A coherent account of geometricity

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June 28, 2014

Abstract

A "bundle endofunctor preserving proneness" for a cartesian category $\mathcal{C}$ is an endofunctor $\mathfrak{T}_\bullet$ of the arrow category $\mathcal{C}^\downarrow$, the identity on codomains, that preserves the property of morphisms of being pullback squares. Such a $\mathfrak{T}_\bullet$ is "slicewise strong", restricting to a strong endofunctor $\mathfrak{T}_\bullet$ on every slice $\mathcal{C}/B$, and with that structure preserved by pullback between slices.

In the other direction, a strong endofunctor T on $\mathcal{C}$ extends to a bundle endofunctor $T_\bullet$ on $\mathcal{C}$, which preserves proneness if T preserves coreflexive equalizers.

If a bundle endofunctor $\mathfrak{T}_\bullet$ preserving proneness also preserves coreflexive equalizers, then it is naturally isomorphic to the $T_\bullet$ arising from $T = \mathfrak{T}_1$.

Combining these, bundle endofunctors for $\mathcal{C}$ preserving proneness and coreflexive equalizers are equivalent to strong endofunctors on $\mathcal{C}$ preserving coreflexive equalizers; this latter structure is inherited by all slices and preserved by pullback between them.

The results extend to situations where $\mathfrak{T}_\bullet$ and T are monads.

We propose that the structure of bundle endofunctor preserving proneness is a satisfactory categorical abstraction of the notion of geometric construction when $\mathcal{C}$ is the category **Loc** of locales. The powerlocales give rise to bundle monads on **Loc** preserving proneness and coreflexive equalizers; likewise for the covariant powerobject monad on any topos.

1 Introduction

The Localic Bundle Theorem, as we shall call it, is an important means by which topos theory gives applications in topology: it says that bundles are equivalent to spaces internal in the topos $\mathcal{S}B$ of sheaves over the base space B. Here "bundle" just means any map (for us the term "map" will always imply continuity), but viewed as a variable space – the fibre – parametrized by the base space point it lies over. To make it work, an important proviso is that spaces should be understood point-free, as locales: then $\mathbf{Loc}/B \cong \mathbf{Loc}_{\mathcal{S}B}$. The theorem is proved in [JT84, FS79].

This implies that any topos-valid construction on locales can also be understood as a construction on bundles. However, in practice we are particularly interested in constructions that work fibrewise. Since the fibres can be got by pulling the bundle back along base space points (as maps from 1), a sufficient condition for this, and one that sits more comfortably with the point-free nature, is that the construction is (up to isomorphism) pullback stable. This property, known as "geometricity", has been proved for various powerlocales [Vic04] and valuation locales ([Vic11]; see also [Vic08, CS09]).

Though geometricity is important, its characterization as "pullback stable up to isomorphism" begs coherence questions – how do the different isomorphisms for different changes of base relate to each other? The purpose of this paper is to supply a strengthened geometricity structure, that includes data for change of base and covers the coherence questions by providing canonical isomorphisms for the pullback stability. We also give sufficient conditions for obtaining this strengthening. We prove our results in the generality of an arbitrary cartesian category $\mathcal{C}$ in place of the category **Loc** of locales.

The strengthened structure is that of "bundle endofunctor preserving proneness". It expands on the structure of slicewise constructions by being defined as an endofunctor on the arrow category $\mathcal{C}^{\downarrow}$, [1] of $\mathcal{C}$, whose objects are bundles and whose morphisms are commutative squares. Thus it contains not only the constructions on the bundles but also the information about change of base. By "bundle endofunctor" we mean that it is over the codomain bifibration $\mathsf{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$, and so restricts to each slice $\mathcal{C}/B$, and "preserving proneness" says that its morphism part preserves pullback squares, and implies the pullback stability. The coherence questions are now answered in its functoriality.

To see more concretely how the morphism part of this bundle endofunctor will arise, suppose we have a pullback-stable, slicewise endofunctor. In other words, for each object B of $\mathcal{C}$ we have an endofunctor $\mathfrak{T}_B$ of the slice category $\mathcal{C}/B$, and for each morphism $f\colon B \to B'$ and each bundle $p'\colon E' \to B'$ we have an isomorphism $\mathfrak{T}_B f^* E' \cong f^* \mathfrak{T}_{B'} E'$ over B.

We shall extend this from the separate slices to the whole of $\mathcal{C}^{\downarrow}$, which contains all the slices as fibres of cod. Given a morphism as on the left below, we can factorize g via a morphism $g'\colon E \to f^* E'$ and define a composite morphism $\mathfrak{T}_f g$ over f as on the right:

$$
\begin{tikzcd}
E \arrow[r, "g"'] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
B \arrow[r, "f"] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{T}_B E \arrow[r, "\mathfrak{T}_B g'"'] \arrow[rrr, bend left, "\mathfrak{T}_f g"] \arrow[dr] & \mathfrak{T}_B f^* E' \arrow[r, "\cong"'] \arrow[d] & f^* \mathfrak{T}_{B'} E' \arrow[r, "{p'}^* f"'] \arrow[dl] & \mathfrak{T}_{B'} E' \arrow[d] \\
& B \arrow[rr, "f"] & & B'
\end{tikzcd}
.
$$

Thus we have defined the data for the object and morphism parts of an endofunctor $\mathfrak{T}_\bullet$ of $\mathcal{C}^{\downarrow}$. Clearly, $\mathfrak{T}_\bullet$ will preserve pullback squares, because then g' is an isomorphism.

We shall require $\mathfrak{T}_\bullet$ to be an endofunctor on $\mathcal{C}^{\downarrow}$, which in effect embodies coherence conditions on the isomorphisms used for the pullback stability.

[1] It would be more usual to write this as $\mathcal{C}^{\to}$, but we use the downward arrow to emphasize that the objects are being viewed as bundles.

Because the bundle endofunctor contains so much structure, it is hard to define directly. We show how, in the surprisingly common case where the construction preserves coreflexive equalizers, the whole of $\mathfrak{T}_\bullet$ can be recovered – up to natural isomorphism – from its global part $\mathfrak{T}_1$ on $\mathcal{C}/1$, together with a strength whose existence is guaranteed by "preservation of proneness".

Central to this is the observation that any bundle $p\colon E \to B$ can be expressed as a coreflexive equalizer in $\mathcal{C}/B$, using

$$E \xrightarrow[\langle p,E\rangle]{} B \times E \underset{\Delta\times E}{\overset{B\times\langle p,E\rangle}{\rightrightarrows}} B \times B \times E\ .$$

(The coreflexivity morphism is $\pi_{13}\colon B \times B \times E \to B \times E$.) If $\mathfrak{T}$ preserves coreflexive equalizers and is stable under pullback, then we obtain an equalizer diagram

$$\mathfrak{T}_B E \longrightarrow \mathfrak{T}_B(B \times E) \xrightarrow{\cong} B \times \mathfrak{T}_1 E \overset{B\times\mathfrak{T}_1\langle p,E\rangle}{\rightrightarrows} B \times \mathfrak{T}_1(B \times E) \xrightarrow{\cong} \mathfrak{T}_1(B \times B \times E)\ .$$

The lower, unlabelled morphism is not of the form $B \times \cdots$, but does have the right type to come from a strength on $\mathfrak{T}_1$, and in Section 4.3 we use this idea to recover a bundle endofunctor from a strong endofunctor on $\mathcal{C}/1$.

Our main result (Theorem 63) is that bundle endofunctors that preserve both proneness and coreflexive equalizers are equivalent to strong endofunctors on $\mathcal{C}$ that preserve coreflexive equalizers. We also show how the equivalence lifts to monads. Finally, we apply the result to the motivating geometric examples from locale theory, such as the powerlocales.

We shall frequently us "tangle diagrams", a 2-dimensional calculus in which vertical lines represent functors, composed horizontally from right to left, and "coupons", nodes in the diagram, represent natural transformations, composed from top to bottom. Most coupons are rectangles, though other shapes are possible. In particular, some, depicted as giant "=" signs, represent equality between functors. Areas in the diagram represent categories, though they are not labelled as such. Thus n-cells in the 2-category of categories are represented by $(2-n)$-dimensional parts of the tangle diagram. The paper [FV14], while not the origin of these ideas, provides a detailed discussion of this particular use of tangle diagrams.

2 Remarks on the arrow category

Throughout the paper we shall take $\mathcal{C}$ to be a cartesian category. By this we mean that each finite diagram has a *canonical* limit cone, although we make no further assumptions about them.

We shall write $\mathcal{C}^{\downarrow}$ for the *arrow category*, whose objects are morphisms $p\colon E \to B$ in $\mathcal{C}$, and in which a morphism f from p to p' is a commutative square

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\underline{f}"'] & B' \end{tikzcd} \quad . \tag{1}$$

Thus $\overline{f}$ and $\underline{f}$ denote the "upstairs" and "downstairs" parts of f, got by applying the domain and codomain functors $\mathsf{dom}, \mathsf{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$. We shall occasionally write an object X as $\overline{X} \xrightarrow{X\downarrow} \underline{X}$. In particular, a functor $\mathfrak{F}\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$ is equivalent to a morphism $\mathfrak{F}\downarrow\colon \overline{\mathfrak{F}} \to \underline{\mathfrak{F}}$ in $[\mathcal{D}, \mathcal{C}]$.

As is well known, the codomain functor $\mathsf{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$ is a bifibration. We shall frequently use the universal properties of the associated prone and supine morphisms.[2] Recall that, given a functor $\mathfrak{F}\colon \mathcal{D} \to \mathcal{C}$, a morphism $g\colon Y \to Z$ in $\mathcal{D}$ is *prone* (with respect to $\mathfrak{F}$) if for every $h\colon X \to Z$ in $\mathcal{D}$, and factorization $\mathfrak{F}h = (\mathfrak{F}g)f'$ in $\mathcal{C}$, there is a unique factorization $h = gf$ in $\mathcal{D}$ such that $\mathfrak{F}f = f'$. Similarly, $f\colon X \to Y$ is *supine* if for every $h\colon X \to Z$, and factorization $\mathfrak{F}h = g'(\mathfrak{F}f)$, there is a unique factorization $h = gf$ such that $\mathfrak{F}g = g'$. We shall also say that $f\colon X \to Y$ is *vertical* if $\mathfrak{F}f$ is an identity morphism. If $\mathcal{C}$ is cartesian and $\mathfrak{F}$ is $\mathsf{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$, then it is well known that the prone morphisms in $\mathcal{C}^{\downarrow}$ are those for which (1) is a pullback square.

Proposition 1 *The supine morphisms f in $\mathcal{C}^{\downarrow}$ are those for which $\overline{f}$ is an isomorphism.*

Proof. ($\Leftarrow$): Given

$$\begin{tikzcd} {} \arrow[r, "\overline{f}"'] \arrow[rr, bend left, "\overline{h}"] \arrow[d, "\pi_1"] & {} \arrow[r, dotted] \arrow[d, "\pi_2"] & {} \arrow[d, "\pi_3"] \\ {} \arrow[r, "\underline{f}"] \arrow[rr, bend right, "\underline{h}"'] & {} \arrow[r, "\underline{g}"] & {} \end{tikzcd}$$

then the unique fillin is $\overline{h}\,\overline{f}^{-1}$.

($\Rightarrow$): Let k be the fillin for

$$\begin{tikzcd} {} \arrow[r, "\overline{f}"'] \arrow[rr, bend left, equal] \arrow[d, "\pi_1"] & {} \arrow[r, dotted, "k"'] \arrow[d, "\pi_2"] & {} \arrow[d, "\pi_2\overline{f}"] \\ {} \arrow[r, "\underline{f}"] \arrow[rr, bend right, "\underline{f}"'] & {} \arrow[r, equal] & {} \end{tikzcd} \quad .$$

[2] We follow Paul Taylor's terminology. They are usually known as *cartesian* and *cocartesian*.

Then $k\overline{f} = \mathsf{Id}$. But, also, $\overline{f}k$ and Id are equal because they both serve as fillin for

$$\begin{tikzcd}
{} \arrow[r, "\overline{f}"'] \arrow[rr, bend left, "\overline{f}"] \arrow[d, "\pi_1"] & {} \arrow[r, dotted] \arrow[d, "\pi_2"] & {} \arrow[d, "\pi_2"] \\
{} \arrow[r, "\underline{f}"] \arrow[rr, bend right, "\underline{f}"'] & {} \arrow[r, equal] & {}
\end{tikzcd} .$$

Hence $\overline{f}$ has inverse k. ■

Now suppose $\mathcal{D}$ is another category, giving a functor $\mathsf{cod}\colon [\mathcal{D}, \mathcal{C}^{\downarrow}] \to [\mathcal{D}, \mathcal{C}]$. It is easy to see that a morphism $\alpha\colon \mathfrak{F} \to \mathfrak{G}$ in $[\mathcal{D}, \mathcal{C}^{\downarrow}]$ is prone or supine with respect to cod if all the morphisms $\alpha(X)$ (for X an object of $\mathcal{D}$) are prone or supine in $\mathcal{C}^{\downarrow}$.

Definition 2 *The categories* Opspan *and* Comp *are defined as pullbacks:*

$$\begin{tikzcd}
\mathrm{Opspan} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{C}^{\downarrow} \arrow[d, "\mathsf{cod}"] \\
\mathcal{C}^{\downarrow} \arrow[r, "\mathsf{cod}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Comp} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{C}^{\downarrow} \arrow[d, "\mathsf{cod}"] \\
\mathcal{C}^{\downarrow} \arrow[r, "\mathsf{dom}"'] & \mathcal{C}
\end{tikzcd} .$$

We think of the objects of Opspan and Comp as (f_1, p) and (f_1, p_1), representing

$$\begin{tikzcd}
 & E \arrow[d, "p"'] \\
B_1 \arrow[r, "f_1"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E_1 \arrow[d, "p_1"'] & \\
B_1 \arrow[r, "f_1"'] & B
\end{tikzcd} .$$

In other words, the first component f_1 will often play the role of change of base.

Definition 3 *The functor* $\mathsf{pb}\colon \mathrm{Opspan} \to \mathrm{Comp}$ *is defined on objects by* $\mathsf{pb}(f_1, p) = (f_1, f_1^* p)$*, with the obvious extension to morphisms.*

The natural transformation $\kappa\colon \pi_2\,\mathsf{pb} \to \pi_2$ *is defined by taking* $\kappa(f_1, p)$ *to be the pullback square*

$$\begin{tikzcd}
f_1^* E \arrow[r, "p^* f_1"] \arrow[d, "f_1^* p"'] & E \arrow[d, "p"] \\
B_1 \arrow[r, "f_1"'] & B
\end{tikzcd} .$$

Definition 4 *The functor* $\Sigma\colon \mathrm{Comp} \to \mathrm{Opspan}$ *is defined on objects by* $\Sigma(f_1, p_1) = (f_1, f_1 p_1)$*, with the obvious extension to morphisms.*

The natural transformation $\lambda\colon \pi_2 \to \pi_2\Sigma$ is defined by taking $\lambda(f_1, p_1)$ to be the commutative square

$$\begin{tikzcd} E_1 \arrow[r, equal] \arrow[d, "p_1"'] & E_1 \arrow[d, "f_1 p_1"] \\ B_1 \arrow[r, "f_1"'] & B \end{tikzcd} .$$

Proposition 5

1. *If $\mathfrak{F}\colon \mathcal{E} \to$ Opspan is a functor, then $\kappa\mathfrak{F}$ is prone with respect to* cod*.*
2. *If $\mathfrak{F}\colon \mathcal{E} \to$ Comp is a functor, then $\lambda\mathfrak{F}$ is supine with respect to* cod*.*

Proof. By our earlier remarks κ is prone and λ supine, and likewise $\kappa\mathfrak{F}$ or $\lambda\mathfrak{F}$. ■

2.1 Notation for change of base

Many of our arguments will depend on chasing bundles around some diagram by changing the base. Corresponding to a diagram shape will be a category $\mathcal{D}$ of diagrams in $\mathcal{C}$ with that shape, and our calculations will be over $\mathcal{D}$, using pullbacks against $\mathcal{C}^{\downarrow}$, Opspan or Comp. We introduce a systematic notation for this, and illustrate it using diagrams of the shape $f_1\colon B_1 \to B$. The corresponding diagram category is, of course, $\mathcal{C}^{\downarrow}$, but we shall write it in the more normal notation of $\mathcal{C}^{\to}$ to emphasize that we treat the objects as morphisms for base change, not as bundles.

