The connected Vietoris powerlocale

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Abstract

The connected Vietoris powerlocale is defined as a strong monad V^c on the category of locales. V^cX is a sublocale of Johnstone's Vietoris powerlocale VX, a localic analogue of the Vietoris hyperspace, and its points correspond to the weakly semifitted sublocales of X that are "strongly connected". A product map $\times : V^cX \times V^cY \to V^c(X \times Y)$ shows that the product of two strongly connected sublocales is strongly connected. If X is locally connected then V^cX is overt. For the localic completion $\overline{Y}$ of a generalized metric space Y, the points of $V^c\overline{Y}$ are certain Cauchy filters of formal balls for the finite power set $\mathcal{F}Y$ with respect to a Vietoris metric.

Application to the point-free real line $\mathbb{R}$ gives a choice-free constructive version of the Intermediate Value Theorem and Rolle's Theorem.

The work is topos-valid (assuming natural numbers object). V^c is a geometric construction.

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1 Introduction

In classical analysis, some of the most basic results follow from the facts that bounded closed intervals $[a, b]$ in the real line are both compact and connected, and that those properties are preserved under continuous direct image.

Let f be a continuous, real-valued function on $[a, b]$. Then the fact that f is bounded and attains its bounds follows from compactness. The Intermediate Value Theorem follows from connectedness, for suppose $f(a) \leq 0 \leq f(b)$ but 0 is not in the image of f (we are arguing classically here). Then $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ gives us a non-trivial disjoint decomposition of $[a, b]$, contradicting connectedness.

In constructive mathematics, however, there are problems. Clearly the argument given above ("suppose 0 is not in the image ...") is a proof by contradiction. The bisection method comes closer to a procedure for finding a solution. We let $c = \frac{a+b}{2}$, and then iterate with $[a, c]$ or $[c, b]$ according as $f(c) \geq 0$ or $f(c) \leq 0$. However, this too uses excluded middle. In fact the problem arises exactly when $f(c) = 0$, for if our computation of $f(c)$ is only within finite error at any finite stage, then we shall never discover either $f(c) \geq 0$ or $f(c) \leq 0$. We are looking for x with $f(x) = 0$. But if, by sheer bad luck, we find it straight away, we shall never recognize our success! A solution is given in [BB85]. Given any $\varepsilon > 0$, we can determine either $f(c) < \varepsilon$ or $f(c) > -\varepsilon$, and the process then allows us to find x with $|f(x)| < \varepsilon$. However, this uses countable dependent choice to find a solution x, since if we have both $f(c) < \varepsilon$ and $f(c) > -\varepsilon$ then we must choose which side to go.

There are various schools of constructivism, some (such as Bishop's form and predicative type theory) allowing some choice, and others (such as topos-valid mathematics) not. (In our "topos-valid" mathematics we shall also assume the existence of a natural numbers object. This then justifies free algebra constructions – see [JW78].) A finding common to many of them is the need to use *point-free* topology, for example locales (in topos-valid mathematics) or formal topologies (in predicative type theory; see [Sam87]). This is sometimes characterized as pointless topology, since it can all too easily be doing just the same as point-set topology but obfuscated by lattice manipulations. However, it regains its point in the constructive (and particularly choice-free) context. The first reason for this is that some important results such as Tychonoff and Heine-Borel, for which a point-set formulation fails constructively, remain true in point-free form. Constructively, point-free topology works better than point-set topology. The second is that within a certain *geometric* fragment of topos-valid mathematics, there are logical techniques that restore a pointwise reasoning style to locale theory. These are discussed in Section 3.

The standard topos-valid treatment, as in [JT84], identifies a point-free topology (a locale) with an internal frame, and has a good body of theorems adapted from standard topology. Moreover, if W is a locale then (see [JT84]) internal locales in its category $\mathcal{S}W$ of sheaves are equivalent to locale maps with W as codomain. It follows that internal topos-valid reasoning about locales can be externalized to obtain information about maps.

A major part of our work here is the *Vietoris powerlocale* VX of [Joh85], a localic analogue of the Vietoris hyperspace. In [Vic03], $V\mathbb{R}$ (or, rather, its "positive" part $V^+\mathbb{R}$) is used to address some of the aspects of the compactness side of closed real intervals, particularly with regard to the Heine-Borel Theorem. Its points are certain sublocales of $\mathbb{R}$, all of them compact (and non-empty in the case of V^+), and the Heine-Borel Theorem is expressed using a continuous map $HB_C : \leq \to V^+\mathbb{R}$ such that $HB_C(a,b)$ corresponds to the closed interval $[a,b]$. (The domain $\leq$ of HB_C is the sublocale of $\mathbb{R}^2$ whose points are those pairs (a,b) for which $a \leq b$.) It is also shown there that there are maps $\sup, \inf : V^+\mathbb{R} \to \mathbb{R}$ that calculate the sup and inf of the points of $V^+\mathbb{R}$, with the reals $\sup K$ and $\inf K$ both in the sublocale corresponding to K.

The constructions V and V^+ are functorial – in fact, they are the functor parts of monads. If $f : X \to Y$ is a map of locales, then Vf calculates the "weakly semifitted closure" of the direct image of the sublocales corresponding to points of VX, and similarly for V^+. We shall discuss in Section 5.1 just what this means, but in the case of the real line $\mathbb{R}$ the direct image is already weakly semifitted and no further closure is needed. Hence for a map $f : \mathbb{R} \to \mathbb{R}$, the boundedness of f on $[a,b]$ follows from the fact that $V^+f(HB_C(a,b))$ is compact. Classically, the fact that f attains its bounds can be deduced from the choice-free, constructive assertion that $V^+f(HB_C(a,b))$ contains both its sup and its inf. However, the nature of localic surjections means that there might be no $x \in K$ for which $f(x)$ is $\sup(Vf)(K)$ or $\inf(Vf)(K)$, and [Vic03] leaves aside questions of how such an x might be found when it does exist.

In this paper we turn to connectedness by defining a sublocale of V^+X, which we call V^cX, whose points are those points of V^+X that are *connected* in a strong sense. (Note: even in the usual sense, we do not admit the empty space as connected.) We show that HB_C factors via $V^c\mathbb{R}$, thus showing that each compact interval $[a,b]$ is connected – indeed, it provides a homeomorphism $\leq \cong V^c\mathbb{R}$. Now suppose $f : \mathbb{R} \to \mathbb{R}$ has $f(a) \leq 0 \leq f(b)$. Classically, the Intermediate Value Theorem can be deduced from $\inf_{a \leq x \leq b} |f(x)| = 0$, and the quantification over x can be eliminated to give the choice-free, constructive, point-free form $\inf(V^c|f|(HB_C(a,b))) = 0$. Again, we leave aside the question of how to find x such that $f(x) = 0$ when it does exist — a thorough discussion of it can be found in [Tay05].

We also consider Rolle's Theorem. This requires some account of differentiation, and for that we use Caratheodory's approach. The basic idea is that f is differentiable at x_0 if the function $g(h) = \frac{f(x_0+h)-f(x_0)}{h}$, defined for $h \neq 0$, can be continuously filled in at $h = 0$, and then $g(0)$ is the derivative $f'(x_0)$. In other words, there is some continuous g with $hg(h) = f(x_0 + h) - f(x_0)$. Localically, we shall find it more tractable to deal with *continuously* differentiable functions f, i.e. those for which there is some continuous $g(x,y)$ such that $(y-x)g(x,y) = f(y) - f(x)$, and then $f'(x) = g(x,x)$. We shall then prove Rolle's Theorem in the form that if $a < b$ and $f(a) = f(b)$, then $\inf(V^c|f'|(HB_C(a,b)) = 0$. From this it is a straightforward consequence that if f' is constant 0 on an open interval, then f is constant on that interval.

The reasoning style in the paper depends a lot on the reader's ability to recognize constructivity in the reasoning, both topos-valid (as in [JT84]) and geometric. This in itself will feel unfamilar to classical mathematicians, though we believe it is a sensible approach to doing constructive mathematics. In addition, for the sections on real analysis, the relevant subspaces (in their localic form as sublocales) are described in an unfamilar form, in terms of how opens cover and meet them, that relates to their status as powerlocale points. (A similar style can be seen in [Tay05].) To ease the burden of dealing with two unfamiliarities at once, Section 7.1 gives a classical account of the powerlocale style of description for compact subspaces of $\mathbb{R}$, thus enabling one to see how the subsequent geometric reasoning can be understood as a classical argument for $\mathbb{R}$ as a topological space.

2 Notes on locales

For the basic notions of locales, we refer to [Joh82] and [Vic89]. If X is a locale we write ΩX for its frame of opens. (A *frame* is a complete lattice in which binary meet $\wedge$ distributes over all joins $\bigvee$. We also write $\top$ and $\bot$ for its top and bottom elements.) If $f : X \to Y$ is a map of locales, we write $f^* : \Omega Y \to \Omega X$ for its inverse image function, a frame homomorphism (preserving finite meets and arbitrary joins). We write **Fr** and $\mathbf{Loc} = \mathbf{Fr}^{op}$ for the categories of frames and locales.

The 1-point locale is written 1. Its frame $\Omega 1$ is Ω, the lattice of truth-values. (We are thinking in non-classical, topos-valid mathematics, where the lattice of truth-values is the subobject classifier Ω in some topos. $\top$ and $\bot$ in Ω correspond to the truth values **true** and **false**.) A *global point* of X is a map $x : 1 \to X$, i.e. a frame homomorphism $\Omega X \to \Omega$. If $U \in \Omega X$ is an open, then we say x is *in* U if $x^*(U) = \top$, and write $x \vDash U$. These are often just called points, but, for reasons explained in Section 3.3, we shall use the word in a more generalized sense: a *point of* X *at stage* W is a map $W \to X$. The *generic* point of X is the identity map $\mathrm{Id} : X \to X$.

The *specialization order* on points is defined as follows. If $x_1, x_2 : W \to X$, then $x_1 \sqsubseteq x_2$ if $x_1^*(U) \leq x_2^*(U)$ for every $U \in \Omega X$. (In the case of global points $x_1, x_2 : 1 \to X$, this says $x_1 \sqsubseteq x_2$ if, for every $U \in \Omega X$, if $x_1 \vDash U$ then $x_2 \vDash U$.) If a family $x_i : W \to X$ $(i \in I)$ is directed with respect to the specialization order, then their join $\bigsqcup_i^{\uparrow} x_i$ also exists, defined by $(\bigsqcup_i^{\uparrow} x_i)^*(U) = \bigvee_i^{\uparrow} x_i^*(U)$.

We shall often use frame presentations by generators and relations, in the form $\mathrm{Fr}\langle G|R\rangle$. Here G is a set of symbolic generators, and R is a set of relations of the form $e_1 = e_2$ or $e_1 \leq e_2$, where each e_i is an expression representing a join of finite meets of generators. (See in particular [Vic89].) A point of the corresponding locale is, using the universal property of presentations, a subset $F \subseteq G$ that respects all the relations in the following sense. Suppose $e_1 \leq e_2$ is a relation (equations are treated similarly) with $e_\lambda = \bigvee_{i \in I_\lambda} \bigwedge E_i^\lambda$, with each E_i^λ a finite subset of G. Then if $E_i^1 \subseteq F$ for some $i \in I_1$, then $E_j^2 \subseteq F$ for some $j \in I_2$. If G has structure that is to be preserved in the frame, then we

use "qua" notation. For example, if G is a poset then $\mathrm{Fr}\langle G \text{ (qua poset)} |\ R\rangle$ indicates that the order in G is to be preserved in the frame.

A locale map f is *surjective* iff f^* is 1-1, an *embedding* (or *inclusion*) if f^* is onto. Categorically, the surjections are the epis and the embeddings are the regular monics. A localic surjection is not necessarily surjective on points, not even classically. For a simple example, if P is a poset then the map from P (discrete) to its ideal completion (treated localically using the Scott topology) is surjective. Equivalence classes of embeddings are *sublocales*, the localic analogues of subspaces, and are described in various ways in the standard accounts. (See also [Vic07b].) Normally the best intuition is that a sublocale is described by adjoining extra relations to a frame presentation. This adds extra constraints on the points, and so corresponds to a subspace.

$f : X \to Y$ is *dense* if f^* reflects $\bot$ (if $f^*V = \bot$ then $V = \bot$), *codense* if it reflects $\top$. f is *strongly dense* if whenever $f^*V \leq !^*_X p$ ($p \in \Omega = \Omega 1$, $!_X : X \to 1$ the unique map) then $V \leq !^*_Y p$.

A locale X is *compact* if it satisfies the usual finite subcover property, and *overt* (Paul Taylor's word, though the concept is much older) if it is open in the sense of [JT84] – the unique map $X \to 1$ is an open map. This holds iff every open is a join of positive opens, where a locale is *positive* if every cover is inhabited (see [Joh84]). Classically, all locales are overt. A useful consequence of overtness of X is that for any $I \subseteq \Omega X$ we have $\bigvee I = \bigvee (I \cap \mathrm{Pos})$, where Pos is the set of positive opens. To see this, take $U \in I$. To show $U \leq \bigvee (I \cap \mathrm{Pos})$, using the fact that U is a join of positive opens, suppose $U' \leq U$ with U' positive. Then U also is positive, so $U \in I \cap \mathrm{Pos}$, and hence $U' \leq \bigvee (I \cap \mathrm{Pos})$.

A sublocale Y of X is *fitted* if it is a meet of open sublocales (this is the localic analogue of *saturated*, i.e. up-closed under the specialization order). It is *weakly closed* if it is a meet of sublocales of the form $C \vee !^* p$ where C is closed, $p \in \Omega$ and $! : X \to 1$ is the unique locale map. Note that $!^* p = \bigvee \{\top \mid p\}$. Classically, where p must be either $\top$ or $\bot$, all weakly closed sublocales are closed. Finally, Y is *weakly semifitted* if it is a meet of a fitted sublocale and a weakly closed sublocale.

X is *regular* if every $U \in \Omega X$ is the join of those U' for which there is V with $U' \wedge V = \bot$, $U \vee V = \top$. Then every sublocale is fitted. This is because any adjoined relation $U_1 \leq U_2$ is equivalent to the set of relations $\top \leq U_2 \vee V$ (where $U_1 \vee V = \top$ in ΩX).

If locales X_i ($i = 1, 2$) are presented by $\Omega X_i = \mathbf{Fr}\langle G_i \mid R_i \rangle$, then their product can be presented by $\Omega(X_1 \times X_2) = \mathbf{Fr}\langle G_1 + G_2 \mid R_1 + R_2 \rangle$ where $+$ denotes disjoint union (understood in the obvious way in the case of the relations). Its points are pairs (x, y) where x and y are points of X_1 and X_2 – in fact, since we are understanding "point" in the generalized sense, this is just a restatement of the categorical characterization of product.

The opens $U \in G_1$ (representing $U \times \top$) and $V \in G_2$ (representing $\top \times V$) form a subbase. Generalizing U and V to arbitrary opens of X_1 and X_2, and taking finite meets, we get a base of opens of the form $U \times V$ just as one would expect from topology. Less familar is the fact that finite joins give a "preframe base" of opens $U \odot V = U \times \top \vee \top \times V$ – that is to say, any open is a directed

join of finite meets of preframe basics. This can be seen by rewriting an open $\bigvee_i U_i \times V_i$ as a directed join of finite joins of basics $U_i \times V_i$, and then using distributivity together with the equations

$$U \times V = U \odot \bot \wedge \bot \odot V$$
$$U_1 \odot V_1 \vee U_2 \odot V_2 = (U_1 \vee U_2) \odot (V_1 \vee V_2)$$

to rewrite each finite join of basics as a finite meet of preframe basics.

3 Topos-validity and geometricity

The internal, intuitionistic mathematics of toposes includes products, fibred products (= pullbacks), disjoint unions (= coproducts), quotients (= coequalizers), function sets (= exponentials) and powersets. For us, toposes will be *Grothendieck* toposes, and these support also infinitary coproducts, a natural number object, and ([JW78]; see also [PV07]) free algebra constructions. We shall call "topos-valid" those constructions that can be performed in Grothendieck toposes and reasoning principles that are valid for them.

A geometric morphism $f : X \to Y$ between toposes comprises an adjoint pair $f^* \dashv f_*$ of functors such that the left adjoint $f^* : \mathcal{S}Y \to \mathcal{S}X$ preserves finite limits. (Following [Vic99], we distinguish notationally between "toposes as generalized topological spaces" X and their "categories of sheaves" $\mathcal{S}X$ – i.e. the categories that are normally taken as embodying the toposes and where the topos-valid reasoning lives. This will blur the distinction between locales and the corresponding localic toposes. Any locale or topos will have both a category of sheaves $\mathcal{S}X$ and a frame of opens ΩX, the lattice of subsheaves of 1. It also blurs the distinction between locale maps and the geometric morphisms between the corresponding localic toposes – in any case, the two notions are equivalent. The symbol f^* as used here agrees with the previous $f^* : \Omega Y \to \Omega X$ when opens are treated as subsheaves of 1.)

Not all topos-valid constructions are preserved by inverse image functors. The ones that are, we shall call *geometric*. By definition, this includes colimits and finite limits, and so also are the free algebra constructions. Non-geometric constructions include exponentials and powersets. Amongst numerical constructions, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ (natural numbers, integers, rationals) are geometric, but the various topos-valid constructions of the reals are not. For this reason, the real line is treated geometrically as a locale, not a set.

