The Born rule as structure of spectral bundles (extended abstract)

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Topos approaches to quantum foundations are described in a unified way by means of spectral bundles, where the base space is a space of contexts and each fibre is its spectrum. Differences in variance are due to the bundle being a fibration or opfibration. Relative to this structure, the probabilistic predictions of the Born rule in finite dimensional settings are then described as a section of a bundle of valuations. The construction uses in an essential way the geometric nature of the valuation locale monad.

1 Introduction

Two topos approaches to quantum foundations, [7] and [12][13], describe a quantum system given algebraically (as von Neumann algebra or C*-algebra respectively) as a topos combined with a space defined internally "in" the topos. However, toposes present many difficulties to the beginner: their basic definitions are non-trivial, and many important parts of topos-theory relate only indirectly to the basic definitions. The aim of this paper is to describe the topos approaches – both existing and prospective – in terms of "spectral bundles" $p : \Sigma \to B$.[1] We shall refer to the base points C in B as *contexts*, or (to use a phrase of [7]) "classical points of view", and each fibre $C^*\Sigma = p^{-1}\{C\}$ as the *spectrum* of C, also written $\Sigma_C = \mathrm{Spec}(C)$. It is a "classical state space" from point of view C. As we shall see below, a canonical realization of this in a quantum situation is where C represents a commuting set of observables and then the state space is the set of their common eigenspaces. Each of the observables can be diagonalized with respect to those eigenspaces, and then the diagonal entries, the corresponding eigenvalues, are the measured values, while the resulting state is got by projecting to the corresponding eigenspace.

In the topos approaches, the topos is the topos $\mathscr{S}B$ of sheaves over B and the bundle corresponds to something internal in that topos: either an object or a point-free topological space (locale). Since our aim is to replace the language of toposes by that of spaces, one might wonder if there is still any need at all for topos theory in this topos approach. The key insight is that the study of *bundles over B* is equivalent to the study of *spaces internal in $\mathscr{S}B$*.[2] In other words, the study of bundles is just a version of topology, but different from ordinary classical topology since it has to be adapted to the non-classical internal mathematics of toposes. As we shall see, under certain logical constraints of *geometricity* this becomes equivalent to treating bundles as "fibrewise topology", ordinary topology but sprinkled with base point parameters C from B.

[1] We shall use the word "bundle" in a very general sense, of arbitrary map between two topological spaces (more precisely, the spaces need to be point-free, but the naive reader can largely ignore this issue). If "bundle" just means the same as "map" one might wonder why we should waste a second word on the same notion. However, they will carry different connotations. A bundle $p : X \to B$ is to be thought of as a family of spaces (the fibres $p^{-1}(\{b\})$) parameterized by base point b. This is exactly the view one has of, for example, the tangent bundle over a differentiable manifold.

[2] "Space" here implicitly means point-free.

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In the present paper this is the key to our description of the Born rule. We use a construction that, for any space X, gives a *valuation space* $\mathfrak{V}X$ whose points are the valuations (regular measures) on X. Since this is geometric, it can be applied fibrewise to any bundle $p : X \to B$ to give a valuation bundle $q : \mathfrak{V}_B X \to B$. A valuation on a fibre of the spectral bundle, in other words a valuation on the spectrum of some context, turns out to be exactly the kind of probability distribution that is expressed in the Born rule. It is topos theory – more precisely, the understanding of the logic of topos-internal mathematics and of the geometricity constraints – that gives us access to this fibrewise valuation space and allows us to infer that it has good properties.

For other treatments of the Born rule in the topos approach see [5, 6] .

1.1 States

Classical physics: Let us be clear about the notions of *state* that will concern us. In classical physics, it is assumed that (given a selection of observables O) there is a set Σ of (classically) *pure states* that determine the values of all the observables. Thus each observable O is realized mathematically as a function $\tilde{O} : \Sigma \to \mathbb{R}$. To measure O in state x is then to discover the value $\tilde{O}(x)$. In practice, as for instance in thermodynamics, we often do not have access to the exact pure states and are reduced to using probabilistic distributions. We shall write $\mathfrak{V}^{(1)}(X)$ for the space of probability valuations (regular measures with total mass 1) on X, assuming that X has appropriate structure (specifically: X will be a point-free topology). In that case O is also realized as $\mathfrak{V}^{(1)}(\tilde{O}) : \mathfrak{V}^{(1)}(\Sigma) \to \mathfrak{V}^{(1)}(\mathbb{R})$, taking *mixed states* (points of $\mathfrak{V}^{(1)}(\Sigma)$) to distributions of reals (points of $\mathfrak{V}^{(1)}(\mathbb{R})$). In other words, once $m \in \mathfrak{V}^{(1)}(\Sigma)$ is given, $\tilde{O}$ becomes a *random variable*. Although in practice the mixed states of this probabilistic approach may be the best we can do, it is nonetheless assumed that they do arise as probability distributions of pure states.