We shall treat the labels in the diagram as canonical, and use them to denote functors such as $B_1\colon \mathcal{C}^{\to} \to \mathcal{C}$ and $f_1\colon \mathcal{C}^{\to} \to \mathcal{C}^{\downarrow}$ (which is just the identity).

We also systematically subscript the labels to denote pullbacks involving the diagram category. First, if X is a node of the diagram, then $\mathcal{C}^{\downarrow}_X$ denotes the pullback of $X\colon \mathcal{D} \to \mathcal{C}$ against $\mathsf{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$. Its objects are diagrams augmented with a bundle over X. We define the functor $\pi\colon \mathcal{C}^{\downarrow}_X \to \mathcal{C}^{\downarrow}$ as the projection.

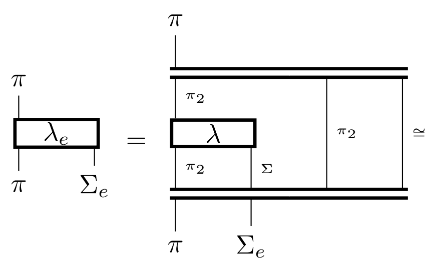
Now suppose $e\colon X \to Y$ is an arrow in the diagram. We can form the pullback Opspan_e of $e\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$ against π_1, but it is clearly isomorphic to $\mathcal{C}^{\downarrow}_Y$. Similarly $\mathrm{Comp}_e \cong \mathcal{C}^{\downarrow}_X$. We shall use these isomorphisms in conjunction with functors and natural transformations related to Opspan and Comp. For example, $\mathsf{pb}_e\colon \mathcal{C}^{\downarrow}_Y \to \mathcal{C}^{\downarrow}_X$ corresponds to

$$\mathcal{D} \times_{\mathcal{C}^{\downarrow}} \mathsf{pb}\colon \mathcal{D} \times_{\mathcal{C}^{\downarrow}} \mathrm{Opspan} \to \mathcal{D} \times_{\mathcal{C}^{\downarrow}} \mathrm{Comp}$$

and likewise for natural transformations.

Thus in our example we have $\mathcal{C}^{\downarrow}_{B_1} \cong$ Comp, $\mathcal{C}^{\downarrow}_B \cong$ Opspan, and, modulo these isomorphisms, pb_{f_1} and Σ_{f_1} correspond to pb and Σ.

For applications of the following proposition, which is a manifestation of the fact that $\Sigma \dashv \mathsf{pb}$ over (first projections to) $\mathcal{C}^{\downarrow}$, we define



Here the isomorphism is $\mathcal{C}^{\downarrow}_X \cong \mathrm{Comp}_e$, and the rightmost π_2 is the projection $\mathrm{Comp}_e \to \mathrm{Comp}$. By Proposition 5, λ_e is supine.

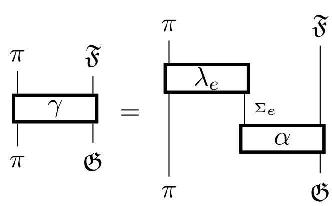
Similarly we can define κ_e, and it is prone by Proposition 5.

Proposition 6 *Let $e\colon X \to Y$ be an arrow in a diagram as above, and let $\mathfrak{F}\colon \mathcal{E} \to \mathcal{C}^{\downarrow}_X$ and $\mathfrak{G}\colon \mathcal{E} \to \mathcal{C}^{\downarrow}_Y$ be functors that agree when projected down to the diagram category $\mathcal{D}$. Then the following structures are equivalent.*

1. *A natural transformation $\alpha\colon \Sigma_e\mathfrak{F} \to \mathfrak{G}$ over Y.*
2. *A natural transformation $\beta\colon \mathfrak{F} \to \mathsf{pb}_e\,\mathfrak{G}$ over X.*
3. *A natural transformation $\gamma\colon \pi\mathfrak{F} \to \pi\mathfrak{G}$ over e.*

"Over Y" means that $\mathsf{cod}\,\alpha$ is the identity, and likewise for "over X". "Over e" means that $\mathsf{cod}\,\gamma$ has each component equal to the morphism got by composing $\mathfrak{F}$ (or $\mathfrak{G}$) with the projection to $\mathcal{D}$ and the functor $e\colon \mathcal{D} \to \mathcal{C}^{\downarrow}$.

Proof. We get from (1) to (3) by taking α to

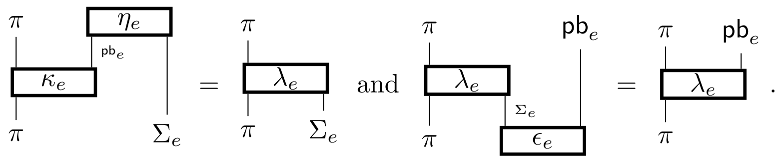


Supineness of λ_e shows that the transformation is invertible.

Similarly we get from (2) to (3) by composing at the bottom with κ_e, and use its proneness. ∎

Roughly speaking, the role of this proposition is as follows. Without π we can use functors such as pb_e to move bundles from one base to another, but each natural transformation is over a given base. With π and κ_e or λ_e we lose the underlying diagram category but gain the flexibility of dealing with change of base in the natural transformations.

Proposition 6 embodies an adjunction $\Sigma_e \dashv \mathsf{pb}_e$ over $\mathcal{D}$. Its unit η and counit ϵ are characterized by



(2)

In fact, the standard "yanking" equations relating η to ϵ are immediate from this.

Let us now use this notation in examining the functoriality of Σ and pb. We define the category CT to be that whose objects are commutative triangles in $\mathcal{C}$, with shape

$$\begin{tikzcd} B_1 \arrow[r, "f_1"'] \arrow[rr, bend left, "f"] & B_2 \arrow[r, "f_2"'] & B_3 \end{tikzcd} \tag{3}$$

Proposition 7

$$\Sigma_{f_2}\Sigma_{f_1} = \Sigma_f .$$

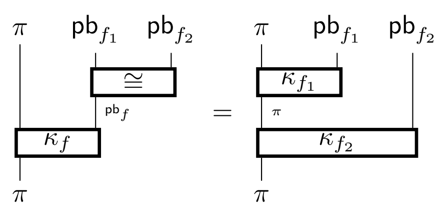
Proof. This is straightforward calculation, using associativity of composition in $\mathcal{C}$. ■

The following proposition is of course well known as an isomorphism $f_1^* f_2^* E \cong (f_2 f_1)^* E$; we show how to characterize the isomorphism using the algebra of proneness.

Proposition 8

$$\mathsf{pb}_{f_1}\,\mathsf{pb}_{f_2} \cong \mathsf{pb}_f .$$

Proof. Using the proneness of κ_e, we define $\cong$ as the unique natural transformation over B_1 such that



If we compose at the top with a formal inverse of $\cong$, we get an equation that defines a natural transformation $\cong^{-1}$. (More carefully, proneness of κ_{f_2} gives us $\alpha\colon \pi\,\mathsf{pb}_f \to \pi\,\mathsf{pb}_{f_2}$ over f_1 such that $\begin{pmatrix}\alpha \\ \kappa_{f_2}\end{pmatrix} = \kappa_{f_1}$, and then proneness of κ_{f_1} gives us $\cong^{-1}$ such that $\begin{pmatrix}\pi \cong^{-1} \\ \kappa_{f_1}\end{pmatrix} = \alpha$.) The two equations then allow us to prove that $\cong^{-1}$ is indeed an inverse of $\cong$. ■

2.2 The Beck-Chevalley condition

We must also examine the Beck-Chevalley condition for pullback squares.

Definition 9 *The category* PB *has as objects the pullback squares*

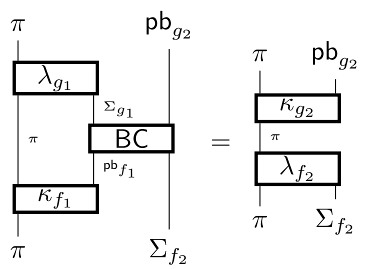
$$\begin{tikzcd} C \arrow[r, "g_2"] \arrow[d, "g_1"'] \arrow[dr, "g"] & B_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "f_1"'] & B \end{tikzcd}$$

in $\mathcal{C}$, with the obvious diagram morphisms.

The Beck-Chevalley condition says that if you start with a bundle over B_2 in the pullback square, and shift it to a bundle over B_1 via two different routes round the square, then the results are isomorphic.

In the rest of this subsection, we work over PB.

Definition 10 *The* Beck-Chevalley transformation $\mathsf{BC}\colon \Sigma_{g_1} \mathsf{pb}_{g_2} \to \mathsf{pb}_{f_1} \Sigma_{f_2}$ *is defined as being over* PB *and such that*



.

The definition combines Propositions 6 and 7. The natural transformation on the right is over g and gives rise to a natural transformation from $\Sigma_{f_1}\Sigma_{g_1} \mathsf{pb}_{g_2} = \Sigma_g \mathsf{pb}_{g_2}$ to Σ_{f_2}, and this in turn gives rise to the natural transformation BC.

Proposition 11 BC *is a natural isomorphism.*

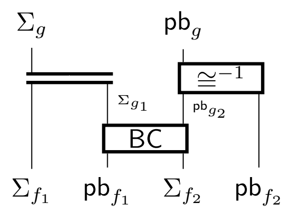
Proof. An object of $\mathcal{C}^{\downarrow}_{B_2}$ is pullback diagram in $\mathcal{C}$ equipped with a bundle $p\colon E \to B_2$ in $\mathcal{C}$. The component of BC at that object is a square

$$\begin{tikzcd} (B_1 \times_B B_2) \times_{B_2} E \arrow[r] \arrow[d] & B_1 \times_B E \arrow[d] \\ B_1 \arrow[r, equal] & B_1 \end{tikzcd}$$

where the top arrow is well known to be an isomorphism. ■

For the following lemma we work over a diagram comprising three morphisms $f_1\colon B_1 \to B$, $f_2\colon B_2 \to B$ and $p\colon E \to B$. We write Opspan_3 for its diagram category. This enables us to work over PB via the functor from Opspan_3 that completes the pullback square for f_1 and f_2, giving a functor from Opspan_3 to $\mathcal{C}_B^{\downarrow}$.

Lemma 12 *The canonical isomorphism* $(B_1 \times_B B_2) \times_B E \cong B_1 \times_B (B_2 \times_B E)$ *is provided by the natural transformation*



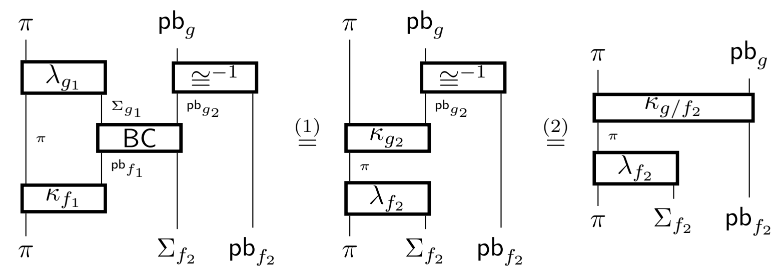
,

where $\cong$ *is that of Proposition 8.*

Proof. Leaving out Σ_{f_1}, we can take the canonical isomorphism over B_1 and we then see that it is characterized by the diagram

$$\begin{tikzcd}
(B_1 \times_B B_2) \times_B E \arrow[r, equal] \arrow[d] \arrow[ddrr, bend left=15, "\pi_2 \times_B E"'] & (B_1 \times_B B_2) \times_B E \arrow[rr, "\cong"] \arrow[d] & & B_1 \times_B (B_2 \times_B E) \arrow[r] \arrow[d] & B_2 \times_B E \arrow[d] \arrow[ddll, equal, bend left=15] \\
B_1 \times_B B_2 \arrow[r, "g_1"'] \arrow[ddrr, bend right=20, "g_2"'] & B_1 \arrow[rr, equal] & & B_1 \arrow[r, "f_1"'] & B \\
& & B_2 \times_B E \arrow[d] & & \\
& & B_2 \arrow[uurr, bend right=20, "f_2"'] & &
\end{tikzcd}$$

Here the leftmost and rightmost squares in the top line are λ_{g_1} and κ_{f_1}, and the curved square lower right is λ_{f_2}. The curved square on the lower left is a natural transformation κ_{g/f_2} over g_2 characterized by $\begin{pmatrix} \kappa_{g/f_2} \\ \kappa_{f_2} \end{pmatrix} = \kappa_g$. To show that the canonical isomorphism is as stated, we calculate



Here equation (1) follows from Definition 10, and equation (2) from Proposition 8. ■

3 Bundle endofunctors

Definition 13 *Let $\mathcal{C}$ be a cartesian category.*

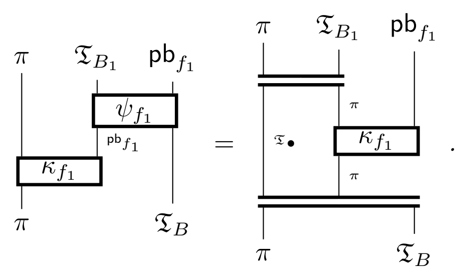
A bundle endofunctor *on $\mathcal{C}$ is an endofunctor $\mathfrak{T}_\bullet$ of $\mathcal{C}^\downarrow$ that restricts to the identity on bases (i.e. via the codomain fibration* cod*).*

A bundle monad *on $\mathcal{C}$ is a monad $(\mathfrak{T}_\bullet, \eta_\bullet, \mu_\bullet)$ on $\mathcal{C}^\downarrow$ in which all three components restrict to the identity on bases.*

The "$\bullet$" is used to indicate the fact that a bundle endofunctor or monad restricts to an endofunctor or monad on each slice category $\mathcal{C}/B$, notated by replacing the $\bullet$ by B. Applying $\mathfrak{T}_\bullet$ to the square (1) will give us the square

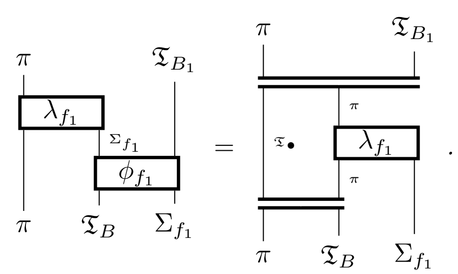
$$\begin{tikzcd}
\mathfrak{T}_B E \arrow[r, "\mathfrak{T}_{\underline{f}}\overline{f}"] \arrow[d] & \mathfrak{T}_{B'} E' \arrow[d] \\
B \arrow[r, "\underline{f}"'] & B'
\end{tikzcd} \quad . \tag{4}$$