In this Section we explain how geometric reasoning enables one to treat locales as spaces of points, and locale maps as transformations of points. A significant part of the paper will be about how to convert topos-valid reasoning (e.g. about connectedness of sublocales) into geometric reasoning (e.g. about points of powerlocales).

3.1 Geometric theories

Geometric theories ([MR77]; see also [Vic07a]) provide a logical basis for the geometric reasoning.

Definition 1 *Let Σ be a first-order, many-sorted signature with sorts, function symbols (possibly including constants) and predicates. Each function and predicate has an arity to stipulate the number and sorts of the arguments, as well as (for a function) the sort of the result.*

The terms over Σ are built in the usual way, and then geometric formulae *are built from terms and predicates using finite conjunction $\wedge$, arbitrary disjunction $\bigvee$, equality $=$ between terms and existential quantification $\exists$. A* geometric sequent *over Σ is of the form $(\forall \vec{x})(\phi \to \psi)$ where ϕ and ψ are geometric formulae whose free variables all appear in the finite list $\vec{x}$. A* geometric theory *is a pair (Σ, T) where Σ is a signature and T is a set of geometric sequents over it.*

Note that a *propositional* geometric theory, one with no sorts (so no terms of any kind) is the same as a frame presentation by generators (the propositional symbols) and relations (the sequents).

The usual notions of *interpretation* and *model* of a theory can be generalized from sets to any Grothendieck topos. Moreover, the constructions needed to interpret geometric theories are all geometric, and it follows that if $f : X \to Y$ is a geometric morphism then f^* transforms models in $\mathcal{S}Y$ to models in $\mathcal{S}X$.

A consequence of having infinitary disjunctions, making a sharp contrast with finitary first-order theories, is that many important constructions can be characterized up to isomorphism by geometric structure and sequents. These will all be geometric constructions, and they include colimits, finite limits, and free algebra constructions. Consequently, we can informally take such "geometric type constructors" as being part of geometric logic, though as yet there is is no formal geometric type theory to make this precise. See [Vic07a].

3.2 Finite sets

When we refer to *finite* sets, we shall mean *Kuratowski finite* – that is to say, finitely enumerable (the elements can be listed in the form $\{x_1, \ldots, x_n\}$ for some n, though there may be repetitions amongst the x_is). We do not assume a decidable equality between elements, but note that emptiness of a finite set is decidable. For any set X we write $\mathcal{F}X$ for its finite powerset, the set of finite subsets of X, and $\mathcal{F}^+X$ for the set of non-empty finite subsets of X.

Since we shall be using finite sets extensively, we make some remarks concerning their geometricity. A more detailed account can be found in [Vic99]. The finite powerset $\mathcal{F}X$ is isomorphic to the free semilattice over X, and, like other free algebra constructions, it is a geometric construction. One particular consequence of this regards universal quantification. Unrestricted universal quantification, although topos-valid, is not geometric. It cannot be used in building geometric formulae, although it appears, at a single level, as part of

geometric sequents. However, geometric formulae can include universal quantification if it is bounded over finite sets. In terms of infinitary disjunction, the formula $(\forall x \in S)\phi$ can be understood as

$$\bigvee_{n\in\mathbb{N}} (\exists x_1 \cdots x_n)(S = \{x_1, \ldots, x_n\} \wedge \bigwedge_{i=1}^{n} \phi[x_i/x]),$$

although in practice it is more convenient to take the bounded universal quantification as part of geometric logic, subject to suitable logical rules.

Geometric sequents $(\forall S)(\phi \to \psi)$, where S is of type $\mathcal{F}X$, can be proved by induction (the "simple $\mathcal{F}$-induction" of [Vic99]). The base case is $\phi[\emptyset/S] \to \psi[\emptyset/S]$, and the induction step is that if we have S satisfying $\phi \to \psi$ then we also have $(\forall x)(\phi[S \cup \{x\}/S] \to \psi[S \cup \{x\}/S])$. The geometric expression of the induction step is as follows. To the base theory in which we are working, adjoin a constant $S : 1 \to \mathcal{F}X$ and a morphism $S^*\phi \to S^*\psi$. In this enlarged theory we have a morphism $S \cup \{-\} : X \to \mathcal{F}X$, and need to construct a morphism $(S \cup \{-\})^*\phi \to (S \cup \{-\})^*\psi$. If we have these ingredients, then the topos-valid induction argument in [Vic99] shows that the geometric sequent holds in the classifying topos for the theory, and hence also in other toposes over it.

3.3 Generalized points

If X is presented by $\Omega X = \mathrm{Fr}\langle G|R\rangle$, then points at stage W are functions $G \to \Omega W$ that respect the relations R. But since the opens in ΩW are equivalent to the subsheaves of 1 in $\mathcal{S}W$, the points are equivalent to models in $\mathcal{S}W$ of the corresponding propositional geometric theory. Hence by using generalized points, we may think of a locale X as the "space of models" of a propositional geometric theory. The generic point Id_X corresponds to the injection of generators $G \to \Omega X$, and it follows that, applying x^* to this model, we get $x^*(\mathrm{Id}_X) = x$.

Note that the logic also topologizes the space: the opens correspond to geometric formulae.

A map $f : X \to Y$ transforms points to points (at any stage) by composition, and this gives a point transformer $F_W : \mathbf{Loc}(W, X) \to \mathbf{Loc}(W, Y)$ at each stage W. The map f can be recovered from this, by applying F_X to the generic point: hence we have a sense in which f can be defined by its effect on points.

Theorem 2 *Let X and Y be locales. We assume that frame presentations are given for them, so that we can identify their points with models of respective propositional geometric theories. If F is a geometric construction that transforms models of the theory for X to models of that for Y, then there is a unique map $f : X \to Y$ such that for every point $x : W \to X$, the composite $f \circ x$ is got by applying F to x in $\mathcal{S}W$.*

Proof. The construction F can be applied in any $\mathcal{S}W$, giving a point transformer $F_W : \mathbf{Loc}(W, X) \to \mathbf{Loc}(W, Y)$. It can be applied to the generic point in $\mathcal{S}X$, giving a map $f : X \to Y$, and this shows uniqueness. (In fact, so

far the argument would work for any topos-valid construction F.) Composition with f gives $f \circ x$ (for $x : W \to X$) as $x^*(F_X(\mathrm{Id}_X))$, and because x^* – as inverse image functor – preserves geometric constructions, by geometricity of F this is $F_W(x^*(\mathrm{Id}_X)) = F_W(x)$. (Actually, in principle this is only up to isomorphism, since the "geometric type constructors" are defined by universal characterizations. However, for for propositional theories the isomorphism becomes equality.) ■

Note the power of this. Not only does it include locales with insufficient global points, it also dispenses with the need for a proof of continuity. We may summarize it in a slogan *continuity = geometricity.*

In practice we shall abuse notation and use the same symbol for both construction and map. Thus we shall define f by defining $f(x)$ geometrically in terms of x.

The Theorem extends to specializations. Suppose we have two constructions f and g, giving two maps $f, g : X \to Y$. Then $f \sqsubseteq g$ if from x we can geometrically show $f(x) \sqsubseteq g(x)$.

Note also that it applies to sublocale inequations. If Y_1 and Y_2 are sublocales of X, then $Y_1 \leq Y_2$ iff there is a map $Y_1 \to Y_2$ over X. Hence to prove it, it suffices to show, geometrically, that any point x of Y_1 is also in Y_2.

3.4 Locale constructions

Geometricity also becomes important when discussing topos-valid constructions of locales, such as powerlocales. For a full discussion of this see [Vic04].

Suppose $f : W_1 \to W_2$ is a locale map. By [JT84] the internal locales in $\mathcal{S}W_2$ are equivalent to locale maps with codomain W_2, say $p : X \to W_2$. The category of these is (by definition) the slice category $\mathbf{Loc}/W_2$. The pullback functor $f^* : \mathbf{Loc}/W_2 \to \mathbf{Loc}/W_1$ generalizes the inverse image functor on sheaves, since when sheaves are considered as local homeomorphisms, the inverse image functor acts by pullback. If F is a topos-valid locale construction, hence applicable in each slice of **Loc**, one can ask whether it "commutes with change of base" – whether $f^*(F(p))$ is (homeomorphic to) $F(f^*(p))$.

A key issue is that f^* on locales is calculated by applying the inverse image functor f^* not to the internal frames, but to the presentations. (See, e.g., [Vic04].) The frames themselves in general require the powerset for their construction, and inverse image functors do not preserve frame structure. Suppose, internally in $\mathcal{S}W_2$, that $\Omega(X \xrightarrow{p} W_2) = \mathrm{Fr}\langle G|R\rangle$ with (G, R) an internal presentation. G is an object in $\mathcal{S}W_2$, and R is an object that is equipped with structure enabling its elements to be understood as pairs of sets of finite sets of elements of G (standing for joins of finite meets). Then $\Omega(f^*(X \xrightarrow{p} W_2)) \cong \mathrm{Fr}\langle f^*(G|R)\rangle$ – here the first f^* is locale pullback, while the second is the inverse image functor, applied to the presentation including all the associated structure of R.

Theorem 3 *Let $F(X)$ be a topos-valid and functorial construction of locales from locales. We write $F_W : \mathbf{Loc}/W \to \mathbf{Loc}/W$ for its operation in $\mathcal{S}W$. Let F' be a geometric construction of frame presentations from frame presentations,*

such that if $\Omega X \cong \mathrm{Fr}\langle G|R\rangle$ *then* $\Omega F(X) \cong \mathrm{Fr}\langle F'(G,R)\rangle$ *holds, topos-validly. Then for any* $f : W_1 \to W_2$ *and locale* X *over* W_2 *we have* $F_{W_1}(f^*(X)) \cong f^*(F_{W_2}(X))$.

Proof. Suppose X is presented (in $\mathcal{S}W_2$) by $\Omega X = \mathrm{Fr}\langle G|R\rangle$. Then in $\mathcal{S}W_1$ we have

$$\begin{aligned}\Omega F_{W_1}(f^*(X)) &\cong \mathrm{Fr}\langle F'(f^*(G,R))\rangle \\ &\cong \mathrm{Fr}\langle f^*(F'(G,R))\rangle \text{ (by geometricity of } F') \\ &\cong \Omega(f^*(F_{W_2}(X))).\end{aligned}$$

■

[Vic04] shows that any frame presentation can be geometrically transformed into an equivalent one in the *DL-site* form $\mathrm{Fr}\langle G \text{ (qua DL)}|R\rangle$, where G is a distributive lattice (DL) and each relation in R is of the form $a \leq \bigvee_i^{\uparrow} b_i$ where the family $(b_i)_{i\in I}$ is directed. A convenient way to show geometricity of a construction F is often to show how geometrically it transforms DL-site presentations into other (possibly more general) presentations.

Our prime examples of geometric constructions are the powerlocales. We shall introduce the connected Vietoris powerlocale as a geometric construction on locales, and show how known topos-valid discussions of sublocales and their properties (compactness, overtness, connectedness) can be related to points and maps involving the powerlocales. This then opens up a geometric discussion. In particular, sublocales of special kinds become models of geometric theories describing them in terms of certain open covers. In terms of these open covers, the manipulations become geometric. We shall make particular use of [Vic05a] and [Vic03], which describe locales corresponding to metric completions, using geometric theories of Cauchy filters of formal balls and extend the approach to their powerlocales.

3.5 Other reading

The importance of geometric theories has been known in topos theory all along, and is seen most clearly in the idea of *classifying topos*. A classifying topos for a (predicate) geometric theory T is essentially defined as one whose generalized points are the models of T. Sites, and Grothendieck topologies, can be viewed as particular forms of presentation for geometric theories. [Wra79] is a good example of a work where notions are systematically expressed in geometric form, and [MW86] one where locales are viewed as spaces of points. [Vic07a] explains in some detail how to understand toposes as spaces of points, and gives a rational reconstruction of the machinery of topos theory from this point of view. [Vic99] and [Vic01] give examples of the technique including non-localic toposes (for predicate theories). [Vic04] discusses geometricity for locale constructions, and in particular for the powerlocales used extensively in the present paper.

Geometricity is also relevant in formal topology (see [Sam87]), another approach to point-free topology, based foundationally on predicative type theory.

The main feature of this is that the powerset is not admitted as a set construction. Consequently, frames cannot in general be carried by sets, and point-free topology is instead described entirely using sites (the "formal topology"). Thus formal topology (i) rejects the non-geometric construction of powerset, and (ii) in effect works entirely with generators and relations. The geometric approach is generally compatible with formal topology, and its techniques have been transferred for sublocales (see [Vic07b]), powerlocales (see [Vic06], [Vic05b]) and connectedness (see [Vic09]). We expect that this will also be possible for the techniques in the present paper.

4 Connectedness

In this section we summarize some of the main topos-valid results about connectedness of locales. These are mostly already known – see, e.g., [Joh84]. However, we shall also find it convenient to define a new and stronger notion of "strong connectedness", which characterizes the points of our connected Vietoris powerlocale. Some related discussions appear in [Tay05]. The standard topos-valid definition is –

Definition 4 *A locale X is* connected *if every map from X to a discrete locale I is constant – that is to say, it factors via* 1*.*

Note that under Definition 4 *the empty space is not connected.* For the empty locale $\emptyset$ (no points, one open) is discrete, but the identity map on it cannot factor via a global point $1 \to \emptyset$ since there are none. (This is in line with the standard convention that $\emptyset$ does not count as a connected *component* of a space. It is analogous to deeming the natural number 1 to be not prime.) In fact (Proposition 5), any connected locale is positive.

If I is a set, we also write I for the corresponding discrete locale, with ΩI the powerset $\mathcal{P}I$. There is a frame presentation of $\mathcal{P}I$ as

$$\mathrm{Fr}\langle s_i \ (i \in I) \mid \top \leq \bigvee_i s_i$$
$$s_i \wedge s_j \leq \bigvee \{s_k \mid i = k = j\} \ (i, j \in I)\rangle$$

where the generating symbol s_i corresponds to the singleton $\{i\}$. Using this presentation, one sees that connectedness translates into the following property of the frame ΩX. Let U_i $(i \in I)$ be a set-indexed family of opens of X, and suppose the U_is are *pairwise disjoint:* if $i, j \in I$ then

$$U_i \wedge U_j \leq \bigvee \{U_k \mid i = k = j\}.$$

(Note that we do not assume I has decidable equality, so we cannot simply say $U_i \wedge U_j \leq \bot$ whenever $i \neq j$.) Then if $\bigvee_i U_i = \top$, we must have $U_i = \top$ for some i.

Proposition 5 *Let X be a connected locale. Then X is positive.*

Proof. Suppose $X \leq \bigvee_{i \in I} U_i$ with each U_i open. If $i \in I$ then (I is inhabited and so) $U_i \leq \bigvee\{\top \mid I \text{ inhabited}\}$. This join is of a family indexed by the subsingleton set $I_0 = \{* \in 1 \mid I \text{ inhabited}\}$. Writing $V_* = \top$, we see $X \leq \bigvee_{i \in I} U_i \leq \bigvee_{* \in I_0} V_*$, a pairwise disjoint open cover. It follows there is some $* \in I_0$ such that $X \leq V_*$, and so I is inhabited. ■

It follows that, in the definition of connectedness, a map from a connected X to a discrete I factors *uniquely* via 1. For if $(U_i)_{i \in I}$ is a pairwise disjoint open cover of X, and $X \leq U_i$, $X \leq U_j$, then $X \leq U_i \wedge U_j \leq \bigvee\{U_k \mid i = k = j\}$. Hence by positivity of X there is some k with $i = k = j$.

These constructive results hold internally in a topos $\mathcal{S}W$ of sheaves over a locale W. Since an internal locale there is equivalent to a locale over W, i.e. a map $p : X \to W$, we can ask what it means for a locale over W to be "connected over W". The "discrete" locales over W are the local homeomorphisms with codomain W, so it follows that $p : X \to W$ is connected over W iff, for every commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \downarrow & \swarrow q & \\ W & & \end{array}$$

with q a local homeomorphism, there is a (unique) $h : W \to Y$ with $q \circ h = \mathrm{Id}_W$ (so h is a global point of Y over W) and $f = h \circ p$. Using the fact that local homeomorphisms are preserved under pullback, it is then not hard to see that p is connected over W iff p is *orthogonal* to local homeomorphisms, i.e. for every commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & W \\ f \downarrow & & \downarrow g \\ Y & \underset{q}{\longrightarrow} & W' \end{array}$$

with q a local homeomorphism there is a unique $k : W \to Y$ such that both triangles commute.

Classically, the connectedness property for maps from X to $I = 2$ implies the cases for all other non-empty I. To see this, for each $i \in I$ define $U_i' = \bigvee\{U_j \mid j \neq i\}$. We see either $X = U_i$ or ($X = U_i'$ and) X is disjoint from U_i. Consider the set $I' = \{i \in I \mid X \text{ disjoint from } U_i\}$. X is disjoint from $\bigvee_{i \in I'} U_i$, so if $I' = I$ then X is empty and so $X = U_i$ for any $i \in I$. On the other hand, if $I' \neq I$ then there is some i with $X = U_i$.

For good enough X we can constructively recover this sufficiency of the binary case.