Quantum physics: In quantum physics, on the other hand, an observable O may be realized as a self-adjoint operator $\hat{O}$ on some Hilbert space $\mathscr{H}$, which we assume, for simplicity, is of finite dimension n. The spectral theorem then tells us that $\hat{O} = \sum_{i=1}^{m} \lambda_i P_i$ where the λ_is are the eigenvalues of $\hat{O}$ and the P_is form a complete (they sum to 1) set of mutually orthogonal projectors (self-adjoint idempotents), projecting onto the corresponding eigenspaces. The *quantum* pure states are now taken to be the non-zero vectors $|\phi\rangle \in \mathscr{H}$, modulo scalar multiplication (in other words the states are 1-dimensional subspaces, or *rays*, in $\mathscr{H}$). To measure O in state $|\psi\rangle$ is probabilistic. Its measured result is one of the eigenvalues λ_i, with probability $\frac{\langle\psi|P_i|\psi\rangle}{\langle\psi|\psi\rangle}$ according to the Born rule, and with resulting state $P_i|\psi\rangle$ according to the Lüders Principle. Quantum mixed states (probabilistic distributions over quantum pure states) are still of use, but now even the pure states have a probabilistic nature.

This raises the question of whether there might be *classically* pure states out of which the quantum pure states are mixed. The answer depends on the observables. If we consider O alone, then we can take $\Sigma = \{1, \ldots, m\}$ (for the m eigenvalues, taken as distinct). Then $\tilde{O}(i) = \lambda_i$ and $|\psi\rangle$ is the mixed state in which i has weight $\frac{\langle\psi|P_i|\psi\rangle}{\langle\psi|\psi\rangle}$. This extends to the situation where we have any collection of commuting observables, since they are simultaneously diagonalizable and there is a complete set of orthogonal projectors P_i ($1 \leq i \leq m$) such that each $\hat{O}$ can be expressed as $\sum_{i=1}^{m} \lambda_i P_i$ (although now we cannot necessarily assume that the λ_is are distinct).

However, the Kochen-Specker Theorem tells us that in general when we have *non*-commuting observables there is no possible space of classically pure states out of which the quantum pure states can be mixed.

We shall in general take "context" to mean some situation, such as a family of commuting observables, in which it is possible to find a classically pure state space – the spectrum of the context. A fundamental aspect (going back to [14] and [15]) of the topos approach to quantum physics is to work in a mathematics that works in contexts but in some sense works in them all at once. In this mathematics, though it is *logically* non-classical, there is some possibility of being *physically* classical.

1.2 Toposes and bundles

The big content of topos theory is that it provides a generalization of topological spaces, new spaces whose topological structure must be given by stipulating the *sheaves*, not just the opens. However, the topos approach to quantum physics as currently conceived uses only ungeneralized spaces (though point-free, as locales), and one of our aims here is to conduct our discussion in terms of those spaces.

What toposes bring is a more conscious use of sheaves, and in particular the ability to manipulate them by interpreting mathematics (subject to constructivist constraints) as the internal mathematics of $\mathscr{S}B$.

To get a feel for how this works, think of an open U of a space B as a "continuously varying truth value", parametrized by points b of B. The value is true when $b \in U$: so U is a generalized truth value that says not *whether* something is true, but *where*. There is an asymmetry, "locality of truth", deriving from the nature of openness: if the value is true at x then it is true throughout some neighbourhood of x. The same does not hold for falsehood, and this asymmetry shows up in the associated logic – negation cannot be a connective. Another way to see the situation, which respects this asymmetry, is as a "continuously varying subsingleton". If we write $*$ for the unique element of some standard singleton set, then the subsingleton is $\{*\}$ where $b \in U$, $\emptyset$ where $b \notin U$. Thus locality of truth translates into locality of existence of $*$. This view is simply that of the inclusion $U \hookrightarrow B$ as bundle, since the fibre at b is exactly the subsingleton described.