Definition 14 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$, and suppose we work over a diagram $f_1 \colon B_1 \to B$. Using proneness of $\kappa_{f_1}\mathfrak{T}_B$ (Proposition 5), we define $\psi_{f_1} \colon \mathfrak{T}_{B_1}\, \mathsf{pb}_{f_1} \to \mathsf{pb}_{f_1}\, \mathfrak{T}_B$ over B_1 by*



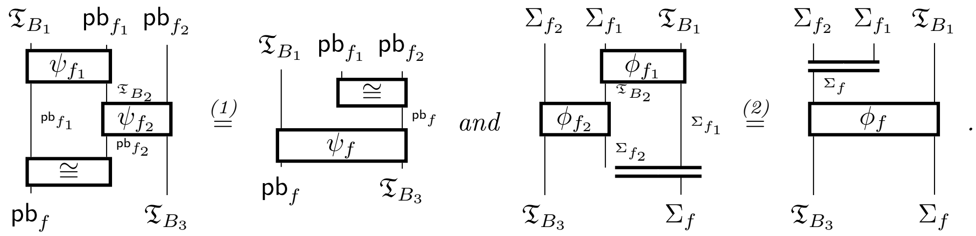
(5)

Definition 15 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$, and suppose we work over a diagram $f_1 \colon B_1 \to B$. Using supineness of $\lambda_{f_1}\mathfrak{T}_{B_1}$ (Proposition 5), we define $\phi_{f_1} \colon \Sigma_{f_1}\mathfrak{T}_{B_1} \to \mathfrak{T}_B \Sigma_{f_1}$ over B by*



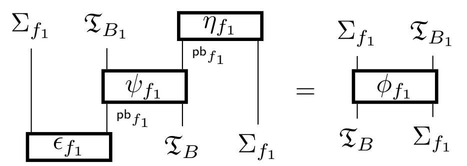
(6)

Lemma 16 *Working over the diagram category* CT, *with shape* (3), *we have*

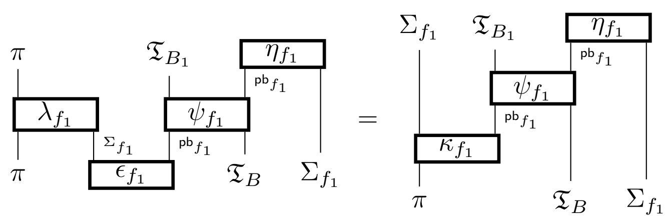


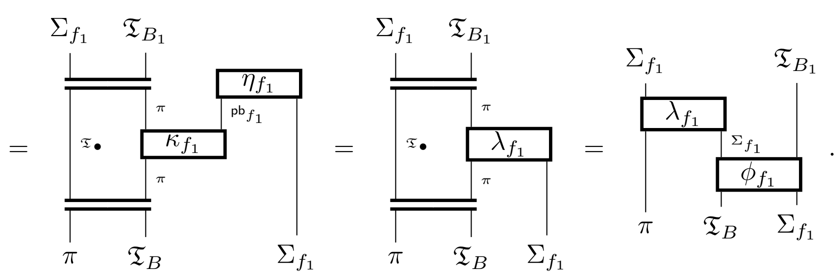
Proof. Compose on the left with π and (1) below with $\kappa_f \mathfrak{T}_{B_3}$, or (2) above with $\lambda_{f_1}\mathfrak{T}_{B_1}$ and $\lambda_{f_2}\Sigma_{f_1}\mathfrak{T}_{B_1}$; it is then a straightforward calculation. ∎

Lemma 17



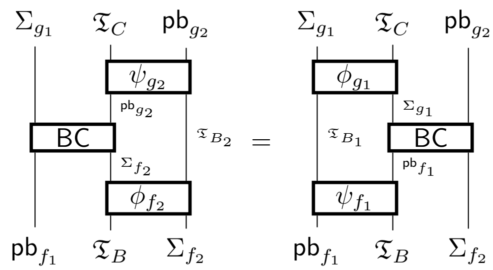
Proof. Composing with π and λ_{f_1}, the calculation reduces to



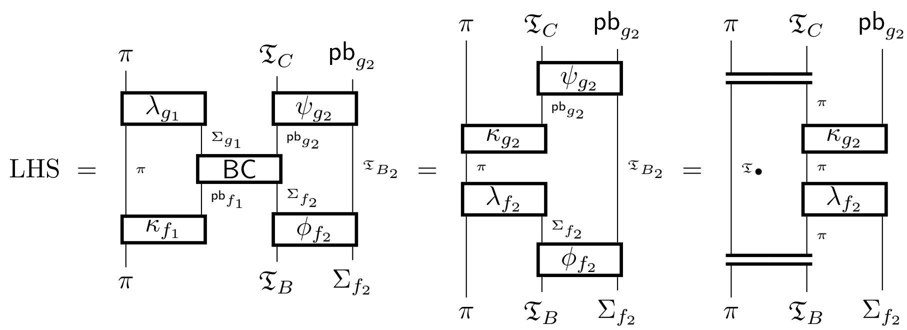


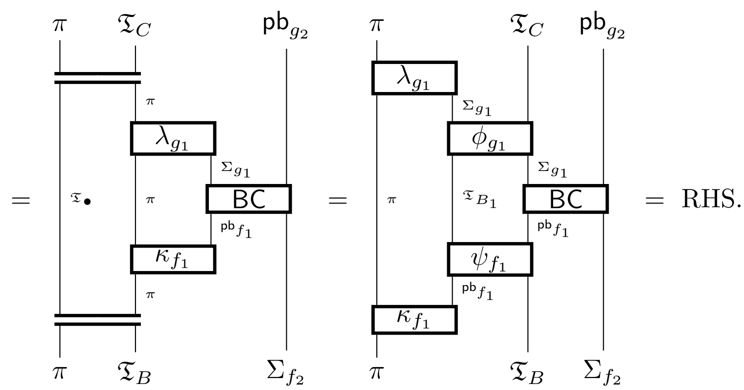
∎

Lemma 18



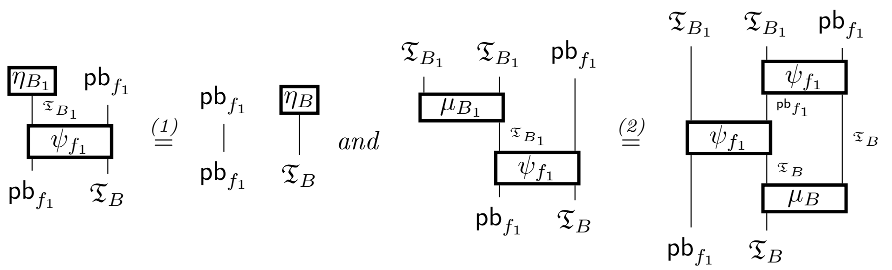
Proof. Since everything is over PB, it remains to show that the expressions are equal when composed with π on the left, and with λ and κ top and bottom. Then –



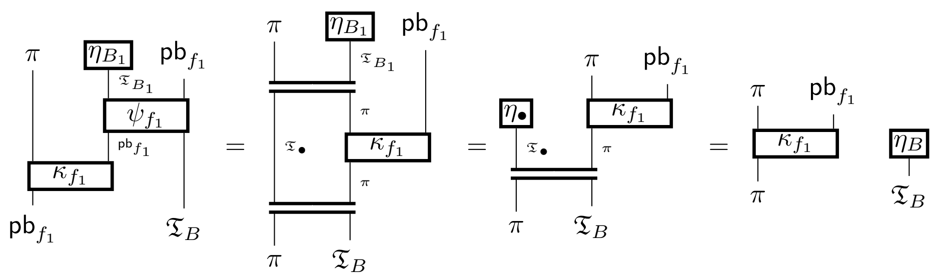


■

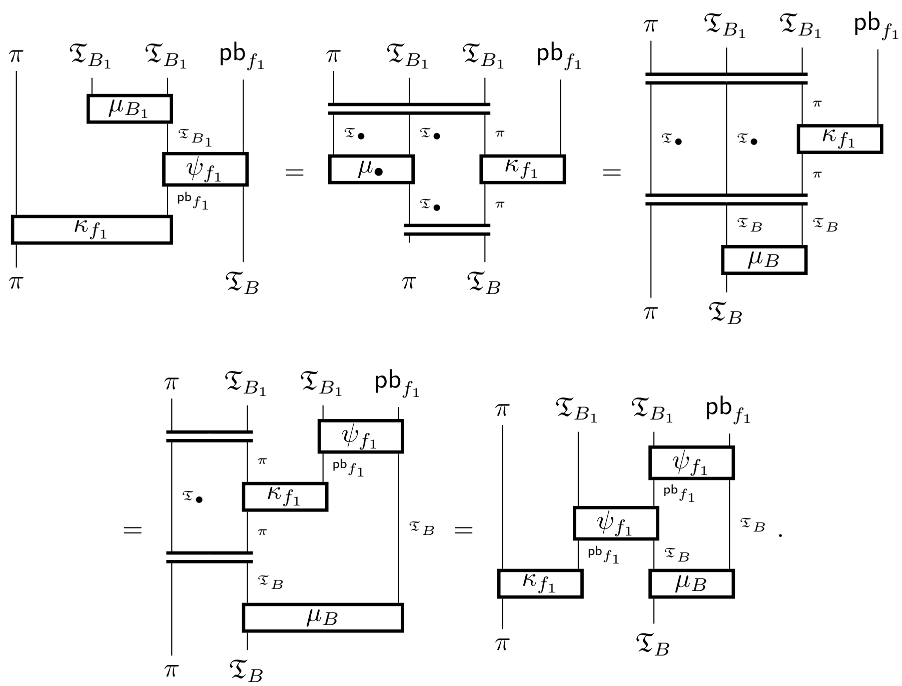
Lemma 19 *Let $\mathfrak{T}_\bullet$ be a bundle monad on $\mathcal{C}$. Then, generically,* $(\mathsf{pb}_{f_1}, \psi_{f_1})$ *is* *a monad functor from $(\mathcal{C}/B, \mathfrak{T}_B)$ to $(\mathcal{C}/B_1, \mathfrak{T}_{B_1})$. More precisely,*



Proof. For each equation we compose on the left with π and at the bottom with κ_{f_1}. The calculation (1) then reduces to

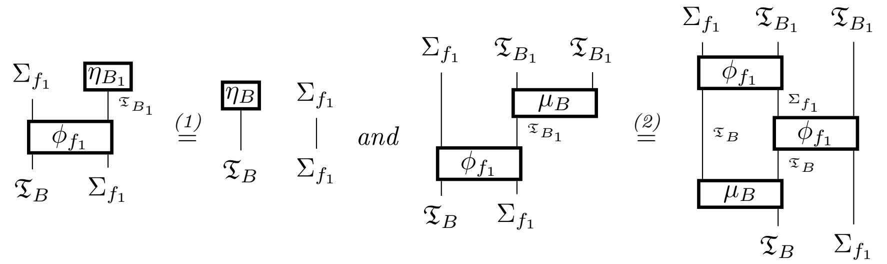


For (2) the calculation reduces to



■

Lemma 20 *Let $\mathfrak{T}_\bullet$ be a bundle monad on $\mathcal{C}$. Then, generically, $(\Sigma_{f_1}, \phi_{f_1})$ is a monad opfunctor from $(\mathcal{C}/B_1, \mathfrak{T}_{B_1})$ to $(\mathcal{C}/B, \mathfrak{T}_B)$. More precisely,*



Proof. Combine Lemmas 19 and 17. ■

3.1 Slicewise strengths

First, we recall the definition of strong endofunctors and monads.

Definition 21 *Let $\mathcal{C}$ be a category with binary products and $T\colon \mathcal{C} \to \mathcal{C}$ an endofunctor. Then a* strength *on T is a natural transformation given by morphisms $t(X,Y)\colon X \times TY \to T(X \times Y)$ for each $X, Y \in Ob(\mathcal{C})$ such that the following diagrams both commute.*

$$\begin{tikzcd} 1 \times TY \arrow[r, "t(1,Y)"] \arrow[d, "\cong"'] & T(1 \times Y) \arrow[d, "T(\cong)"] \\ TY \arrow[r, equal] & TY \end{tikzcd} \tag{7}$$

and

$$\begin{tikzcd} (X_1 \times X_2) \times TY \arrow[rr, "t(X_1 \times X_2, Y)"] \arrow[d, "\cong"'] & & T((X_1 \times X_2) \times Y) \arrow[d, "\cong"'] \\ X_1 \times (X_2 \times TY) \arrow[r, "X_1 \times t(X_2,Y)"'] & X_1 \times T(X_2 \times Y) \arrow[r, "t(X_1, X_2 \times Y)"'] & T(X_1 \times (X_2 \times Y)) \end{tikzcd} \tag{8}$$

We say that T, thus equipped, is a strong *endofunctor.*

Note that by composing (7) with a naturality square for t, we get a commutative triangle

$$\begin{tikzcd} X \times TY \arrow[r, "t(X,Y)"] \arrow[dr, "\pi_2"'] & T(X \times Y) \arrow[d, "T(\pi_2)"] \\ & TY \end{tikzcd} \tag{9}$$

that generalizes (7) to the case with X instead of 1.

It is important to note that this definition of strength for an endofunctor is different from that of strength for a monad ([Koc72].

Definition 22 *Let T be an endofunctor on $\mathcal{C}$ with strength t, and suppose in addition T is a monad with unit η and multiplication μ. Then t is a strength for the monad if the following diagrams commute.*

$$\begin{tikzcd}
X \times Y \arrow[d, "X\times\eta"'] \arrow[drr, "\eta"] & & \\
X \times TY \arrow[rr, "t(X,Y)"] & & T(X \times Y) \\
X \times T^2 Y \arrow[u, "X\times\mu"] \arrow[r, "t(X,TY)"'] & T(X \times TY) \arrow[r, "T(t(X,Y))"'] & T^2(X \times Y) \arrow[u, "\mu"']
\end{tikzcd} \quad . \tag{10}$$

The strength is a natural transformation between two functors out of $\mathcal{C} \times \mathcal{C}$, and using the isomorphism of this with Comp_1 we can view the strength t as a natural transformation $t_! \colon \Sigma_! \, \mathsf{pb}_! \, T_1 \to T_1 \Sigma_! \, \mathsf{pb}_!$. We can generalize this to a strength for an endofunctor T of a slice $\mathcal{C}/B$, replacing $! \colon B \to 1$ by $f_1 \colon B_1 \to B$. Our aim in the present subsection is to do this for generic f_1.

