Cosheaves and connectedness in formal topology

Steven Vickers
School of Computer Science, University of Birmingham,
Birmingham, B15 2TT, UK
s.j.vickers@cs.bham.ac.uk

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Abstract

The localic definitions of cosheaves, connectedness and local connectedness are transferred from impredicative topos theory to predicative formal topology. A formal topology is locally connected (has base of connected opens) iff it has a cosheaf π_0 together with certain additional structure and properties that constrain π_0 to be the connected components cosheaf. In the inductively generated case, complete spreads (in the sense of Bunge and Funk) corresponding to cosheaves are defined as formal topologies. Maps between the complete spreads are equivalent to homomorphisms between the cosheaves. A cosheaf is the connected components cosheaf for a locally connected formal topology iff its complete spread is a homeomorphism, and in this case it is a terminal cosheaf.

A new, geometric proof is given of the topos-theoretic result that a cosheaf is a connected components cosheaf iff it is a "strongly terminal" point of the symmetric topos, in the sense that it is terminal amongst all the *generalized* points of the symmetric topos. It is conjectured that a study of sites as "formal toposes" would allow such geometric proofs to be incorporated into predicative mathematics.

Key words: Formal topology, predicative, locally connected, cosheaf, symmetric topos, complete spread

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1 Introduction

Topos theory has its own abstruse definition of the topological notion of local connectedness: a (Grothendieck) topos $\mathcal{E}$ is locally connected if a particular functor $!^* : \mathbf{Set} \to \mathcal{E}$ has a left adjoint π_0. To make sense of this, consider the case where $\mathcal{E}$ is the category $\mathcal{S}X$ of sheaves over a space (or locale) X, with also

Set $= \mathcal{S}1$. The functor $!^*$ is the inverse image part of the map $! : X \to 1$.[1] For each set S, $!^*S$ is the sheaf with constant stalk S, corresponding to the local homeomorphism given by the projection map $X \times S \to X$. Then X is locally connected iff $\mathcal{S}X$ satisfies the topos theoretic definition. The set $\pi_0(a)$ should be thought of as the set of connected components of a, and we can see why in the special case where the sheaf a is an open U (equivalent to a subsheaf of $!^*1$). The open U is connected iff any map to a discrete space S is constant (factors through a singleton), and thus taking $\pi_0(U) = 1$ gives a bijection between maps $U \to S$ and functions $\pi_0(U) \to S$. The maps $U \to S$ can also be seen as sheaf morphisms $U \to !^*S$; this is most clearly seen using the local homeomorphisms $U \hookrightarrow X$ and $X \times S \to X$. More generally, for disconnected U we can use local connectedness of X to get U as a disjoint union of connected open components, and then we still see a bijection between maps $U \to S$ and functions $\pi_0(U) \to S$. The notion of "set of connected components" $\pi_0(U)$ for opens U extends to arbitrary sheaves in a way that preserves colimits, using the fact that any sheaf is a colimit of opens.

Somewhat separate from this approach to local connectedness, there is a study of functors $\mathcal{S}X \to$ **Set** that – like π_0 – preserve colimits. These are *cosheaves* on the topos $\mathcal{S}X$. The reason for this name is seen better with regard to sites presenting the toposes. If $\mathcal{C}$ is the category for the site, then a sheaf is a contravariant functor $\mathcal{C} \to$ **Set** that satisfies certain "sheaf pasting" conditions with respect to the coverage. By contrast, a cosheaf is a covariant functor $\mathcal{C} \to$ **Set** that satisfies certain "colimit preservation" conditions with respect to the coverage. We shall see these more explicitly in the localic case, where X is a locale. Bunge and Funk [BF06] have explained the topological role of cosheaves by showing the equivalence between cosheaves over X and "complete spreads" over X – certain locale maps with codomain X. They have also [BF96] related this to a similar, but simpler, bijection in the theory of the lower powerlocale. This is between join-preserving functions $\Omega X \to \Omega$ (ΩX is the frame of opens for X), and overt, weakly closed sublocales of X – see also [Vic07] for an account of this in formal topology.

A key part of the topos-theoretic approach to cosheaves is that, for each topos X, there is a topos MX whose points are the cosheaves over X. This is an interesting example of a "topos as generalized space". We know the points of MX – they are the cosheaves. It has "topology", but of a kind that cannot be fully described by the opens. Instead, sheaves must be used, and in topos theory a space is described through its category of sheaves just as in locale theory a space is described through its lattice of opens. Even when X is localic, MX might not be. (Compare this with the related lower powerlocale $P_L X$, which *is* localic if X is. In fact [BF96] show that $P_L X$ is the localic reflection of MX, in other words the locale defined by the opens of MX.) It turns out that X is locally connected iff MX has a terminal point in a certain strong sense (the

[1] We shall use similar notation for arbitrary toposes, distinguishing between the "topos as generalized space" X and the "Giraud frame" (or "topos frame") $\mathcal{S}X$. See [Vic99]. By *map* between toposes we shall mean a geometric morphism, which corresponds to an ordinary continuous map in the case of sheaves over locales.

map – i.e. geometric morphism – $MX \to 1$ has a right adjoint, namely the point $1 \to MX$ that is the connected components cosheaf). We shall compare this with the result [Vic95] that X is overt iff $P_L X$ has a greatest point in a similar strong sense.

An important summary of these topos-theoretic approaches to topological ideas is [BF06]. It is highly impredicative in its topos-theoretic foundations, but the aim of these notes is to show how some of the ideas can be imported into predicative formal topology. At the same time, we hope also to present an easier introduction to [BF06] by focusing on the localic case. Our techniques also have benefit even within topos-valid mathematics. A formal topology (more particularly in the inductively generated case) is a localic site (or, a frame presentation by generators and relations), and the predicative manipulation of these gives access to the "geometric" techniques described in [Vic04].

First, we shall define the notion of cosheaf over formal topology. We shall also describe in particular the inductively generated case, since the predicative parts of topos-valid locale theory generally translate first into inductively generated formal topology. They come out of the technology of sites, and presentations by generators and relations, and the full cover is not normally explicitly present in those workings. The theory of cosheaves over a site is geometric. On the one hand, this is why there is a topos MX whose points are the cosheaves – the topos classifies the geometric theory of cosheaves. On the other hand, it is the geometricity that guarantees predicativity. "Geometric" essentially means "preserved by inverse image functors of geometric morphisms", and those functors do not preserve the impredicative topos construction of powersets.

Next, we define connectedness for an arbitrary formal topology and show its equivalence, in the overt case, to an earlier definition. Our definition uses an impredicative quantification over covers, and we do not know if this can be reduced to predicative structure, even in the inductively generated case.

Next, we look at local connectedness. The simplest understanding is that there is a base of connected opens, and we characterize those formal topologies for which every basic open is connected as "formally locally connected". However, this depends on having a very special base and we next move to a general characterization of when a cosheaf is the connected components cosheaf making a formal topology locally connected. We describe structure and properties for local connectedness, and show that a formal topology has these iff it is homeomorphic to a formally locally connected formal topology. As far as we know, this characterization is new both in formal topology and in topology. It is worth noting that one role of the cosheaf is to provide witnesses for a positivity predicate: a is positive iff $\pi_0(a)$ is inhabited. This also shows that a locally connected formal topology is overt. (Of course, the converse is not true. Classically, all locales are overt, but not all are locally connected.)

After that, in the inductively generated case, we look at the notion of *complete spread*. This is a central topic in [BF06], and we show how some of the ideas there appear in inductively generated formal topology. Each cosheaf F over X gives rise to a complete spread, a map $CS(X; F) \to X$, and X is locally connected, with F the connected components cosheaf, iff this map is a homeomorphism.

We conclude with some remarks on the *symmetric* topos MX, and how one might hope for an analogue in generalized formal topology. We illustrate the discussion with a new proof of some results in [BF06], showing that, if X is a space equipped with a cosheaf π_0, then X is locally connected with π_0 the connected components cosheaf iff π_0 is a "strongly terminal" cosheaf.

1.1 Geometricity

Some remarks are in order regarding the *geometricity* arguments that can make locale theory much more painless.

The starting point is that topos-valid constructions and arguments about locales can also be carried through in toposes $\mathcal{S}W$ of sheaves. A fundamental result [JT84] is that locales internal in $\mathcal{S}W$ are equivalent to locale maps with codomain W (i.e. objects of the slice category $\mathbf{Loc}/W$, or *locales over* W), and so topos-valid results about locales also give us information about maps. In fact this works in two ways. A map $X \to W$ is equivalent to an internal locale in $\mathcal{S}W$ and is then thought of as a locale over W. A good example [Ver86] is the notion of *proper* map, corresponding to compactness of locales.

However, we can also regard a map $f : W \to X$ as a *generalized point* of X *at stage* W. (An ordinary point $1 \to X$ is then a *global* point.) It is equivalent to a commutative triangle

$$\begin{array}{ccc} W & \xrightarrow{\langle W,f\rangle} & W \times X \\ & \mathrm{Id} \searrow & \downarrow p \\ & & W \end{array}$$

where $\mathrm{Id} : W \to W$ is the terminal locale over W, and the projection p is X converted to a locale over W. (This conversion can be done by taking a formal topology describing X and converting the sets involved into constant sheaves over W to give an internal formal topology in $\mathcal{S}W$.) Hence, over W, the generalized point f becomes a global point $\langle W, f\rangle$. Thus topos-valid reasoning about global points also tells us about generalized points.