Sheaves can be understood as generalizing "continuously varying subsingletons" to "continuously varying sets". In bundle form, when one incorporates not only locality of existence of elements but also locality of equality between them, one gets the notion of *local homeomorphism* $X \to B$, the fibres being the continuously varying sets. (Note that the definition implies that the fibres, as subspaces of X, all have the discrete topology.)

The topos of sheaves over B is (equivalent to) the topos of local homeomorphisms with codomain B. Many mathematical constructions can also be carried through on sheaves, and this interpretation gives an *internal mathematics* of the topos. For reasons such as the locality of truth, it does not obey all reasoning principles of classical logic and set theory, but nonetheless it is intuitionistic. The methodology of manipulating sheaves by reasoning intuitionistically has proved an effective one. Moreover, there is a particularly important *geometric* fragment comprising those constructions that work fibrewise on the local homeomorphisms. For these the intuitionistic features are less obtrusive.

The final step is to move to general bundles as "continuously varying spaces". Just as local homeomorphisms are (by definition) "sets" in the internal mathematics, we should also like general bundles to be spaces there. This idea works well, with two provisos. First, the spaces need to be point-free, and, second, the logic needs to be geometric in order for it to work fibrewise, so that the whole construction varies continuously with the base points. Getting the mathematics to work within the geometricity constraints is non-trivial, and the present work provides a case study in a more general geometrization programme. However, it has the beneficial effect of enabling point-based reasoning for point-free topology.

The key result is what we shall call the *Localic Bundle Theorem* ([8], [18]), which states that frames in the internal mathematics of a topos are dual equivalent to localic geometric morphisms with that topos as codomain. Restricted to the case where the topos is that of sheaves over a locale B, these are equivalent to locale maps with codomain B, in other words localic bundles over B.

This has an important consequences for constructions on locales. If the construction is topos-valid then it can be applied to internal frames in toposes of sheaves and hence gives a construction on localic bundles. We shall be particularly interested in constructions that are *geometric* in the sense that, when applied to bundles, are preserved by pullback, since this implies that they work fibrewise.

To summarize: In this work we shall be working with bundles, and with constructions on spaces that can be applied fibrewise to the bundles. Although the details will be largely hidden, the justification for the fibrewise construction, and for choosing the appropriate topology on the bundle space, will be that the constructions work point-free in a way that is topos-valid and moreover geometric. This perspective on geometric logic is summarized in [23].

1.3 Bundles as fibrations and opfibrations

Given a bundle $p : \Sigma \to B$ one can ask how the fibres interact with specialization order amongst the base points: if $C \sqsubseteq D$, is there a corresponding map, in either direction, between the fibres $C^*\Sigma$ and $D^*\Sigma$? In general there is no such map, but there are special classes of bundles, known as fibrations and opfibrations, for which there are. The general theory [21] works in an arbitrary 2-category and has been examined in [16] in the 2-category of toposes. Our own interest is in the restriction to the 2-category of locales, where the category enrichment is in fact order enrichment, the specialization order on each homset: if $f, g : X \to Y$ then $f \sqsubseteq g$ if $f^*V \leq g^*V$ for all V open in Y.

For a fibration p, for $C \sqsubseteq D$ in B there is a map contravariantly between the fibres, $D^*\Sigma \to C^*\Sigma$, while for an opfibration it is covariant. Moreover, in both cases the fibre maps are characterized universally in a way that determines them uniquely. We shall describe this in a way that deals with the points $C \sqsubseteq D$ generically, allowing for generalized points. As we shall see later, these notions provide a fundamental explanation for the difference in variance seen in two topos approaches to quantum theory.

Such pairs $C \sqsubseteq D$ are classified by the exponential locale $B^{\mathbb{S}}$ where $\mathbb{S}$ is the Sierpinski locale with points $\bot \sqsubseteq \top$.[3] Given $f : \mathbb{S} \to B$, a point of $B^{\mathbb{S}}$, we have $f(\bot) \sqsubseteq f(\top)$ in B, and this gives two maps $\pi_\bot \sqsubseteq \pi_\top : B^{\mathbb{S}} \to B$. They are generic. For any other $C \sqsubseteq D : W \to B$, there is a unique $f : W \to B^{\mathbb{S}}$ such that $C = \pi_\bot \circ f$ and $D = \pi_\top \circ f$. This allows us to understand the points of $B^{\mathbb{S}}$ as the pairs $C \sqsubseteq D$. Because of this we don't ask about fibre maps for arbitrary pairs $C \sqsubseteq D$, but just for the generic pair $\pi_\bot \sqsubseteq \pi_\top$. A fibre map found there can then be pulled back for arbitrary $C \sqsubseteq D$.