Definition 6 *A locale X is* strongly connected *if –*

1. *X is compact and overt.*
2. *If $X \leq \bot$ then a contradiction follows.*

3. *If $X \leq U \vee V$ with U and V open, then either $X \leq U$ or $X \leq V$ or $U \wedge V$ is positive.*

(In [Tay05] Taylor has proved these properties of the closed real intervals $[x, y]$ in the context of his Abstract Stone Duality.) Note that condition (2), combined with compactness, implies that X is positive. For suppose $X \leq \bigvee_{i \in I} U_i$. By compactness we can assume I is finite. For Kuratowski finite sets, emptiness is decidable. However, the empty case is impossible by condition (2), and we deduce that I is inhabited.

Theorem 7 *Any strongly connected locale is connected.*

Proof. Let X be strongly connected, and let U_i $(i \in I)$ be a pairwise disjoint open cover. By compactness, we can assume I is Kuratowski finite, say $I = \{i_1, \ldots i_n\}$ (possibly with repetitions). Then $X \leq \bigvee_{j=1}^{n} V_j$ where $V_j = U_{i_j}$. The case $n = 0$ is impossible, by positivity of X, so $n \geq 1$ and $X \leq \bigvee_{j=1}^{n-1} V_j \vee V_n$. Hence either $X \leq \bigvee_{j=1}^{n-1} V_j$, and we can use induction, or $X \leq V_n$, and we are done, or $(\bigvee_{j=1}^{n-1} V_j) \wedge V_n = \bigvee_{j=1}^{n-1}(V_j \wedge V_n)$ is positive. By overtness of X, each open U is covered by the subsingleton set whose sole element is U, provided that it is positive. It follows that

$$\bigvee_{j=1}^{n-1} (V_j \wedge V_n) \leq \bigvee \{V_j \wedge V_n \mid 1 \leq j \leq n-1 \text{ and } V_j \wedge V_n \text{ positive}\}$$

and we deduce that $V_j \wedge V_n$ is positive for some j between 1 and $n - 1$. By pairwise disjointness of the U_is we have

$$V_j \wedge V_n \leq \{U_k \mid i_j = k = i_n\}$$

and it follows that $i_j = i_n$. Hence $\bigvee_{j=1}^{n} V_j = \bigvee_{j=1}^{n-1} V_j$ and we can use induction. ■

Classically, the Theorem has a partial converse. Suppose X is compact and connected. Classically, every locale is overt, with each open positive iff it is non-zero. If X is connected, then it also satisfies condition (3) of Definition 6. For suppose $X \leq U \vee V$, with U and V open. If $U \wedge V = \emptyset$ then by connectedness either $X \leq U$ or $X \leq V$. On the other hand, if $U \wedge V \neq \emptyset$ then $U \wedge V$ is positive. Constructively, I do not know whether there are compact, overt, connected locales that fail to be strongly connected.

5 The Vietoris powerlocale

The Vietoris powerlocale, the localic analogue of the Vietoris hyperspace, was introduced in [Joh85]; see also [Joh82]. For further remarks on its history see [Vic97]. For its technical development we shall largely follow [Vic97], [Vic95] and (for completions of metric spaces) [Vic03].

Our main discussion here will be topos-valid, in terms of frames. In particular, the results identifying the global points with certain sublocales, and so justifying the analogy with hyperspaces, are topos-valid. However, a key result (Theorem 11) is that the powerlocale constructions are geometric. The Section therefore provides a geometric approach to discussing those sublocales.

Definition 8 *Let X be a locale. Then the* Vietoris powerlocale VX *is defined by its frame*

$$\begin{aligned}\Omega VX = \mathrm{Fr}\langle \Box U, \Diamond U \ (U \in \Omega X) \mid & \Box \text{ preserves finite meets and directed joins,} \\ & \Diamond \text{ preserves all joins,} \\ & \Diamond U \wedge \Box V \leq \Diamond(U \wedge V) \\ & \Box(U \vee V) \leq \Box U \vee \Diamond V \rangle.\end{aligned}$$

Theorem 9 *Let X be a locale. Then the global points of VX are in 1-1 correspondence with the compact, overt, weakly semifitted sublocales of X.*

Proof. [Vic97] gives the full topos-valid argument, though its essence is already present in [Joh85]. Let us sketch some of its main steps.

If X has presentation in DL-site form, $\Omega X = \mathrm{Fr}\langle G \text{ (qua DL)} | R \rangle$, then the sublocale for a powerlocale point K can be presented by extra relations

$$\begin{aligned} & \top \leq U \quad (U \in G, K \vDash \Box U) \\ & U \leq \bigvee \{ \top \mid K \vDash \Diamond U \} \quad (U \in G) \end{aligned}$$

(The first kind gives a fitted sublocale, the second a weakly closed sublocale, so combining them gives weakly semifitted.) Hence the sublocale can be derived geometrically from the powerlocale point.

Starting from a sublocale K (we shall usually use the same symbol for point and sublocale), the corresponding point is in $\Box U$ iff $K \leq U$, and is in $\Diamond U$ iff U is positive modulo K – in other words, if $K \wedge U$ is positive. Note that preservation of finite meets by $\Box$ is then obvious, and preservation of directed joins is compactness of K. Preservation of joins by $\Diamond$ follows from overtness of K, since if a join $\bigvee_i U_i$ of opens is positive in an overt locale then so is one of the U_is. The relation $\Diamond U \wedge \Box V \leq \Diamond(U \wedge V)$ is obvious, since if $K \leq V$ then $K \wedge U = K \wedge U \wedge V$. For the remaining axiom, suppose $K \leq U \vee V$. By overtness of K, in ΩK we have that $K \wedge V = \bigvee I$ where I is the subsingleton set $\{W \in \Omega K \mid W = K \wedge V \text{ and } W \text{ is positive}\}$. By compactness, the cover $\{U\} \cup I$ has a finite subcover $\{U\} \cup I_0$ where I_0 is a Kuratowski finite subset of I. (Note that subsingletons are not in general Kuratowski finite, even though the singleton set 1 is.) Emptiness of Kuratowski finite sets is decidable. If $I_0 = \emptyset$ then $\{U\}$ covers K and the point is in $\Box U$, while if I_0 is inhabited then its only possible element is $K \wedge V$, and that is in I only if it is positive, which means the point is in $\Diamond V$. ■

Definition 10 *Let X be a locale. The* positive Vietoris powerlocale V^+X *is the sublocale of VX presented by an extra relation $\top \leq \Diamond \top$, or, equivalently, $\Box \bot \leq \bot$.*

To see the equivalence, note, for instance, that given $\top \leq \Diamond\top$ we have

$$\Box\bot = \Diamond\top \wedge \Box\bot \leq \Diamond(\top \wedge \bot) = \Diamond\bot = \bot.$$

Note also a more general consequence in V^+X, that

$$\Box U = \Diamond\top \wedge \Box U \leq \Diamond(\top \wedge U) = \Diamond U.$$

The global points of V^+X are those points of VX that are, as sublocales, positive.

[Joh85] shows that V is the functor part of a monad (V, η, μ), with $\eta^*(\Box U) = U$, $\mu^*(\Box U) = \Box\Box U$ and similarly for $\Diamond$. The monad structure restricts to V^+.

Our discussion will also involve the upper and lower powerlocales P_U and P_L (as well as their positive parts P_U^+ and P_L^+), which originated in [Smy78], [Rob86], [Win85]. (See [Vic97] for details and more on the history of these constructions.)

$$\Omega P_U X = \mathrm{Fr}\langle \Box U \ (U \in \Omega X) \mid \Box \text{ preserves finite meets and directed joins}\rangle,$$
$$\Omega P_L X = \mathrm{Fr}\langle \Diamond U \ (U \in \Omega X) \mid \Diamond \text{ preserves all joins}\rangle.$$

These too give monads $(P_U, \uparrow, \prod)$ and $(P_L, \downarrow, \bigsqcup)$, in a directly analogous way, and they also have positive parts.

The global points of $P_U X$ are in bijection with the compact, fitted sublocales of X, and those of $P_L X$ with the overt, weakly closed sublocales. The points of $P_U^+ X$ and $P_L^+ X$ are those of $P_U X$ and $P_L X$ whose corresponding sublocales are positive.

Clearly VX embeds as a sublocale of $P_U X \times P_L X$. We write $\langle \Uparrow, \Downarrow \rangle$ for this embedding, so $\Uparrow^* (\Box U) = \Box U$ and $\Downarrow^* (\Diamond U) = \Diamond U$. If K is a point of VX then $\Uparrow K$ is the *fitted hull* of K, i.e. the meet of all its open neighbourhoods, and $\Downarrow K$ is the *weak closure* of K, the smallest weakly closed sublocale bigger than K. K can be recovered as the sublocale meet $\Uparrow K \wedge \Downarrow K$.

Theorem 11 *All three powerlocales, as well as their positive versions, are geometric.*

Proof. [Vic04] shows geometricity of the upper and lower powerlocales. If $\Omega X = \mathrm{Fr}\langle G \text{ (qua DL)} | R\rangle$ then $\Omega P_U X = \mathrm{Fr}\langle G \text{ (qua } \wedge\text{-semilattice)} | R\rangle$ and $\Omega P_L X = \mathrm{Fr}\langle G \text{ (qua } \vee\text{-semilattice)} | R\rangle$. A presentation of $P_U X \times P_L X$ is got with a disjoint union of those for $P_U X$ and $P_L X$, with the two copies of generators G labelled with $\Box$ and $\Diamond$. Since every $U \in \Omega X$ is a directed join of generators, it follows that for the mixed relations for VX it suffices to take $U, V \in G$. thus we can construct the relations for VX geometrically. ■

Remark 12 *The presentational techniques of the proof of the Theorem also show that all three powerlocales preserve embeddings.*

Sublocales of X corresponding to points of $P_U X$ or $P_L X$ can be presented geometrically as for VX, but another geometric description is to use the specialization order. Suppose x, K and L are (generalized) points of $X, P_U X, P_L X$. Then by [Vic95], x is in K iff $\uparrow x \sqsupseteq K$, and is in L iff $\downarrow x \sqsubseteq L$.

Geometricity allows us to generalize the above analysis of global points of powerlocales. Let F be a powerlocale construction, with $F_W : \mathbf{Loc}/W \to \mathbf{Loc}/W$ its action on locales at stage W. A generalized point $K : W \to F(X)$ is equivalent to a map $\langle W, K\rangle : W \to W \times F(X) \cong F_W(W \times X)$ over W, and hence a global point of $F_W(W \times X)$. Since the results about powerlocale points are topos-valid, these points $\langle W, K\rangle$ are equivalent to certain sublocales of $W \times X$. (Note that a map in $\mathbf{Loc}/W$ is an embedding (i.e., for locales, a regular monic) iff it is an embedding in $\mathbf{Loc}$. This is because an equalizer in $\mathbf{Loc}$ of maps $g, h : X \to Y$, with X in $\mathbf{Loc}/W$ by $p : X \to W$, is also an equalizer in $\mathbf{Loc}/W$ of $\langle p, g\rangle$ and $\langle p, h\rangle : X \to W \times Y$.) [Vic97] gives fuller details about the conditions that correspond to compact, overt and weakly semifitted when one is working over W. For a generalized Vietoris point $K : W \to VX$, the pair (w, x) is in the corresponding sublocale of $W \times X$ iff $\uparrow x \sqsupseteq \Uparrow K(w)$ and $\downarrow x \sqsubseteq \Downarrow K(w)$.

A good example of pointwise reasoning is the Heine-Borel map $HB_C : \leq \to V^+\mathbb{R}$ (Section 7.2).

5.1 Direct images of Vietoris points

Since V is functorial, if $f : X \to Y$ and K is a global point of VX, we also have $Vf(K)$ a global point of VY. We should therefore ask how the sublocale $Vf(K)$ of Y is determined by K. We shall see that the composite $K \hookrightarrow X \xrightarrow{f} Y$ factors via $Vf(K) \hookrightarrow Y$, so we should investigate the map $K \to Vf(K)$.

The arguments of this section are largely topos-valid in their discussion of sublocales and direct images, but by relating them to powerlocale points they provide a way to discuss them geometrically.

Lemma 13 *Let K be a global point of VX, U an open in X and $p \in \Omega$. Then $K \leq (X - U) \vee\, !^* p$ iff $(K \vDash \Diamond U) \to p$.*

Proof. First, note that $K \leq (X - U) \vee\, !^* p$ iff $K \wedge U \leq\, !^* p$, i.e. $U \leq\, !^* p$ modulo K. ($X - U$ is the closed complement of the open sublocale U.)

$\Rightarrow$: If $K \vDash \Diamond U$ then U is positive modulo K and the join $!^* p = \bigvee\{\top \mid p\}$ must be inhabited. Hence p holds.

$\Leftarrow$: By overtness of K, U is, modulo K, a join of positive (modulo K) opens. This can be rephrased as

$$U = \bigvee\{U' \mid U' = U,\, K \vDash \Diamond U'\} \text{ modulo } K.$$

Thus to show $U \leq\, !^* p$ modulo K it suffices to assume $K \vDash \Diamond U$. But then p holds, so $!^* p$ is X and $U \leq\, !^* p$. ■

Proposition 14 *Let $f : X \to Y$ and let K be a global point of VX. Then $Vf(K)$ is the weakly semifitted closure of the direct image of K under f.*

Proof. The point $\Uparrow \circ Vf(K)$ of $P_U Y$ is the fitted hull of $Vf(K)$, i.e. the meet of all the open sublocales U for which $Vf(K)$ is in $\Box U$. We have

$$\begin{aligned} Vf(K) \vDash \Box U &\Leftrightarrow K \vDash \Box f^* U \Leftrightarrow K \leq f^* U \\ &\Leftrightarrow f \text{ maps } K \text{ into } U. \end{aligned}$$

It follows that $\Uparrow \circ Vf(K)$ is the fitted hull of the image of K under f.

Similarly, $\Downarrow \circ Vf(K)$ is the weak closure of $Vf(K)$, i.e. the meet of all the weakly closed sublocales containing $Vf(K)$. A sublocale of Y is weakly closed iff it is a meet of sublocales of the form $(Y - U) \vee !^* p$ (U open in Y, $p \in \Omega$), and using Lemma 13 we find

$$\begin{aligned} Vf(K) \leq (Y - U) \vee !^* p &\Leftrightarrow (Vf(K) \vDash \Diamond U) \to p \\ &\Leftrightarrow (K \vDash \Diamond f^* U) \to p \\ &\Leftrightarrow K \leq (X - f^* U) \vee !^* p = f^*((Y - U) \vee !^* p). \end{aligned}$$

It follows that $\Downarrow \circ Vf(K)$ is the weak closure of the image of K under f.

As a point of VY, $Vf(K)$ is weakly semifitted, in other words a meet of opens and weakly closed sublocales. It follows that it is the meet of such sublocales containing the image of K, in other words the weakly semifitted closure of the image. ■

In general, this weakly semifitted closure is bigger than the image. For an example, let $\mathbb{S}$ be the Sierpiński locale, the ideal completion of the 2-element poset $\{\bot \sqsubseteq \top\}$, and let $\mathbb{S}_2$ be the ideal completion of $\{\bot \sqsubseteq 0 \sqsubseteq \top\}$. Let $f : \mathbb{S} \to \mathbb{S}_2$ be the map suggested by the notation, an embedding. It is not surjective, but the weakly semifitted closure is the whole of $\mathbb{S}_2$.

Proposition 15 *Let $f : X \to Y$ and let K be a global point of VX. Then the restricted map $f : K \to Vf(K)$ is dense and codense.*

Proof. In fact it is strongly dense. From Lemma 13 we have f strongly dense iff for all p, U if $(K \vDash \Diamond f^* U) \to p$ then $(Vf(K) \vDash \Diamond U) \to p$. But this is clear.

For codenseness we must show that if $\top \leq f^* U$ (modulo K) then $\top \leq U$ (modulo $Vf(K)$). This is clear from the proof of Proposition 14. ■

Proposition 16 *Let $f : X \to Y$ with Y regular, and let K be a global point of VX. Then the restricted map $f : K \to Vf(K)$ is a surjection.*

Proof. K is compact and $Vf(K)$ regular, so the image of K under f is closed in $Vf(K)$. Density then implies that the image is the whole of $Vf(K)$. ■

We shall be interested in the situation where X and Y are the real line $\mathbb{R}$. In the presence of countable dependent choice, the argument in [BB85] shows

that there is then at least approximate surjectivity on points: if y is a point of $Vf(K)$ and $\varepsilon > 0$, then there is some x in K such that $f(x)$ is within ε of y. However, we do not wish to assume any choice and so we take localic surjectivity as the basic way of stating the existence principle that would more normally appear as surjectivity on points.

Let us briefly outline how generalized points can be dealt with. For clarity, we temporarily write $\widetilde{K} \hookrightarrow W \times X$ for the sublocale corresponding to $K : W \to VX$. Working over W, we find then $Vf \circ K$ corresponds to the "weakly semifitted over W" closure of the image of $\widetilde{K} \hookrightarrow W \times X \stackrel{W\times f}{\longrightarrow} W \times Y$, and that $\widetilde{K} \to \widetilde{Vf \circ K}$ is dense and codense. If Y is regular then $W \times Y$ is regular over W, so $\widetilde{K} \to \widetilde{Vf \circ K}$ is surjective over W. It follows that it is surjective in **Loc**. This is because any locale map $g : \widetilde{Vf \circ K} \to Z$ can be converted to a map $\langle \alpha, g \rangle : \widetilde{Vf \circ K} \to W \times Z$ over W, where α is the map $\widetilde{Vf \circ K} \hookrightarrow W \times Y \to W$.