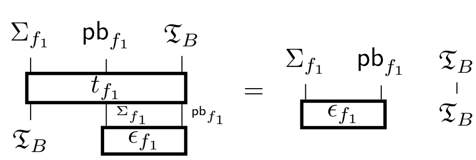
Definition 23 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$. Then a* slicewise strength *for $\mathfrak{T}_\bullet$ is a natural transformation between two endofunctors of* Opspan*,*

$$t_\bullet \colon \Sigma \, \mathsf{pb} \, \mathfrak{T}_\bullet \to \mathfrak{T}_\bullet \Sigma \, \mathsf{pb}$$

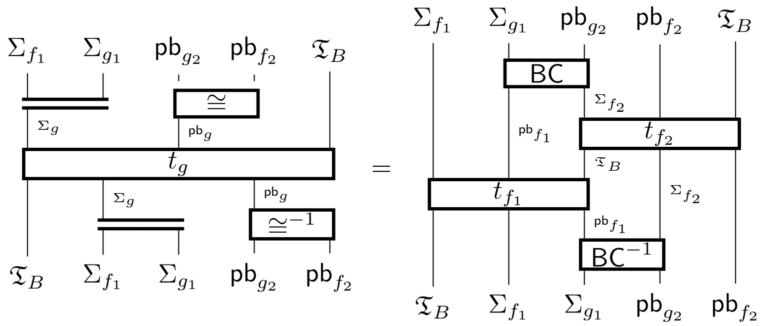
over $\pi_1 \colon$ Opspan $\to \mathcal{C}^\to$. (We are writing $\mathfrak{T}_\bullet$ for the induced endofunctor on Opspan *over $\mathcal{C}^\to$.) The $\bullet$ indicates that $t_\bullet$ will restrict to a strength t_B for T_B on each slice $\mathcal{C}/B$.*

$t_\bullet$ will usually appear with subscript notation for change of base: so if we treat Opspan *as $\mathcal{C}_B^\downarrow$ over a diagram $f_1 \colon B_1 \to B$, then $t_\bullet$ is $t_{f_1} \colon \Sigma_{f_1} \, \mathsf{pb}_{f_1} \, \mathfrak{T}_B \to \mathfrak{T}_B \Sigma_{f_1} \, \mathsf{pb}_{f_1}$. (Thinking slicewise, this t_{f_1} is really $t_B(B_1, -)$.)*

We require $t_\bullet$ to satisfy the following two equations.

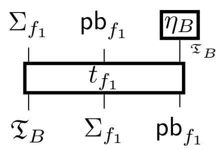


(11)

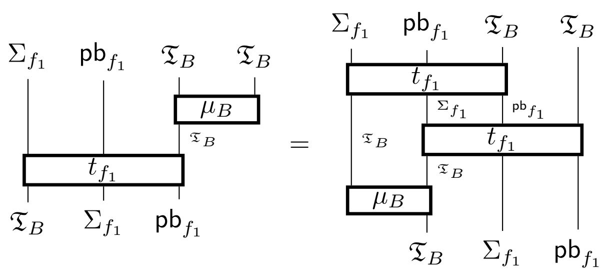


(12)

If in addition $\mathfrak{T}_\bullet$ is the functor part of a bundle monad, then $t_\bullet$ is a strength for the monad if it also satisfies the following two conditions.



$= \eta_B \Sigma_{f_1}\, \mathsf{pb}_{f_1}$ (13)

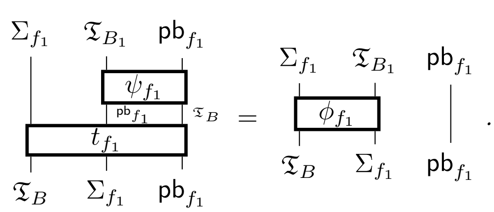


(14)

It is not hard to see that when the generic $f_1 \colon B_1 \to B$ is specialized to $! \colon B \to 1$, these conditions become those expressed in diagrammatic form in Definitions 21 and 22. (Use Lemma 12 for condition (8).)

Although slicewise strength naturally expresses the strength conditions, we shall commonly also need the following property.

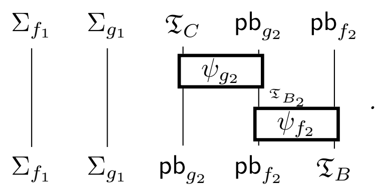
Definition 24 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor and let $t_\bullet$ be a natural transformation with the right type for a slicewise strength. Then $t_\bullet$ has the ψ-ϕ-*condition *if*



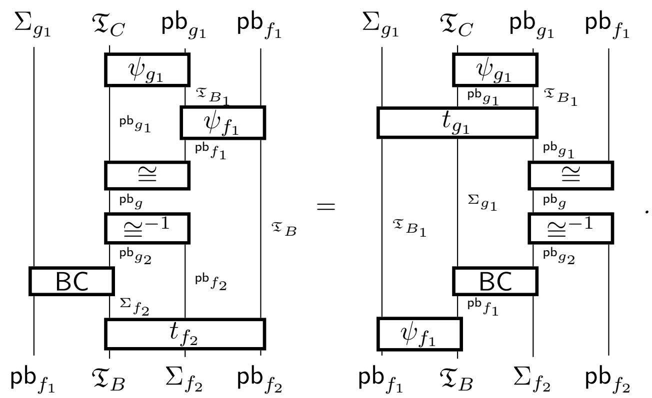
The following lemma shows something of the power of the ψ-ϕ-condition. For, in the case (Section 5) where ψ is invertible, it is already enough to show equation (12).

Lemma 25 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$, and let $t_\bullet$ be a natural transformation, of the type needed for a slicewise strength, that satisfies the ψ-ϕ-condition. Then, working over* PB*, we have the following.*

1. *$t_\bullet$ satisfies equation* (11) *composed at the top with $\Sigma_{f_1}\psi_{f_1}$.*
2. *$t_\bullet$ satisfies equation* (12) *composed at the top with*



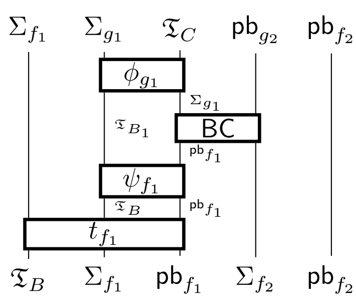
3.



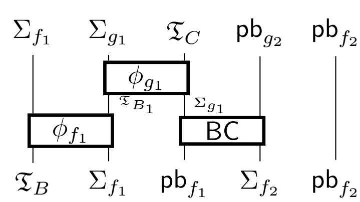
Proof. (1) After applying the ψ-ϕ-condition, the result is immediate from Lemma 17.

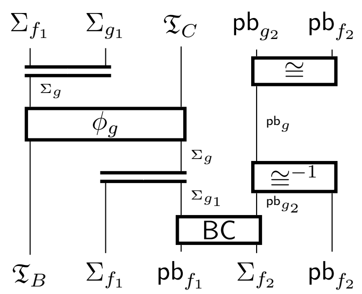
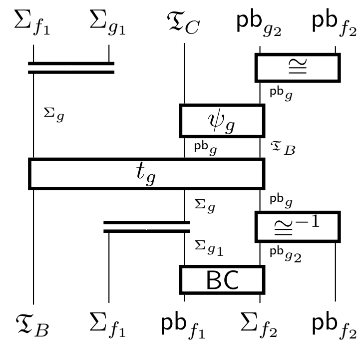
(2) Composing also at the bottom with $\mathfrak{T}_B\Sigma_{f_1}\,\mathsf{BC}\,\mathsf{pb}_{f_2}$, we have

RHS $\overset{(1)}{=}$



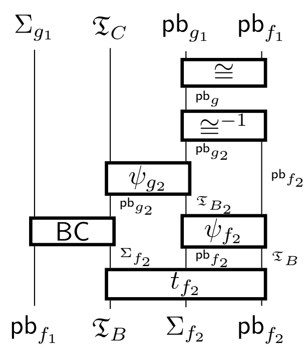
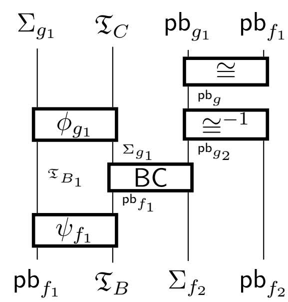
$\overset{(2)}{=}$



$\overset{(3)}{=}$  $\overset{(4)}{=}$  $\overset{(5)}{=}$ LHS.

Equation (1) comes from the ψ-ϕ-condition and Lemma 18, equation (2) from the ψ-ϕ-condition, equation (3) from Lemma 16 part (2), equation (4) from the ψ-ϕ-condition and equation (5) from Lemma 16 part (1).

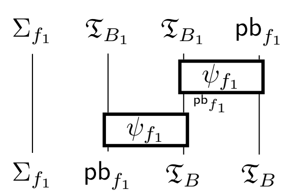
For part (3), we have

LHS $\overset{(1)}{=}$  $\overset{(2)}{=}$  $\overset{(3)}{=}$ RHS.

Here equation (1) follows by applying Lemma 16 twice, equation (2) by the ψ-ϕ-condition and Lemma 18, and equation (3) by the ψ-ϕ-condition. ■

Lemma 26 *Let $\mathfrak{T}_\bullet$ be a bundle monad on $\mathcal{C}$, and let $t_\bullet$ be a natural transformation, of the type needed for a slicewise strength, that satisfies the ψ-ϕ-condition. Then, working over* CT*, we have the following.*

1. *$t_\bullet$ satisfies equation* (13)*.*
2. *$t_\bullet$ satisfies equation* (14) *composed at the top with*



.

Proof. The proof is a straightforward calculation that is conceptually the same in both cases. We describe it for part (2). First, Lemma 19 is used to move any ψs at the top down to below the μ (or the η in part (1)). Next, apply the ψ-ϕ-condition. Next, use Lemma 20 to move the ϕ up, giving two ϕs above the μ. Finally, apply the ψ-ϕ-condition to those ϕs. ■

4 Constructing some bundle endomorphisms

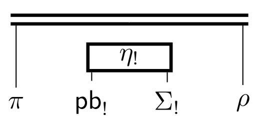
4.1 The bundle monad $U_\bullet$

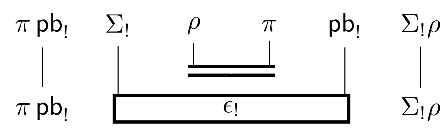
Let us define categories Comp_1 and Opspan_1 as the subcategories of Comp and Opspan in which $B = 1$. Then we have an isomorphism $\rho\colon \mathcal{C}^\downarrow \to \mathrm{Comp}_1$ whose inverse is π. We shall use our usual subscript notation, replacing $f_1\colon B_1 \to B$ by $!\colon B \to 1$.

Now it is easy to see that we have a bundle monad

$$U_\bullet = \pi\,\mathsf{pb}_!\,\Sigma_!\rho$$

on $\mathcal{C}$, with unit and multiplication defined by

$\eta^U_\bullet =$ 

and $\mu^U_\bullet =$ 

.

An important use of $U_\bullet$ will be to express each bundle as a *coreflexive equalizer* – that is to say, an equalizer expressed as that of a pair $f_i\colon X \to Y$ $(i = 1, 2)$ for which there is a map $g\colon Y \to X$ with $g \circ f_i = \mathsf{Id}_X$. Given the existence of binary products, any equalizer of a pair $f_i\colon X \to Y$ can be expressed as a coreflexive equalizer, by replacing f_i by $\langle f_i, X\rangle\colon X \to Y \times X$.

The following proposition states that the monad $U_\bullet$ has the *equalizer property.*

Proposition 27 *The following coreflexive diagram of functors is an equalizer at every object of $\mathcal{C}^\downarrow$.*

$$\begin{tikzcd} \mathsf{Id} \arrow[r, "\eta^U_\bullet"] & U_\bullet \arrow[r, shift left=2, "U_\bullet\eta^U_\bullet"] \arrow[r, shift right=2, "\eta^U_\bullet U_\bullet"'] & U_\bullet^2 \arrow[l, "\mu^U_\bullet"' description] \end{tikzcd} .$$

Proof. For a bundle $p\colon E \to B$, the diagram appears more concretely in $\mathcal{C}^\downarrow$ as

$$\begin{tikzcd} E \arrow[r, "\langle p,E\rangle"] \arrow[d, "p"'] & B \times E \arrow[r, shift left, "B\times\langle p,E\rangle"] \arrow[r, shift right, "\Delta\times E"'] \arrow[d, "\pi_1"] & B \times B \times E \arrow[d, "\pi_1"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd} .$$

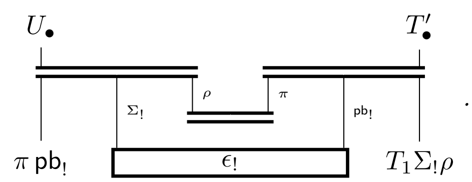
A generalized element of the equalizer is a pair (b, e) such that $(b, pe, e) = (b, b, e)$, i.e. $b = pe$. Such pairs are equivalent to elements e of E. ■

4.2 From T to $T'_\bullet$

Definition 28 *Let T be a strong endofunctor on $\mathcal{C}$, t its strength. We define a bundle endofunctor*

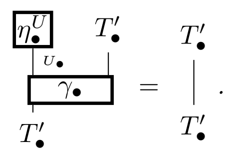
$$T'_\bullet = \pi\,\mathsf{pb}_!\,T_1\Sigma_!\rho.$$

(We are writing T also for the endofunctor of $\mathcal{C}/1$ that arises from the isomorphism with $\mathcal{C}$. T_1 is then the corresponding endofunctor of Comp_1 *using our suffix notation.) As an analogue of $\mu^U_\bullet$, we also define a natural transformation $\gamma_\bullet\colon U_\bullet T'_\bullet \to T'_\bullet$ as*



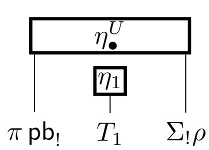
Similarly we define $\overline{\gamma}_\bullet\colon T'_\bullet U_\bullet \to T'_\bullet$.

Lemma 29

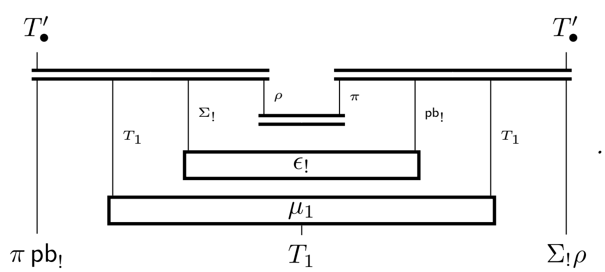


Proof. Simple calculation. ∎

Proposition 30 *Let (T, η, μ) be a monad on $\mathcal{C}$. Then $(T'_\bullet, \eta'_\bullet, \mu'_\bullet)$ is a bundle monad, with unit and multiplication defined as*



and

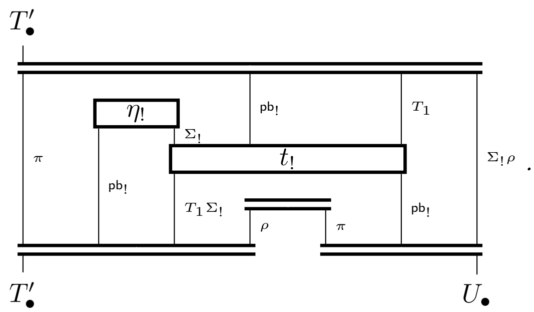


Proof. Routine calculation. ∎

4.3 From strong T to $T_\bullet$

We now turn our central construction of a bundle endofunctor $T_\bullet$ from a strong endofunctor T, which will be used in our main results.