Over $B^{\mathbb{S}}$ we have two bundles $\pi_\bot^*\Sigma$ and $\pi_\top^*\Sigma$, and so far there is no reason why there should be a map between them. However, we do have a span over them from the bundle $p^{\mathbb{S}} : \Sigma^{\mathbb{S}} \to B^{\mathbb{S}}$. For example, the commutative square

$$\begin{array}{ccc} \Sigma^{\mathbb{S}} & \xrightarrow{\pi_\bot} & \Sigma \\ p^{\mathbb{S}} \downarrow & & \downarrow p \\ B^{\mathbb{S}} & \xrightarrow[\pi_\bot]{} & B \end{array}$$

gives us a map $\rho_\bot : \Sigma^{\mathbb{S}} \to \pi_\bot^*\Sigma$ over $B^{\mathbb{S}}$. Similarly, for $\top$ we get a map $\lambda_\top : \Sigma^{\mathbb{S}} \to \pi_\top^*\Sigma$ over $B^{\mathbb{S}}$.

Definition 1 *With the notation as above, p is a* fibration *if $\lambda_\top$ has a right adjoint $\rho_\top$ over $B^{\mathbb{S}}$, with its counit an equality, and is an* opfibration *if $\rho_\bot$ has a left adjoint $\lambda_\bot$ over $B^{\mathbb{S}}$, with its unit an equality.*

[3] Potentially it has other points too, but they arise only in non-classical mathematics.

To say that $\rho_\top$ is right adjoint of $\lambda_\top$ is to say that $\mathrm{Id}_{\Sigma^{\mathbb{S}}} \sqsubseteq \rho_\top \circ \lambda_\top$ (the unit) and $\lambda_\top \circ \rho_\top \sqsubseteq \mathrm{Id}_{\pi_\top^*\Sigma}$ (the counit), and similarly for $\rho_\perp$ and $\lambda_\perp$. In these two cases we get fibre maps. For a fibration we get, contravariantly, $\rho_\perp \circ \rho_\top : \pi_\top^*\Sigma \to \pi_\perp^*\Sigma$, while for an opfibration we get, covariantly, $\lambda_\top \circ \lambda_\perp : \pi_\perp^*\Sigma \to \pi_\top^*\Sigma$.

$$\begin{tikzcd} \pi_\perp^*\Sigma \arrow[r, dashed, shift left, "\lambda_\perp"] & \Sigma^{\mathbb{S}} \arrow[l, shift left, "\rho_\perp"] \arrow[r, shift left, "\lambda_\top"] & \pi_\top^*\Sigma \arrow[l, dashed, shift left, "\rho_\top"] \end{tikzcd}$$

Theorem 2 *Let $p : \Sigma \to B$ be a bundle as above.*

1. *If p is a local homeomorphism, thus corresponding to an object of $\mathscr{S}B$, then it is an opfibration.*
2. *If p corresponds to a compact regular locale in $\mathscr{S}B$, then it is a fibration.*

Proof. [16] □

1.4 Valuation locales

Standard measure theory works badly in toposes, suffering from deep set-theoretic problems. For many purposes a satisfactory replacement can be found by replacing measurable spaces and measures by locales X and *valuations* m on them. Such an m is a Scott continuous map from the frame of opens ΩX to the lower reals $\overrightarrow{[0,\infty]}$, satisfying the *modular law* $m(U \vee V) + m(U \wedge V) = m(U) + m(V)$ and also $m(\emptyset) = 0$. (The *lower* reals differ constructively from the Dedekind reals in being approximable from below but not from above. For present purposes it is best to understand them as being given the Scott topology instead of the usual Hausdorff topology.) *Probability valuations* are those for which $m(X) = 1$.

For every locale X there can be constructed a *valuation locale* $\mathfrak{V}X$ whose points are the valuations on X; it has a sublocale $\mathfrak{V}^{(1)}X$ whose points are the probability valuations. These were first defined in [10], following ideas of the probabilistic powerdomain of [17], and were further developed in [22] and [4]. In particular the results of [4] were central in the quantum treatment of [12].

More recent work [24] has shown that $\mathfrak{V}$ and $\mathfrak{V}^{(1)}$ are the functor parts of monads, localic analogues of the Giry monad in measure theory [9] and the distribution monad of [1]. The monads are commutative, meaning that product valuations exist and a Fubini Theorem holds.