To be well-behaved under change of base W, the topos-valid constructions must in fact be geometric – preserved under inverse image functors for geometric morphisms. Hence they will be predicative, since the powerset construction is not geometric. This shows up clearly when defining locale maps $f : X \to Y$ by their action on points – a procedure which at first sight appear illegitimate, since there might not be enough global points. Suppose a construction Φ takes points of X and returns points of Y. If Φ is topos-valid then it can be applied to the generic point x_g of X in $\mathcal{S}X$ (given by the diagonal $\Delta : X \to X \times X$) to give $\Phi(x_g) : X \to Y$. We now have two ways to convert generalized points of X, $x : W \to X$, to points of Y: we can apply Φ, or we can compose with $\Phi(x_g)$. These will coincide provided Φ is geometric, so geometricity amounts to uniformity of the definition.

Since the argument relies on internal frames in a topos $\mathcal{S}W$, it has no immediate counterpart in predicative mathematics. On the other hand, since predicativity lies at the heart of the technique, we hope that it can still be justified. We shall continue to use pointwise definition of maps here, since in practice it is not hard to convert the definitions into concrete definitions of maps between formal topologies. This will be discussed further following Definition 3.

A deeper issue arises with constructions such as powerlocales, where the same construction must be compared over different bases W. This is discussed in some detail in [Vic04], but we have no ready substitute in formal topology for the geometricity argument. This will impinge on Section 7.

2 Formal topologies

We summarize the background of formal topology and fix notation. The most successful transfer from topos theory to predicative formal topology is in the inductively generated case, where the axiom sets correspond to sites. However, where possible we shall also address the more general situation.

Definition 1 *A formal topology $X = (P, \leq, \lhd)$ is a preordered base $(P, \leq)$ paired with a cover relation $\lhd$ satisfying the usual properties:*

$$\begin{aligned} a \in U &\Longrightarrow a \lhd U \\ a \leq b &\Longrightarrow a \lhd b \\ a \lhd U_1, a \lhd U_2 &\Longrightarrow a \lhd U_1 \downarrow U_2 \\ a \lhd U \lhd V &\Longrightarrow a \lhd V. \end{aligned}$$

For subsets A and B of P, we write

$$\begin{aligned} \downarrow A &= \{c \in P \mid (\exists a \in A)\ c \leq a\}, \\ A \downarrow B &= (\downarrow A) \cap (\downarrow B). \end{aligned}$$

Informally, we shall often refer to a formal topology as a *space*.

Our use (following [CSSV03]) of an explicit order $\leq$ is not strictly necessary. The default order is $\lhd$. However, in certain situations (e.g. [Vic07]) it is useful to have a fixed order.

We shall often refer to an arbitrary subset $A \subseteq P$ as an *open* of X, but this is modulo equivalence. The opens are preordered by $\lhd$. On occasion we shall want to complete A to give the *formal open* $\lhd A$, the set of all basics covered by A, and this is the greatest open equivalent to A.

In [Vic06] a *flat site* $(P, \leq, \lhd_0)$ is defined as alternative notation for the data required for an inductively generated formal topology with the localization condition [CSSV03].[2] The base $(P, \leq)$ is a preorder. $\lhd_0$ is the data for an

[2] In predicative mathematics, not every formal topology can be inductively generated. In topos-valid mathematics the distinction is less essential because every frame can be set-presented (with sets of generators and relations). The flat site described here is a mild generalization of the site as described in [Joh82] (where P is required to be a meet-semilattice).

axiom set generating the full cover relation $\lhd$. An instance $a \lhd_0 U$ is a *basic cover*. The localization condition says that if $b \leq a \lhd_0 U$ then $b \lhd_0 V$ for some $V \subseteq b \downarrow U$.

Definition 2 *A formal topology $X = (P, \leq, \lhd)$ is* overt *(also known as* open*) if it is equipped with a positivity predicate* Pos *on P, satisfying*

$$\mathrm{Pos}(a), a \lhd U \Longrightarrow (\exists u \in U)\,\mathrm{Pos}(u)$$
$$a \lhd \{a\}^+ \textit{ for all } a \in P.$$

(For any $A \subseteq P$ we write $A^+ = A \cap \mathrm{Pos}$.)

As an immediate consequence of these conditions, if $a \lhd b$ (so, in particular, if $a \leq b$) and $\mathrm{Pos}(a)$, then $\mathrm{Pos}(b)$.

See [Neg02] for more discussion on the relationship with overt (open) locales (as described in [JT84], [Joh84]); also [Vic07] for the relationship with the lower powerlocale. A predicate satisfying the first two conditions given for Pos is called a *lower powerpoint* of X. In the inductively generated case, the second condition can be restricted to basic covers; the condition for general covers can then be deduced by induction on the proof of $a \lhd U$.

Definition 3 *A* continuous map *(or just* map*) from $(P, \leq, \lhd)$ to $(Q, \leq, \lhd)$ is a relation $f \subseteq P \times Q$ satisfying the following conditions.*

1. *If $a \lhd U f b$ ($U f b$ means $u f b$ for every $u \in U$) then $a f b$.*
2. *If $a f b \lhd V$ then $a \lhd f^- V$.*
3. *$P \lhd f^- Q$.*
4. *If $a f b_i$ $(i = 1, 2)$ then $a \lhd f^-(b_1 \downarrow b_2)$.*

(We write $f^- V$ for the inverse image of V under f.)

The first of these says that $f^- b$ is a formal open for each b. The intention is that it should be the inverse image under the map of the basic open b. Since inverse image preserves joins, the inverse image of an open $B \subseteq Q$ must be the formal open $\lhd f^- B$. Condition (2) says that this assignment respects the covers, and conditions (3) and (4) say that it preserves finite meets.

In the case where the formal topologies are inductively generated by flat sites $(P, \leq, \lhd_0)$ and $(Q, \leq, \lhd_0)$, we can replace conditions (1) and (2) above by the following.

1a If $a \leq a' f b$ then $a f b$.

1b If $a \lhd_0 U f b$ then $a f b$. ($U f b$ means that $u f b$ for every $u \in U$.)

2a If $a f b \lhd_0 V$ then $a \lhd f^- V$.

It is usually convenient to define a map by afb if $a \lhd \Phi(b)$ where $\Phi(b)$ is some subset of P. This will automatically give condition (1). One then needs to check the following conditions:

2b If $b \lhd_0 V$ then $\Phi(b) \lhd \bigcup_{v \in V} \Phi(v)$.

3a $P \lhd \bigcup_{b \in Q} \Phi(b)$.

4a $\Phi(b_1) \downarrow \Phi(b_2) \lhd \bigcup_{b \in b_1 \downarrow b_2} \Phi(b)$.

As mentioned in Section 1.1, in practice we shall define maps by their action on points using a geometricity argument. Although we do not at present have a general metatheorem to justify this in predicative mathematics, nonetheless, a geometric pointwise construction will embody the reasoning needed to construct a map. The construction will first of all describe $f(x)$ as a formal point, given x. In other words it must show how $b \in f(x)$ depends on the basics in x. This amounts to describing the inverse image f^*b as a union of a set $\Phi(b)$ of basics. The pointwise argument must also explain how pointhood of $f(x)$ follows from that of x being a point, and this can be transformed into a proof of conditions (2b), (3a) and (4a) above.

Composition $f; g$ or $g \circ f$ of maps $f : X \to Y$ and $g : Y \to Z$ is defined by $a(f; g)c$ if $a \lhd f^- g^- c$. The identity map on X is $a \,\mathrm{Id}\, a'$ if $a \lhd a'$.

The *specialization order* on maps $X \to Y$ is defined by $f_1 \sqsubseteq f_2$ if for all b, basic open for Y, we have $f_1^- b \lhd f_2^- b$.

If $f : X \to Y$ and $g : Y \to X$ are maps, then we say (f, g) is an *adjoint pair*, with f and g the *left* and *right* adjoints respectively, if $\mathrm{Id}_X \sqsubseteq f; g$ and $g; f \sqsubseteq \mathrm{Id}_Y$ – in other words, $a \lhd f^-(g^- a)$ for all basic opens a of X, and $g^-(f^- b) \lhd b$ for all basic opens b of Y.

3 Cosheaves

Central to our discussion of connectedness and (more particularly) local connectedness is the notion of *cosheaf* over a space, which Bunge and Funk [BF06] trace back to the Borel-Moore homology [BM60]. It is known in topos-valid mathematics that for a locally connected space X, the assignment to each open U of its set π_0 of connected components is a cosheaf. This is a most important example, but we shall also find it useful to consider more general cosheaves.

Technically, a cosheaf can be seen in a variety of ways. A fundamental one is as a particular kind of covariant assignment of sets to opens, but this can be reduced to data on a site for the space, and this will be our initial definition (Definition 4) when we transfer the notion to formal topology. A cosheaf over X can also be extended to a covariant assignment of sets to *sheaves*, specifically a colimit-preserving functor from $\mathcal{S}X$ to **Set**, and this enables one to extend the notion to general toposes (i.e. toposes that do not necessarily arise as categories of sheaves over spaces). This connection was made in [Pit85] and discussed extensively in [BF06] where such a functor is usually called a *distribution*. Finally

[BF06], a cosheaf over X can also be understood as a particular kind of map into X, a "complete spread with locally connected domain", and we shall describe this in the context of inductively generated formal topology.