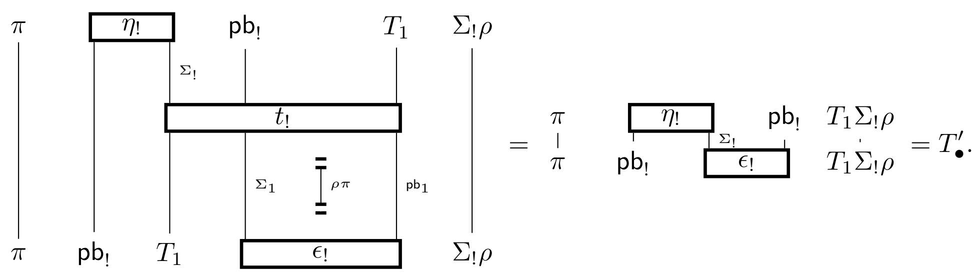
Definition 31 *Let T be a strong endofunctor on $\mathcal{C}$, with strength t. We define $\beta_\bullet \colon T'_\bullet \to T'_\bullet U_\bullet$ as*



($t_!$ here is really $t(B, -)$.)

Proposition 32 *The natural transformations $T'_\bullet \eta^U_\bullet, \beta_\bullet \colon T'_\bullet \to T'_\bullet U_\bullet$ form a coreflexive pair, with coreflexivity morphism $\overline{\gamma}_\bullet$.*

Proof. For $T'_\bullet \eta^U_\bullet$ we have a simple yanking. For $\beta_\bullet$ we have, using the first strength condition,



■

Definition 33 *Let T be an endofunctor on a cartesian category $\mathcal{C}$, and let t be a strength for it. We define a bundle endofunctor $T_\bullet$ over $\mathcal{C}$ as the equalizer (constructed objectwise)*

$$T_\bullet \xrightarrow{e_\bullet} T'_\bullet \underset{\beta_\bullet}{\overset{T'_\bullet \eta^U_\bullet}{\rightrightarrows}} T'_\bullet U_\bullet \ .$$

Since the equalizer diagram is all over cod, we can and will define $e_\bullet$ also to be over cod. Thus the whole diagram restricts to each slice $\mathcal{C}/B$, with T_B etc. Since the equalizer is constructed objectwise, it is also an equalizer on the slice.

Proposition 34 *On a bundle $p \colon E \to B$, then $T_\bullet(p)$ is the morphism $T_B E \to B$ where $T_B E$ is the equalizer of two morphisms*

$$B \times TE \underset{t_{BE}}{\overset{T\langle p, E\rangle \pi_2}{\rightrightarrows}} T(B \times E) \ .$$

In other words, the (generalized) elements of $T_B E$ are the pairs $(b,u) \in B \times TE$ such that $t(b,u) = T\langle p, E\rangle(u)$.

Proof. $T'_\bullet(p)$ and $T'_\bullet U_\bullet(p)$ are $B \times TE$ and $B \times T(B \times E)$, both as bundles over B by the first projection. In terms of generalized elements, $T'_\bullet \eta^U_\bullet(b,u) = (b, T\langle p, E\rangle(u))$ and $\beta_\bullet(b,u) = (b, t_{BE}(u))$. Since both are over B, we get the same equalizer if we project down to $T(B \times E)$. ■

We next examine the case $B = 1$. We shall write T equally for the endofunctor on $\mathcal{C}$ and that on $\mathcal{C}/1$ derived from the isomorphism, but distinguish them from the endomorphism T_1 on $\mathcal{C}/1$ obtained by restricting $T_\bullet$. To void confusion with our suffix notation for change of base, we shall write $(T)_1$ and $(T_1)_1$.

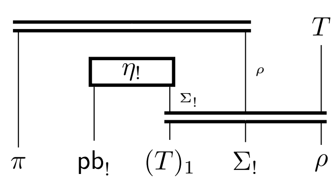
Proposition 35 *The transformations $T'_1 \eta^U_1$ and β_1 are equal.*

Proof. The transformation $\epsilon_! \Sigma_! \rho$ is invertible on $\mathcal{C}/1$, hence so also is the coreflexivity morphism $\overline{\gamma}_1$ (Proposition 32). ■

Corollary 36 *The transformation $e_1 \colon T_1 \to T'_1$ is an isomorphism.*

Definition 37 *The natural isomorphism $e' \colon T \to T'_1$ is defined in diagrammatic form with components as on the left here, or in tangle form on the right (restricted to $\mathcal{C}/1$).*

$$\begin{tikzcd} TE \arrow[r, "\langle !, TE\rangle"] \arrow[d] & 1 \times TE \arrow[d] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$



Using equation (2), and noting that $\lambda_! \rho$ is equality when restricted to $\mathcal{C}/1$, we see that $\begin{pmatrix} e' \\ \kappa_!(T)_1\Sigma_!\rho \end{pmatrix}$ is the equality between T and $\pi(T)_1\Sigma_!\rho$.

Definition 38 *Because e' is an isomorphism, so too is its factorization via T_1, which we shall write as $\cong \colon T \to T_1$.*

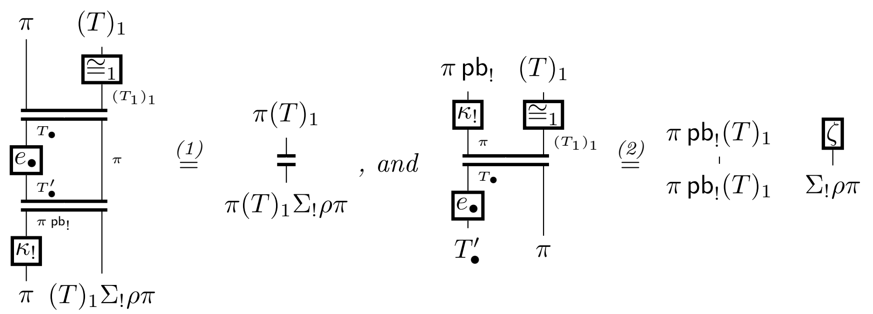
Definition 39 *We define the natural transformation ζ from $\mathsf{Id}_{\mathrm{Opspan}_1}$ to $\Sigma_!\rho\pi$ (restricted to* Opspan$_1$*) to have components*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & 1 \arrow[d, equal] & E \arrow[l] \arrow[d, equal] \\ 1 \arrow[r, equal] & 1 & E \arrow[l] \end{tikzcd} .$$

Lemma 40 *(1) $\pi(T)_1\zeta$ is equality, and (2) $\zeta\Sigma_!\rho = \Sigma_!\rho \begin{pmatrix} \eta^U_\bullet \\ \kappa_!\Sigma_!\rho \end{pmatrix}$.*

Proof. Both easily calculated. ■

Lemma 41

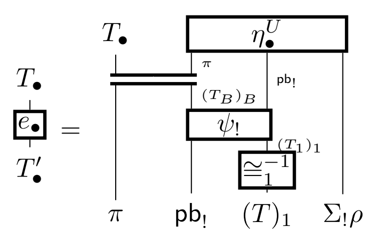


Proof. (1) If we write $i\colon \mathcal{C}/1 \to \mathcal{C}^{\downarrow}$ for the obvious inclusion, then we find that the $\cong_1$ and $e_\bullet$ can be brought together as $i\begin{pmatrix}\cong\\ e_1\end{pmatrix}\pi = ie'\pi$. We can now use the tangle characterization of e'.

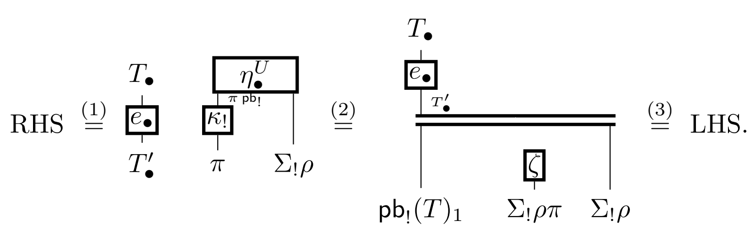
(2) Compose at bottom with κ, then use part (1) and properties of ζ. ■

The following proposition will have an analogue in Definition 53.

Proposition 42



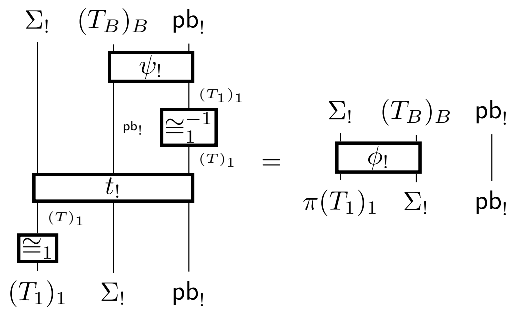
Proof. Using the fact that $\kappa_!$ is prone and $e_\bullet$ is monic, it suffices to compose below with the tangle on the left in Lemma 41 (2). Then –



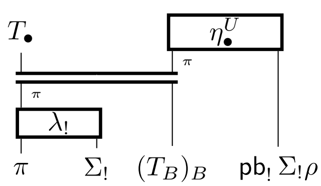
Here equation (1) uses the definition of ψ, (2) uses Lemma 40 (2) and (3) uses Lemma 41 (2). ■

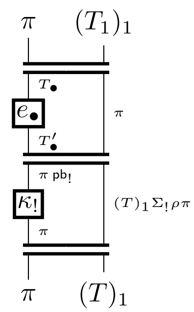
Although we lack a slicewise strength for $T_\bullet$, the following proposition provides the analogue of the ψ-ϕ-condition at $\mathcal{C}/1$.

Proposition 43



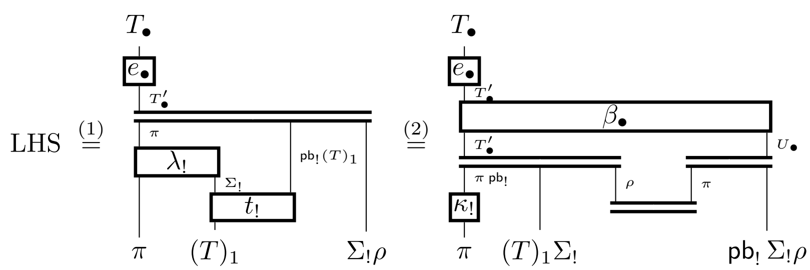
Proof. It suffices to prove the equation when composed on the left with π, on the right with $\Sigma_!\rho$, and at top and bottom with

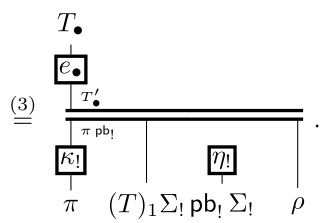
(top) 

and (bottom) 

.

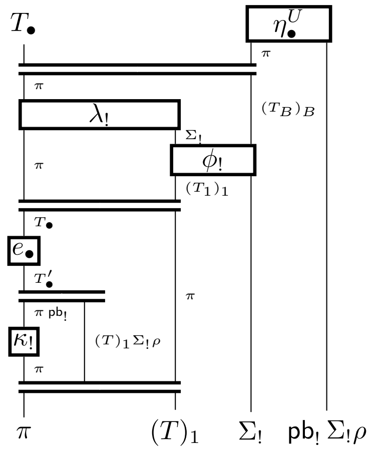
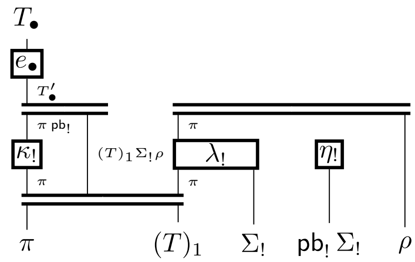
At the top this uses the supineness of $\lambda_!$ and the fact that $\eta^U_\bullet$ can be yanked out. At the bottom it uses the fact that $e_\bullet$ is mono and $\kappa_!$ is prone. We now calculate





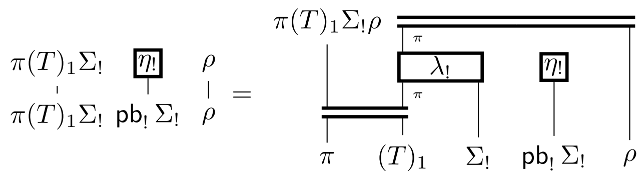
.

Here, equation (1) uses Proposition 42 and Lemma 41, equation (2) uses Definition 31, and equation (3) uses the equalizer property of $e_\bullet$ and the definition of $\eta^U_\bullet$. On the other hand,

RHS $\overset{(1)}{=}$  $\overset{(2)}{=}$  .

Here, equation (2) has used the definitions of ϕ and $\eta^U_\bullet$.

Now it suffices to prove the equation without the $e_\bullet$ or the $\kappa_!$, i.e.



Both sides can be calculated as having (upstairs) $T\langle p, E\rangle \colon TE \to T(B \times E)$. ■

4.4 From strong monad T to $T_\bullet$

In this subsection we assume that (T, η, μ) is a strong monad, with strength t. We show that $T_\bullet$ is the functor part of a bundle monad $(T_\bullet, \eta_\bullet, \mu_\bullet)$.

Lemma 44 *Let (T, η, μ) be a strong monad on $\mathcal{C}$, with strength t. Then each of the following diagrams commutes in the two evident ways.*

1.

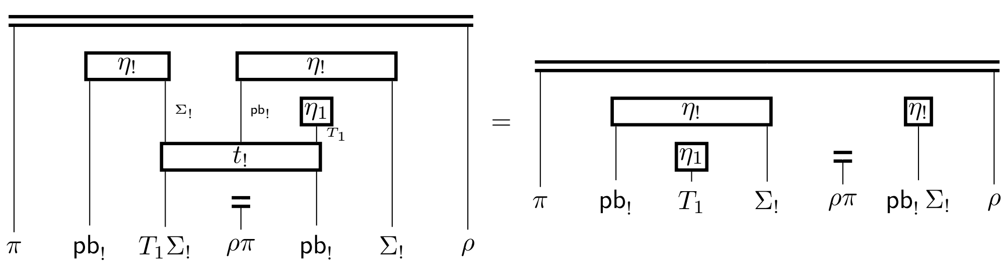
$$\begin{tikzcd}
\mathsf{Id} \arrow[r, shift left, "\eta^U_\bullet"] \arrow[r, shift right, "\eta^U_\bullet"'] \arrow[d, "\eta'_\bullet"'] & U_\bullet \arrow[d, "\eta'_\bullet U_\bullet"] \\
T'_\bullet \arrow[r, shift left, "T'_\bullet \eta^U_\bullet"] \arrow[r, shift right, "\beta_\bullet"'] & T'_\bullet U_\bullet
\end{tikzcd} .$$

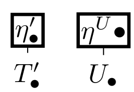
2.

$$\begin{tikzcd}
T'_\bullet T'_\bullet \arrow[r, "\beta_\bullet T'_\bullet"'] \arrow[rrr, bend left, "T'_\bullet T'_\bullet \eta^U_\bullet"] \arrow[d, "\mu'_\bullet"'] & T'_\bullet U_\bullet T'_\bullet \arrow[r, "T'_\bullet \gamma_\bullet"'] & T'_\bullet T'_\bullet \arrow[r, "T'_\bullet \beta_\bullet"'] & T'_\bullet T'_\bullet U_\bullet \arrow[d, "\mu'_\bullet U_\bullet"] \\
T'_\bullet \arrow[rrr, "\beta_\bullet"'] \arrow[rrr, bend left, "T'_\bullet \eta^U_\bullet"] & & & T'_\bullet U_\bullet
\end{tikzcd} .$$

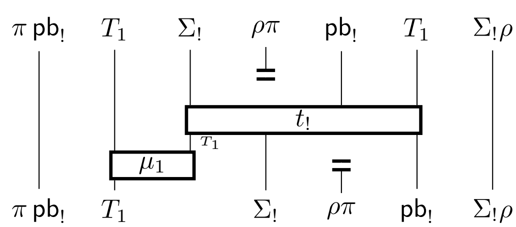
Proof. In each part, commutativity of the upper square is obvious. It remains to discuss the lower squares, involving $\beta_\bullet$.