[24] also describes in some detail the geometricity of $\mathfrak{V}$ (and likewise $\mathfrak{V}^{(1)}$), and this will be key to our development here. The topos-validity of $\mathfrak{V}$ tells us that for any bundle $p : \Sigma \to B$ we can also construct a bundle $q : \mathfrak{V}_B\Sigma \to B$, by applying $\mathfrak{V}$ to an internal locale in the topos of sheaves over B got using the Localic Bundle Theorem. But $\mathfrak{V}$ is also geometric, and this tells us that the bundle construction works fibrewise: in other words, each fibre $b^*\mathfrak{V}_B^{(1)}\Sigma = q^{-1}(\{b\})$ is homeomorphic to $\mathfrak{V}^{(1)}(b^*\Sigma)$. We shall not need to dwell on the topos theory here, but it is the topos theory that tells us how the valuation locales of the fibres of p can be bundled together, with an appropriate topology on the bundle space, to make q.

Remark 3 *In general, to define a valuation on X involves defining its values for all* opens *of X, or at least for a generating lattice of opens. However, in the particular case where X is discrete and moreover has decidable equality, it is enough to define the values for all* points. *(The issue is similar to that well known for Lebesgue measure, where the points all have measure 0 but opens have non-zero measure.) The frame of opens, the powerset $\mathscr{P}X$, is the ideal completion of the Kuratowski finite powerset $\mathscr{F}X$. Each $S \in \mathscr{F}X$ is a finite disjoint union of singletons (we need decidability of equality to remove duplicates and hence achieve disjointness), and so its measure is determined by that of the singletons.*

2 Spectral and related bundles

We briefly summarize some topos approaches and how they lead to bundles. For convenience we shall refer to the *Imperial* approach of Isham and Butterfield [14], [15] and subsequently Döring and Isham [7]; and the *Nijmegen* approach of Heunen, Landsman and Spitters [12] (see also [11]).

Both Imperial and Nijmegen start with a C*-algebra $\mathscr{A}$ (or, more specifically in the Imperial case, a von Neumann algebra; in the finite dimensional case these notions are equivalent) and then take a "classical point of view" (to use the Imperial phrase) to be a commutative C*-subalgebra C. By Gelfand-Naimark duality, C is isomorphic to the algebra of continuous maps $\Sigma_C \to \mathbb{C}$ where Σ_C is the spectrum, and it follows that Σ_C provides a classically pure state space for the self-adjoint elements of C, considered as observables. They are all represented as maps $\Sigma_C \to \mathbb{R}$. Thus C is a context in which the physics of those observables is classical.

Let us write $\mathscr{C}(\mathscr{A})$ for the poset of commutative C*-subalgebras (or, for Imperial, of commutative von Neumann subalgebras). The base space of the bundle is constructed out of $\mathscr{C}(\mathscr{A})$, and the fibre over a context C is its spectrum. A significant difference between the two approaches lies in how those fibres are topologized.

For *Imperial*, the topos is the category of contravariant functors from $\mathscr{C}(\mathscr{A})$ to **Set** (i.e. presheaves over $\mathscr{C}(\mathscr{A})$). For a correct point-free approach one should take the base space B to be $\mathrm{Idl}(\mathscr{C}(\mathscr{A})^{op})$, the space of filters of $\mathscr{C}(\mathscr{A})$, with its Scott topology. However, for many purposes it suffices to consider only the *principal* filters, of the form $\{D \in \mathscr{C}(\mathscr{A}) \mid C \subseteq D\}$ for some C. The Imperial workers seek the spectrum as an *object* of the topos, i.e. a sheaf, corresponding to a bundle that is a local homeomorphism over $\mathrm{Idl}(\mathscr{C}(A)^{op})$, and so the fibres are all discrete.

By Theorem 2 the spectral bundle is therefore an opfibration, giving fibre maps covariant with respect to the specialization order. Since the fibre maps are of necessity contravariant with respect to context inclusion (by a "coarse-graining" argument), it follows that the specialization order is the opposite of context inclusion. This is achieved in [7] by taking the base space to be the filter completion of the context poset, and its sheaves are then presheaves – contravariant set-valued functors – on the context poset. Any approach that seeks a spectral object, a discrete space internally, will be subject to this argument, so it is a consequence of using point-set topology internally.

