers:
"Sketches for arithmetic universes"
(arXiv)

T = "sketch for arithmetic universes"
- interpretable in any elementary topos with nno

Have strict or non-strict models
- depending on interpretation of pullbacks etc.
- (canonical or arbitrary?)

Also T a "context"
- built so every non-strict model has a canonical strict isomorph

E-Mod-T is strict models.
Reindexing first makes a non-strict model
(f* doesn't preserve structure on the nose)
then takes strict isomorph.

E |-> E-Mod-T is strict indexation over Top^op
- 2-functor Top^op -> Cat

Grothendieck construction

$$
\begin{tikzcd}
\mathbf{Top}\text{-}\mathbb{T} \arrow[d, "p^{co}"'] \\
\mathbf{Top}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & g^*M \\
M' \arrow[ur, "v"] \arrow[dr, "u"'] & \\
 & f^*M \arrow[u, Rightarrow, "\alpha^*M"']
\end{tikzcd}
\qquad M
$$

$$
\begin{tikzcd}
\mathcal{E}' \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Uparrow\alpha"] & & \mathcal{E}
\end{tikzcd}
$$

Extend to GTop. Base toposes S don't interact with models M.

$$
\begin{tikzcd}
\mathbf{GTop}\text{-}\mathbb{T} \arrow[d, "p^{co}"'] \\
\mathbf{GTop}
\end{tikzcd}
$$

Split 2-fibration
(in 2-category sense)

Classifying toposes

Classifying topos
Generic model

Fix S. Classifier (S[T],G) is representing object for indexed category

$$\mathrm{GTop}(\mathcal{S})^{op} \xrightarrow{\bullet\text{-Mod-}\mathbb{T}} \mathrm{Cat}$$

$$\begin{aligned}\mathrm{GTop}(\mathcal{S})(\mathcal{E}, \mathcal{S}[\mathbb{T}]) &\longrightarrow \mathcal{E}\text{-Mod-}\mathbb{T} \\ f &\longmapsto f^*G\end{aligned}$$

is an equivalence
(full, faithful, essentially surjective).

Hence: classifying topos describes models everywhere (all E).
Hence: it represents "space of models of T".

Viewed via fibration?

Fix S. Classifier (S[T],G) is representing object for indexed category

$$\begin{array}{c} (GTop(S)\text{-}T)^{co} \\ \Big\downarrow p_S \\ (GTop(S))^{co} \end{array}$$

$(S[T], G)$ is 0-cell

$$GTop(S)(\mathcal{E}, S[T]) \longrightarrow \mathcal{E}\text{-Mod-}T$$
$$f \longmapsto f^* G$$

is an equivalence
(full, faithful, essentially surjective).

$\forall (\mathcal{E}, N)\ \exists f.\ N \cong f^* G$

0-cell of $(GTop(S)\text{-}T)^{co}$

1-cell, $u : N \to f^* G$

$\forall (\mathcal{E}, N)\ \exists (f, u) : (\mathcal{E}, N) \to (S[T], G)$
with (f, u) cartesian

u an iso

Viewed via fibration?

~~Fix S. Classifier~~ (S[T],G) is representing object for indexed category

$$\begin{array}{c} (\mathbf{GTop}(\mathcal{S})\text{-}\mathbb{T})^{co} \\ {\scriptstyle p_{\mathcal{S}}}\big\downarrow \\ (\mathbf{GTop}(\mathcal{S}))^{co} \end{array}$$

$(\mathcal{S}[\mathbb{T}], G)$ is 0-cell

$$\mathbf{GTop}(\mathcal{S})(\mathcal{E}, \mathcal{S}[\mathbb{T}]) \longrightarrow \mathcal{E}\text{-}\mathbf{Mod}\text{-}\mathbb{T}$$
$$f \longmapsto f^*G$$

is an equivalence
(full, faithful, essentially surjective).