6 The connected Vietoris powerlocale

We now introduce our new powerlocale. This section is topos-valid, and we also show the geometricity of the powerlocale. As we shall see (Theorem 22), its points are those points of the Vietoris powerlocale whose corresponding sublocales are strongly connected. In fact, the last axiom in the presentation corresponds directly to condition (3) in Definition 6.

Definition 17 *Let X be a locale. Then the* connected Vietoris powerlocale *V^cX is defined by*

$$\begin{aligned}\Omega V^c X = \mathrm{Fr}\langle \Box U, \Diamond U \ (U \in \Omega X) \mid & \Box \text{ preserves finite meets and directed joins,} \\ & \Diamond \text{ preserves all joins,} \\ & \top \leq \Diamond \top \\ & \Diamond U \wedge \Box V \leq \Diamond(U \wedge V) \\ & \Box(U \vee V) \leq \Box U \vee \Box V \vee \Diamond(U \wedge V) \rangle.\end{aligned}$$

From the relations we can deduce (as in V^+X) that

$$\Box V = \Diamond \top \wedge \Box V \leq \Diamond V$$

and hence

$$\Box(U \vee V) \leq \Box U \vee \Box V \vee \Diamond(U \wedge V) \leq \Box U \vee \Diamond V.$$

It follows that V^cX is a sublocale of V^+X.

We also deduce

$$\begin{aligned}\Box(U \vee V) \wedge \Diamond U \wedge \Diamond V &\leq (\Box U \vee \Box V \vee \Diamond(U \wedge V)) \wedge \Diamond U \wedge \Diamond V \\ &\leq \Diamond(U \wedge V),\end{aligned}$$

which is a dual of the new relation (got by reversing the order and exchanging $\wedge$ with $\vee$ and $\Box$ with $\Diamond$). In fact,

Proposition 18 *V^cX can be equivalently presented using the relations to say $\Box$ preserves directed joins and finite meets, $\Diamond$ preserves joins, and*

$$\begin{aligned}\Box\bot &\leq \bot\\ \Box(U\vee V) &\leq \Box U\vee\Diamond V\\ \Box(U\vee V)\wedge\Diamond U\wedge\Diamond V &\leq \Diamond(U\wedge V).\end{aligned}$$

Lemma 19 *Let $U_i\in\Omega X$ $(1\leq i\leq n)$. Then in ΩV^cX,*

$$\Box(\bigvee_{i=1}^n U_i)\leq\bigvee_{i=1}^n\Box U_i\vee\bigvee_{i\neq j}\Diamond(U_i\wedge U_j).$$

Proof. The case $n=0$ says $\Box\bot\leq\bot$, which we have from the V^+ axiom. For the induction step,

$$\begin{aligned}\Box(\bigvee_{i=1}^{n+1}U_i)&\leq\Box(\bigvee_{i=1}^n U_i)\vee\Box U_{n+1}\vee\Diamond(\bigvee_{i=1}^n U_i\wedge U_{n+1})\\ &\leq\bigvee_{i=1}^n\Box U_i\vee\bigvee\{\Diamond(U_i\wedge U_j)\mid 1\leq i,j\leq n\text{ and }i\neq j\}\\ &\qquad\vee\Box U_{n+1}\vee\bigvee_{i=1}^n\Diamond(U_i\wedge U_{n+1})\\ &=\bigvee_{i=1}^{n+1}\Box U_i\vee\bigvee_{i\neq j}\Diamond(U_i\wedge U_j).\end{aligned}$$

■

Remark 20 *Essentially the same proof as in Theorem 7 also shows that if U_i $(i\in I)$ is a pairwise disjoint family of opens in X, then $\Box(\bigvee_i U_i)\leq\bigvee_i\Box U_i$.*

Theorem 21 *The monad structure on V restricts to V^c.*

Proof. We know already that the monad structure restricts to V^+, so there is no need to consider the relation $\top\leq\Diamond\top$.

First we show V^c is a functor. Suppose $f:X\to Y$ is a map of locales. $(Vf)^*$ takes $\Box U$ and $\Diamond U$ to $\Box f^*U$ and $\Diamond f^*U$. Modulo V^cX we have

$$\begin{aligned}(Vf)^*(\Box(U\vee V))&=\Box(f^*(U)\vee f^*(V))\\ &\leq\Box(f^*(U))\vee\Box(f^*(V))\vee\Diamond(f^*(U)\wedge f^*(V))\\ &=(Vf)^*(\Box U\vee\Box V\vee\Diamond(U\wedge V)).\end{aligned}$$

Hence Vf restricts to a map $V^cf:V^cX\to V^cY$. V^c is functorial because V is.

Next we show the unit $\eta:X\to VX$ factors via V^cX.

$$\begin{aligned}\eta^*(\Box(U\vee V))&=U\vee V=U\vee V\vee(U\wedge V)\\ &=\eta^*(\Box U\vee\Box V\vee\Diamond(U\wedge V)).\end{aligned}$$

Finally, we show that the multiplication $\mu : VVX \to VX$ restricts to V^cX. In V^cV^cX we have

$$\begin{aligned}\mu^*(\Box(U \vee V)) &= \Box\Box(U \vee V) \\ &\leq \Box(\Box U \vee \Box V \vee \Diamond(U \wedge V)) \\ &\leq \Box\Box U \vee \Box\Box V \vee \Box\Diamond(U \wedge V) \vee \Diamond(\Box U \wedge \Box V) \\ &\quad \vee \Diamond(\Box U \wedge \Diamond(U \wedge V)) \vee \Diamond(\Box V \wedge \Diamond(U \wedge V)) \\ &\leq \Box\Box U \vee \Box\Box V \vee \Diamond\Diamond(U \wedge V) = \mu^*(\Box U \vee \Box V \vee \Diamond(U \wedge V)).\end{aligned}$$

■

Theorem 22 *Let X be a locale. Then the correspondence of Theorem 9 restricts to a 1-1 correspondence between the global points of V^cX and the weakly semifitted, strongly connected sublocales of X.*

Proof. A strongly connected sublocale is compact and overt, and so the weakly semifitted, strongly connected sublocales of X already correspond to certain points of VX. The extra axioms in V^cX are direct translations of the remaining conditions in Definition 6. ■

Theorem 23 *The V^c construction is geometric.*

Proof. Similar to Theorem 11. ■

Proposition 24 *A locale X is strongly connected iff there is a global point K_X of V^cX such that $\Downarrow K_X : 1 \to P_LX$ and $\Uparrow K_X : 1 \to P_UX$ are respectively right and left adjoint to the unique maps to* 1.

Proof. The adjointness conditions say that $\Downarrow K_X$ and $\Uparrow K_X$ are greatest and least amongst all the *generalized* points of P_LX and P_UX, and by [Vic95] a powerlocale point has the respective condition iff its corresponding sublocale is the whole of X.

$\Leftarrow$: The sublocale for K_X is the meet of those for $\Downarrow K_X$ and $\Uparrow K_X$, i.e. X, which is therefore strongly connected by Theorem 22.

$\Rightarrow$: X is a weakly semifitted, strongly connected sublocale of itself, and so corresponds to a point K_X. Its weak closure and fitted hulls are both X. ■

The condition in Proposition 24 is geometric, and so we see that strong connectedness is preserved by change of base (pullback) functors.

6.1 Strength and product maps

This section discusses products of strongly connected locales, showing in particular (Theorem 28) that they are again strongly connected. Access to geometric methods is given by a product map $\times : V^cX \times V^cY \to V^c(X \times Y)$, and this is related to the strength of the V^c monad.

Let (T, η, μ) be a monad on a category $\mathcal{C}$ with finite products. A *strength* for the monad (T, η, μ) ([Koc72]; see also [Mog89]) is defined as a natural transformation $\tau_{X,Y} : X \times TY \to T(X \times Y)$ that satisfies the following conditions. (Here $r_X : 1 \times X \to X$ and $\alpha_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$ denote the natural isomorphisms.)

$$\begin{aligned} r_{TX} &= \tau_{1,X}; T(r_X) \\ \tau_{X\times Y,Z}; T(\alpha_{X,Y,Z}) &= \alpha_{X,Y,TZ}; (X \times \tau_{Y,Z}); \tau_{X,Y\times Z} \\ (X \times \eta_Y); \tau_{X,Y} &= \eta_{X\times Y} \\ (X \times \mu_Y); \tau_{X,Y} &= \tau_{X,TY}; T(\tau_{X,Y}); \mu_{X\times Y} \end{aligned}$$

To see the existence of strengths for P_L and P_U, first note from [Vic95] that for both the lower and upper powerlocales, there are product maps

$$\times : P_\bullet X \times P_\bullet Y \to P_\bullet(X \times Y)$$

(where $\bullet$ stands for either L or U). Each takes a pair of sublocales (K, L) to the product $K \times L$. Moreover, $\times$ is an adjoint (right for P_L, left for P_U) of the map $\langle P_\bullet p, P_\bullet q\rangle : P_\bullet(X \times Y) \to P_\bullet X \times P_\bullet Y$, where p and q are the projection maps.

For the lower powerlocale P_L, the product map is defined by

$$\times^*(\Diamond(U \times V)) = \Diamond U \times \Diamond V.$$

Since the opens $U \times V$ form a base of opens for the product, and $\Diamond$ preserves all joins, this formula enables us to calculate $\times^*(W)$ for any open W of $P_L(X \times Y)$. (This does not in itself guarantee that the result is well-defined, independent of representation of W, but that is proved in [Vic95].)

Now we can define a map

$$\tau = (\downarrow \times P_L Y); \times : X \times P_L Y \to P_L X \times P_L Y \to P_L(X \times Y),$$

and it will have

$$\tau^*(\Diamond(U \times V)) = U \times \Diamond V.$$

From this it is easy to show that τ has the properties required of a strength.

Similarly for the upper powerlocale P_U, the product map is defined by

$$\times^*(\Box(U \odot V)) = \Box U \odot \Box V$$

(Recall the notation $U \odot V$ from Section 2.) Because $\Box$ preserves directed joins and finite meets, it follows that from the above formula for $\times^*(\Box(U \odot V))$ we can calculate $\times^*(W)$ for any open W of $P_U(X \times Y)$. Again the strength can be defined as $(\uparrow \times P_U Y); \times$, and then

$$\tau^*(\Box(U \odot V)) = U \odot \Box V.$$

In each case, the product map can be derived from the strength τ of the monad as (writing μ for the multiplication for the monad)

$$\tau; P_\bullet \tau; \mu : P_\bullet X \times P_\bullet Y \to P_\bullet(P_\bullet X \times Y) \to P_\bullet P_\bullet(X \times Y) \to P_\bullet(X \times Y).$$

(Note that in each case this is equal to the analogous composite $P_\bullet X \times P_\bullet Y \to P_\bullet(X \times P_\bullet Y) \to P_\bullet P_\bullet(X \times Y) \to P_\bullet(X \times Y)$.)

We can extend these results to the Vietoris powerlocale.

Proposition 25 *The same formulae give a strength $\tau : X \times VY \to V(X \times Y)$.*

Proof. We must check that it respects the mixed Vietoris axioms. Let us consider $\Box(A \vee A') \leq \Box A \vee \Diamond A'$, where A and A' are opens for the product locale $X \times Y$. Because both $\Box$ and $\Diamond$ preserve directed joins, it suffices to consider the case where A and A' are finite joins of basics $U \times V$, or, equivalently, finite meets of preframe basics $U \odot V$. We shall use induction on n where $A' = \bigvee_{j=1}^{n} U'_j \times V'_j$ – the base case $n = 0$ is obvious.

We first prove the result for the case $n = 1$. Suppose we have $A = \bigwedge_{i=1}^{m} U_i \odot V_i$, so

$$\begin{aligned} A \vee U' \times V' &= (\bigwedge_{i=1}^{m} U_i \odot V_i) \vee (U' \odot \bot \wedge \bot \odot V') \\ &= \bigwedge_{i=1}^{m} (U_i \vee U') \odot V_i \wedge \bigwedge_{i=1}^{m} U_i \odot (V_i \vee V'). \end{aligned}$$

Then

$$\begin{aligned} \tau^*(\Box(A \vee U' \times V')) &= \bigwedge_{i=1}^{m} (U_i \vee U') \odot \Box V_i \wedge \bigwedge_{i=1}^{m} U_i \odot \Box(V_i \vee V') \\ &\leq \bigwedge_{i=1}^{m} (U_i \vee U') \odot \Box V_i \wedge \bigwedge_{i=1}^{m} U_i \odot (\Box V_i \vee \Diamond V') \\ &= (\bigwedge_{i=1}^{m} U_i \odot \Box V_i) \vee (U' \times \Diamond V') \\ &= \tau^*(\Box A) \vee \tau^*(\Diamond(U' \times V')). \end{aligned}$$

Now for the induction step. If $A' = B' \vee U' \times V'$ then

$$\begin{aligned} \tau^*(\Box(A \vee A')) &= \tau^*(\Box(A \vee B' \vee U' \times V')) \\ &\leq \tau^*(\Box(A \vee B')) \vee \tau^*(\Diamond(U' \times V')) \\ &\leq \tau^*(\Box A) \vee \tau^*(\Diamond B') \vee \tau^*(\Diamond(U' \times V')) \qquad \text{by induction} \\ &= \tau^*(\Box A) \vee \tau^*(\Diamond B' \vee \Diamond(U' \times V')) = \tau^*(\Box A) \vee \tau^*(\Diamond A'). \end{aligned}$$

The other mixed axiom is similar.

The fact that it is a strength is easily deduced from the corresponding fact for the upper and lower powerlocales. ■

The strength now gives a product map $\times : VX \times VY \to V(X \times Y)$, with $\langle x, y \rangle \in K \times L$ iff $x \in K$ and $y \in L$. We can see this by geometric reasoning. By definition $\langle x, y \rangle \in K \times L$ iff $\uparrow \langle x, y \rangle \sqsupseteq \Uparrow (K \times L) = (\Uparrow K) \times (\Uparrow L)$ and

$\downarrow \langle x, y\rangle \sqsubseteq \Downarrow (K \times L) = (\Downarrow K) \times (\Downarrow L)$. For the first (upper) part of this, the adjunction property of the product map shows that the condition is equivalent to $\langle P_U p(\uparrow \langle x, y\rangle), P_U q(\uparrow \langle x, y\rangle)\rangle = \langle \uparrow x, \uparrow y\rangle \sqsupseteq \langle \Uparrow K, \Uparrow L\rangle$, i.e. $\uparrow x \sqsupseteq \Uparrow K$ and $\uparrow y \sqsupseteq \Uparrow L$. Similarly, the second (lower) condition is equivalent to $\downarrow x \sqsubseteq \Downarrow K$ and $\downarrow y \sqsubseteq \Downarrow L$. These give the result.

It is also clear that the strengths and product maps restrict to the positive powerlocales P_L^+, P_U^+ and V^+. We now show that they also restrict to the connected Vietoris powerlocale. We first prove a lemma.

Lemma 26 *Let $\tau : X \times P_U Y \to P_U(X \times Y)$ be the strength for the upper powerlocale. Let $I = \{1, \ldots, n\}$, and for each $i \in I$ let U_i and V_i be opens for X and Y respectively. Then*

$$\tau^*(\Box(\bigvee_{i\in I} U_i \times V_i)) = \bigvee_{I_0 \in \mathcal{F}I} (\bigwedge_{i\in I_0} U_i \times \Box \bigvee_{i\in I_0} V_i).$$

Proof. It is proved in [VT04, Theorem 19] that for any locales X, Y and Z, if $q : \Omega Y \to \Omega Z$ is a dcpo morphism (preserving directed joins), then a dcpo morphism $q_X : \Omega(X \times Y) \to \Omega(X \times Z)$ can be defined by

$$q_X(\bigvee_{i\in I} U_i \times V_i) = \bigvee_{I_0 \in \mathcal{F}I} (\bigwedge_{i\in I_0} U_i \times q(\bigvee_{i\in I_0} V_i)).$$

Moreover, if q preserves finite meets then so does q_X. Applying this with $Z = P_U Y$ and $q = \Box$, we obtain a preframe homomorphism (preserving finite meets and directed joins) $\Box_X : \Omega(X \times Y) \to \Omega(X \times P_U Y)$. We have

$$\begin{aligned}\Box_X(U \odot V) &= \Box_X(U \times \top \vee \top \times V)\\ &= \top \times \Box\bot \vee U \times \Box\top \vee \top \times \Box V \vee U \times \Box\top\\ &= U \times \top \vee \top \times \Box V = U \odot \Box V\end{aligned}$$

and it follows that $\Box_X = \tau^* \circ \Box$. ■

Theorem 27 *The Vietoris strength restricts to a connected Vietoris strength $\tau : X \times V^c Y \to V^c(X \times Y)$.*

Proof. We must show that τ^* respects the relation $\Box(A \vee B) \leq \Box A \vee \Box B \vee \Diamond(A \wedge B)$. As before, it suffices to assume A and B are finite joins of basic opens $U \times V$. Let us take $A = \bigvee_{i\in I_1} U_i \times V_i$ and $B = \bigvee_{i\in I_2} U_i \times V_i$, with $I = I_1 \cup I_2$.