Obviously the name *cosheaf* comes from analogy with sheaves, and the most obvious comparison is that, respectively, sheaves and cosheaves give *contra*variant and *co*variant assignments of sets to opens. However, this apparently simple categorical duality hides a deep difference between their natures.

Bunge and Funk [BF06, Section 1.3] describe this distinction using Lawvere's terminology (adopted from physics) of *intensive* and *extensive*. In physics an intensive quantity is one such as density that varies from point to point, while an extensive quantity is one such as volume or mass that depends on the extent of what is measured. Integration pairs the two, since the integrand is intensive while the measure is extensive. Although measures are defined as "measuring" the measurable sets, integration enables us to use measures to "measure measurable functions". Riesz's Representation Theorem says that integration allows us to identify the extensive quantities with linear functionals on the intensives.

Lawvere's insight was that a similar theory is obtained if the real line is replaced by the category of sets, and linear maps are replaced by cocontinuous (colimit preserving) functors. Sheaves are then the intensive quantities, since a sheaf on X is a "continuous set-valued map" on X taking each point of X to its stalk. (X here might be a generalized space in Grothendieck's sense, i.e. a topos.) A technical manifestation of this is that a sheaf is a geometric morphism from X to the object classifier $[set]$, the topos whose points are sets. (See [Vic99] for our notation. It is often written $\mathcal{S}[U]$, and Bunge and Funk [BF06] call it $\mathcal{R}$, by deliberate analogy with the reals. Note that its objects as a topos are more complicated. In fact they are functors from $\mathbf{Set}_{\mathrm{fin}}$ to $\mathbf{Set}$, where $\mathbf{Set}_{\mathrm{fin}}$ is the category of sets that are finite in the strong sense of being isomorphic to some finite cardinal.) Cosheaves are the extensive quantities. Although they may be defined at first – at least for localic X – as set-valued functors on the opens, this can be extended to a cocontinuous functor from sheaves to sets. This pairing of sheaf with cosheaf is the analogue of integration, and the fact that cosheaves can be identified with those cocontinuous functors is the analogue of Riesz's Theorem.

One sharp contrast between sheaves and cosheaves is that the cosheaves over X are the points of a (generalized) space MX, the *symmetric topos*. This is because the cosheaves over X are the models of a geometric theory. We shall discuss in Section 7 how MX might be viewed as an example of a generalized formal topology. By contrast, the sheaves, obviously the objects of a topos, are not in general the *points* of a topos, except in those cases where an exponential topos $[set]^X$ can be found (see [JJ82], or [Joh02a, B4.3]).

Our first definition of cosheaf is the adaptation to formal topologies of [BF06, Definition 1.4.1] for sites.

Definition 4 *Let $X = (P, \leq, \lhd)$ be a formal topology. A* cosheaf *over X is an indexed set $F(a)$ ($a \in P$), together with* corestriction *functions $F_{a,b} : F(a) \to F(b)$ ($a \leq b$ in P) such that the following properties hold.*

1. $F_{a,a}(\gamma) = \gamma$.
2. If $c \leq b \leq a$ and $\varepsilon \in F(c)$ then $F_{b,a}(F_{c,b}(\varepsilon)) = F_{c,a}(\varepsilon)$.
3. If $a \lhd U$ then the functions $F_{b,a} : F(b) \to F(a)$ form a colimit cocone over the diagram with nodes $F(b)$ for $b \in a \downarrow U$ and edges $F_{b,b'}$ for $b \leq b'$ in $a \downarrow U$.

Example 5 *Let x be a point of $X = (P, \leq, \lhd)$. The* point cosheaf $\delta(x)$ *is defined by*

$$\delta(x)(a) = \{* \in 1 \mid x \models a\}.$$

(Here $*$ denotes the unique element of a singleton 1. *We write $x \models a$ to mean that a is a neighbourhood of x. Conceptually this means that x is in the open a, but formally, since x is defined as a subset of P, it means $a \in x$.) The corestrictions are obvious.*

Fixing a cosheaf F over $X = (P, \leq, \lhd)$, if $\gamma \in F(a)$ and $\delta \in F(b)$, let us write $(b, \delta) \leq (a, \gamma)$ to mean $b \leq a$ and $F_{b,a}(\delta) = \gamma$. This is a preorder on $\sum_{a \in P} F(a)$.

The requirement in condition (3) that those functions $F_{b,a}$ form a colimit cocone can be analysed in two parts. First, for every $\gamma \in F(a)$ there is some (not necessarily unique) $(b, \delta) \leq (a, \gamma)$ with $b \in a \downarrow U$. Second, if $b_i \in a \downarrow U$ and $\delta_i \in F(b_i)$ $(i = 1, 2)$, then $F_{b_1,a}(\delta_1) = F_{b_2,a}(\delta_2)$ if and only if there is a sequence $(c_0, \varepsilon_0), \ldots, (c_n, \varepsilon_n)$ with each c_j in $a \downarrow U$, such that $(c_0, \varepsilon_0) = (b_1, \delta_1)$, $(c_n, \varepsilon_n) = (b_2, \delta_2)$ and for each j with $0 \leq j < n$ we have either $(c_j, \varepsilon_j) \leq (c_{j+1}, \varepsilon_{j+1})$ or $(c_{j+1}, \varepsilon_{j+1}) \leq (c_j, \varepsilon_j)$. Let us call this a "connection from (b_1, δ_1) to (b_2, δ_2) in $a \downarrow U$".

Proposition 6 *Let $X = (P, \leq, \lhd_0)$ be a flat site and let F be an indexed set with corestrictions satisfying conditions (1) and (2) in Definition 4. Then for F to be a cosheaf, it suffices to have condition (3) in the restricted case of basic covers $a \lhd_0 U$.*

Proof. Fixing U, let us define the property $\Phi(a)$ as for all $a' \leq a$, $F(a') \cong \operatorname{colim}_{b \in a' \downarrow U} F(b)$ in the way described. If $a \in U$ then $\Phi(a)$, for if $a' \leq a$ then $a' \in a' \downarrow U$ and so $F(a')$ is a terminal node in the diagram. Also, if $\Phi(a)$ and $b \leq a$ then obviously $\Phi(b)$. Now suppose $a \lhd_0 V$, and for every $v \in V$ we have $\Phi(v)$. We wish to show $\Phi(a)$. If $a' \leq a$ then $a' \lhd_0 V'$ for some $V' \subseteq a' \downarrow V$, and $\Phi(v')$ for every $v' \in a' \downarrow V$, so we might as well take a' to be a. But note also that $F(a') \cong \operatorname{colim}_{b \in a' \downarrow V} F(b)$. Now

$$\begin{aligned}
F(a) &\cong \operatorname*{colim}_{b \in a \downarrow V} F(b) \cong \operatorname*{colim}_{b \in a \downarrow V} \operatorname*{colim}_{c \in b \downarrow U} F(c) \\
&\cong \operatorname*{colim}_{c \in a \downarrow U \downarrow V} F(c) \\
&\cong \operatorname*{colim}_{d \in a \downarrow U} \operatorname*{colim}_{c \in d \downarrow V} F(c) \cong \operatorname*{colim}_{d \in a \downarrow U} F(d).
\end{aligned}$$

We deduce that $\Phi(a)$. It follows that from any proof of $a \lhd U$ we can derive a proof of $\Phi(a)$. ∎

It follows that in the inductively generated case, the data required to describe a cosheaf is predicatively small.

Definition 7 *Let F be a cosheaf on $X = (P, \leq, \lhd)$. If $A \subseteq P$ then we define*

$$F(A) = \operatorname*{colim}_{a' \in \downarrow A} F(a').$$

If $A \lhd B$ in X then we define $F_{A,B} : F(A) \to F(B)$ as follows. For each $a' \in \downarrow A$ we have $a' \lhd B$, and so we have a function

$$F(a') \cong \operatorname*{colim}_{b \in a' \downarrow B} F(b) \to F(B).$$

Clearly these respect the morphisms $F_{a'',a'}$ for $a'' \leq a'$ in $\downarrow A$, so we obtain a cocone from the diagram over $\downarrow A$ to $F(B)$. This then defines $F_{A,B}$.

$F_{A,B}$ can be calculated as follows. Suppose $a' \in \downarrow A$ and $\gamma \in F(a')$. Then there is some $(b, \delta) \leq (a', \gamma)$ with $b \in a' \downarrow B$. Let $\mathrm{in}_{a'} : F(a') \to F(A)$ and $\mathrm{in}_b : F(b) \to F(B)$ be the colimit injections. Then $F_{A,B}(\mathrm{in}_{a'}(\gamma)) = \mathrm{in}_b(\delta)$. The previous results show that this is independent of the choices of a' and b. From this it is easy to show the following.

Lemma 8

1. *$F(\{a\}) \cong F(a)$, and modulo these isomorphisms $F_{\{a\},\{b\}} = F_{a,b}$.*
2. *$F_{A,A}$ is the identity on $F(A)$.*
3. *If $A \lhd B \lhd C$ then $F_{A,C} = F_{B,C} \circ F_{A,B}$.*

In particular, if $A \lhd B \lhd A$ (so A and B generate the same formal open), then $F(A) \cong F(B)$.

Definition 9 *Let F, G be two cosheaves on $X = (P, \leq, \lhd)$. A* homomorphism *from F to G is a natural transformation from F to G considered as functors from $(P, \leq)$ to* **Set**.