(1): $\begin{pmatrix} \eta'_\bullet \\ \beta_\bullet \end{pmatrix}$ is the left-hand side of the following equation, which then uses one of the monad laws for the strength $t_!$.



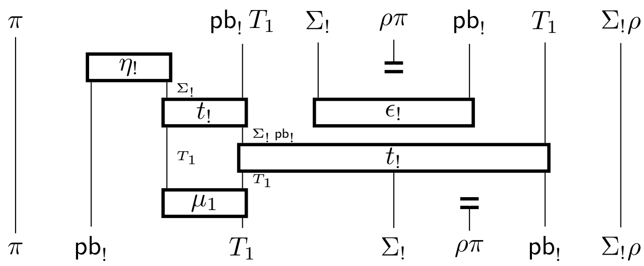
The right-hand side then equals  $= \begin{pmatrix} \eta^U_\bullet \\ \eta'_\bullet U_\bullet \end{pmatrix}$.

(2): First note that by yanking the $\eta_!$ in $\beta_\bullet$ against the $\epsilon_!$ in $\mu'_\bullet$, we obtain

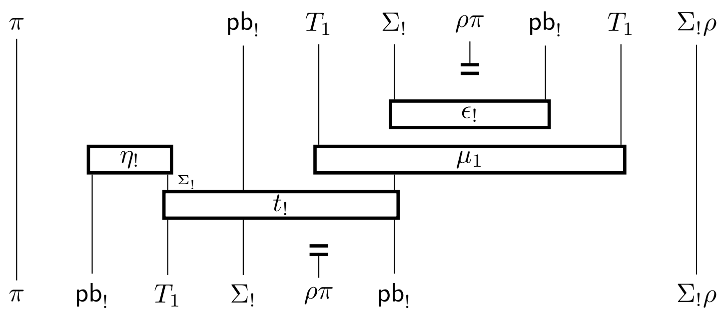
$\begin{pmatrix} T'_\bullet \beta_\bullet \\ \mu'_\bullet U_\bullet \end{pmatrix} =$  .

Next, note that $\beta_\bullet$ and $\gamma_\bullet$ can both be decomposed horizontally, as $\beta_\bullet = \pi\beta'\Sigma_!\rho$ and $\gamma_\bullet = \pi\,\mathsf{pb}_!\,\gamma' T_1 \Sigma_! \rho$. It follows that the β' can be slipped down past the γ', and its equality cancels the upper equality in the above diagram. We obtain

$$\begin{pmatrix}\beta_\bullet T'_\bullet\\ T'_\bullet\gamma\\ T'_\bullet\beta_\bullet\\ \mu'_\bullet U_\bullet\end{pmatrix} =$$



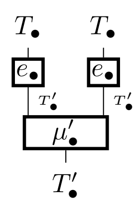
$$=$$



$$= \begin{pmatrix}\mu'_\bullet\\ \beta_\bullet\end{pmatrix}$$

Here the second equation uses one of the monad laws for the strength $t_!$. ■

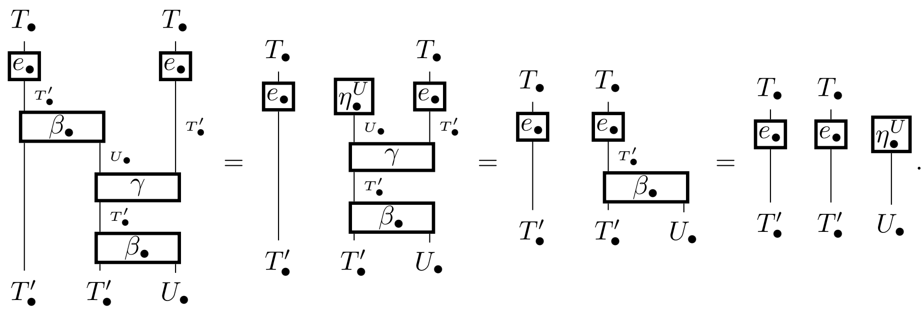
Definition 45 *We define natural transformations $\eta_\bullet : \mathcal{C}^\downarrow \to T_\bullet$ and $\mu_\bullet : T_\bullet T_\bullet \to T_\bullet$ over $\mathcal{C}$ by using the following equations.*

$$\begin{pmatrix}\eta_\bullet\\ e_\bullet\end{pmatrix} \overset{(1)}{=}$$  *and* $$\begin{pmatrix}\mu_\bullet\\ e_\bullet\end{pmatrix} \overset{(2)}{=}$$  .

Note that by definition this will make $e_\bullet$ a monad morphism.

Lemma 46 *$\eta_\bullet$ and $\mu_\bullet$ are well defined.*

Proof. Using Lemma 44, it suffices to show that the identity transformation (for $\eta_\bullet$) or $e_\bullet e_\bullet$ (for $\mu_\bullet$) composes equally with the transformations in the upper pair of the appropriate diagram. For $\eta_\bullet$ this is obvious. For $\mu_\bullet$ we have



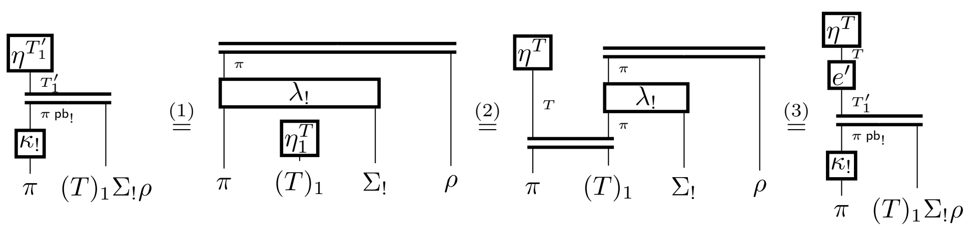
Here we have used the equalizer property twice, and Lemma 29 once. ∎

Theorem 47 *Suppose (T, η, μ) is a monad on $\mathcal{C}$, strengthened by t. Then $(T_\bullet, \eta_\bullet, \mu_\bullet)$ is a bundle monad. The isomorphism $T \cong T_1$ is an isomorphism of monads.*

Proof. The monad laws for $\eta_\bullet$ and $\mu_\bullet$ are easily proved: compose them at the bottom with $e_\bullet$, use the definitions to move $e_\bullet$ to the top, apply the laws for $\eta'_\bullet$ and $\mu'_\bullet$, and bring $e_\bullet$ back down to the bottom.

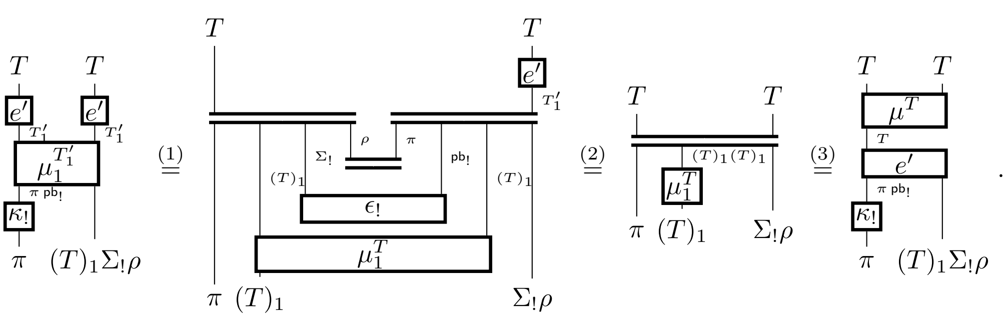
To show that the isomorphism is one of monads, it suffices to prove the conditions when composed below with $e_1 \colon T_1 \to T'_1$: in other words, we show that $e' \colon T \to T'_1$ is a monad morphism.

For the unit,



Equation (1) here expands $\eta^{T'_1}$ and uses Equation (2) from earlier, and equation (3) here uses the fact that $\lambda_! \rho$ restricts to equality on $\mathcal{C}/1$.

For the multiplication,



In equation (1), expanding $\mu_1^{T_1'}$ shows us we can move $\kappa_!$ past it to interact with the left-hand e'; in equation (2) we yank ϵ against the other e'. ∎

5 Bundle endofunctors preserving proneness

In this section we introduce our prime notion of geometric construction: a bundle endofunctor $\mathfrak{T}_\bullet$ that preserves proneness of morphisms (in other words, if it is applied to a morphism of $\mathcal{C}^\downarrow$ that is a pullback square, then the result is still a pullback square). A fundamental consequence (Theorem 50) is that $\mathfrak{T}_\bullet$ is slicewise strong.

Clearly, such a $\mathfrak{T}_\bullet$ restricts to an endofunctor $\mathfrak{T}_B$ on each slice $\mathcal{C}/B$ in a way that is pullback-stable. We can make precise the isomorphisms for pullback stability.

Proposition 48 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$. Then $\mathfrak{T}_\bullet$ preserves proneness iff ψ_{f_1} (Definition 14) is invertible.*

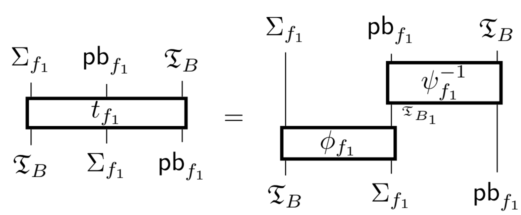
Proof. ($\Rightarrow$): First, note that, because $\mathfrak{T}_\bullet$ preserves proneness in $\mathcal{C}^\downarrow$, $\mathfrak{T}_\bullet\kappa_{f_1}\mathfrak{F}$ is prone for any $\mathfrak{F}\colon \mathcal{E} \to \mathcal{C}^\downarrow_B$. This follows by the same reasoning as in Proposition 5. Using this, we can define ψ^{-1} by a diagram analogous to that of (5), and also prove that it is a 2-sided inverse.

($\Leftarrow$): If square (1) is a pullback, then invertibility of ψ tells us that $\mathfrak{T}_B E \to B \times_{B'} \mathfrak{T}_{B'} E'$ is an isomorphism. ∎

5.1 Slicewise strength for $\mathfrak{T}_\bullet$

Throughout this subsection, $\mathcal{C}$ is a cartesian category and $\mathfrak{T}_\bullet$ is a bundle endofunctor that preserves proneness. We shall define a slicewise strength $t_\bullet$ for $\mathfrak{T}_\bullet$ that satisfies the ψ-ϕ-condition, and clearly that already requires the following definition.

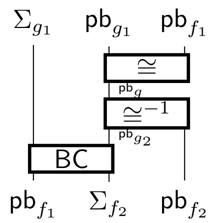
Definition 49 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$ that preserves proneness. The natural transformation $t_\bullet\colon \Sigma\,\mathsf{pb}\,\mathfrak{T}_\bullet \to \mathfrak{T}_\bullet\Sigma\,\mathsf{pb}$ is defined as the following composite.*



Theorem 50 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor on $\mathcal{C}$ that preserves proneness. Then it has a unique slicewise strength $t_\bullet$ that satisfies the ψ-ϕ-condition, and it is preserved by pullback between slices.*

Proof. For the ψ-ϕ-condition $t_\bullet$ has to be as in Definition 49, and then it is a strength from Lemma 25, parts (1) and (2), because ψ is invertible.

For preservation by pullback, we work over PB, with f_2 being the parameter of t and pullback being along f_1 – so the parameter f_2 pulls back to g_1. Thus we wish to show that, up to isomorphism, $\mathsf{pb}_{f_1}\, t_{f_2} = t_{g_1}\, \mathsf{pb}_{f_1}$. Allowing occurrences of ψ_{f_1} for the preservation of $\mathfrak{T}_\bullet$ itself, the remaining isomorphism is



.

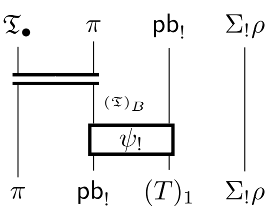
Once these are inserted, we find that the desired result follows by cancelling $\Sigma_{g_1}\psi_{g_1}\,\mathsf{pb}_{f_1}$ from the top of Lemma 25(3). ■

Theorem 51 *Let $\mathfrak{T}_\bullet$ be a bundle monad, with unit $\eta_\bullet$ and multiplication $\mu_\bullet$, such that $\mathfrak{T}_\bullet$ preserves proneness. Then the bundle monad has a unique slicewise strength $t_\bullet$ that satisfies the ψ-ϕ-condition. Pullback between slices preserves the strong monad structure.*

Proof. Theorem 50 already covers $t_\bullet$ as strength for the endofunctor, and the monad strength conditions follow from Lemma 26 by cancelling ψs. Preservation by pullback of the monad structure is the content of Lemma 19. ■

We now know that $T = \mathfrak{T}_1$ has a strength t_1, and can apply the constructions of Sections 4.3 and (for a bundle monad $\mathfrak{T}_\bullet$) 4.4. We show that there is a natural transformation $i_\bullet \colon \mathfrak{T}_\bullet \to T_\bullet$.

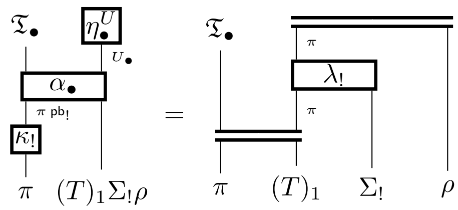
Lemma 52 *Let $\mathfrak{T}_\bullet$ be a bundle endofunctor that preserves proneness, and let $T = \mathfrak{T}_1$. Let $\alpha_\bullet \colon \mathfrak{T}_\bullet U_\bullet \to T'_\bullet$ be defined as*

$\alpha_\bullet =$ 

.

Here $(\mathfrak{T})_B$ is the endofunctor of Comp$_1$ *derived from $\mathfrak{T}_\bullet$, and $(T)_1$ that of* Opspan$_1$ *derived from $T = \mathfrak{T}_1$. $\psi_!$ is derived from $\mathfrak{T}_\bullet$, not $T_\bullet$.*

1.



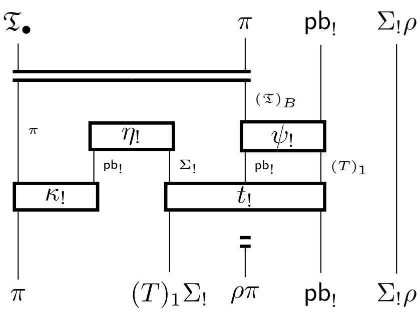
2. The two obvious squares in the following diagram both commute.

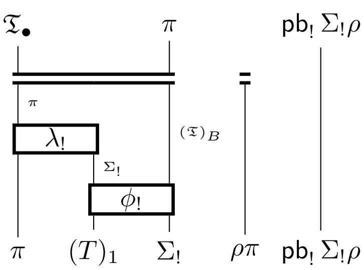
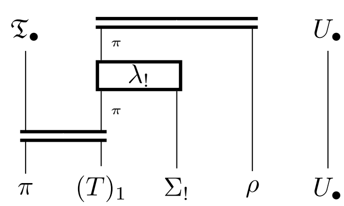
$$\begin{tikzcd}
\mathfrak{T}_\bullet U_\bullet \arrow[r, shift left, "\mathfrak{T}_\bullet U_\bullet \eta^U_\bullet"] \arrow[r, shift right, "\mathfrak{T}_\bullet \eta^U_\bullet U_\bullet"'] \arrow[d, "\alpha_\bullet"'] & \mathfrak{T}_\bullet U^2_\bullet \arrow[d, "\alpha_\bullet U_\bullet"] \\
T'_\bullet \arrow[r, shift left, "T'_\bullet \eta^U_\bullet"] \arrow[r, shift right, "\beta_\bullet"'] & T'_\bullet U_\bullet
\end{tikzcd} . \tag{15}$$

Proof. (1) is a straightforward calculation.