Using Lemma 26, we see that

$$\tau^*(\Box(A \vee B)) = \bigvee_{I_0 \in \mathcal{F}I} (\bigwedge_{i\in I_0} U_i \times \Box \bigvee_{i\in I_0} V_i).$$

Given I_0, we can find $I_0 = I_1' \cup I_2'$ with each $I_\lambda' \subseteq I_\lambda$, and then

$$\Box \bigvee_{i\in I_0} V_i \leq \Box \bigvee_{i\in I_1'} V_i \vee \Box \bigvee_{i\in I_2'} V_i \vee \bigvee_{(i,j)\in I_1' \times I_2'} \Diamond(V_i \wedge V_j).$$

For the first of these three disjuncts we have

$$\bigwedge_{i\in I_0} U_i \times \Box \bigvee_{i\in I_1'} V_i \leq \bigwedge_{i\in I_1'} U_i \times \Box \bigvee_{i\in I_1'} V_i \leq \tau^*(\Box A)$$

and similarly for the second disjunct. For the third, if we have $i \in I_1'$ and $j \in I_2'$ then

$$\begin{aligned}\bigwedge_{k\in I_0} U_k \times \Diamond(V_i \wedge V_j) &\leq (U_i \wedge U_j) \times \Diamond(V_i \wedge V_j) \\ &\leq \bigvee_{(i,j)\in I_1\times I_2} (U_i \wedge U_j) \times \Diamond(V_i \wedge V_j) \\ &= \tau^*(\Diamond(A \wedge B)).\end{aligned}$$

■

By composing strengths in the same way as before, we find a product map $\times : V^cX \times V^cY \to V^c(X \times Y)$. We already know from the case of V that this gives the product of the corresponding sublocales, but we now know also that our strong connectedness is preserved under binary products.

Theorem 28 *If X and Y are strongly connected locales, then so is $X \times Y$.*

Proof. Let K_X and K_Y be the corresponding points of V^cX and V^cY, as in Proposition 24. Then $K_X \times K_Y$, as calculated by the map $\times : V^cX \times V^cY \to V^c(X \times Y)$, corresponds to the locale product $X \times Y$. ■

Classically, of course, it is well known that binary products of connected spaces are still connected.

6.2 Overtness of the connected Vietoris powerlocale

Classically all locales are overt, but constructively overtness becomes a significant issue as it provides a positive way of asserting the non-emptiness of opens. In formal topology it is common to take overtness as a standard assumption (in the form of a positivity predicate) – see [Neg02].

We give a sufficient condition for V^cX to be overt, namely that X is locally connected. This is far from being necessary, as is clear from the classical situation. However, it gives a direct characterization of the positive opens of V^cX. Consider the basic open $\Box U \wedge \bigwedge_{i=1}^n \Diamond V_i$. If this is to contain a point K of V^cX, then K must lie in one of the connected components γ of U. It also meets each $U \wedge V_i$, and that must be in a connected component of $U \wedge V_i$ that lies in γ. Hence for $\Box U \wedge \bigwedge_{i=1}^n \Diamond V_i$ to be positive there must be connected components γ of U and δ_i of each $U \wedge V_i$ such that $\delta_i \subseteq \gamma$. We show that this necessary condition for positivity is also sufficient and shows the overtness of V^cX.

Topos-theoretically (see [Joh02, C1.5.9]), X is locally connected iff the "constant sheaf" functor $!^* : \mathbf{Set} \to \mathcal{S}X$ has a left adjoint π_0. (We write $\mathcal{S}X$ for the category of sheaves over X.) From the adjunction property one can deduce that

maps from a sheaf S to any discrete locale I are equivalent to functions from $\pi_0(S)$ to I, and it follows that $\pi_0(S)$ is in bijection with the set of connected components of the display locale (the domain of the local homeomorphism) of the sheaf S. As a left adjoint, π_0 preserves all colimits of sheaves. In fact, this implies that the general action of π_0 is determined by its action on opens, i.e. subsheaves of 1. From this point of view, π_0 is a covariant functor from ΩX to **Set** and in fact a *cosheaf* (see [BF06]; also [Vic09], for a more localic summary). We shall write its functorial part as a "corestriction" – if $U \leq V$ in ΩX and $\gamma \in \pi_0(U)$, then we write $\gamma|^V$ for $\pi_0(U \leq V)(\gamma)$. As connected components this means that if γ is a connected component of U then $\gamma|^V$ is the connected component of V that includes γ.

We can also describe how π_0 acts on joins $U = \bigvee_i U_i$ of opens – in fact, this description is the cosheaf property of π_0. As a sheaf, U is a colimit of a diagram with nodes U_i for each i and $U_i \wedge U_j$ for each pair (i, j) – this is essentially a description of the sheaf pasting property. Hence $\pi_0(U)$ can be described as the disjoint union $\coprod_i \pi_0(U_i)$, factored by the equivalence relation generated by pairs $(\gamma|^{U_i}, \gamma|^{U_j})$ for $\gamma \in U_i \wedge U_j$. Note also the case where the join is directed. Each proof of $\gamma|^U = \gamma'|^U$ ($\gamma \in \pi_0(U_i)$, $\gamma' \in \pi_0(U_j)$) involves only finitely many U_ks, and by taking an upper bound of them we can find some k such that $\gamma|^{U_k} = \gamma'|^{U_k}$. It follows that $\pi_0(U)$ is the directed colimit of the $\pi_0(U_i)$s.

If $\gamma \in \pi_0(U)$, we write U_γ for the pullback (of sheaves)

$$\begin{array}{ccc} U_\gamma & \longrightarrow & U \\ \downarrow & & \downarrow \\ 1 & \underset{!^*\gamma}{\longrightarrow} & !^*\pi_0(U) \end{array}$$

where the down arrow on the right is the unit of the adjunction $\pi_0 \dashv !^*$. Since $!^*\gamma$ is monic, so too is $U_\gamma \to U$. Hence U_γ is an open, and it is included in U. Since $\pi_0(U)$ is the coproduct of the maps $1 \xrightarrow{\gamma} \pi_0(U)$ (and $!^*$, as a left adjoint, preserves coproducts), we get that U is the coproduct, hence the pairwise disjoint join, of the U_γs. Because each $\pi_0(U_\gamma \leq U)$ maps $\pi_0(U_\gamma)$ to $\{\gamma\}$, we can deduce that $\pi_0(U_\gamma)$ is a singleton. In other words, each U_γ is connected – it is the connected component of U corresponding to γ.

Lemma 29 *If X is a locale then*

$$\begin{aligned} \Omega V^c X \cong \mathrm{Fr}\langle \Omega X \times \mathcal{F}\Omega X \ &\textit{(qua } \wedge\textit{-semilattice)} \mid \\ &(\textstyle\bigvee_i^\uparrow U_i, T) \leq \bigvee_i^\uparrow (U_i, T) \\ &(U, \{\textstyle\bigvee A\} \cup T) \leq \bigvee_{U' \in A} (U, \{U'\} \cup T) \\ &(\bot, T) \leq \bot \\ &(U \vee V, T) \leq (U, T) \vee (U \vee V, \{V\} \cup T) \\ &(U \vee V, \{U, V\} \cup T) \leq (U \vee V, \{U, V, U \wedge V\} \cup T)\rangle. \end{aligned}$$

"Qua $\wedge$-semilattice" is shorthand for further relations to say that the $\wedge$-semilattice structure of $\Omega X \times \mathcal{F}\Omega X$ is preserved in $\Omega V^c X$. The $\wedge$ operation on ΩX is as expected; that on $\mathcal{F}\Omega X$ here is $\cup$.

Proof. Going from $\Omega V^c X$ to the frame presented, we map $\Box U \mapsto (U, \emptyset)$ and $\Diamond U \mapsto (\top, \{U\})$. Using the presentation in Proposition 18, it is routine to verify that the relations are respected.

In the opposite direction, we map $(U, T) \mapsto \Box U \wedge \bigwedge_{V \in T} \Diamond V$. Again, it is routine to verify the relations are respected.

Thus we get homomorphisms between the two frames. To see they are mutually inverse, we have

$$\begin{aligned}
\Box U &\mapsto (U, \emptyset) \mapsto \Box U \wedge \bigwedge_{V \in \emptyset} \Diamond V = \Box U \\
\Diamond V &\mapsto (\top, \{V\}) \mapsto \Box \top \wedge \Diamond V = \Diamond V \\
(U, T) &\mapsto \Box U \wedge \bigwedge_{V \in T} \Diamond V \mapsto (U, \emptyset) \wedge \bigwedge_{V \in T} (\top, \{V\}) = (U, T).
\end{aligned}$$

■

The significance of Lemma 29 is as follows. The presentation given there has the structure of a *site* in the sense of [Joh82]. Each relation describes a family of covers in the semilattice $\Omega X \times \mathcal{F}\Omega X$; for example, the second says that $(U, \{\bigvee A\} \cup T)$ is covered by the set $\{(U, \{U'\} \cup T) \mid U' \in A\}$. Moreover, the coverage is meet-stable. (If a is covered by C, then $a \wedge b$ is covered by $\{c \wedge b \mid c \in C\}$ for every b.) The presentation then describes the universal property proved for the frame of C-ideals. The *coverage theorem* (proved explicitly in [AV93]) shows how a frame presented in this way can also be presented by generators and relations *as a suplattice*, i.e. a complete join semillatice (hence a complete lattice, but the homomorphisms are only required to preserve all joins). Explicitly,

$$\Omega V^c X \cong \mathbf{SupLat}\langle \Omega X \times \mathcal{F}\Omega X \text{ (qua poset)} \mid \text{the same relations} \rangle.$$

This enables us readily to define suplattice homomorphisms out of $\Omega V^c X$.

Theorem 30 *Let X be a locally connected locale. Then $V^c X$ is overt.*

Proof. We first construct the suplattice homomorphism from ΩX to Ω that will turn out to be the positivity predicate. Referring to the suplattice presentation derived from Lemma 29, we define $\theta(U, T) \in \Omega$ to hold if there are $\gamma \in \pi_0(U)$ and, for each $V \in T$, some $\delta_V \in \pi_0(U \wedge V)$ such that $\delta_V|^U = \gamma$. We must check that this respects all the relations.

For the first relation, we use the remark above that π_0 transforms directed joins to directed colimits. Let $U = \bigvee_i^{\uparrow} U_i$. Suppose we have $\gamma \in \pi_0(U)$ and $\delta_V \in \pi_0(U \wedge V)$ for each $V \in T$, with the required property. We can find some $\gamma' \in \pi_0(U_i)$ and, for each $V \in T$, $\delta'_V \in \pi_0(U_{j_V} \wedge V)$ such that $\gamma = \gamma'|^U$, $\delta_V = \delta'_V|^{U \wedge V}$. By directedness we can find some k such that U_k is an upper bound for U_i and the U_{j_V}s and $\delta'_V|^{U_k} = \gamma'|^{U_k}$. It follows that $\theta(U_k, T)$ holds.

The second relation is clear, since if $\delta \in \pi_0(U \wedge \bigvee A)$ then $\delta = \delta'|^{U \wedge \bigvee A}$ for some $U' \in A, \delta' \in \pi_0(U \wedge U')$. The third also is clear, since $\pi_0(\bot) = \emptyset$.

For the fourth, suppose we have $\gamma \in \pi_0(U \vee V)$ and, for each $W \in T$, $\delta_W \in \pi_0((U \vee V) \wedge W)$ such that $\delta_W|^{U\vee V} = \gamma$. If $\gamma = \gamma'|^{U\vee V}$ for some $\gamma' \in \pi_0(V)$, then we have $\theta(U \vee V, \{V\} \cup T)$. Likewise if we have $\delta_W = \delta'_W|^{(U\vee V)\wedge W}$ for some $\delta'_W \in \pi_0(V \wedge W)$, by considering $\gamma' = \delta'_W|^V$. There remains the possibility of having $\gamma = \gamma'|^{U\vee V}$ for some $\gamma' \in \pi_0(U)$, and, for each $W \in T$, $\delta_W = \delta'_W|^{(U\vee V)\wedge W}$ for some $\delta'_W \in \pi_0(U \wedge W)$. Consider γ' and $\delta'_W|^U$ in $\pi_0(U)$. They both corestrict to γ in $\pi_0(U \vee V)$. By considering the diagram of which $\pi_0(U \vee V)$ is the colimit, we find that either $\gamma' = \delta'_W|^U$ in $\pi_0(U)$, or there is some chain of equations between them involving some $\gamma'' \in \pi_0(V)$ such that $\gamma = \gamma''|^{U\vee V}$. In that latter case we get $\theta(U \vee V, \{V\} \cup T)$ as before. We thus reduce to the case where we have $\gamma' = \delta'_W|^U$ for all $W \in T$, and this implies $\theta(U, T)$.

For the fifth, the essential part is that we have $\delta_1 \in \pi_0(U)$ and $\delta_2 \in \pi_0(V)$ such that $\delta_1|^{U\vee V} = \delta_2|^{U\vee V}$. The proof of equivalence of these elements within $\pi_0(U) + \pi_0(V)$ must involve an element of $\pi_0(U \wedge V)$, and this tells us that $\theta(U \vee V, \{U, V, U \wedge V\} \cup T)$ holds.

We now know θ extends to a suplattice homomorphism $\overline{\theta} : \Omega V^c X \to \Omega$. We must show this is left adjoint to $!^* : \Omega \to \Omega V^c X$.

Suppose $p \in \Omega$, i.e. p is a truth value. $!^*(p) = \bigvee\{\top \mid p\} = \bigvee\{\Box\top \mid p\}$, so $\overline{\theta}(!^*(p)) \equiv (p \wedge \exists\gamma \in \pi_0(X)) \leq p$. It remains to show that

$$\Box U \wedge \bigwedge_{i=1}^{n} \Diamond V_i \leq !^*(\overline{\theta}(\Box U \wedge \bigwedge_{i=1}^{n} \Diamond V_i)).$$

U is a pairwise disjoint join $\bigvee_{\gamma\in\pi_0(U)} U_\gamma$, so by the Remark 20 it follows that $\Box U \leq \bigvee_{\gamma\in\pi_0(U)} \Box U_\gamma$. Hence,

$$\begin{aligned}
\Box U \wedge \bigwedge_{i=1}^{n} \Diamond V_i &\leq \bigvee_{\gamma\in\pi_0(U)} (\Box U_\gamma \wedge \bigwedge_{i=1}^{n} \Diamond(U_\gamma \wedge V_i)) \\
&= \bigvee_{\gamma\in\pi_0(U)} \bigvee_{\delta_1\in\pi_0(U_\gamma\wedge V_1)} \cdots \bigvee_{\delta_n\in\pi_0(U_\gamma\wedge V_n)} (\Box U_\gamma \wedge \bigwedge_{i=1}^{n} \Diamond(U_\gamma \wedge V_i)_{\delta_i}) \\
&\leq !^*(\overline{\theta}(\Box U \wedge \bigwedge_{i=1}^{n} \Diamond V_i)).
\end{aligned}$$

The final inequality is shown as follows. Suppose we have $\gamma \in \pi_0(U)$ and $\delta_i \in \pi_0(U_\gamma \wedge V_i)$ $(1 \leq i \leq n)$. By connectedness of U_γ, we see that

$$\delta_i|^U = \gamma$$

so by taking $\delta'_i = \delta_i|^{U\wedge V_i}$ we get the data required to show $\overline{\theta}(\Box U \wedge \bigwedge_{i=1}^{n} \Diamond V_i)$. ■

6.3 Generalized metric space completions

The geometricity of the powerlocale constructions has been used indirectly so far, to show that they commute with change of base. It can also be used more

directly to describe powerlocale points as models of geometric theories. We have not done this explicitly so far, although the idea was implicit in our use of Lemma 29. There are special situations, however, where it is relatively easy, and we now describe one that will be useful in discussing the real line $\mathbb{R}$.

If X is a metric space (possibly asymmetric) then [Vic05a] describes the points of its completion $\overline{X}$ geometrically as Cauchy filters of formal balls. [Vic03] shows that the same can also be done for the powerlocales P_U, P_L and V of $\overline{X}$ (and their positive parts). The same is not true for V^c, but nonetheless we shall be able to exploit the techniques to describe the points of $V^c\overline{X}$ as certain Cauchy filters.

Definition 31 *[Vic05a] A* generalized metric space *(*gms*) is a set X equipped with a metric $X(-,-) : X^2 \to \overleftarrow{[0,\infty]}$ satisfying zero self-distance ($X(x,x) = 0$) and the triangle inequality ($X(x,z) \leq X(x,y) + X(y,z)$).*

This is based on the definition in [Law73], but generalizes it in that the metric takes its values in the *upper reals* (which we treat as a locale $\overleftarrow{[0,\infty]}$) rather than the Dedekind sections. An upper real is a rounded upper set of rationals. (Classically these are equivalent to Dedekind sections, but even classically we see a difference in the topologies. The topology on $\overleftarrow{[0,\infty]}$ is that of upper semicontinuity, whose opens are of the form $[0,x)$. This is also the Scott topology on $([0,\infty],\geq)$.) Compared with ordinary metric spaces, we see here that the distance may be infinite, need not be symmetric, and need not satisfy the axiom $X(x,y) = 0 \Rightarrow x = y$.