For *Nijmegen*, the topos is the category of covariant functors from $\mathscr{C}(\mathscr{A})$ to **Set** and the corresponding base space B is $\mathrm{Idl}\mathscr{C}(\mathscr{A})$, the space of ideals of $\mathscr{C}(\mathscr{A})$, with its Scott topology. The Nijmegen workers seek the spectrum as an internal compact regular – actually, completely regular – space in the topos, as expected from the Gelfand-Naimark duality. In fact, they can construct an internal commutative C*-algebra (for which the fibre over C is just C itself) and then use a topos-valid version [2] of Gelfand-Naimark duality to get an internally compact regular spectrum. In the corresponding bundle the fibres are all compact regular. Now by Theorem 2 the spectral bundle is a fibration, and the same reasoning shows that the specialization order will agree with context inclusion. This is achieved in [11] by taking the base to be the ideal completion of the context poset, and the sheaves are the covariant set-valued functors on the context poset.

Note how the Nijmegen approach had to adopt a point-free approach to topology in order to use the topos-valid form of Gelfand-Naimark. Whereas a point-set approach always gives an opfibration, the change to point-free does not in itself bring any consequences for the variance because a general bundle need be neither fibration nor opfibration. However, any approach that seeks a compact, regular, point-free spectrum will get a fibration with the same variance as for Nijmegen.

Although these two approaches are the best worked out so far, they are not the only possible. As we shall see in Section 3, in finite dimensional systems parts of the base and bundle spaces have natural manifold structure and can therefore be given non-discrete Hausdorff topologies. Our bundle description is intended to allow a wide range of such possibilities, when described point-free.

2.1 Valuation bundles and Born sections

Suppose $p : \Sigma \to B$ is our spectral bundle, giving rise to a probability valuation bundle $q : \mathfrak{V}_B^{(1)}\Sigma \to B$.

For the Nijmegen situation this was discussed in [12] (referring to the development in[4]), and the paper proves (their Theorem 14) that its global points – the global sections of q – are equivalent to quasistates of the C*-algebra. Thus in particular each pure state $|\psi\rangle$ gives a global section of q, and it is continuous. This is already interesting, since Kochen-Specker tells us that global sections of p should normally be impossible. If we try to extract external mathematics from the topos internal by taking global sections, then the spectrum p loses all its points, but the valuation space q retains points and we are familiar with them through quantum pure states.

Once a section σ is given for q, we can infer the probabilities that arise in the Born rule. (See also [5], [6] for some other discussions of the Born rule.) Suppose C is a context, a point of B, and for simplicity consider the simple case where C is a commutative subalgebra (rather than a filter or ideal). The fibre $C^*\Sigma$ is the Gelfand spectrum $\mathrm{Spec}(C)$, and any self adjoint in C is represented as a map $\tilde{O} : C^*\Sigma \to \mathbb{R}$. Using geometricity to see $C^*\mathfrak{V}_B^{(1)}\Sigma \cong \mathfrak{V}^{(1)}C^*\Sigma$, we can then get the probabilistic result of observing O in state $|\psi\rangle$ as the probabilistic valuation $\mathfrak{V}^{(1)}(\tilde{O})(\sigma(C))$ on $\mathbb{R}$, a random variable. The core probability is determined by C and $|\psi\rangle$, and all that remains is to allow for the way the observable O labels the eigenstates (in $C^*\Sigma$) with real numbers.

Our next step is to generalize the state $|\psi\rangle$. In the finite dimensional case it corresponds to a projector $|\psi\rangle\langle\psi|$ of rank 1 and it is possible to generalize to higher rank projectors by summing over basis states. Thus for each context D, the elements of $D^*\Sigma$ can play a role similar to that of states $|\psi\rangle$. However, we do not attempt to normalize – to do so presents problems when allowing for refinement of the context D – and so we no longer have probability valuations in general. Thus we get sections of $q : \mathfrak{V}_B\Sigma \to B$. Bundling these together we postulate a *Born section* of the bundle $\mathfrak{V}_{B^2}\Sigma^2 \to B^2$. For each pair (D,C) of contexts it gives a valuation Born_{DC} of $D^*\Sigma \times C^*\Sigma$.

At present we have the Born sections defined only in finite dimensional situations. Nonetheless, the formal notion makes sense more generally and would appear to have a good phenomenological footing in that it describes probabilities. We therefore hope that the structure of bundle together with Born maps, appropriately axiomatized, will prove a good foundation for topos-based contextual physics.