Suppose $\alpha : F \to G$ is a homomorphism of cosheaves. If $A \subseteq P$ then clearly we get a function $\alpha_A : F(A) \to G(A)$. Now suppose $A \lhd B$. It is easily checked that $G(A, B) \circ \alpha_A = \alpha_B \circ F(A, B)$, so α extends to a natural transformation between F and G considered as functors from formal opens to **Set**.

It is clear that we get a category $\mathrm{Cosh}\, X$ of cosheaves over X.

Example 10 *If x and y are points of X then there is at most one homomorphism from $\delta(x)$ to $\delta(y)$. It exists iff $x \sqsubseteq y$.*

Definition 11 *Let $f : X \to Y$ be a map, where $X = (P, \leq, \lhd)$ and $Y = (Q, \leq, \lhd)$. Let F be a cosheaf over X. Then we define a cosheaf* $\mathrm{Cosh}\, f(F)$ *over Y by*

$$\mathrm{Cosh}\, f(F)(b) = F(f^- b),$$
$$\mathrm{Cosh}\, f(F)_{b,b'} = F_{f^- b, f^- b'}.$$

Condition (3) in Definition 4 follows from the fact that if $b \lhd V$ in Y then $f^{-}b \lhd f^{-}V$.

If $\alpha : F \to G$ is a cosheaf homomorphism over X, then we also get a homomorphism $\mathrm{Cosh}\, f(\alpha) : \mathrm{Cosh}\, f(F) \to \mathrm{Cosh}\, f(G)$, defined by $\mathrm{Cosh}\, f(\alpha)_b = \alpha_{f^{-}b}$. It is then readily checked that we have a functor $\mathrm{Cosh}\, f : \mathrm{Cosh}\, X \to \mathrm{Cosh}\, Y$. Moreover, $\mathrm{Cosh}\,\mathrm{Id}_X \cong \mathrm{Id}_{\mathrm{Cosh}\, X}$, and if $g : Y \to Z$ then $\mathrm{Cosh}(f; g) \cong \mathrm{Cosh}\, f; \mathrm{Cosh}\, g$.

4 Connectedness

The classical definition says that a space X is connected if, whenever $X = U \cup V$ with U and V disjoint opens, then either $X = U$ or $X = V$. An alternative way to express this is that any map $X \to 2$, where 2 is the discrete space on two elements, is constant (i.e. factors through an element of 2).

Following the standard topos-theoretic treatment (e.g. [Joh84]), we shall modify this is two ways.

First, we shall require X to be non-empty. *We do not admit $\emptyset$ as a connected space.* A constructive way to express this is that, if X is connected, then any map $X \to \emptyset$ is constant. This rules out $\emptyset$ as a connected space, since the identity map on $\emptyset$ does not factor through any element of $\emptyset$.

If all maps from X to $\emptyset$ or 2 are constant, then by induction the same follows for maps to any finite cardinal $\{1, \ldots, n\}$ $(n \in \mathbb{N})$. *Classically*, it follows that maps to any discrete space I are constant. The second modification now is to require this stronger condition explicitly. We say that X is connected if any map $X \to I$, with I discrete, factors through an element of I.

Example 12 *We give an example of a space for which every map to* 2 *or to* 0 *is constant, but not every map to a discrete space is. We first describe this in the topos-valid context of sheaves over Sierpiński space* $\mathbb{S}$*. For this particular case, a sheaf, most conveniently represented as a local homeomorphism over* $\mathbb{S}$*, is given by a function* $X_\perp \to X_\top$ *where the two sets are the stalks for the bottom and top points* $\perp$ *and* $\top$ *of* $\mathbb{S}$*. Sheaves are, of course, exactly the notion of discrete space in this context. The discrete 2-element space is given by the projection* $\mathbb{S} \times 2 \to \mathbb{S}$*. Let* X *be the sheaf over* $\mathbb{S}$ *whose stalks are* $X_\perp = \{0, 1\}$*,* $X_\top = \{*\}$*. A map (sheaf morphism) from* X *to* 2 *must choose one or other element of* 2 *for the image of* $*$*, and this then forces the same choice of element of* 2 *for the images of* 0 *and* 1*. Hence any map* $X \to 2$ *is constant. However, the identity map on* X *(which is a map from* X *to a discrete space) is not constant.*

We can view this alternatively as a Brouwerian counterexample. Let ϕ *be some proposition for which* $\neg\neg\phi$ *is known, but* ϕ *is not. (In* $\mathcal{S}\mathbb{S}$*,* ϕ *could be the subsheaf of* 1 *with stalks* $\phi_\perp = \emptyset$*,* $\phi_\top = 1$*.) Let* X *be the set* $\{0, 1\}$ *equipped with an imposed equality such that* $0 = 1$ *if* ϕ*. Suppose we have a map* $f : X \to 2$*. If* ϕ *then* $0 = 1$ *and so we must have* $f(0) = f(1)$*. Hence* $f(0) \neq f(1)$ *implies* $\neg\phi$*, which is impossible, and so, since equality on* 2 *is decidable, we must have* $f(0) = f(1)$*. It follows that* f *is constant. However, the identity map on* X *is not.*

We can express this notion of connectedness for formal topologies as follows. Note that, in type theoretic terms, by *set* we mean what is often called a *setoid*, i.e. a type equipped with an equivalence relation that serves as a defined equality. (This already appeared in the example.) The discrete space on a set I is presented as a formal topology with base $(I, =)$ and cover $i \lhd U$ if $i \in U$. The cover relation is inductively generated from the empty set of basic covers. It follows that a map f from $X = (P, \leq, \lhd)$ to I is given by a family of subsets $A_i \subseteq P$ $(i \in I)$ with afi iff $a \lhd A_i$. We need at least $A_i \lhd A_j$ if $i = j$ (so by symmetry also $A_j \lhd A_i$), though in practice we commonly have $A_i = A_j$. We also need that the A_is cover X (i.e. $P \lhd \bigcup_i A_i$) and are *pairwise disjoint* in the sense that $A_i \downarrow A_j \lhd \{b \in P \mid i = j\}$ (this is a constructive way of saying $A_i \downarrow A_j \lhd \emptyset$ if $i \neq j$).

Definition 13 *Let $X = (P, \leq, \lhd)$ be a formal topology. Then X is* connected *if it has the following property. Suppose $A_i \subseteq P$ $(i \in I)$ are pairwise disjoint and cover X. Then $P \lhd A_i$ for some i.*

From the form of definition here, which relies on opens (subsets of P modulo $\lhd$), it is clear that this is presentation independent – if X is connected then so is any formal topology homeomorphic to X.

The following Proposition for the overt case shows the equivalence of our definition with another, essentially that given in [NV97]. The "connecting sequences" are also used (as "chains") in [Cur03] to describe, in a metric context, the distance between basic opens.

Proposition 14 *Let $X = (P, \leq, \lhd)$ be an overt formal topology, with positivity predicate* Pos*. Then X is connected iff it is positive and has the property that whenever $P \lhd U$ and $a, b \in U$ are both positive, then there is a "connecting sequence" $c_k \in U$ $(0 \leq k \leq n)$ such that $c_0 = a$, $c_n = b$, and if $0 \leq k < n$ then $c_k \downarrow c_{k+1}$ is positive.*

Proof. $\Longrightarrow$: Suppose $P \lhd U$. By overtness we have $P \lhd U^+ = U \cap \mathrm{Pos}$. Define a relation $\sim$ on U^+ by $x \sim y$ if $x \downarrow y$ is positive, and let $\sim^*$ be its transitive, reflexive closure: in other words, $x \sim^* y$ if there is a connecting sequence from x to y. Write U_x for the equivalence class of x in U^+. Then these give a pairwise disjoint open cover of X indexed by $U^+/\sim^*$. For suppose $z \in U_x \downarrow U_y$, with $z \leq x' \sim^* x$ and $z \leq y' \sim^* y$. If z is positive then the connecting sequences from x to x' and from y' to y concatenate to give a connecting sequence from x to y, and it follows that

$$U_x \downarrow U_y \lhd (U_x \downarrow U_y)^+ \lhd \{b \in P \mid x \sim^* y\}.$$

Note from this that if $x, y \in U^+$ and $y \lhd U_x$, then $y \lhd U_x \downarrow U_y$ and by positivity of y we deduce that $x \sim^* y$.

By connectedness we deduce that $P \lhd U_x$ for some $x \in U^+$, so from the particular case $U = P$ we see that X is positive. More generally for $P \lhd U$, if $a, b \in U^+$ then $\{a, b\} \lhd U_x$ and so $a \sim^* x \sim^* b$.