Given a gms X, a *formal ball*, written symbolically as $B_\delta(x)$, is a pair $(x,\delta) \in X \times Q_+$, where Q_+ is the set of positive rationals. We call x and δ the *centre* and *radius* of the formal ball. A formal order is defined on these by

$$B_\varepsilon(y) \subset B_\delta(x) \text{ if } X(x,y) + \varepsilon < \delta.$$

The *localic completion* $\overline{X}$ of X is then defined as a locale whose points are the Cauchy filters of formal balls. ("Filter" is with respect to $\subset$, and "Cauchy" means the filter has balls of arbitrarily small radius.)

For each powerlocale $P_\bullet$ (upper, lower or Vietoris; it is convenient here to write P_C for V, with C standing for *convex*), the powerlocale of the localic completion of X is again a localic completion, of the finite powerset $\mathcal{F}X$ with an appropriate generalized metric. Specifically, $P_\bullet\overline{X} \cong \overline{\mathcal{F}_\bullet X}$ where

$$\begin{aligned}
\mathcal{F}_U X(S,T) &= \max_{t\in T}\min_{s\in S} X(s,t),\\
\mathcal{F}_L X(S,T) &= \max_{s\in S}\min_{t\in T} X(s,t),\\
\mathcal{F}_C X(S,T) &= \max(\mathcal{F}_U X(S,T), \mathcal{F}_L X(S,T)).
\end{aligned}$$

The metric on $\mathcal{F}_C X$ is analogous to the Vietoris metric on compact subspaces. Here, however, it is restricted to finite subsets.

We do not find a similar construction of the connected Vietoris powerlocale of a localic completion as itself a localic completion. However, we can (in

Lemma 34) identify the Cauchy filters for $V\overline{X}$ that lie in the sublocale $V^c\overline{X}$. We shall need this result when we turn to the case of $\mathbb{R} \cong \overline{\mathbb{Q}}$. The proof relies on examining the proof of $V\overline{X} \cong \overline{\mathcal{F}_C X}$, so we first review some aspects of that from [Vic03].

A key tool is that $\overline{X}$ embeds in the *ball domain* $\mathrm{Ball}(X)$, the ideal completion of $X \times Q_+$ under $\supset$. It is a continuous dcpo, whose points are the rounded filters of formal balls. It follows that $P_\bullet\overline{X}$ embeds in $P_\bullet(\mathrm{Ball}(X))$ (because $P_\bullet$ preserves embeddings – see Remark 12) and $\overline{\mathcal{F}_\bullet X}$ embeds in $\mathrm{Ball}(\mathcal{F}_\bullet X)$. The construction of powerlocales for continuous dcpos is described in [Vic93].

Specifically, a continuous dcpo can be expressed as the ideal completion $\mathrm{Idl}\, D$ of a set D equipped with an idempotent (transitive, interpolative) relation $<$ (and $\supset$ is such on $\mathrm{Ball}(X)$). Each $a \in D$ then gives a basic open $\uparrow a$ of $\mathrm{Idl}\, D$, with I in $\uparrow a$ iff $a \in I$. The powerlocales $P_\bullet(\mathrm{Idl}\, D)$ are then again continuous dcpos, the ideal completions of the finite powerset $\mathcal{F}D$ ordered by $<_\bullet$, with

$$\begin{aligned} S <_U T &\text{ if } (\forall t \in T)(\exists s \in S) s < t, \\ S <_L T &\text{ if } (\forall s \in S)(\exists t \in T) s < t, \\ S <_C T &\text{ if } S <_U T \text{ and } S <_L T. \end{aligned}$$

For $P_U(\mathrm{Idl}\, D)$, an ideal I of $\mathcal{F}D$ (with respect to $<_U$) is in $\Box(\bigvee_{i=1}^n \uparrow a_i)$ iff $\{a_1, \dots a_n\} \in I$. For $P_L(\mathrm{Idl}\, D)$, an ideal I of $\mathcal{F}D$ (with respect to $<_L$) is in $\Diamond(\uparrow a)$ iff $\{a\} \in I$. It follows that I is in $\bigwedge_{i=1}^n \Diamond(\uparrow a_i)$ iff $\{a_1, \dots a_n\} \in I$. For $V(\mathrm{Idl}\, D)$, an ideal I of $\mathcal{F}D$ is with respect to $<_C$. Its images $\Uparrow I$ and $\Downarrow I$ are the down closures $<_U I$ and $<_L I$. (If R is a relation from A to B, then for $B' \subseteq B$ we write RB' for the inverse image of B' under R.) It follows that I is in $\Box(\bigvee_{i=1}^n \uparrow a_i)$ iff $\{a_1, \dots a_n\} \in <_U I$, and in $\bigwedge_{i=1}^n \Diamond(\uparrow a_i)$ iff $\{a_1, \dots a_n\} \in <_L I$.

A map $\phi' : \mathrm{Ball}(\mathcal{F}_\bullet X) \to P_\bullet(\mathrm{Ball}(X))$ is defined by

$$\phi'(I) = \supset_\bullet \{\phi(B_\delta(S)) \mid B_\delta(S) \in I\}$$

where $\phi(B_\delta(S)) = \{B_\delta(s) \mid s \in S\}$. Then ϕ' restricts to a homeomorphism from $\overline{\mathcal{F}_\bullet X}$ to $P_\bullet\overline{X}$. We know the points of $\overline{\mathcal{F}_\bullet X}$ can be expressed as Cauchy filters of balls of $\mathcal{F}_\bullet X$, so a central technical question is how these relate to the opens $\Box U$ and $\Diamond U$ of $P_\bullet\overline{X}$.

Proposition 32 *Let B be a finite subset of $X \times Q_+$.*

1. *If I is a Cauchy filter for $\mathcal{F}_\bullet X$ ($\bullet = U$ or C), then I is in $\Box(\bigvee B) = \Box\bigvee\{B_\delta(x) \mid (x,\delta) \in B\}$ iff there is some $B_\varepsilon(S) \in I$ such that $B \supset_U \phi(B_\varepsilon(S))$.*

2. *If I is a Cauchy filter for $\mathcal{F}_\bullet X$ ($\bullet = L$ or C), then I is in $\bigwedge\{\Diamond B_\delta(x) \mid (x,\delta) \in B\}$ iff there is some $B_\varepsilon(S) \in I$ such that $B \supset_L \phi(B_\varepsilon(S))$.*

Proof. In each case we consider I as a point of $\mathrm{Ball}(\mathcal{F}_\bullet X)$, and ask when $\phi'(I)$ is in the corresponding open of $P_\bullet(\mathrm{Ball}(X))$. The answer can be derived from the case of continuous dcpos, which is addressed in [Vic03].

(1): $\phi'(I)$ is in $\Box(\bigvee B)$ iff $B \in (\supset_U \ \phi'(I)) = (\supset_U \ (\supset_\bullet \ \{\phi(B_{\varepsilon'}(S')) \mid B_{\varepsilon'}(S') \in I\}))$. Clearly this implies $B \supset_U \phi(B_\varepsilon(S))$ for some $B_\varepsilon(S) \in I$. For the converse, if $B_\varepsilon(S) \in I$ then $B_{\varepsilon'}(S) \in I$ for some $\varepsilon' < \varepsilon$. Then $\phi(B_\varepsilon(S)) \supset_C \phi(B_{\varepsilon'}(S))$ so $\phi(B_\varepsilon(S)) \in \phi'(I)$.

(2): I is in $\bigwedge\{\Diamond B_\delta(x) \mid (x,\delta) \in B\}$ iff $B \in (\supset_L \phi'(I))$ and then the argument is much as for part (1). ■

Note that when choosing $B_\varepsilon(S)$ for the "only if" direction, ε can be made arbitrarily small by the Cauchy property of I. This can be important when covering the possibility of empty sets.

Proposition 33 *Let I be a Cauchy filter for $\mathcal{F}_C X$. Then $B_\delta(S) \in I$ iff I is in both $\Box(\bigvee_{x\in S} B_\delta(x))$ and $\bigwedge_{x\in S} \Diamond B_\delta(x)$.*

Proof. $\Rightarrow$ is clear. For $\Leftarrow$, suppose $\phi(B_\delta(S)) \supset_U \phi(B_\alpha(A))$ and $\phi(B_\delta(S)) \supset_L \phi(B_\beta(B))$ with $B_\alpha(A), B_\beta(B) \in I$ and $\alpha, \beta < \delta$. We deduce that $B_\delta(S) \supset B_\alpha(A)$ in $\mathcal{F}_U X$ and $B_\delta(S) \supset B_\beta(B)$ in $\mathcal{F}_L X$. Choose $B_\varepsilon(T) \in I$ such that $B_\alpha(A) \supset B_\varepsilon(T)$ and $B_\beta(B) \supset B_\varepsilon(T)$ in $\mathcal{F}_C X$. Then we have $B_\delta(S) \supset B_\varepsilon(T)$ in both $\mathcal{F}_U X$ and $\mathcal{F}_L X$ and hence also in $\mathcal{F}_C X$, so $B_\delta(S) \in I$. ■

Now we prove our main Lemma in this Section.

Lemma 34 *Let X be a generalized metric space, and I a Cauchy filter for $\mathcal{F}_C X$. Then the point of $V\overline{X}$ corresponding to I is in $V^c\overline{X}$ iff the following conditions hold.*

1. *Every $B_\delta(S)$ in I has S non-empty.*
2. *If $B_\delta(S_1 \cup S_2) \in I$ then either $B_\delta(S_1) \in I$ or $B_\delta(S_2) \in I$ or there is $B_\varepsilon(T) \in I$ with some $t \in T$ and $s_i \in S_i$ $(i = 1, 2)$ such that $B_\varepsilon(t) \subset B_\delta(s_i)$ $(i = 1, 2)$.*

Proof. By [Vic03], condition (1) is equivalent to the point being in $V^+\overline{X}$. Given this, it remains to show that condition (2) is equivalent to respecting the relation

$$\Box(U_1 \vee U_2) \leq \Box U_1 \vee \Box U_2 \vee \Diamond(U_1 \wedge U_2).$$

Because the open balls $B_\delta(x)$ form a base of opens, and $\Box$ and $\Diamond$ preserve directed joins, it suffices to restrict to the case where each U_i is a finite join $\bigvee B_i$ of open balls, where $B_i \in \mathcal{F}(X \times Q_+)$. If I is in $\Box(U_1 \vee U_2) = \Box(\bigvee(B_1 \cup B_2))$ then we can find $B_\delta(S) \in I$ such that

$$B_1 \cup B_2 \supset_U \phi(B_\delta(S)).$$

We can find a decomposition $S = S_1 \cup S_2$ with $B_i \supset_U \phi(B_\delta(S_i))$. Calling on condition (2), we now find that if $B_\delta(S_i) \in I$ then I is in $\Box(\bigvee B_i)$. In the remaining possibility, find $\varepsilon' < \varepsilon$ such that $B_{\varepsilon'}(T) \in I$ and it follows that I is in $\Diamond B_\varepsilon(t)$. But

$$B_\varepsilon(t) \subset B_\delta(s_i) \leq \bigvee B_i$$

so $\Diamond B_\varepsilon(t) \leq \Diamond(U_1 \wedge U_2)$ which is thus satisfied by I.

For the converse, suppose $B_\delta(S_1 \cup S_2) \in I$. We can then find $\delta' < \delta$ with $B_{\delta'}(S_1 \cup S_2) \in I$, and it follows that I is in $\Box(\bigvee \phi(B_{\delta'}(S_1 \cup S_2)))$, which is equal to $\Box(\bigvee \phi(B_{\delta'}(S_1)) \vee \bigvee \phi(B_{\delta'}(S_2)))$. The relation gives us three possibilities. For the first two, suppose I is in one of $\Box(\bigvee \phi(B_{\delta'}(S_i)))$. Then there is some $B_\varepsilon(T) \in I$ such that $\phi(B_{\delta'}(S_i)) \supset_U \phi(B_\varepsilon(T))$. By choosing a common refinement of $B_{\delta'}(S_1 \cup S_2)$ and $B_\varepsilon(T)$ in I, we can assume without loss of generality that $\phi(B_{\delta'}(S_1 \cup S_2)) \supset_C \phi(B_\varepsilon(T))$. Since $\phi(B_\delta(S_i)) \supset_L \phi(B_{\delta'}(S_1 \cup S_2))$, it follows that $\phi(B_\delta(S_i)) \supset_C \phi(B_\varepsilon(T))$. From this we deduce that $B_\delta(S_i) \supset B_\varepsilon(T)$ and so $B_\delta(S_i) \in I$. In the third case I is in

$$\begin{aligned}
&\Diamond(\bigvee \phi(B_{\delta'}(S_1)) \wedge \bigvee \phi(B_{\delta'}(S_2))) \\
&= \bigvee \{\Diamond(B_{\delta'}(s_1) \wedge B_{\delta'}(s_2)) \mid s_1 \in S_1, s_2 \in S_2\} \\
&= \bigvee \{\Diamond B_\alpha(x) \mid \exists s_1 \in S_1, s_2 \in S_2.\ B_\alpha(x) \subset B_{\delta'}(s_1), B_\alpha(x) \subset B_{\delta'}(s_2)\}.
\end{aligned}$$

In this case we have I satisfying some such $\Diamond B_\alpha(x)$, and then there is some $B_\varepsilon(T) \in I$ and $t \in T$ with $B_\varepsilon(t) \subset B_\alpha(x)$. ■

7 Real intervals

We now apply the machinery of the connected Vietoris powerlocale to the real line $\mathbb{R}$ to obtain forms of standard analytic results.

$\mathbb{R}$ here is the localic reals as described in [Joh82]. Its points are Dedekind sections of rationals (see, e.g., [Vic07a]), but by [Vic05a] it is also the localic completion of the rationals $\mathbb{Q}$ with the usual metric. Let us summarize the results of Section 6.3 in this context. For formal balls for $\mathbb{R}$, refinement of formal balls is defined by $B_\delta(s) \subset B_\varepsilon(t)$ if $|s - t| < \varepsilon - \delta$. For $V\mathbb{R}$, refinement is $B_\delta(S) \subset B_\varepsilon(T)$ $(S, T \in \mathcal{F}\mathbb{Q})$ if –

$$\begin{aligned}
&\delta < \varepsilon, \\
&(\forall s \in S)(\exists t \in T) B_\delta(s) \subset B_\varepsilon(t), \\
&(\forall t \in T)(\exists s \in S) B_\delta(s) \subset B_\varepsilon(t).
\end{aligned}$$

Theorem 35 *1. There is a bijection between compact, overt sublocales K of $\mathbb{R}$ and Cauchy filters I of formal balls $B_\delta(S)$ $(0 < \delta \in \mathbb{Q}$, $S \in \mathcal{F}\mathbb{Q})$.*

2. Given K, we have $B_\delta(S) \in I$ iff K is covered by the opens $(s - \delta, s + \delta)$ $(s \in S)$, and each of these opens is positive modulo K.

3. K is positive iff every $B_\delta(S)$ in I has S non-empty.

4. K is strongly connected iff I has the properties given in Lemma 34.

Proof. (1) Because $\mathbb{R}$ is regular, every sublocale is fitted and hence weakly semifitted. Hence Theorem 9 characterizes the compact, overt sublocales. This is then combined with the homeomorphism $V\overline{\mathbb{Q}} \cong \overline{\mathcal{F}_C\mathbb{Q}}$ ([Vic03]; see Section 6.3 here).

(2) is from Proposition 33, (3) from [Vic03] and (4) from Lemma 34. ■

The impact of this theorem is that – in the localic setting – it can describe suitable subspaces of $\mathbb{R}$ not as sets of reals but as sets of pairs $(S,\delta) \in \mathcal{F}\mathbb{Q}\times Q_+$. In effect we are describing the subspace by its covers of a particular form. The constructive advantage is that it is geometric – the Cauchy filters are the points of a geometric theory. Hence it allows us to deduce constructive results about point-free analysis in a pointwise way. This will look very like ordinary topology, except for the unorthodox representation of subspaces.

7.1 Real subspaces as Cauchy filters, classically

In this section we shall make a more direct link with ordinary topology by giving a classical proof of the spatial result corresponding to Theorem 35. The results are not part of the constructive, localic development, but are included to give classical topologists an independent entry point to the techniques in the remainder of Section 7.

Classically, the localic real line is spatial, its frame isomorphic to the usual topology on the set of reals. We shall apply the Hofmann-Mislove Theorem ([HM81]; or see [Vic89]). This applies to general locales, and in the particular case of $\mathbb{R}$ it shows a 1-1 correspondence between compact subspaces of $\mathbb{R}$ and Scott open filters of opens of $\mathbb{R}$. A compact subspace corresponds to its open neighbourhood filter, of which it is the intersection.

If x,δ are reals with $\delta > 0$, let us write $\mathsf{b}_\delta(x)$ for the concrete open ball $\{y \in \mathbb{R} \mid |y-x| < \delta\}$, and if $S \subseteq \mathbb{R}$ we write $\mathsf{b}_\delta(S)$ for $\bigcup_{x\in S} \mathsf{b}_\delta(x)$. Note that if $\mathsf{b}_\delta(s) \subseteq \mathsf{b}_\varepsilon(t)$ and $\delta' < \delta$ then $B_{\delta'}(s) \subset B_\varepsilon(t)$; likewise if $\varepsilon < \varepsilon'$ then $B_\delta(s) \subset B_{\varepsilon'}(t)$. Also, $\mathsf{b}_\delta(x) = \bigcup\{\mathsf{b}_{\delta/2}(s) \mid s \in \mathbb{Q}, |x-s| < \delta/2\}$.