3 Finite dimensional quantum systems

3.1 Bundles for finite dimensional quantum systems

In the following we propose a fibrational bundle in which the spaces B and Σ contain manifolds. This contrasts with the original Imperial and Nijmegen constructions, where the topology on B arises solely from the order structure on $\mathscr{C}(\mathscr{A})$, though it accords with the use of flag manifolds in [3]. We fix an algebra $\mathscr{A} = M_n(\mathbb{C})$ and all constructions are relative to this algebra. Any commutative sub-C*-algebra C is also finite dimensional[4] and so, by Gelfand-Naimark duality, isomorphic as unital C*-algebra to $\mathbb{C}^m$ for some m. It has m indecomposable projectors (self-adjoint idempotents) C_i corresponding up to

[4]Constructively this is not quite true. But we shall construct our bundle in terms of the subalgebras that *are* finite dimensional.

isomorphism to the elements of $\mathbb{C}^m$ that have a 1 in a single component, 0 elsewhere, and these are also projectors in $\mathscr{A}$ because C is a sub-C*-algebra. They are orthogonal ($C_iC_j = 0$ if $i \neq j$) and complete (they sum to 1). Note also that the trace of each is a positive integer, equal to the rank, since the eigenvalues are 0 or 1.

We call a complete orthogonal sequence $\overrightarrow{C}$ of projectors a *projector system*, and define its *type* to be the sequence of traces of the projectors, an ordered partition of n. Generally we are only interested in the *set* of projectors (because this is what characterizes the subalgebra C) and the set of traces as type; but for setting up the bundle it is useful to remember the automorphisms. With this in mind, we define –

Definition 4 *Let $(\mu_i)_{i=1}^l$ and $(\nu_j)_{j=1}^m$ be two partitions of n. A* refinement *from $\overrightarrow{\nu}$ to $\overrightarrow{\mu}$ is a function $r : \{1,\dots,l\} \to \{1,\dots m\}$ such that $\nu_j = \sum_{r(i)=j} \mu_i$ for every j. (Note that the reindexing function is in the opposite direction to the refinement.)*

For each partition $\overrightarrow{\mu}$ we define the space $\mathrm{Proj}(\overrightarrow{\mu})$ of projector systems of type $\overrightarrow{\mu}$. This can clearly be done localically, since it is defined as a subspace of $\mathbb{C}^{ln^2}$, where l is the length of $\overrightarrow{\mu}$, by a system of equations. We also have a trivial bundle over it, $\mathrm{Proj}(\overrightarrow{\mu}) \times l \to \mathrm{Proj}(\overrightarrow{\mu})$. Each fibre has exactly l elements, which is the Gelfand spectrum for all subalgebras of type $\overrightarrow{\mu}$. For any refinement $r : \overrightarrow{\nu} \to \overrightarrow{\mu}$ we now have a map $\mathrm{Proj}(r) : \mathrm{Proj}(\overrightarrow{\mu}) \to \mathrm{Proj}(\overrightarrow{\nu})$ given by $\mathrm{Proj}(r)(\overrightarrow{C})_j = \sum_{r(i)=j} C_i$. This extends to a bundle map using $\mathrm{Proj}(r)(\overrightarrow{C}, i) = (\mathrm{Proj}(r)(\overrightarrow{C}), r(i))$. We thus have a diagram of bundles, whose shape is the opposite of the category of partitions and refinements. Our bundle $\Sigma \to B$ is now defined as a lax colimit of this diagram. More precisely, it has images of all the spaces $\mathrm{Proj}(\overrightarrow{\mu})$, subject to $\mathrm{Proj}(r)(\overrightarrow{C}) \sqsubseteq \overrightarrow{C}$ and $(\mathrm{Proj}(r)(\overrightarrow{C}), r(i)) \sqsubseteq (\overrightarrow{C}, i)$.

The imposition of this specialization order has two effects. The first, and perhaps less obvious one, is with regard to invertible refinements. These permute equal values in partitions, and the effect of the imposed specialization is to make two projector systems equal if they generate the same subalgebra – because they have the same projectors, but permuted. This makes B a space of contexts as required. After that, the specialization agrees with context inclusion as required for a fibrational bundle. The action on the bundle spaces ensures that states are kept track of correctly under permuting of the matrices.