$\Longleftarrow$: Suppose $(A_i)_{i\in I}$ is a pairwise disjoint open cover of X. Let $a \in P$ be positive. Since $a \lhd \bigcup_i A_i$, we can find $j \in I$ with A_j positive. By choosing a positive element of A_j, we can assume $a \in A_j$. By the property, if $b \in \bigcup_i A_i$ is positive, then there is a connecting sequence $(c_k)_{0\leq k\leq n}$ from a to b in $\bigcup_i A_i$. We show by induction on n that $b \lhd A_j$. The case $n = 0$ is trivial, since $b = a \in A_j$. For $n \geq 1$, we have $c_{n-1} \lhd A_j$ by induction and $c_{n-1} \downarrow b$ is positive. If $b \in A_i$ then $c_{n-1} \downarrow b \lhd A_j \downarrow A_i$, and it follows that $i = j$ and $b \lhd A_j$. Hence $X \lhd \bigcup_i A_i \lhd A_j$. ■

5 Local connectedness

We now characterize local connectedness in formal topology. It is known in topos theory that a topos is locally connected iff it has a site in which every cover is inhabited and connected (see, e.g., [Moe86]). This translates into formal topology and hence gives a characterization that relies on finding a special base – it is our condition (3) in Proposition 15. We shall call such a formal topology *formally locally connected*. Next, we shall use cosheaf technology to describe local connectedness in terms of an arbitrary base – we believe this approach is new. Our Theorem 23 validates our new definition by showing it is equivalent to there being a formally locally connected presentation, and hence to the topos definition.

Our development is predicative in the inductively generated case. More generally it is not, because the definition of cosheaf involves quantification over subsets.

From the introductory remarks, we expect a locally connected space $X = (P \leq, \lhd)$ to come equipped with a cosheaf π_0 such that $\pi_0(U)$ is a set indexing the connected components of U. Our central question is how, given a space X and cosheaf π_0, to characterize the situation in which X is locally connected and π_0 is its connected components cosheaf.

The standard definition of local connectedness is that there is a base of connected opens, so we first look at the special case in which every basic open is connected.

Proposition 15 *Let $X = (P, \leq, \lhd)$ be a formal topology. Then the following are equivalent.*

1. *Every basic open is connected.*
2. *The assignment $\pi_0(a) = 1$ for every $a \in P$ is a cosheaf. (1 denotes a singleton set $\{*\}$, and the corestrictions are obvious.)*
3. *If $a \lhd U$, then $a \downarrow U$ has exactly one equivalence class for the equivalence relation $\sim$ generated by $\leq$. (In the inductively generated case it suffices to have this for basic covers $a \lhd_0 U$.)*

Proof. Condition (3) is just a rephrasing of the colimit property for the functor π_0 that makes it a cosheaf. Hence (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3): Suppose $a \lhd U$. Let I be an indexing set for the equivalence classes A_i ($i \in I$). (More precisely, I is $a \downarrow U$ equipped with $\sim$ as defined equality.) Then $A_i \downarrow A_j \lhd \{b \in P \mid i = j\}$ and it follows that the A_is are a pairwise disjoint cover of a. By connectedness, $a \lhd A_i$ for some i. Now suppose $a' \in a \downarrow U$, with $a' \in A_j$. Then

$$a' \lhd A_i \downarrow A_j \lhd \{b \in P \mid i = j\}.$$

Applying the above argument to this cover, we deduce that $a' \downarrow \{b \in P \mid i = j\}$ is inhabited, so $i = j$. Hence A_i is the only equivalence class in $a \downarrow U$.

(3) $\Rightarrow$ (1): Let $A_i \subseteq (\downarrow a)$ ($i \in I$) be pairwise disjoint as opens, with $a \lhd \bigcup_i A_i$, and let the single equivalence class of $\downarrow (\bigcup_i A_i)$ be that of $a' \in \downarrow A_j$. It suffices to show that $A_k \lhd A_j$ for every k. If $b \in A_k$ then $b \sim a'$, i.e. there is a sequence $c_0, \ldots, c_n$ in $\downarrow (\bigcup_i A_i)$ such that $c_0 = a'$, $c_n = b$ and for each r either $c_r \leq c_{r+1}$ or $c_r \geq c_{r+1}$. If $n \geq 1$ then by induction $c_{n-1} \in \downarrow A_j$. Let c be the smaller of c_{n-1} and b. Then $c \in A_j \downarrow A_k \lhd \{d \in P \mid j = k\}$, so by considering the single equivalence class of $c \downarrow \{d \in P \mid i = j\}$ it follows that $k = j$. ■

Let us call a formal topology *formally locally connected* if it satisfies the conditions of the Proposition – "formally" because it is a property of the formal presentation.

We now describe extra structure and properties that, for an arbitrary cosheaf π_0, constrain it to be the connected components cosheaf making X locally connected. Suppose a is a basic open and $\gamma \in \pi_0(a)$. Let $A_\gamma \subseteq P$ be the (open) connected component of a corresponding to γ. Then $A_\gamma \lhd a$ and so $A_\gamma \lhd a \downarrow A_\gamma$, and it follows that A_γ is covered by a set $\leq^\gamma a$ of basics in $\downarrow a$. We write $b \leq^\gamma a$ if $b \in (\leq^\gamma a)$. Canonically, one can define $b \leq^\gamma a$ if $b \leq a$ and b included in the γ-component of a, though our axiomatization will not force this (Example 18). In any event, the γ-component of a will be the open $(\leq^\gamma a)$. At least for the canonical $\leq^\gamma$, we see that we should expect the four properties 2(a)–(d) listed below in Definition 16. All are fairly clear, bearing in mind that $(\pi_0)_{b,a}(\delta)$ is expected to be the component of a that includes the δ-component of b, and (for (d)) that by local connectedness every open in covered by its connected components, also open.

We take this as motivation for Definition 16: the structure and properties should be necessary for the usual notion of local connectedness. It is perhaps not obvious that the collection of connected components of each open should be a *set*, and thus give a cosheaf, but we shall accept this following the example of the topos-theoretic account. The Definition will eventually be validated in Theorem 24: a space has the structure and properties iff it is homeomorphic to a formally locally connected space. We shall also describe the connected components of an arbitrary open.

Definition 16 *A formal topology* $X = (P, \leq, \lhd)$ *is* locally connected *if it is equipped with the following structure.*

1. *A cosheaf* π_0.

2. *For each $(a,\gamma) \in \sum_{a\in P} \pi_0(a)$, a subset $\leq^\gamma a \subseteq {\downarrow} a$ (we write $b \leq^\gamma a$ if b is in this subset) satisfying the following conditions.*

   (a) *If $b' \leq b \leq^\gamma a$ then $b' \leq^\gamma a$.*

   (b) *If $b \leq^\gamma a$ and $\delta \in \pi_0(b)$ then $(b,\delta) \leq (a,\gamma)$.*

   (c) *If $c \leq^\delta b$ and $(b,\delta) \leq (a,\gamma)$ then $c \leq^\gamma a$.*

   (d) *If $a \in P$ then $a \lhd \bigcup_{\gamma\in\pi_0(a)}(\leq^\gamma a)$.*

Note that in the inductively generated case the structure and properties for local connectedness are expressed without quantification over subsets. (For general formal topologies, such a quantification is needed in saying that π_0 is a cosheaf.) In this respect it seems better behaved than connectedness.

It is obvious that a formally locally connected formal topology is locally connected, with the relation $\leq^*$ ($* \in 1$) being just $\leq$.

Condition 2 (d) says that to prove $a \lhd U$ it suffices to show $(\leq^\gamma a) \lhd U$ for every $\gamma \in \pi_0(a)$ – or even that $a \lhd U$ for every $\gamma \in \pi_0(a)$ (in other words, in proving $a \lhd U$ we may assume that $\pi_0(a)$ is inhabited). We prove some further simple properties.

Lemma 17 *Let $X = (P, \leq, \lhd)$, equipped with π_0 and $\leq^\gamma$, be locally connected.*

1. *If $\gamma \in \pi_0(a)$ then there is some $a' \leq^\gamma a$ and $\gamma' \in \pi_0(a')$ with $(a',\gamma') \leq (a,\gamma)$.*

2. *$(\leq^\gamma a) \downarrow (\leq^\delta b) \lhd \bigcup\{\leq^\varepsilon c \mid (c,\varepsilon) \in (a,\gamma) \downarrow (b,\delta)\}$.*

Proof. (1) We have $a \lhd \bigcup_{\alpha\in\pi_0(a)}(\leq^\alpha a)$ and so there is some $a' \leq^\alpha a$ and $\gamma' \in \pi_0(a')$ with $(a',\gamma') \leq (a,\gamma)$. But then $\alpha = (\pi_0)_{a',a}(\gamma') = \gamma$ so $a' \leq^\gamma a$.

(2) Suppose $c \leq^\gamma a$ and $c \leq^\delta b$. We have $c \lhd \bigcup\{\leq^\varepsilon c \mid \varepsilon \in \pi_0(c)\}$, and given such an ε we know $(c,\varepsilon) \in (a,\gamma) \downarrow (b,\delta)$. ■

Example 18 *The relations $\leq^\gamma$ are not uniquely determined by X and π_0. For a counterexample, consider X the partial order $\{1, 2, \top\}$, with $\top$ a top point and 1 and 2 mutually incomparable. Let P have the four elements $\{\top, 1\top, 2\top, 12\top\}$ with the obvious (subset inclusion) order and basic cover $12\top \lhd \{1\top, 2\top\}$. All the basic opens are connected, so π_0 is constant singleton $\{*\}$. The order $\leq^*$ can be equal to $\leq$, but it is also possible to omit $12\top \leq 12\top$ from it. As long as $1\top \leq^* 12\top$ and $2\top \leq^* 12\top$ we have $12\top \lhd \{b \mid b \leq^* 12\top\}$.*

Proposition 19 *If $X = (P, \leq, \lhd)$ is locally connected then it is overt. Its positivity predicate is defined as $\mathrm{Pos}(a)$ iff $\exists\gamma \in \pi_0(a)$.*

Proof. Suppose $\gamma \in \pi_0(a)$. If $a \lhd U$ then we have $(b,\delta) \leq (a,\gamma)$ for some $b \in a \downarrow U$, and so U is inhabited. Hence a is positive. Since for every γ we have $a \lhd \{a\}^+$, it follows that $a \lhd \{a\}^+$. ■

The following theorem (Corollary 1.4.11 in [BF06]) will be examined more closely in Section 7, where its converse will be discussed.