Lemma 36 *(Classically) Let $K \subseteq \mathbb{R}$ be compact, and let $K \subseteq \bigcup_{i\in I} U_i$ be an open cover. Then there is a* Lebesgue number *for the cover, i.e. some $\delta > 0$ such that for each $x \in K$ there is some i for which $\mathsf{b}_\delta(x) \subseteq U_i$.*

Proof. For any δ, let $U_i^\delta = \{x \mid (\exists \delta' > \delta)\ \mathsf{b}_{\delta'}(x) \subseteq U_i\}$, which is open. Then $K \subseteq \bigcup_{0<\delta\in\mathbb{Q}} \bigcup_i U_i^\delta$ and by compactness we deduce $K \subseteq \bigcup_i U_i^\delta$ for some δ, which is then a Lebesgue number. ■

Theorem 37 *(Classically) There is a 1-1 correspondence between non-empty compact subspaces of $\mathbb{R}$, and Cauchy filters of pairs $(S,\delta) \in \mathcal{F}^+\mathbb{Q} \times Q_+$. We write $B_\delta(S)$ for the pair (S,δ), and order them by $B_\varepsilon(T) \subset B_\delta(S)$ if $\varepsilon < \delta$ and*

$$(\forall s \in S)(\exists t \in T)|s-t| < \delta - \varepsilon,$$
$$(\forall t \in T)(\exists s \in S)|s-t| < \delta - \varepsilon.$$

(In other words each $B_\delta(s)$, $s \in S$, is refined by some $B_\varepsilon(t)$, $t \in T$, and each $B_\varepsilon(t)$ refines some $B_\delta(s)$.)

Proof. Let K be a non-empty compact subspace. We define

$$I_K = \{B_\varepsilon(S) \mid K \subseteq \mathsf{b}_\varepsilon(S) \text{ and } K \text{ meets } \mathsf{b}_\varepsilon(s) \text{ for every } s \in S\}.$$

Suppose $B_\varepsilon(S) \in I_K$. We have $K \subseteq \mathsf{b}_\varepsilon(S) = \bigcup_{0<\varepsilon'<\varepsilon} \mathsf{b}_{\varepsilon'}(S)$, so, by compactness $K \subseteq \mathsf{b}_{\varepsilon'}(S)$ for some $\varepsilon' < \varepsilon$. Also there is some $\varepsilon'' < \varepsilon$ such that K meets $\mathsf{b}_{\varepsilon''}(s)$ for every $s \in S$. It follows that there is some $\varepsilon''' < \varepsilon$ such that $B_{\varepsilon'''}(S) \in I_K$.

We shall often make use of the following argument. Suppose K has an open cover $\bigcup_i U_i$, with a Lebesgue number α. Then by a remark above

$$K \subseteq \bigcup_{x \in K} \mathsf{b}_\alpha(x) = \bigcup \{\mathsf{b}_{\alpha/2}(s) \mid s \in \mathbb{Q} \text{ and } \mathsf{b}_{\alpha/2}(s) \text{ meets } K\}.$$

Hence there is some $B_{\alpha/2}(S) \in I_K$ such that for each $s \in S$ there is some i with $\mathsf{b}_{\alpha/2}(s) \subseteq U_i$.

K is the intersection of its open neighbourhoods, since if $x \notin K$ then $\mathbb{R} - \{x\}$ is an open neighbourhood of K that does not contain x. However, taking a Lebesgue number α for an open cover $K \subseteq U$, we find $B_{\alpha/2}(S) \in I_K$ such that $K \subseteq \mathsf{b}_{\alpha/2}(S) \subseteq U$. It follows that $K = \bigcap \{\mathsf{b}_\varepsilon(S) \mid B_\varepsilon(S) \in I_K\}$.

I_K is a Cauchy filter. For the filter property, suppose $B_\delta(S)$ and $B_\varepsilon(T)$ are both in I_K and find $\delta' < \delta$ and $\varepsilon' < \varepsilon$ such that $B_{\delta'}(S), B_{\varepsilon'}(T) \in I_K$. Then $K \subseteq \bigcup_{s \in S} \bigcup_{t \in T} \mathsf{b}_{\delta'}(s) \cap \mathsf{b}_{\varepsilon'}(t)$. Let α be a Lebesgue number for the cover, and find $\beta < \min(\alpha, (\delta - \delta')/2, (\varepsilon - \varepsilon')/2)$. Then we can find $B_{\beta/2}(R) \in I_K$ such that if $r \in R$ then $\mathsf{b}_{\beta/2}(r) \subseteq \mathsf{b}_{\delta'}(s) \cap \mathsf{b}_{\varepsilon'}(t)$ for some $s \in S$, $t \in T$, from which it follows that $B_{\beta/2}(r)$ refines both $B_\delta(s)$ and $B_\varepsilon(t)$. On the other hand, if $s \in S$ then we can find $x \in \mathsf{b}_{\delta'}(s) \cap K$. There is some $r \in R$ with $x \in \mathsf{b}_{\beta/2}(r)$, and then $\mathsf{b}_{\beta/2}(r) \subseteq \mathsf{b}_\beta(x) \subseteq \mathsf{b}_{(\delta+\delta')/2}(s)$. It follows that $B_{\beta/2}(R) \subset B_\delta(S)$, and similarly $B_{\beta/2}(R) \subset B_\varepsilon(T)$.

In the reverse direction, given a Cauchy filter I of formal balls, let us write $K_I = \bigcap \{\mathsf{b}_\varepsilon(S) \mid B_\varepsilon(S) \in I\}$. We must show that K_I is compact, and $I = I_{K_I}$.

Let us write F_I for $\{U \in \Omega\mathbb{R} \mid \mathsf{b}_\delta(S) \subseteq U \text{ for some } B_\delta(S) \in I\}$, a Scott open filter. To show Scott openness, if $U = \bigcup_i^\uparrow U_i$ is a directed union with $U \in F_I$, and $\mathsf{b}_\delta(S) \subseteq U$ for some $B_\delta(S) \in I$, then find $\delta' < \delta$ with $B_{\delta'}(S) \in I$. Then $\mathsf{b}_{\delta'}(S) \subseteq \mathrm{Cl}(\mathsf{b}_{\delta'}(S)) \subseteq \mathsf{b}_\delta(S)$ and $\mathrm{Cl}(\mathsf{b}_{\delta'}(S))$ is compact, from which it follows that $\mathsf{b}_{\delta'}(S) \subseteq U_i$ for some i. By the Hofmann-Mislove Theorem (see above) we deduce that $K_I = \bigcap F_I$ is compact; moreover, for any open U we have $K_I \subseteq U$ iff $U \in F_I$.

Now let us define $G_I = \{U \in \Omega\mathbb{R} \mid \mathsf{b}_\delta(s) \subseteq U \text{ for some } B_\delta(S) \in I, s \in S\}$. This is up-closed, and it is also inaccessible by unions. For suppose $\mathsf{b}_\delta(s) \subseteq \bigcup_i U_i$ for some $B_\delta(S) \in I, s \in S$. We can find $\delta' < \delta$ with $B_{\delta'}(S) \in I$, and then $\mathsf{b}_{\delta'}(s) \subseteq \mathrm{Cl}(\mathsf{b}_{\delta'}(s)) \subseteq \mathsf{b}_\delta(s) \subseteq \bigcup_i U_i$. Let α be a Lebesgue number for this cover of the compact set $\mathrm{Cl}(\mathsf{b}_{\delta'}(s))$, and find $B_\varepsilon(T) \in I$ with $B_\varepsilon(T) \subset B_{\delta'}(S)$ and $\varepsilon < \alpha$. Choose $t \in T$ with $B_\varepsilon(t) \subset B_{\delta'}(s)$. Then $\mathsf{b}_\varepsilon(t) \subseteq U_i$ for some i, and it follows that $U_i \in G_I$.

Let V be the union of the opens not in G_I. By inaccessibility by unions, V is not in G_I – and is the greatest such. In fact for any open U we have $U \in G_I$ iff $U \nsubseteq V$. It is clear also that $V = \bigcup \{\mathsf{b}_\delta(s) \mid (\forall B_\delta(S) \in I)\ s \notin S\}$.

We now show that V is the complement of K_I. First, if $x \in V$ then $x \in \mathsf{b}_\delta(s)$ where $(\forall B_\delta(S) \in I)\ s \notin S$. Find ε such that $B_\varepsilon(x) \subset B_\delta(s)$, and then find $B_{\varepsilon'}(T) \in I$ with $\varepsilon' < \varepsilon/2$. If $x \in K_I$ then $x \in B_{\varepsilon'}(t)$ for some $t \in T$. Then

$B_{\varepsilon'}(t) \subset B_\varepsilon(x) \subset B_\delta(s)$, from which it follows that $B_{\varepsilon'}(T) \subset B_\delta(\{s\} \cup T)$ and so $B_\delta(\{s\} \cup T) \in I$, a contradiction. Hence V and K_I are disjoint. Now take any $x \in \mathbb{R} - K_I$. There is some $B_\delta(S) \in I$ with $x \notin \mathsf{b}_\delta(S)$. Choose $\delta' < \delta$ with $B_{\delta'}(S) \in I$; then there is some $B_\varepsilon(t)$ such that $x \in \mathsf{b}_\varepsilon(t)$ and $\mathsf{b}_\varepsilon(t)$ is disjoint from every $\mathsf{b}_{\delta'}(s)$ $(s \in S)$. If $B_\varepsilon(\{t\} \cup T) \in I$ then it and $B_{\delta'}(S)$ have a common refinement $B_\gamma(R)$ in I. But then $B_\gamma(r) \subset B_\varepsilon(t)$ for some $r \in R$, and $B_\gamma(r) \subset B_{\delta'}(s)$ for some $s \in S$, and this contradicts disjointness of $\mathsf{b}_\varepsilon(t)$ and $\mathsf{b}_{\delta'}(s)$. Hence $x \in V$.

It follows that an open U meets K_I iff there are some $B_\delta(S) \in I$ and $s \in S$ such that $\mathsf{b}_\delta(s) \subseteq U$.

We can now complete our proof that $I = I_{K_I}$. The $\subseteq$ direction is clear. For $\supseteq$, using the above discussion of F_I and G_I, we must show that $B_\delta(S) \in I$ if (i) there is some $B_\varepsilon(T) \in I$ with $\mathsf{b}_\varepsilon(T) \subseteq \mathsf{b}_\delta(S)$, and (ii) for each $s \in S$ there are some $B_\gamma(R) \in I$ and $r \in R$ such that $\mathsf{b}_\gamma(r) \subseteq \mathsf{b}_\delta(s)$. In (ii), by taking a common refinement in I for the $B_\gamma(R)$s, we can assume that a single $B_\gamma(R)$ does for all the s's. In (i) find $\varepsilon' < \varepsilon$ such that $B_{\varepsilon'}(T) \in I$ and let β be a Lebesgue number for the cover $\mathrm{Cl}(\mathsf{b}_{\varepsilon'}(T)) \subseteq \bigcup_{s \in S} \mathsf{b}_\delta(s)$. Let $B_\alpha(P)$ be a common refinement in I for $B_{\varepsilon'}(T)$ and $B_\gamma(R)$, with $\alpha < \beta$. We show $B_\alpha(P) \subset B_\delta(S)$. If $p \in P$ then $B_\alpha(p) \subset B_{\varepsilon'}(t)$ for some $t \in T$, and then by the Lebesgue number property $\mathsf{b}_\beta(p) \subseteq \mathsf{b}_\delta(s)$ for some $s \in S$, so $B_\alpha(p) \subset B_\delta(s)$. Conversely, if $s \in S$ then $\mathsf{b}_\gamma(r) \subseteq \mathsf{b}_\delta(s)$ for some $r \in R$, and then $B_\alpha(p) \subset B_\gamma(r)$ for some $p \in P$, giving $B_\alpha(p) \subset B_\delta(s)$. Hence $B_\alpha(P) \subset B_\delta(S)$ and $B_\delta(S) \in I$ as required. ■

Having established this classical correspondence between the compact subspaces of $\mathbb{R}$ and the Cauchy filters of formal balls for $\mathcal{F}^c\mathbb{Q}$, we now return to the constructive account. In constructive generality it deals with compact sub*locales* of $\mathbb{R}$. However, the working is in terms of the Cauchy filters and so the classical reader can relate those to the subspaces.

7.2 The Heine-Borel map

For the rest of Section 7 the reasoning is geometric.

For $\mathbb{R}$, we show that each closed interval $[x, y]$ is in $V^c\mathbb{R}$, and hence is strongly connected. More precisely, we show that the *Heine-Borel* map $HB_C : \leq \to V^+\mathbb{R}$, defined in [Vic03] so that $HB_C(x, y)$ corresponds to $[x, y]$, factors via $V^c\mathbb{R}$. Indeed, we show that the map defines a homeomorphism $\leq \cong V^c\mathbb{R}$. For each point K in $V^c\mathbb{R}$, we can calculate its inf and sup and show that K is the corresponding interval.

If $x \leq y$ then $HB_C(x, y)$ is defined as a point of $V^+\overline{\mathbb{Q}} \cong \overline{\mathcal{F}^+_C\mathbb{Q}}$ as follows. First, if S is a non-empty finite subset of $\mathbb{Q}\times Q_+$, we say that $\{B_\varepsilon(s) \mid (s, \varepsilon) \in S\}$ *covers* $[x, y]$ iff there is some non-empty finite sequence (s_i, ε_i) $(1 \leq i \leq n)$, with each $B_{\varepsilon_i}(s_i) \in S$, such that

$$\begin{aligned} s_1 - \varepsilon_1 &< x \\ s_{i+1} - \varepsilon_{i+1} &< s_i + \varepsilon_i \qquad (1 \leq i < n) \\ y &< s_n + \varepsilon_n \end{aligned}$$

We say that $B_\delta(s)$ *meets* $[x,y]$ if $x < s+\delta$ and $s-\delta < y$. Then $HB_C(x,y)$ comprises those balls $B_\delta(S)$ for which $\{B_\delta(s) \mid s \in S\}$ covers $[x,y]$, and for each $s \in S$ the ball $B_\delta(s)$ meets $[x,y]$. It is shown in [Vic03] that, as a point of $V^+\mathbb{R}$, $HB_C(x,y)$ corresponds to the closed interval $[x,y]$: geometrically, one shows for all x,y,z that $x \leq z \leq y$ iff z is in the sublocale for $HB_C(x,y)$. Moreover, the particular definitions above of "covers" and "meets" match the more general definitions using $\Box$ and $\Diamond$ (cf. Propositions 32 and 33). This provides a constructive proof of the localic Heine-Borel Theorem.

Lemma 38 *Let $x \leq y$ be reals, and let $\{B_{\varepsilon_i}(s_i) \mid 1 \leq i \leq n\}$ cover $[x,y]$ with the conditions holding as above. Then there is a subsequence $\{B_{\varepsilon'_j}(s'_j) \mid 1 \leq j \leq m\}$ such that*

$$\begin{aligned} &s'_1 - \varepsilon'_1 < x \\ &s'_j - \varepsilon'_j < s'_{j+1} - \varepsilon'_{j+1} < s'_j + \varepsilon'_j < s'_{j+1} + \varepsilon'_{j+1} \qquad (1 \leq j < m) \\ &\qquad y < s'_m + \varepsilon'_m \end{aligned}$$

Proof. First, note the following. Suppose we have $l \leq n$ such that

$$\forall i.\ (l < i \leq n \rightarrow s_l + \varepsilon_l \leq s_i - \varepsilon_i \text{ or } s_i + \varepsilon_i \leq s_l + \varepsilon_l) \qquad (*).$$

In other words, none of the open balls $B_{\varepsilon_i}(s_i)$ $(l < i \leq n)$ contains $s_l + \varepsilon_l$. Then by induction on $n-l$ we see that $s_n + \varepsilon_n \leq s_l + \varepsilon_l$. For $s_n - \varepsilon_n < s_{n-1} + \varepsilon_{n-1} \leq s_l + \varepsilon_l$ (by induction), so from the case $i = n$ we see $s_n + \varepsilon_n \leq s_l + \varepsilon_l$.