(2): For the square taking the upper in each pair of transformations, the result is immediate. For the lower we compose at the bottom with κ and calculate –

$$\begin{pmatrix} \alpha_\bullet \\ \beta_\bullet \\ \kappa_!(T)_1\Sigma_!\rho U_\bullet \end{pmatrix} =$$



$=$  $=$  ,

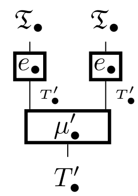
after which we can use part (1). ■

Definition 53 *Let $\mathfrak{T}_\bullet$ be a bundle endomorphism that preserves proneness. By Lemma 52(2) we can define $i_\bullet\colon \mathfrak{T}_\bullet \to T_\bullet$ such that* $\begin{pmatrix} i_\bullet \\ e_\bullet \end{pmatrix} = \begin{pmatrix} \mathfrak{T}_\bullet \eta^U_\bullet \\ \alpha_\bullet \end{pmatrix}$.

Proposition 54 *Let $\mathfrak{T}_\bullet$ be a bundle monad that preserves proneness, and suppose in addition it is a bundle monad with unit $\eta_\bullet$ and $\mu_\bullet$. Then $i_\bullet\colon \mathfrak{T}_\bullet \to T_\bullet$ (where $T = \mathfrak{T}_1$) is a morphism of monads.*

Proof. We have equations to check involving $\begin{pmatrix}\eta_\bullet\\ i_\bullet\end{pmatrix}$ and $\begin{pmatrix}\eta_\bullet\\ i_\bullet\end{pmatrix}$, but it suffices to prove them when composed below with the monad morphism $e_\bullet\colon T_\bullet \to T'_\bullet$. This comes down to

$$\begin{pmatrix}\eta_\bullet\\ i_\bullet\\ e_\bullet\end{pmatrix} = \begin{pmatrix}\eta_\bullet\eta^U_\bullet\\ \alpha_\bullet\end{pmatrix} \overset{(1)}{=} \boxed{\eta'_\bullet}\ T'_\bullet \quad\text{and}\quad \begin{pmatrix}\mu_\bullet\\ e_\bullet\end{pmatrix} = \begin{pmatrix}\mu_\bullet\eta^U_\bullet\\ \alpha_\bullet\end{pmatrix} \overset{(2)}{=}$$

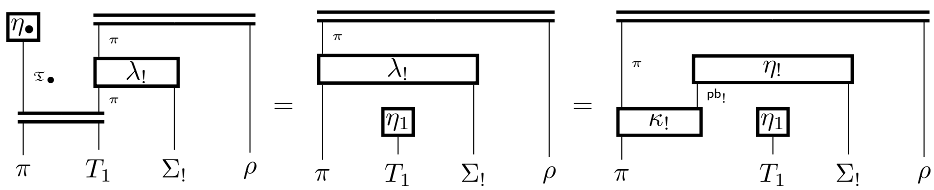


.

For both parts it suffices to check the equations when composed with κ at the bottom. We shall frequently use part (1) of Lemma 15.

For (1) we have

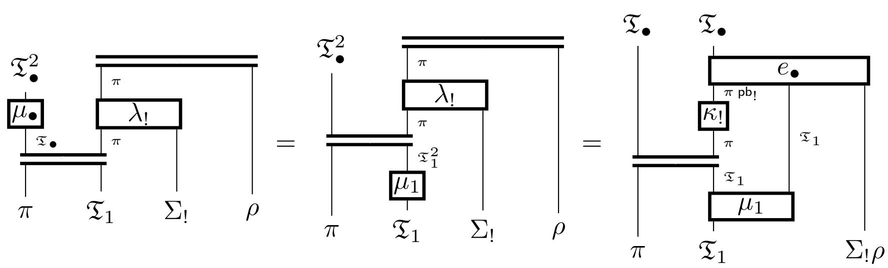
LHS =



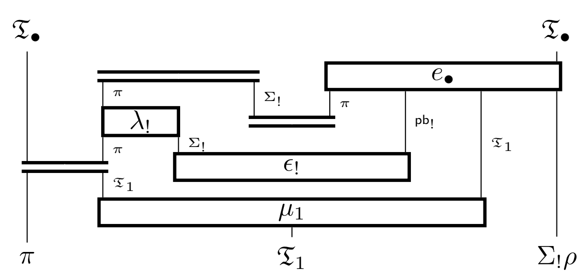
= RHS.

For (2) we have

LHS =



=



= RHS.

■

We now turn to the situation where $\mathfrak{T}_\bullet$ preserves coreflexive equalizers (in addition to proneness). Our main result is that $\mathfrak{T}_\bullet$ is then naturally isomorphic to the bundle endofunctor $T_\bullet$ got from the strong endofunctor $T = \mathfrak{T}_1$.

Proposition 55 *Let $\mathfrak{T}_\bullet$ be an arrow endofunctor on $\mathcal{C}^\downarrow$ that preserves coreflexive equalizers. Then so does each $\mathfrak{T}_B$ on $\mathcal{C}/B$.*

Proof. Equalizers in $\mathcal{C}/B$ can be calculated in $\mathcal{C}^\downarrow$. ■

Theorem 56 *Let $\mathcal{C}$ be a cartesian category and $\mathfrak{T}_\bullet$ a bundle endofunctor on it that preserves proneness and coreflexive equalizers. Then $i_\bullet$ is a natural isomorphism.*

Proof. Since the diagram of Proposition 27 gives a coreflexive equalizer, we can apply $\mathfrak{T}_\bullet$ and obtain that $\mathfrak{T}_\bullet \eta^U_\bullet$ gives equalizers of the top line in (15). Now since $\alpha_\bullet$ and $\alpha_\bullet U_\bullet$ are isomorphisms, we see that $\begin{pmatrix} i_\bullet \\ e_\bullet \end{pmatrix}$ gives equalizers for the bottom line. ■

Theorem 57 *Let $\mathcal{C}$ be a cartesian category and $\mathfrak{T}_\bullet$ a bundle monad on it that preserves proneness and coreflexive equalizers. Then $i_\bullet$ is a natural isomorphism of monads.*

Proof. Combine Theorem 56 with Proposition 54. ■

6 Sliceable endofunctors

We have already shown (Section 4.3) how from any strong endofunctor T on $\mathcal{C}$ we can construct a bundle endofunctor $T_\bullet$. We now show that $T_\bullet$ preserves proneness if T preserves coreflexive equalizers: we call such a T "sliceable". This holds for the powerlocales and valuation locales.

Definition 58 *Let $\mathcal{C}$ be a cartesian category.*

A sliceable endofunctor *on $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ that is strong and preserves coreflexive equalizers.*

A sliceable monad *on $\mathcal{C}$ is a monad on $\mathcal{C}$ that is strong and whose functor part preserves coreflexive equalizers.*

Throughout the rest of this section $\mathcal{C}$ is a cartesian category with sliceable endofunctor T and strength t. Our main result (Theorem 61) is that the derived $T_\bullet$ preserves proneness and coreflexive equalizers, and that T and t can be recovered from them. In Section 7 we shall show that the two structures are equivalent.

Proposition 59 *$T_\bullet$ preserves coreflexive equalizers.*

Proof. Between Opspan_1 and Comp_1 we have that $\mathsf{pb}_!$ preserves finite limits (because it is a right adjoint) and so does $\Sigma_!$ (by calculation). Hence so does $U_\bullet$. Since T preserves coreflexive equalizers, so do $T'_\bullet$ and $T'_\bullet U_\bullet$. Now consider a coreflexive equalizer in $\mathcal{C}^\downarrow$. Applying $T_\bullet$ gives us the left hand column in a 3×3 diagram in which the rows are the equalizer diagrams for 3 applications of $T_\bullet$, while the middle and right hand columns are equalizers got by applying $T'_\bullet$ and $T'_\bullet U_\bullet$. A straightforward diagram chase show that the left hand column too is an equalizer. ■

Proposition 60 *$T_\bullet$ preserves proneness.*

Proof. Consider the two squares (1) and (4). We must show that if (1) is a pullback, then so is (4).

First, note that square (1) being a pullback is equivalent to the following diagram (which is coreflexive, by morphism π_{13}) being an equalizer.

$$E \xrightarrow{\langle p, \overline{f} \rangle} B \times E' \underset{\langle B, \underline{f} \rangle \times E'}{\overset{B \times \langle p', E' \rangle}{\rightrightarrows}} B \times B' \times E'$$

This is because an element of the equalizer is a pair (b, e') such that $(b, p'e', e') = (b, \underline{f}b, e')$.

Given $(b, v) \in B \times TE'$ such that $(\underline{f}b, v) \in T_{B'}E'$, we want there to be a unique $u \in TE$ such that $(b, u) \in T_B E$ and $T\overline{f}(u) = v$.

Let $r = \langle p, \overline{f} \rangle \colon E \to B \times E'$. Then Tr is the coreflexive equalizer of

$$T(B \times \langle p', E' \rangle), T(\langle B, \underline{f} \rangle \times E') \colon T(B \times E') \to T(B \times B' \times E')$$

and so is monic.

Uniqueness: Given a u as required, we have

$$Tr(u) = T(B \times \overline{f})T\langle p, E \rangle(u) = T(B \times \overline{f})t(b, u) = t(b, T\overline{f}(u)) = t(b, v).$$

Now we show existence.

$$\begin{aligned} T(B \times \langle p', E' \rangle)t(b, v) &= t(b, T\langle p', E' \rangle(v)) = t(b, t(\underline{f}b, v)) = t((\underline{f}b, b), v) \\ &= T(\langle \underline{f}, B \rangle \times E')t(b, v). \end{aligned}$$

It follows that there is a unique $u \in TE$ such that $Tr(u) = t(b, v)$. Then

$$T\overline{f}(u) = T\pi_2 Tr(u) = T\pi_2(t(b, v)) = v.$$

To show that $(b, u) \in T_B E$ we use the fact that $T(r \times B)$ is monic:

$$\begin{aligned} T(B \times r)T\langle p, E \rangle(u) &= T(\Delta \times E')Tr(u) = T(\Delta \times E')t(b, v) \\ &= t((b, b), v) = t(b, t(b, v)) = t(b, Tr(u)) \\ &= T(B \times r)t(b, u). \end{aligned}$$

■

Theorem 61 *Let T be a sliceable endofunctor on a cartesian category $\mathcal{C}$. Then Definition 33 provides a bundle endofunctor $T_\bullet$ that preserves proneness and coreflexive equalizers. Moreover, T and its strength t are naturally isomorphic to the T_1 and t_1 got by restricting $\mathfrak{T}_\bullet$ and $t_\bullet$ to $\mathcal{C}/1 \cong \mathcal{C}$.*

Proof. Preservation of proneness and coreflexive equalizers is done in Propositions 60 and 59. The isomorphism $T \cong T_1$ is in Definition 38. Since $t_\bullet$ has the ψ-ϕ-condition, it follows from Proposition 43 that t_1 corresponds to t under the isomorphism. ■

Theorem 62 *Let T be a sliceable monad on a cartesian category $\mathcal{C}$. Then Definition 45 makes $T_\bullet$ a bundle monad, with $t_\bullet$ a slicewise strength. Moreover, the monad structure on T is isomorphic to that on T_1 got by restricting that on $\mathfrak{T}_\bullet$.*

Proof. After Theorem 61, the rest is covered by Theorems 47 and 51. ■

7 Summary of results

The results show an interplay between two different kinds of structures on a cartesian category $\mathcal{C}$: endofunctors T on $\mathcal{C}$, and bundle endofunctors $\mathfrak{T}_\bullet$ – which give endofunctors on each slice of $\mathcal{C}$, and also embody some information about change of base between slices.

If $\mathfrak{T}_\bullet$ preserves proneness, that is enough to imply (Theorem 50) that it has a unique slicewise strength with the ψ-ϕ-property. (In fact, with preservation of proneness, the ψ-ϕ-property for a t of the right type is already enough to show that t is a slicewise strength.)

That gives us a strong endofunctor $T = \mathfrak{T}_1$ on $\mathcal{C}/1$, and then Section 4.3 shows us how to construct a bundle endofunctor $T_\bullet$ from which T and its strength can be recovered. Moreover, if we started from $\mathfrak{T}_\bullet$, then Definition 53 gives us a natural transformation $i_\bullet \colon \mathfrak{T}_\bullet \to T_\bullet$.

Theorem 56 tells us that if $\mathfrak{T}_\bullet$ preserves coreflexive equalizers then $i_\bullet$ is an isomorphism. Theorem 61 shows that if T preserves coreflexive equalizers then $T_\bullet$ preserves proneness and coreflexive equalizers.

Putting these ingredients together, we obtain –

Theorem 63 *Let $\mathcal{C}$ be a cartesian category. Then the following structures are equivalent.*

1. *A strong endofunctor T on $\mathcal{C}$ that preserves coreflexive equalizers.*
2. *A bundle endofunctor $\mathfrak{T}_\bullet$ for $\mathcal{C}$ that preserves proneness and coreflexive equalizers.*

Combining this theorem with Theorem 50 yields an interesting fact: from a sliceable endofunctor (strong and preserving coreflexive equalizers) on a single slice $\mathcal{C}/1$, we obtain the same structure on every slice, and it is preserved up to isomorphism by the pullback functors between slices.

We also discuss situations where $\mathfrak{T}_\bullet$ and T are monads. If $\mathfrak{T}_\bullet$, preserving proneness, is a monad then Theorem 51 shows that the monad is slicewise strong, using the same strength as defined for the endofunctor $\mathfrak{T}_\bullet$. Theorem 47 shows that T, being a strong monad, induces monad structure on $T_\bullet$, with $T \cong T_1$ as monad. Proposition 54 shows that $i_\bullet : \mathfrak{T}_\bullet \to T_\bullet$ is a morphism of monads, hence (Theorem 57) an isomorphism if $\mathfrak{T}_\bullet$ preserves coreflexive equalizers. Thus, –

Theorem 64 *Let $\mathcal{C}$ be a cartesian category. Then the following structures are equivalent.*

1. *A strong monad T on $\mathcal{C}$ that preserves coreflexive equalizers.*
2. *A bundle monad for $\mathcal{C}$ that preserves proneness and coreflexive equalizers.*

8 Examples

Our first example is the covariant powerobject functor $\mathcal{P}$ on a topos $\mathcal{E}$. This is well-known to be a strong monad. In the internal language of $\mathcal{E}$ we have $\eta(x) = \{x\}$, $\mu(\mathcal{U}) = \bigcup \mathcal{U}$ and $t(X,Y)(x,U) = \{(x,u) \mid u \in U$.

Now suppose we have a coreflexive equalizer

$$X \xrightarrow{e} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \quad (h : Z \to Y) . \tag{16}$$

To show this is preserved by $\mathcal{P}$, suppose we have $V \subseteq Y$ such that $\mathcal{P}f(U) = \mathcal{P}g(U)$. if $y \in U$ then $fy = gy'$ for some $y' \in U$, so $y = hfy = hgy' = y'$, and it follows that $fy = gy$ and $y \in X$, so $U \in \mathcal{P}X$.