Theorem 20 *Let $X = (P, \leq, \lhd)$ be locally connected, equipped with π_0 and $\leq^\gamma$. Then π_0 is a terminal cosheaf – in other words, for every cosheaf F over X there is a unique homomorphism $\alpha : F \to \pi_0$.*

Proof. We first prove uniqueness of the homomorphism α. If $a \in P$ then $a \lhd \bigcup_{\gamma \in \pi_0(a)} (\leq^\gamma a)$, and it follows that any $x \in F(a)$ is $F_{b,a}(y)$ for some $y \in F(b)$, $b \leq^\gamma a$, $\gamma \in \pi_0(a)$. Then

$$\alpha_a(x) = \alpha_a(F_{b,a}(y)) = (\pi_0)_{b,a}(\alpha_b(y)) = \gamma.$$

This last equation follows from condition 2(b) in Definition 16.

It remains to prove existence. We have

$$F(a) \cong \operatorname*{colim}_{\substack{\gamma \in \pi_0(a) \\ b \leq^\gamma a}} F(b)$$

and by the above argument α_a has to map everything in $F(b)$ to γ. Condition 2(a) in Definition 16 tells us that this respects all the edges in the diagram, and so defines a function $\alpha_a : F(a) \to \pi_0(a)$. In other words, $\alpha_a(x) = \gamma$ where $x = F_{b,a}(y)$ for some $b \leq^\gamma a$. Condition 2(c) now tells us that these functions are natural in a, and so give a homomorphism α. ■

Our next aim is to show that existence of the structure for local connectedness is homeomorphism invariant. In fact, we can be slightly more general. In a topos-theoretic setting, local connectedness of X is equivalent to the existence of π_0 left adjoint to $!^* : \mathbf{Set} \to \mathcal{S}X$. If we have $f : X \to Y$ and $g : Y \to X$ with (f, g) an adjoint pair, then $f^* : \mathcal{S}Y \to \mathcal{S}X$ is left adjoint to g^*, and so $\pi_0 \circ f^* = \operatorname{Cosh} f(\pi_0)$ is left adjoint to $g^* \circ !^*$. It follows that Y is also locally connected. We now show this in formal topology.

Lemma 21 *Let $X = (P, \leq, \lhd)$ be locally connected, equipped with π_0 and $\leq^\gamma$. Let $Y = (Q, \leq, \lhd)$, and let $f : X \to Y$ and $g : Y \to X$ be maps such that (f, g) is an adjoint pair. Then Y is locally connected, using cosheaf* $\operatorname{Cosh} f(\pi_0)$*.*

Proof. Suppose $b \in Q$ and $\delta \in \operatorname{Cosh} f(\pi_0)(b) = \operatorname{colim}_{afb} \pi_0(a)$. Define $b' \leq^\delta b$ if $b' \leq b$ and

$$b' \lhd \bigcup \{ g^-(\leq^\gamma a) \mid afb, \mathrm{in}_a(\gamma) = \delta \}.$$

Now consider the conditions in Definition 16. 2(a) is obvious.

For 2(b), suppose $b' \leq^\delta b$ and $\delta' = \mathrm{in}_{a'}(\gamma') \in \operatorname{Cosh} f(\pi_0)(b')$, with $a' f b'$. From $b' \leq^\delta b$ we deduce

$$\begin{aligned} f^- b' &\lhd f^- \bigcup \{ g^-(\leq^\gamma a) \mid afb, \mathrm{in}_a(\gamma) = \delta \} \\ &= \bigcup \{ f^-(g^-(\leq^\gamma a)) \mid afb, \mathrm{in}_a(\gamma) = \delta \} \\ &\lhd \bigcup \{ \leq^\gamma a \mid afb, \mathrm{in}_a(\gamma) = \delta \} \text{ by the adjunction.} \end{aligned}$$

Since $a'fb'$ we can find $(a'',\gamma'') \leq (a',\gamma')$ with $a'' \leq^\gamma afb$. Using the fact that $a'fb$, we have that $\mathrm{Cosh}\, f(\pi_0)_{b',b}(\delta')$ is $\mathrm{in}_{a'}(\gamma')$ considered as element of $\mathrm{Cosh}\, f(\pi_0)(b)$, and that is

$$\mathrm{in}_{a'}(\gamma') = \mathrm{in}_{a''}(\gamma'') = \mathrm{in}_a((\pi_0)_{a'',a}(\gamma'')) = \mathrm{in}_a(\gamma) = \delta.$$

For 2(c), suppose $b'' \leq^{\delta'} b'$ and $(b',\delta') \leq (b,\delta)$. We want $b'' \leq^\delta b$, so it suffices to show that if $a'fb'$ and $\mathrm{in}_a(\gamma') = \delta'$ then

$$g^-(\leq^{\gamma'} a') \lhd \bigcup\{g^-(\leq^\gamma a) \mid afb, \mathrm{in}_a(\gamma) = \delta\}.$$

But in fact we have $\subseteq$ here, for $a'fb$ and, considered as element of $\mathrm{Cosh}\, f(\pi_0)(b)$, we have $\mathrm{in}_{a'}(\gamma') = \mathrm{Cosh}\, f(\pi_0)_{b',b}(\delta') = \delta$.

Now we look at 2(d). If $b \in Q$, then by the adjunction $b \lhd g^-(f^-b)$. Hence it suffices to show that if afb then

$$b \downarrow g^- a \lhd \bigcup_{\delta \in \mathrm{Cosh}\, f(\pi_0)(b)} (\leq^\delta b).$$

Now $a \lhd \bigcup_{\gamma\in\pi_0(a)}(\leq^\gamma a)$, so $b \downarrow g^-a \lhd \bigcup_{\gamma\in\pi_0(a)} b \downarrow g^-(\leq^\gamma a)$, and by definition if afb and $\mathrm{in}_a(\gamma) = \delta$ then $b \downarrow g^-(\leq^\gamma a) \subseteq (\leq^\delta b)$. ■

Corollary 22 *If $X = (P, \leq, \lhd)$ is locally connected, then so too is any formal topology homeomorphic with X.*

Proposition 23 *Let $X = (P, \leq, \lhd)$ be locally connected, equipped with π_0 and $\leq^\gamma$. Let us define covers on $\sum_{a\in P}\pi_0(a)$ by*

$$(a,\gamma) \lhd V \text{ if } (\leq^\gamma a) \lhd \bigcup\{\leq^\delta b \mid (b,\delta) \in V\}.$$

1. *$(\sum_{a\in P}\pi_0(a), \leq, \lhd)$ is a formally locally connected formal topology. (Here we shall write π_0' for its connected components cosheaf.)*
2. *There is a homeomorphism $p : (\sum_{a\in P}\pi_0(a), \leq, \lhd) \to X$ defined by $(a,\gamma)pb$ if $(\leq^\gamma a) \lhd b$.*
3. *Its inverse s is defined by $bs(a,\gamma)$ if $b \lhd (\leq^\gamma a)$.*
4. *The unique homomorphism $\mathrm{Cosh}\, p(\pi_0') \to \pi_0$ is an isomorphism.*

Proof. We shall not prove the parts in the order of the statement. First, let us define relations $p \subseteq (\sum_{a\in P}\pi_0(a)) \times P$ and $s \subseteq P \times (\sum_{a\in P}\pi_0(a))$ as stated in parts (2) and (3). Then Lemma 17 says that $s^-(a,\gamma) \downarrow s^-(b,\delta) \lhd s^-((a,\gamma)\downarrow (b,\delta))$ and we can deduce that $s^-V_1 \downarrow s^-V_2 \lhd s^-(V_1 \downarrow V_2)$ for any subsets V_1, V_2 of $\sum_{a\in P}\pi_0(a)$. Also, we see that $(a,\gamma) \lhd V$ iff $s^-(a,\gamma) \lhd s^-V$ (although we have not yet proved that this gives a formal topology).

Next, we prove that $c \lhd s^-p^-c \lhd c$ for each c. When we know that p and s are both maps, this will tell us that $s;p = \mathrm{Id}_X$. For $s^-p^-c \lhd c$, if $bs(a,\gamma)pc$

then $b \lhd (\leq^\gamma a) \lhd c$. For $c \lhd s^- p^- c$, we use that $(\leq^\varepsilon c) \subseteq s^- p^- c$ for every ε. It follows that $V \lhd s^- p^- V \lhd V$ for each $V \subseteq P$. Putting $V = s^-(a, \gamma)$ we deduce that $s^-(a, \gamma) \lhd s^- p^- s^-(a, \gamma) \lhd s^-(a, \gamma)$, which is equivalent to $(a, \gamma) \lhd p^- s^-(a, \gamma) \lhd (a, \gamma)$. Now, when we know that p and s are both maps, it will follow that they are mutually inverse.