$\forall\ \omega : f^*G \to g^*G$ in $\mathcal{E}$

$\exists!\ \alpha : f \to g\ .\ \omega = \alpha^*G$

If $(f,u) : (\mathcal{E}, N) \longrightarrow (\mathcal{S}[\mathbb{T}], G)$ cartesian, then it's terminal in

$$(\mathbf{GTop}(\mathcal{S})\text{-}\mathbb{T})^{co}((\mathcal{E}, N), (\mathcal{S}[\mathbb{T}], G))$$

Fibration representable:

$$\begin{array}{c} (\mathcal{G}\mathrm{Top}(\mathcal{S})\text{-}\pi)^{co} \\ \quad \downarrow p_{\mathcal{S}} \\ (\mathcal{G}\mathrm{Top}(\mathcal{S}))^{co} \end{array}$$

$\exists$ 0-cell $(\mathcal{S}[\mathbb{T}], \mathcal{G})$

representing object

$\forall\, (\mathcal{E}, N)\ \exists\, (f,u) : (\mathcal{E}, N) \to (\mathcal{S}[\mathbb{T}], \mathcal{G})$
with (f,u) cartesian

If $(f,u) : (\mathcal{E}, N) \to (\mathcal{S}[\mathbb{T}], \mathcal{G})$ cartesian,
then it's terminal in
$(\mathcal{G}\mathrm{Top}(\mathcal{S})\text{-}\mathbb{T})^{co}((\mathcal{E}, N), (\mathcal{S}[\mathbb{T}], \mathcal{G}))$

What if S varies?

$$
\begin{array}{c}
(GTop\text{-}\Pi)^{co} \\
P \big\downarrow \\
(GTop)^{co} \\
Q \big\downarrow \\
Top_{\simeq}
\end{array}
\qquad\qquad
\begin{tikzcd}
 & \boxed{\begin{array}{c} (GTop(\mathcal{S})\text{-}\Pi)^{co} \\ P_{\mathcal{S}} \big\downarrow \\ (GTop(\mathcal{S}))^{co} \end{array}} \\
\mathcal{S} \arrow[ur, mapsto] &
\end{tikzcd}
$$

fibre over $\mathcal{S}$

$P_{\mathcal{S}}$ representable

Want good behaviour under change of base f: S' -> S:

classifiers transform by pseudopullback.

"Local representability" for P over Q

$$\begin{tikzcd} (\mathrm{GTop}\text{-}\mathbb{T})^{co} \arrow[d, "P"'] \\ (\mathrm{GTop})^{co} \arrow[d, "Q"'] \\ \mathrm{Top}_{\cong} \end{tikzcd}$$

- $\forall \mathcal{S}$, $P_{\mathcal{S}}$ is representable
- Suppose:
  $(\mathcal{S}[\mathbb{T}], G)$ representing object for $P_{\mathcal{S}}$
  $f: \mathcal{S}' \to \mathcal{S}$
  $f: (\mathcal{E}, N) \to (\mathcal{S}[\mathbb{T}], G)$ over f is cartesian with respect to $P;Q$
  Then $(\mathcal{E}, N)$ representing object for $P_{\mathcal{S}'}$

Theorem If T is an AU context, then P is locally representable over Q.

"Classifiers exist and are preserved by pseudopullback."

More generally $\mathbb{T}_0 \subset \mathbb{T}_1$ "extension of contexts"

For $p \downarrow$ consider models of $\mathbb{T}_1$ / $\mathbb{T}_0$ such that $\mathbb{T}_0$ reduct of N $= p^*M$

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"] \arrow[r, dotted, no head] & N \\
\mathcal{F} \arrow[r, dotted, no head] & M
\end{tikzcd}$$

Given M, get geometric theory $\mathbb{T}_1/M$ of N's

Theorem Pseudopullback along f transforms $\mathcal{E}[\mathbb{T}_1/M]$ to $\mathcal{E}'[\mathbb{T}_1/f^*M]$

Example

Joyal and Tierney

$\mathbb{T}_0$ theory of frame presentations

$\mathbb{T}_1$ theory of frame presentation + point of corresponding locale

Vickers: The double powerlocale and exponentiation

$\mathcal{S}[\mathbb{T}_1/M]$ = topos of sheaves for internal frame $Fr\langle M\rangle$ in $\mathcal{S}$

Theorem ⇒ geometricity of presentations

Two processes equivalent:

1. Make topos of sheaves then pseudopullback
2. Apply inverse image functor to presentation then make topos of sheaves

Conclusions

For a geometric theory T that comes from an AU context:

"Space of models" as classifying topos S[T] is not well defined - depends on choice of base topos S.

Can fibre it over 2-category of possible S's.
Well behaved under change of base - pseudopullback of classifiers.

However: alternative lesson is that the AU workings already provide a good notion of "space of models".
Context T presents AU<T>, analogous to the Grothendieck topos S[T], but without depending on choice of S.