Thus $\mathcal{P}$ is a sliceable monad and gives rise to a bundle monad $\mathcal{P}_\bullet$ that preserves proneness (and coreflexive equalizers). Categorically this gives a geometric character to $\mathcal{P}_\bullet$, covering the coherence issues for the isomorphisms that arise when the slicewise constructions are pulled back. However, there is one remaining issue. The good behaviour, with coherent isomorphisms, is for the slicewise constructions $\mathcal{P}_B$ calculated as equalizers from the sliceable monad $\mathcal{P}$. But we thought we already knew how to calculate each $\mathcal{P}_B$, for the slice $\mathcal{E}/B$ is also a topos and so has its own powerobject functor. Does this agree with the $\mathcal{P}_B$ derived from $\mathcal{P}$? The answer is Yes, and this follows from the fact that the pullback functors between the slice toposes are logical, and hence preserve (up to isomorphism) the powerobject construction.

This final issue becomes important in our main motivating examples, the powerlocales. We shall show for each one – lower, upper, Vietoris and double – that it is a sliceable monad, and so gives rise to a bundle monad on **Loc** preserving proneness. This structure on its own is good, and has been used (following the ideas of this paper) as the categorical definition of geometricity in [FV14]. However, the powerlocale constructions are topos-valid and hence give rise (by working in $\mathcal{S}B$) to constructions on each $\mathbf{Loc}/B$. Again we must ask, Do these agree with the slicewise constructions derived from the sliceable monads? If not, then the sliceable monads are giving us the wrong slicewise constructions: for we rely on properties of the topos-valid constructions in $\mathcal{S}B$.

Fortunately, the constructions do agree, but this does not come from the results in this paper. Instead it relies on applying the construction to presentations of locales, the technique prominent in [Vic04]. As long the construction on presentations is preserved by the inverse image parts of geometric morphisms, this is enough to show that the constructions derived from the sliceable monads are, up to isomorphism, as expected.

Thus the question of geometricity for localic constructions is answered in two parts. First, the results of this paper show that it is sufficient to have a sliceable monad (or endofunctor). This provides a bundle monad or endofunctor with good properties, categorically abstract. Second, the established presentation techniques are used to show that the topos-valid slicewise constructions are preserved by pullback, up to incoherent isomorphism. In the following subsections we deal with the various powerlocales. The paper [Vic11] deals with the valuation locale.

8.1 The lower powerlocale P_L

Definition 65 *If X is a locale then its* lower powerlocale *$P_L X$ is defined by*

$$\Omega P_L X = \mathbf{Fr}\langle \Omega X \text{ (qua SupLat) } \rangle.$$

If U is an element of ΩX then we shall write $\lozenge U$ for the corresponding generator in $\Omega P_L X$.

If $f\colon X \to Y$ is a map, then $P_L f\colon P_L X \to P_L Y$ is defined by $(P_L f)^*(\lozenge U) = \lozenge(f^* U)$.

The maps $\eta\colon X \to P_L X$ and $\mu\colon P_L^2 X \to P_L X$ are defined by $\eta^*(\lozenge U) = U$ and $\mu^*(\lozenge U) = \lozenge\lozenge U$.

As is well known, and easy enough to prove, these make P_L an endofunctor of $\mathbf{Loc}$, η and μ natural transformations, and (P_L, η, μ) a monad. For its strength, recall that the frame for $X \times Y$ is the suplattice tensor of the frames for X and Y: that is to say, it is generated as a suplattice by symbols $U \otimes V$ subject to "suplattice bilinearity" relations that say $\otimes$ distributes on both sides over all joins. This suplattice presentation allows us to deduce that $\Omega P_L(X \times Y)$ is presented *as a frame* by symbols $\lozenge(U \otimes V)$ subject to suplattice bilinearity relations such as $\lozenge(U \otimes \bigvee_i V_i) = \bigvee_i \lozenge(U \otimes V_i)$. We can now define $t(X,Y)\colon X \times P_L Y \to P_L(X \times Y)$ by $t(X,Y)^* \lozenge(U \otimes V) = U \otimes \lozenge V$. The proofs that this is a strength are straightforward.

Proposition 66 *P_L preserves coreflexive equalizers.*

Proof. Consider a coreflexive equalizer as in (16). Then

$$\Omega X \cong \mathbf{Fr}\langle \Omega Y \text{ (qua Fr) } \mid f^*W = g^*W \quad (W \in \Omega Z)\rangle. \tag{17}$$

In order to calculate $\Omega P_L X$, we must convert this frame presentation into a suplattice presentation, which we do using the coverage theorem – see [Vic04] for more details. Now

$$\begin{aligned}\Omega X &\cong \mathbf{Fr}\langle \Omega Y \text{ (qua } \wedge\text{-semilattice)} \mid \text{ joins preserved,} \\ &\qquad\qquad f^*W = g^*W \quad (W \in \Omega Z)\rangle \\ &\cong \mathbf{SupLat}\langle \Omega Y \text{ (qua poset)} \mid \text{ joins preserved,} \\ &\qquad\qquad f^*W = g^*W \quad (W \in \Omega Z)\rangle,\end{aligned}$$

where the final isomorphism follows from the coverage theorem *provided* that the relations are meet stable. For the preservations of joins this follows from frame distributivity in ΩY, while for the other relations it follows from coreflexivity using $f^*W \wedge V = f^*(W \wedge h^*V)$ and so on. Hence,

$$\begin{aligned}\Omega P_L X &\cong \mathbf{Fr}\langle \Omega Y \text{ (qua suplattice)} \mid f^*W = g^*W \quad (W \in \Omega Z)\rangle \\ &\cong \mathbf{Fr}\langle \Omega P_L Y \text{ (qua frame)} \mid (P_L f)^*W = (P_L g)^*W \quad (W \in \Omega P_L Z)\rangle.\end{aligned}$$

Hence the equalizer is preserved by P_L. ■

8.2 The upper powerlocale P_U

The upper powerlocale is broadly similar to the lower, but with suplattice structure replaced by preframe structure.

Definition 67 *If X is a locale then its* upper powerlocale *$P_U X$ is defined by*

$$\Omega P_U X = \mathbf{Fr}\langle \Omega X \text{ (qua preframe) } \rangle.$$

If U is an element of ΩX then we shall write $\Box U$ for the corresponding generator in $\Omega P_U X$.

If $f\colon X \to Y$ is a map, then $P_U f\colon P_U X \to P_U Y$ is defined by $(P_U f)^*(\Box U) = \Box(f^* U)$.

The maps $\eta\colon X \to P_U X$ and $\mu\colon P_U^2 X \to P_U X$ are defined by $\eta^*(\Box U) = U$ and $\mu^*(\Box U) = \Box\Box U$.

As is well known, and easy enough to prove, these make P_U an endofunctor of **Loc**, η and μ natural transformations, and (P_U, η, μ) a monad. For its strength, recall from [JV91] that the frame for $X \times Y$ is also the preframe tensor of the frames for X and Y: that is to say, it is generated as a preframe by symbols $U \odot V$ subject to "preframe bilinearity" relations that say $\odot$ distributes on both sides over finite meets and directed joins. We have $U \odot V = U \otimes Y \vee X \otimes V$ and $U \otimes V = U \odot \wedge \odot V$. This preframe presentation allows us to deduce that $\Omega P_U(X \times Y)$ is presented *as a frame* by symbols $\Box(U \odot V)$ subject to preframe bilinearity relations such as $\Box(U \odot \bigwedge_{i=1}^n V_i) = \bigwedge_{i=1}^n \Box(U \odot V_i)$. We can now define $t(X,Y)\colon X \times P_U Y \to P_U(X \times Y)$ by $t(X,Y)^*\Box(U \odot V) = U \odot \Box V$. The proofs that this is a strength are, again, straightforward.

Proposition 68 *P_U preserves coreflexive equalizers.*

Proof. Consider a coreflexive equalizer as in (16), giving a frame presentation (17). In order to calculate $\Omega P_U X$, we must this time convert the frame presentation into a preframe presentation, which we do using the preframe coverage theorem of [JV91] – see [Vic04] for more details. Now

$$\begin{aligned} \Omega X \cong \mathbf{Fr}\langle \Omega Y \text{ (qua } \vee\text{-semilattice)} \mid &\text{ finite meets and directed joins preserved,} \\ & f^*W = g^*W \quad (W \in \Omega Z)\rangle \\ \cong \mathbf{PreFr}\langle \Omega Y \text{ (qua poset)} \mid &\text{ finite meets and directed joins preserved,} \\ & f^*W = g^*W \quad (W \in \Omega Z)\rangle, \end{aligned}$$

where the final isomorphism follows from the preframe coverage theorem *provided* that the relations are join stable. For the preservations of finite meets and directed joins this follows from frame distributivity in ΩY, while for the other relations it follows from coreflexivity using $f^*W \vee V = f^*(W \vee h^*V)$ and so on. Hence,

$$\begin{aligned} \Omega P_U X &\cong \mathbf{Fr}\langle \Omega Y \text{ (qua preframe)} \mid f^*W = g^*W \quad (W \in \Omega Z)\rangle \\ &\cong \mathbf{Fr}\langle \Omega P_U Y \text{ (qua frame)} \mid (P_U f)^*W = (P_U g)^*W \quad (W \in \Omega P_U Z)\rangle. \end{aligned}$$

Hence the equalizer is preserved by P_U. ∎

8.3 The Vietoris powerlocale V

The Vietoris powerlocale V originated in [Joh85]. It combines features of the lower and upper powerlocales (although, historically, the Vietoris powerlocale came first).

Definition 69 *If X is a locale then its* Vietoris powerlocale *VX is defined by*

$$\begin{aligned} \Omega VX = \mathbf{Fr}\langle \Diamond U, \Box U \quad (U \in \Omega X) \mid &\ \Diamond \textit{ preserves all joins} \\ &\ \Box \textit{ preserves finite meets and directed joins} \\ &\ \Diamond U \wedge \Box V \leq \Diamond (U \wedge V) \\ &\ \Box (U \vee V) \leq \Diamond U \vee \Box V\rangle. \end{aligned}$$

If $f\colon X \to Y$ is a map, then Vf is defined by combining the clauses for the lower and upper powerlocales, and likewise for the monad unit and multiplication η, μ.

[Joh85] proved that these make V an endofunctor of **Loc**, η and μ natural transformations, and (V, η, μ) a monad. For its strength, we again combine the definitions for the lower and upper powerlocales. We must check that the mixed relations are respected. For example,

$$
\begin{aligned}
&t(X,Y)^*(\Diamond(U_1 \otimes V_1) \wedge \Box(U_2 \odot V_2)) \\
&\qquad = (U_1 \otimes \Diamond V_1) \wedge (U_2 \odot \Box V_2) \\
&\qquad = (U_1 \otimes \Diamond V_1) \wedge (U_2 \otimes VY \vee X \otimes \Box V_2) \\
&\qquad = (U_1 \wedge U_2) \otimes \Diamond V_1 \vee U_1 \otimes (\Diamond V_1 \wedge \Box V_2) \\
&\qquad \leq (U_1 \wedge U_2) \otimes \Diamond V_1 \vee U_1 \otimes \Diamond(V_1 \wedge V_2) \\
&\qquad = t(X,Y)^* \Diamond((U_1 \wedge U_2) \otimes V_1 \vee U_1 \otimes (V_1 \wedge V_2)) \\
&\qquad = t(X,Y)^* \Diamond(U_1 \otimes V_1 \wedge U_2 \odot V_2).
\end{aligned}
$$

It is then straightforward to check that it is a strength.

Proposition 70 *V preserves coreflexive equalizers.*

Proof. Consider a coreflexive equalizer as in (16), giving a frame presentation (17). The frame ΩVX is presented by two sets of generators $\Diamond U$ and $\Box U$ (for $U \in \Omega X$) subject to the Vietoris relations; and then the calculations in Propositions 66 and 68 show that these can be replaced by generators $\Diamond V$ and $\Box V$ for $V \in \Omega Y$, subject to the Vietoris relations together with $\Diamond f^* W = \Diamond g^* W$ and $\Box f^* W = \Box g^* W$ for $W \in \Omega Z$. These last ones can be replaced by $(Vf)^* W = (Vg)^* W$ for $W \in \Omega VZ$, which now tells us that VX is the desired equalizer. ■

The fact that V is preserved by pullback is remarked on in [Vic09].

8.4 The double powerlocale

The double powerlocale monad $\mathbb{P}X$ is defined in [Vic04] as the composite $P_L P_U X \cong P_U P_L X$; this relies on the fact that the lower and upper powerlocale monads distribute over each other. That paper also proves that it is preserved by pullback. Its strength is defined in [VT04]. The fact that it preserves coreflexive equalizers follows immediately from Propositions 66 and 68.

9 Conclusions

The work leading to this paper started with a less ambitious aim: to show how to use the strength of an endofunctor T on **Loc**, together with preservation of coreflexive equalizers, to construct a strong endofunctor T_B on each slice **Loc**$/B$ using an equalizer as in Proposition 34, with the structure preserved by pullback. This formed a natural part of Townsend's programme of finding an axiomatic description of structure for *Loc*. Already the slice endofunctors raised the question of coherence, and exploring this gradually led to a feeling that the slice endofunctors T_B needed somehow to act between slices as well as within them, and hence to the idea of bundle endofunctor, preserving proneness.

Later, the work leading to [FV14] investigated preservation of fibrations and opfibrations in **Loc** by geometric constructions. Again this started as a purely localic investigation, but it quickly led to techniques of abstract category theory and so raised the question of how to formulate geometricity in an abstract way. The notion of bundle endofunctor preserving proneness then turned out to give our required results in a satisfactorily abstract way.

The experience there motivated the work of the present paper to explore the bundle endofunctors more carefully under various conditions. It seems, particularly in the light of [FV14], that preservation of proneness is the decisive property. Its good behaviour is illustrated by the fact that it is sufficient to imply slicewise strength. However, the total structure is complex, and the role of preservation of coreflexive equalizers seems to be to ensure that the bundle endofunctor can be reconstructed from its action on $\mathcal{C}/1$ (and the strength there). We have shown that the original motivating examples of the powerlocales can be handled this way, and [Vic11] does the same for the valuation locale.

We also introduced other properties of bundle endofunctors, namely slicewise strength and the ψ-ϕ-condition. These are certainly useful for proving what we need on the slices, though it is less clear how independently useful they are in their own right without preservation of proneness.

As mentioned before, the treatment of pullback stability of the powerlocales in [Vic04] relied on assuming that each locale came along with a presentation, and it was in terms of the presentations that a uniform homeomorphism $f^*(P_Y(Z_g)) \cong P_X(f^*Z_g)$ was given. It would be interesting to show this more explicitly in the example of the category of formal topologies. The reduction to presentations is a valuable technical tool. However, it would be difficult to make it a part of an abstract categorical account of **Loc**. This is the programme of [3], with the aim of elucidating categorically the reasoning principles underlying constructive locale theory and – one may hope also – constructive analysis. It is also analogous to the ASD programme of Taylor[4]. In the axiomatization presented here one is able to express the pullback stability (or geometricity) of powerlocale constructions without relying on the extra-categorical structure of presentations.

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