Next, $\lhd$ is a formal topology. If $(a', \gamma') \leq (a, \gamma)$ then $(\leq^{\gamma'} a') \subseteq (\leq^\gamma a)$, so $(a', \gamma') \lhd (a, \gamma)$. If $(a, \gamma) \lhd V_i$ $(i = 1, 2)$, then

$$s^-(a, \gamma) \lhd s^- V_1 \downarrow s^- V_2 \lhd s^-(V_1 \downarrow V_2)$$

so $(a, \gamma) \lhd V_1 \downarrow V_2$. If $(a, \gamma) \lhd V \lhd W$, then $s^-(a, \gamma) \lhd s^- V \lhd s^- W$ so $(a, \gamma) \lhd W$.

Next, s and p are maps. The case of s is straightforward. For p, if $(a, \gamma) \lhd Vpb$ then $s^-(a, \gamma) \lhd s^- V \lhd b$ so $(a, \gamma) pb$. If $(a, \gamma) pb \lhd U$, then $s^-(a, \gamma) \lhd b \lhd U \lhd s^- p^- U$ and so $(a, \gamma) \lhd p^- U$. The remaining two conditions are simple.

At this point, we have proved that $(\sum_{a \in P} \pi_0(a), \leq, \lhd)$ is a formal topology, and (parts (2) and (3)) it is homeomorphic to X by maps p and $s = p^{-1}$.

To show it is formally locally connected, suppose $(a, \gamma) \lhd V$. We must show that $(a, \gamma) \downarrow V$ has exactly one equivalence class with respect to the equivalence relation $\sim$ generated by $\leq$. By Lemma 17 we can find $(a', \gamma') \leq (a, \gamma)$ with $a' \leq^\gamma a$. Since $a' \lhd s^-(a, \gamma) \downarrow s^- V \lhd s^-((a, \gamma) \downarrow V)$, and a' is positive, $(a, \gamma) \downarrow V$ is inhabited.

Now suppose that $(b_i, \delta_i) \in (a, \gamma) \downarrow V$ $(i = 1, 2)$; we must show that $(b_1, \delta_1) \sim (b_2, \delta_2)$. We can find $(b'_i, \delta'_i) \leq (b_i, \delta_i)$ with $b'_i \leq^{\delta_i} b_i$ so that $b'_i \leq^\gamma a$; also, $(b_i, \delta_i) \sim (b'_i, \delta'_i)$. Hence without loss of generality we can assume $b_i \in s^-((a, \gamma) \downarrow V)$. Since $a \lhd \bigcup \{\leq^\alpha a \mid \alpha \in \pi_0(a)\}$, there is a connection from (b_1, δ_1) to (b_2, δ_2) in $\bigcup \{\leq^\alpha a \mid \alpha \in \pi_0(a)\}$. If we have $(c_1, \varepsilon_1) \leq (c_2, \varepsilon_2)$ with $c_i \leq^{\alpha_i} a$ then $\alpha_1 = (\pi_0)_{c_1, a}(\varepsilon_1) = (\pi_0)_{c_2, a}(\varepsilon_2) = \alpha_2$, and it follows that the connection from (b_1, δ_1) to (b_2, δ_2) must be in $\leq^\gamma a$. If (c, ε) is an element of the connection, then because $(\leq^\gamma a) \lhd s^- V$ we can find $(c', \varepsilon') \leq (c, \varepsilon)$ with $c' \in s^- V$. Suppose now we have $(c_1, \varepsilon_1) \leq (c_2, \varepsilon_2)$ as a link in the connection from (b_1, δ_1) to (b_2, δ_2). Then there is a connection from (c'_1, ε'_1) to (c'_2, ε'_2) in $c_2 \downarrow s^- V \subseteq s^-((a, \gamma) \downarrow V)$ and so $(c'_1, \varepsilon'_1) \sim (c'_2, \varepsilon'_2)$. Putting these together, and remembering that (b_1, δ_1) and (b_2, δ_2) are already in $(a, \gamma) \downarrow V$, we find $(b_1, \delta_1) \sim (b_2, \delta_2)$.

This completes the proof of part (1). For part (4), since π'_0 is a terminal cosheaf for Y, and p is a homeomorphism, $\mathrm{Cosh}\, p(\pi'_0)$ is a terminal cosheaf for X. ∎

In Proposition 31 we shall see the above construction of $(\sum_{a \in P} \pi_0(a), \leq, \lhd)$ and p as a special example of the "complete spread" construction (for cosheaves other than π_0), which can be described more generally in the inductively generated case.

Theorem 24 *A formal topology is locally connected iff it is homeomorphic to a formally locally connected formal topology.*

Proof. The $\Rightarrow$ direction is Proposition 23. The $\Leftarrow$ direction comes from Corollary 22. ∎

We shall next show how to find connected components for arbitrary opens in a locally connected formal topology.

Definition 25 *Let $X = (P, \leq, \lhd)$ be locally connected, and suppose $A \subseteq P$ and $\gamma \in \pi_0(A) = \mathrm{colim}_{a \in \downarrow A} \pi_0(a)$. Define*

$$\leq^\gamma A = \bigcup \{ \leq^\delta a \mid \gamma = \mathrm{in}_a(\delta) \}.$$

Proposition 26 *Let $X = (P, \leq, \lhd)$ be locally connected, and suppose $A \subseteq P$.*

1. *$A \lhd \bigcup_{\gamma \in \pi_0(A)} (\leq^\gamma A)$.*
2. *The opens $\leq^\gamma A$ ($\gamma \in \pi_0(A)$) are pairwise disjoint.*
3. *Each $\leq^\gamma A$ is connected.*

Proof. (1) This follows from the fact that $a \lhd \bigcup_{\gamma \in \pi_0(a)} \leq^\gamma a$.

(2) Suppose $b \in (\leq^{\gamma_1} A) \downarrow (\leq^{\gamma_2} A)$. Then $b \leq^{\delta_i} a_i$ with $\mathrm{in}_{a_i}(\delta_i) = \gamma_i$. If $\varepsilon \in \pi_0(b)$ then $(b, \varepsilon) \leq (a_i, \delta_i)$ $(i = 1, 2)$ so $\gamma_i = \mathrm{in}_b(\varepsilon)$ and $\gamma_1 = \gamma_2$. Hence $(\leq^{\gamma_1} A) \downarrow (\leq^{\gamma_2} A) \lhd \{ d \in P \mid \gamma_1 = \gamma_2 \}$.

(3) Let $B_i \subseteq (\leq^\gamma A)$ $(i \in I)$ be a family of pairwise disjoint opens covering $\leq^\gamma A$. We must show $(\leq^\gamma A) \lhd B_{i_0}$ for some (unique) i_0.

Since $\gamma \in \pi_0(A)$, we can find $a \in \downarrow A$ and $\delta \in \pi_0(a)$ such that $\gamma = \mathrm{in}_a(\delta)$, and then (by Lemma 17) some $(a', \delta') \leq (a, \delta)$ with $a' \leq^\delta a$. Now $a' \in (\leq^\gamma A)$, so $a' \lhd a' \downarrow \bigcup_{i \in I} B_i$. Hence for some $i_0 \in I$ there is some $b \in a' \downarrow B_{i_0}$ and $\varepsilon \in \pi_0(b)$ with $(b, \varepsilon) \leq (a', \delta')$. We shall show that i_0 is unique once γ is given, and it will follow that $(\leq^\gamma A) \lhd B_{i_0}$.

First, suppose a, δ, a' and δ' are given but we have $(b_\lambda, \varepsilon_\lambda) \leq (a', \delta')$ $(\lambda = 1, 2)$ with $b_\lambda \in a' \downarrow B_{i_\lambda}$. Then there is a connection from (b_1, ε_1) to (b_2, ε_2) in $a' \downarrow \bigcup_{i \in I} B_i$. Therefore, to show $i_1 = i_2$, we can assume without loss of generality that $(b_1, \varepsilon_1) \leq (b_2, \varepsilon_2)$. Then $b_1 \in B_{i_1} \downarrow B_{i_2} \lhd \{ c \in P \mid i_1 = i_2 \}$ and from positivity of b_1 (as witnessed by ε_1) we deduce $i_1 = i_2$.

Now suppose a and δ are given, but we have a'_λ, δ'_λ, b_λ, ε_λ and i_λ $(\lambda = 1, 2)$. There is a connection from (a'_1, δ'_1) to (a'_2, δ'_2) in $\bigcup \{ \leq^\alpha a \mid \alpha \in \pi_0(a) \}$, but clearly this must be in $\leq^\delta a$. Hence to show $i_1 = i_2$ we can assume without loss of generality that $(a'_1, \delta'_1) \leq (a'_2, \delta'_2)$. Then $b_1 \in a'_2 \downarrow B_{i_1}$ and $\varepsilon_1 \in \pi_0(b_1)$ with $(b_1, \varepsilon_1) \leq (a'_2, \delta'_2)$ and it follows by uniqueness with respect to (a'_2, δ'_2) that $i_1 = i_2$.

Finally, suppose just γ is given, and we have a_λ, δ_λ, a'_λ, δ'_λ, b_λ, ε_λ and i_λ $(\lambda = 1, 2)$. There is a connection from (a_1, δ_1) to (a_2, δ_2) in $\downarrow A$ and so to show $i_1 = i_2$ we can assume without loss of generality that $(a_1, \delta_1) \leq (a_2, \delta_2)$. But then $a'_1 \leq^{\delta_2} a$ and $(a'_1, \delta'_1) \leq (a_2, \delta_2)$ and it follows from uniqueness with respect to (a_2, δ_2) that $i_1 = i_2$. ∎

From the Proposition we see that (i) the opens $\leq^\gamma A$ are the connected components of A, and (ii) the connected components of the open A are in bijection with the elements of $\pi_0(A)$, so π_0 is indeed the connected components cosheaf.