To define the Born section, we define it first for projector systems, and show that the definition respects refinements. Suppose $\overrightarrow{C}$ and $\overrightarrow{D}$ have types $\overrightarrow{\mu}$ and $\overrightarrow{\nu}$, of lengths l and m. At this level (before imposing the specialization) the spectra are finite discrete with decidable equality, cardinalities l and m, and so by Remark 3 a valuation on the product can be defined by the values on its elements (i, j). We define

$$\beta_{\overrightarrow{C}\overrightarrow{D}}(i, j) = \mathrm{Tr}(C_i D_j),$$

a non-negative real, and then let Born_{CD} be the image of $\beta_{\overrightarrow{C}\overrightarrow{D}}$ in $\mathfrak{V}_{B^2}(\Sigma^2)$. Now suppose $\overrightarrow{C} \sqsubseteq \overrightarrow{C}'$ and $\overrightarrow{D} \sqsubseteq \overrightarrow{D}'$, with refinements $r : \overrightarrow{\mu} \to \overrightarrow{\mu}'$ and $s : \overrightarrow{\nu} \to \overrightarrow{\nu}'$. Then

$$\mathrm{Tr}(C_i D_j) = \mathrm{Tr}(\sum_{r(i')=i} C'_{i'} \sum_{s(j')=j} D'_{j'}) = \sum_{r(i')=i, s(j')=j} \mathrm{Tr}(C'_{i'} D'_{j'})$$

and so $\beta_{\overrightarrow{C}\overrightarrow{D}} = \mathfrak{V}\left(\mathrm{Proj}(r) \times \mathrm{Proj}(s)\right)\left(\beta_{\overrightarrow{C}'\overrightarrow{D}'}\right)$; it follows that $\mathsf{Born}_{CD} \sqsubseteq \mathsf{Born}_{C'D'}$.

In the case where C_i has trace 1 it is of the form $|\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle$ (an eigenvector for eigenvalue 1). Then

$$\mathrm{Tr}(C_i D_j) = \mathrm{Tr}(|\psi\rangle\langle\psi|D_j) = \mathrm{Tr}(\langle\psi|D_j|\psi\rangle) = \langle\psi|D_j|\psi\rangle$$

and so the probability agrees with that obtained from the Born rule for state $|\psi\rangle$. More generally, C_i is a sum $\sum_k |\psi_k\rangle\langle\psi_k|$ for orthonormal vectors, and $\mathsf{Born}_{CD}(i, j)$ is the sum $\sum_k \langle\psi_k|D_j|\psi_k\rangle$.

3.2 Bundles for the qubit

The qubit Hilbert space $\mathscr{H}$ is $\mathbb{C}^2$, and the C*-algebra is the full matrix algebra $\mathscr{A} = M_2(\mathbb{C})$. The commutative context $*$-subalgebras come in two types, (2) (dimension 1) and $(1,1)$ (dimension 2). The sole type (2) algebra is the centre $\mathbb{C}1$, which we denote by $\perp$; $\mathrm{Proj}(2)$ is the 1-point space. The 2-dimensional subalgebras are generated by two projectors C_1, C_2 such that $C_1 + C_2 = 1$ and $\mathrm{Tr}(C_i) = 1$. Projectors P of trace 1 are in bijection with self-adjoint unitaries $U = 2P - 1$ of trace 0, and such unitary can be written as a real linear combination $a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ of the Paulis such that $a_x^2 + a_y^2 + a_z^2 = 1$. (Any self-adjoint is a real linear combination of the Paulis and 1; for trace 0 the coefficient of 1 is 0; and the further condition says that the matrix is an involution.) Since each C_i is determined by the other, it follows that $\mathrm{Proj}(1,1)$ is the 2-sphere S^2 – this is the Bloch sphere. In the context space B antipodes will be identified, giving the real projective plane $\mathbb{R}\mathrm{P}^2$, and $\perp$ is adjoined as a bottom point in the specialization order. Thus B is $\mathbb{R}\mathrm{P}^2$ lifted, Σ is S^2 lifted.

For both Imperial and Nijmegen (but see also [20]), both S^2 and $\mathbb{R}P^2$ are given their discrete topologies. For Imperial, the trivial points are adjoined *above* the rest in the specialization order as a top $\top$, so that each trivial point is open. For Nijmegen they are adjoined *below* as a bottom $\perp$, so that the only open containing a trivial point is the whole space.

An interesting consequence of using the natural, manifold topologies, is that the bundle has no continuous cross-sections. As [14] have pointed out, for dimensions 3 or more the lack of cross-sections for the Imperial bundle is a manifestation of the Kochen-Specker Theorem [19]. That theorem does not apply directly to dimension 2, and indeed the corresponding Imperial bundle has many cross-sections, albeit discontinuous with respect to the manifold topologies. Essentially the Kochen-Specker Theorem as normally formulated is a combinatorial one, relying on having sufficient complexity in the order structure amongst the contexts. In the 2-dimensional case that order structure (one trivial point related to everything, all other points incomparable with each other) is too simple.

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