Now suppose we have a nonempty subsequence for $1 \leq j \leq k$, satisfying the first two of the conditions in the statement. (To get this for $k = 1$, just take $s'_1 = s_1$ and $\varepsilon'_1 = \varepsilon_1$.) Let l be the index in the overall sequence of the end of the subsequence: $(s'_k, \varepsilon'_k) = (s_l, \varepsilon_l)$. If (*) holds then $y < s_n + \varepsilon_n \leq s'_k + \varepsilon'_k$ and so the subsequence already satisfies all three conditions in the statement. Otherwise, let l' be the least index greater than l such that

$$s_{l'} - \varepsilon_{l'} < s_l + \varepsilon_l < s_{l'} + \varepsilon_{l'}.$$

Let k', with $0 \leq k' \leq k$, be such that

$$\begin{aligned} 1 \leq j \leq k' &\rightarrow s'_j - \varepsilon'_j < s_{l'} - \varepsilon_{l'} \\ k' < j \leq k &\rightarrow s'_j - \varepsilon'_j \geq s_{l'} - \varepsilon_{l'}. \end{aligned}$$

We take a new subsequence that comprises the old one up to index k', and then a new term $(s_{l'}, \varepsilon_{l'})$. This now extends further in the main sequence, and it satisfies the first two conditions in the statement. To see this, consider three cases. If $k' = k$ then we have

$$s'_k - \varepsilon'_k < s_{l'} - \varepsilon_{l'} < s'_k + \varepsilon'_k < s_{l'} + \varepsilon_{l'}.$$

If $1 \leq k' < k$ then

$$s'_{k'} - \varepsilon'_{k'} < s_{l'} - \varepsilon_{l'} \leq s'_{k'+1} - \varepsilon'_{k'+1} < s'_{k'} + \varepsilon'_{k'} < s'_k + \varepsilon'_k < s_{l'} + \varepsilon_{l'}.$$

If $k' = 0$ then

$$s_{l'} - \varepsilon_{l'} \leq s'_1 - \varepsilon'_1 < x.$$

■

Theorem 39 *$HB_C : \leq \to V^+\mathbb{R}$ factors via $V^c\mathbb{R}$.*

Proof. We verify condition (2) of Lemma 34. Let $x \leq y$ be reals, and suppose $B_\delta(S_1 \cup S_2) \in HB_C(x, y)$. Then we can find a sequence of elements s_i $(1 \leq i \leq n)$ in $S_1 \cup S_2$ such that

$$\begin{aligned} &s_1 - \delta < x \\ &s_i - \delta < s_{i+1} - \delta < s_i + \delta < s_{i+1} + \delta \qquad (1 \leq i < n) \\ &\quad y < s_n + \delta \end{aligned}$$

If all the s_is are in S_1 then $B_\delta(S_1) \in HB_C(x, y)$, and similarly if they are all in S_2. But otherwise we can find i such that (without loss of generality) $s_i \in S_1$ and $s_{i+1} \in S_2$. Let $\varepsilon' = (s_i + \delta - s_{i+1} + \delta)/2 > 0$, and choose T such that $B_{\varepsilon'}(T) \in HB_C(x, y)$. Let $t = (s_i + s_{i+1})/2$. Then $B_{\varepsilon'}(t)$ meets $[x, y]$, since $t + \varepsilon' = s_i + \delta$ and $t - \varepsilon' = s_{i+1} - \delta$, and it follows that $B_{\varepsilon'}(T \cup \{t\}) \in HB_C(x, y)$. Finding $B_\varepsilon(T \cup \{t\}) \in HB_C(x, y)$ with $\varepsilon < \varepsilon'$, we find the remaining possibility is satisfied in condition (2) of Lemma 34. ■

For the following Theorem, we use the inf and sup maps discussed in [Vic03]. The functions $\max, \min : \mathcal{F}^+_C\mathbb{Q} \to \mathbb{Q}$ are non-expansive and so lift to give maps $\inf = \overline{\min}$ and $\sup = \overline{\max}$ from $V^+\mathbb{R} \cong \overline{\mathcal{F}^+_C\mathbb{Q}}$ to $\mathbb{R} \cong \overline{\mathbb{Q}}$. They are defined by

$$\sup(I) = \supset \{B_\delta(\max S) \mid B_\delta(S) \in I\}$$

and similarly for inf.

The discussion in [Vic03] shows that the points $\inf K$ and $\sup K$ are in the sublocale K. Also, any (generalized) point x in K satisfies $\inf K \leq x \leq \sup K$, so as sublocales $K \leq [\inf K, \sup K]$.

Theorem 40 *The map $HB_C : \leq \to V^c\mathbb{R}$ is a homeomorphism.*

Proof. The maps $\inf, \sup : V^+\mathbb{R} \to \mathbb{R}$ pair to give a map $\langle \inf, \sup \rangle : V^+\mathbb{R} \to \mathbb{R}^2$ that factors via $\leq$ $(\inf K \leq \sup K)$. We show that the restriction of this to $V^c\mathbb{R}$ gives the required inverse to HB_C. Clearly $\inf HB_C(a, b) = a$ and $\sup HB_C(a, b) = b$, so it remains only to show that $K = HB_C(\inf K, \sup K)$ for every K in $V^c\mathbb{R}$.

If $B_\varepsilon(S) \in HB_C(\inf K, \sup K)$ (treated as a Cauchy filter of balls) then we know that $\{B_\varepsilon(s) \mid s \in S\}$ covers K, so by Propositions 32 and 33 we just need to show that each $B_\varepsilon(s)$ meets K (i.e. is positive modulo K). We know that it meets $HB_C(\inf K, \sup K)$, i.e. $\inf K < s + \varepsilon$ and $\sup K > s - \varepsilon$. Let $S_1 = \{s' \in S \mid s' \leq s\}$ and $S_2 = \{s' \in S \mid s' \geq s\}$, so $S = S_1 \cup S_2$. By strong connectedness, either K is covered by $\{B_\varepsilon(s') \mid s' \in S_1\}$ or K is covered by $\{B_\varepsilon(s') \mid s' \in S_2\}$ or for some $s_i \in S_i$ $(i = 1, 2)$ there is a common

refinement $B_\delta(t)$ of the $B_\varepsilon(s_i)$s that meets K. In the first case we have $\sup K$ in $B_\varepsilon(s')$ for some $s' \leq s$ from which we deduce $\sup K$ in $B_\varepsilon(s)$ which therefore meets K. Similarly, in the second case $\inf K$ is in $B_\varepsilon(s)$. In the third case $t + \delta < s_1 + \varepsilon \leq s + \varepsilon$ and $t - \delta > s_2 - \varepsilon \geq s - \varepsilon$, so $B_\delta(t) \subset B_\varepsilon(s)$ and it follows that $B_\varepsilon(s)$ meets K. We have now proved that, as Cauchy filters, $B_\varepsilon(S) \in HB_C(\inf K, \sup K) \Rightarrow B_\varepsilon(S) \in K$.

For the reverse inclusion, suppose $B_\varepsilon(S) \in K$. Every $B_\varepsilon(s)$ ($s \in S$) meets K and hence $HB_C(\inf K, \sup K)$. It remains to show that $\{B_\varepsilon(s) \mid s \in S\}$ covers $HB_C(\inf K, \sup K)$. Since $\mathbb{Q}$ is decidably ordered, we can write $S = \{s_1, \ldots, s_n\}$ with $s_1 < \cdots < s_n$. For each i ($1 \leq i < n$) K is covered by $\{B_\varepsilon(s_j) \mid 1 \leq j \leq i\} \cup \{B_\varepsilon(s_{j'}) \mid i+1 \leq j' \leq n\}$. Hence either K is covered by $\{B_\varepsilon(s_j) \mid 1 \leq j \leq i\}$, or K is covered by $\{B_\varepsilon(s_{j'}) \mid i + 1 \leq j' \leq n\}$, or there are j and j' such that $1 \leq j \leq i$ and $i + 1 \leq j' \leq n$ and $B_\varepsilon(s_j)$ and $B_\varepsilon(s_{j'})$ have a common refinement that meets K. In either of the first two cases we can use induction on n, while in the third case we have $s_{i+1} - \varepsilon < s_i + \varepsilon$. Hence we reduce to the situation where for every i we have $s_{i+1} - \varepsilon < s_i + \varepsilon$, in which case we know that $B_\varepsilon(S)$ covers $HB_C(\inf K, \sup K)$. ■

7.3 The Intermediate Value Theorem

Theorem 41 (Intermediate Value Theorem) *Suppose $f : \mathbb{R} \to \mathbb{R}$, $a \leq b$ in $\mathbb{R}$, and $f(a) \leq 0 \leq f(b)$. Then*

$$\inf(V^c|f|(HB_C(a,b))) = 0.$$

Proof. First, since $|f|$ maps $\mathbb{R}$ to $[0, \infty)$, we have $\forall x \in HB_C(a,b).\ |f(x)| \geq 0$ and so $\forall y \in V^c|f|(HB_C(a,b)).\ y \geq 0$. Hence $\inf(V^c|f|(HB_C(a,b))) \geq 0$.

It remains to show for any positive rational q that $\inf(V^c|f|(HB_C(a,b))) < q$. Since either this holds or $\inf(V^c|f|(HB_C(a,b))) > q/2$, it suffices to show that $\inf(V^c|f|(HB_C(a,b))) > q$ is impossible for every positive rational q. The inequality is equivalent to $V^c|f|(HB_C(a,b)) \vDash \square(q, \infty)$, i.e. $V^c f(HB_C(a,b))) \vDash \square((-\infty, -q) \vee (q, \infty)) \leq \square(-\infty, -q) \vee \square(q, \infty)$ (in $V^c\mathbb{R}$). But this is impossible, since being in $\square(-\infty, -q)$ or $\square(q, \infty)$ contradicts $f(b) \geq 0$ or $f(a) \leq 0$ respectively. ■

7.4 Rolle's Theorem

Rolle's Theorem (44) states that if f is differentiable on an interval and has equal values at the endpoints, then it must have zero derivative somewhere in between. We shall state and prove this in a similar fashion to the Intermediate Value Theorem. However, we must first discuss differentiation in the localic context. Since all localic maps are continuous, we shall restrict ourselves to the C_1 case – in other words, the derivative too is continuous. We use a characterization that is often associated with Carathéodory. It has been worked out in some generality in [BGN04] and we follow their notation and summarize some of their results.[1]

[1] My thanks to Martín Escardó for drawing my attention to this work, and also for a number of discussions on the topic.

Definition 42 *Let $f : \mathbb{R} \to \mathbb{R}$. We say f is* differentiable *if there is some $f^{\langle 1\rangle} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$f(y) - f(x) = (y - x)f^{\langle 1\rangle}(y, x).$$

The derivative *of f is then $f'(x) = f^{\langle 1\rangle}(x, x)$.*

Of course it is the implicit continuity of $f^{\langle 1\rangle}$ that makes this work, for

$$f^{\langle 1\rangle}(x, x) = \lim_{y\to x} f^{\langle 1\rangle}(y, x) = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}.$$

It follows also that $f^{\langle 1\rangle}$ is uniquely determined by f. The formula certainly defines $f^{\langle 1\rangle}$ uniquely on the dense open sublocale $\neq$ of $\mathbb{R}\times\mathbb{R}$. If ϕ and ψ are two parallel locale maps whose codomain is regular, then their equalizer is closed. Hence in the present situation if ϕ and ψ are two candidates for $f^{\langle 1\rangle}$ then their equalizer is closed and contains $\neq$, and hence is the whole of $\mathbb{R} \times \mathbb{R}$.

Proposition 43 *Suppose f_i is differentiable ($i = 1, 2$), and let c be a real.*

1. $(f_1 + f_2)' = f_1' + f_2'$.
2. $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$.
3. $\mathrm{Id}' = 1$.
4. $c' = 0$.
5. $(cf_1)' = cf_1'$.

Proof. 1. Define $(f_1 + f_2)^{\langle 1\rangle} = f_1^{\langle 1\rangle} + f_2^{\langle 1\rangle}$.
2. We have

$$\begin{aligned} f_1(y)f_2(y) - f_1(x)f_2(x) &= (f_1(y) - f_1(x))f_2(y) + f_1(x)(f_2(y) - f_2(x)) \\ &= (y - x)(f_1^{\langle 1\rangle}(y, x)f_2(y) + f_1(x)f_2^{\langle 1\rangle}(y, x)) \end{aligned}$$

so we can define $(f_1 f_2)^{\langle 1\rangle}(y, x) = f_1^{\langle 1\rangle}(y, x)f_2(y) + f_1(x)f_2^{\langle 1\rangle}(y, x)$. Then

$$(f_1 f_2)^{\langle 1\rangle}(x, x) = f_1'(x)f_2(x) + f_1(x)f_2'(x).$$

3, 4 and 5 are obvious. ■

Theorem 44 (Rolle's Theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and let $a < b$ be reals such that $f(a) = f(b)$. Then*

$$\inf(V^c|f'|(HB_C(a, b))) = 0.$$

Proof. As in the Intermediate Value Theorem, the main requirement is to show that if q is a positive rational then $\inf(V^c|f'|(HB_C(a,b))) > q$ is impossible. For from that it would follow that $V^c f'(HB_C(a,b))$ is in $\Box((-\infty,-q) \vee (q,\infty))$ and hence by connectedness it is in either $\Box(-\infty,-q)$ or $\Box(q,\infty)$ – let us take the latter case. We thus have $V^c\Delta(HB_C(a,b))$ in $\Box f^{\langle 1\rangle *}(q,\infty)$. The diagonal map $\Delta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is the lift $\overline{\Delta}$, where this second Δ is the diagonal function for $\mathbb{Q}$. It follows that $V^+\Delta$ is the lift $\overline{\mathcal{F}_C\Delta}$. We can therefore calculate $V^c\Delta(HB_C(a,b))$ as a Cauchy filter. It contains the ball $B_\varepsilon(T)$ ($T \in \mathcal{F}(\mathbb{Q}\times\mathbb{Q})$) iff there is some ball $B_\delta(S)$ in $HB_C(a,b)$ such that $B_\varepsilon(T) \supset B_\delta(\{(s,s) \mid s \in S\})$ with respect to the Vietoris metric.

It follows from Proposition 32 that if $V^c\Delta(HB_C(a,b))$ is in $\Box f^{\langle 1\rangle *}(q,\infty)$ then there is some $B_\varepsilon(T)$ in $V^c\Delta(HB_C(a,b))$ such that for each $(t_1,t_2) \in T$, $B_\varepsilon((t_1,t_2)) \leq f^{\langle 1\rangle *}(q,\infty)$. It then follows that there is some $B_\delta(S)$ in $HB_C(a,b)$ such that for each $s \in S$, $B_\delta((s,s)) \leq f^{\langle 1\rangle *}(q,\infty)$.

By Lemma 38 we can find $s_1 < \ldots < s_n$ in S, with $n \geq 1$, such that

$$\begin{aligned} s_1 - \delta &< a \\ s_{i+1} - \delta &< s_i + \delta \qquad (1 \leq i < n) \\ b &< s_n + \delta. \end{aligned}$$

Define $t_i = (s_i + s_{i+1})/2$ if $1 \leq i < n$, so that $s_{i+1} - \delta < t_i < s_i + \delta$. If $n > 1$ then either $a < t_1$ or $s_2 - \delta < a$, and in the latter case we can omit s_1 from the list. It follows that we can assume without loss of generality that, if $n > 1$, then $a < t_1$ and, similarly, $t_{n-1} < b$.

If $n = 1$, then (b,a) is in $B_\delta((s_1,s_1)) \leq f^{\langle 1\rangle *}(q,\infty)$ and so $f^{\langle 1\rangle}(b,a) > q$. Hence

$$f(b) - f(a) = (b-a)f^{\langle 1\rangle}(b,a) > 0,$$

a contradiction. Similarly, if $n > 1$ then

$$\begin{aligned} f(b) - f(a) &= f(t_1) - f(a) + \sum_{i=1}^{n-2}(f(t_{i+1}) - f(t_i)) + f(b) - f(t_{n-1}) \\ &> (t_1 - a)q + \sum_{i=1}^{n-2}(t_{i+1} - t_i)q + (b - t_{n-1})q \\ &= (b-a)q > 0 \end{aligned}$$

and again we have a contradiction. ■

Corollary 45 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose f' is zero on some closed interval $[a,b]$ with $a < b$. Then $f(a) = f(b)$.*

Proof. Define

$$g(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a).$$

Then $g(a) = g(b) = f(a)$. Also, g is differentiable, with

$$\begin{aligned} g'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a} \\ &= -\frac{f(b) - f(a)}{b - a} \text{ on } [a, b]. \end{aligned}$$

By Rolle's Theorem $\inf(V^c|g'|(HB_C(a, b))) = -\frac{f(b)-f(a)}{b-a} = 0$, and it follows that $f(a) = f(b)$. ■

8 Conclusions

Technically speaking, our investigation has been into topos-valid, point-free constructive analysis, with the main part being the study of a new powerlocale (localic analogue of hyperspace) whose points are strongly connected sublocales. It has pleasing properties, including the existence of "product" maps by which it is seen that the product of two such sublocales is again strongly connected, and the fact that over the reals its points are just the compact intervals.

However, a powerful driving force was the desire to use geometric reasoning in order to restore the points to point-free topology. Powerlocales are a useful tool in that programme, since they make sublocales points of a locale and hence models of a geometric theory. The new connected powerlocale helps to geometrize questions of connectedness for sublocales, and that underlay our applications to the Intermediate Value Theorem and Rolle's Theorem, both of which are related to connectedness.

An unfamiliar feature of the geometric working is the way sublocales are described indirectly in terms of their covers. This made the final work entirely elementary, using calculations with rationals and finite sets of rationals.

The work should be compared with that done in Taylor's formal system Abstract Stone Duality (ASD), and in particular [Tay05] and [BT09], which deal with real analysis and connectedness of the intervals. The foundational postulates of ASD base it on locally compact spaces, and that seems to make it harder to give a general notion of subspaces. Nonetheless, the approach uses many similar techniques to ours, in particular the technique from powerlocales of describing subspaces – at least in the Hausdorff context – in terms of opens that cover them (using $\Box$) or meet them (using $\diamondsuit$). The importance of overtness also comes through very strongly. In [Tay05] our strong connectedness property is proved as a property of closed real intervals $[x, y]$, and that paper follows the result up with a much more thorough investigation than we have given of the Intermediate Value Theorem, and conditions on the map f that allow its zeros to be found.

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