6 Complete spreads

In [BF06, Section 2.4] a key use of a cosheaf F over X is to define a locale over X, by "amalgamation". The locale maps obtained in this way are called *complete spreads.* The domain is always locally connected, with $F(U)$ isomorphic to the set of connected components of the inverse image of U. (When X is itself locally connected, the connected components cosheaf π_0 gives as complete spread the identity map on X.) This is analogous to the way ([BF96], [Vic97]; see also [Vic07]) a sup-preserving function $F : \Omega X \to \Omega$ gives rise to a sublocale of X.

This Section is restricted to the inductively generated case. In the following Definition, I do not know how to describe the full cover relation for the complete spread (except in the particular case where F is a connected components cosheaf π_0 – see Proposition 31).

In [BF06], the Definition here is best compared with their Proposition 2.4.1. They have a site $(\mathbb{C}, J)$ in which $\mathbb{C}$ is a category corresponding to the poset P here. (Their sites are for toposes, i.e. *generalized* spaces.) The J-sieves then correspond to the basic covers $a \lhd_0 U$. (Or, rather, to $a \lhd a \downarrow U$. This is because a sieve at c comprises morphisms with codomain c and in the poset context a sieve at a is a subset of $\downarrow a$.) For a sieve R at c they use sieves G^*R at (c, x) that are made in the same way as our basic covers $(a, \gamma) \lhd_0 \cdots$ are made from $a \lhd a \downarrow U$.

Definition 27 *Let $X = (P, \leq, \lhd_0)$ be a flat site and F a cosheaf over it. Then the corresponding* complete spread $CS(X; F)$ *over X is the flat site $(\sum_{a \in P} F(a), \leq, \lhd_0)$, where $\leq$ is as defined before ($(b, \delta) \leq (a, \gamma)$ if $b \leq a$ and $\gamma = F_{b,a}(\delta)$), and $\lhd_0$ is defined by*

$$(a, \gamma) \lhd_0 \{(b, \delta) \leq (a, \gamma) \mid b \in a \downarrow U\}$$

whenever $a \lhd_0 U$, $\gamma \in F(a)$.

We must check the localization condition. Suppose $a \lhd_0 U$, $\gamma \in F(a)$ and $(a', \gamma') \leq (a, \gamma)$. Then $a' \lhd_0 V$ for some $V \subseteq a' \downarrow U$, so

$$\begin{aligned}(a', \gamma') \lhd_0 \{(c, \varepsilon) \leq (a', \gamma') \mid c \in V\} \\ \subseteq (a', \gamma') \downarrow \{(b, \delta) \leq (a, \gamma) \mid b \in a \downarrow U\}.\end{aligned}$$

In Definition 27, basic covers in X give rise to basic covers for $CS(X; F)$. An induction on proofs shows that this extends to arbitrary covers.

The following Proposition corresponds to the topological description of complete spreads given in [BF06, Section 2.1].

Proposition 28 *Let $X = (P, \leq, \lhd_0)$ be a flat site and F a cosheaf over it. Then a point of $CS(X; F)$ is a pair (x, γ), where x is a point of X, and $\gamma : \delta(x) \to F$ is a homomorphism.*

Proof. Note that such a homomorphism γ is a family $(\gamma_a)_{x \vDash a}$ where $\gamma_a \in F(a)$ for every basic open neighbourhood a of x, and if $x \vDash a \leq b$ then $(a, \gamma_a) \leq (b, \gamma_b)$. Such a family (γ_a) is called a *cogerm* of F at x.

A point of $CS(X;F)$ is a filter A of $\sum_{a\in P} F(a)$ such that if $a \lhd_0 U$ and $(a,\gamma) \in A$ then there is some $(b,\delta) \in A$ such that $b \in a \downarrow U$ and $(b,\delta) \leq (a,\gamma)$. Suppose A contains both (a,γ) and (a,γ'). By the filter property they have a lower bound (b,δ) in A, but then $\gamma = F_{b,a}(\delta) = \gamma'$. Hence A has at most one (a,γ_a) for each $a \in P$. Let $x = \{a \mid (\exists\gamma)\ (a,\gamma) \in A\}$. The filter property for A implies the filter property for x. We also see that if $a \lhd_0 U$ and $a \in x$ then x meets U, and so x is a point. The elements γ_a give a cogerm of F at x. Conversely, given a pair (x,γ), then the pairs (a,γ_a) form a point of $CS(X;F)$ and this gives the bijection required. ■

Proposition 29 *Let F be a cosheaf on $X = (P,\leq,\lhd_0)$. Then $CS(X;F)$ is formally locally connected.*

Proof. Referring to Proposition 15, suppose $a \lhd_0 U$ and $\gamma \in F(a)$. We must show that $\{(b,\delta) \leq (a,\gamma) \mid b \in a \downarrow U\}$ has a single equivalence class under $\sim$. Because $a \lhd a \downarrow U$ there is some such (b,δ). Now suppose (for $i = 1,2$) we have $b_i \in a \downarrow U$ and $(b_i,\delta_i) \leq (a,\gamma)$. Then there is some connection from (b_1,δ_1) to (b_2,δ_2) in $a \downarrow U$, and this makes $(b_1,\delta_1) \sim (b_2,\delta_2)$. ■

Definition 30 *We define a map $p : CS(X;F) \to X$ on points by $p(x,(\gamma_a)) = x$ – in other words, forget the cogerm of F.*

By Proposition 28 we see that the inverse image $p^*b = \{(b,\delta) \mid \delta \in F(b)\}$, so $(a,\gamma)pb$ iff $(a,\gamma) \lhd \{(b,\delta) \mid \delta \in F(b)\}$.

Proposition 31 *Let $X = (P,\leq,\lhd_0)$ be a locally connected formal topology, equipped with π_0 and $\leq^\gamma$. Then $CS(X;\pi_0)$ presents the cover defined in Proposition 23, and $p : CS(X;\pi_0) \to X$ is the same as the map p defined there.*

Proof. We must show that $(a,\gamma) \lhd V$ in $CS(X;\pi_0)$ iff $(\leq^\gamma a) \lhd s^- V$. For the $\Rightarrow$ direction we can assume $(a,\gamma) \lhd_0 V$, so suppose $a \lhd_0 U$ and $V = \{(b,\delta) \leq (a,\gamma) \mid b \in a \downarrow U\}$. Suppose $a' \leq^\gamma a$. We have $a' \lhd a' \downarrow U$, and if $b \in a' \downarrow U$ then $b \lhd \bigcup_{\delta\in\pi_0(b)} s^-(b,\delta) \subseteq s^- V$.

For the $\Leftarrow$ direction we have $a \lhd \bigcup\{\leq^\alpha a \mid \alpha \in \pi_0(a)\}$, and so

$$\begin{aligned}(a,\gamma) &\lhd \{(a',\gamma') \leq (a,\gamma) \mid (\exists\alpha) a' \leq^\alpha a\}\\ &= \{(a',\gamma') \leq (a,\gamma) \mid a' \leq^\gamma a\}.\end{aligned}$$

By hypothesis, if $a' \leq^\gamma a$ then $a' \lhd s^- V$ and so

$$(a',\gamma') \lhd \{(b',\delta') \leq (a',\gamma') \mid b' \in a' \downarrow s^- V\} \subseteq (\downarrow V)$$

and so $(a,\gamma) \lhd V$.

To show that the two definitions of p are equivalent, we must show that $(\leq^\gamma a) \lhd b$ iff $(a,\gamma) \lhd \{(b,\delta) \mid \delta \in \pi_0(b)\}$, i.e. $(\leq^\gamma a) \lhd s^-\{(b,\delta) \mid \delta \in \pi_0(b)\} = \bigcup_{\delta\in\pi_0(b)} (\leq^\delta b)$. This is immediate from condition 2(d) in Definition 16. ■

The following result is a special case of [BF06, Proposition 2.4.2].

Theorem 32 *Let $X = (P, \leq, \lhd_0)$ be inductively generated and $Y = (Q, \leq, \lhd)$ be locally connected, and let $q : Y \to X$ be a map. Let G be a cosheaf on X. Then there is a bijection between homomorphisms $\mathrm{Cosh}\, q(\pi_0) \to G$ and maps $Y \to CS(X; G)$ over X. ("Over X" means that it makes a commutative triangle with the maps q and p.)*

Proof. First, let $f : Y \to CS(X; G)$ be a map over X.

We show that if $b \in P$ and $A \subseteq q^- b$, then for each $\gamma \in \pi_0(A)$ there is a unique $\delta \in G(b)$ such that $(\leq^\gamma A) \lhd f^-(b, \delta)$. Because $q = f; p$ we have $A \lhd \bigcup\{f^-(b, \delta) \mid \delta \in G(b)\}$. If $\gamma \in \pi_0(A)$ then $\leq^\gamma A$ is covered by the opens $f^-(b, \delta)$ ($\delta \in G(b)$). But since the basic opens (b, δ) of $CS(X; G)$ are pairwise disjoint, so too are their inverse images $f^-(b, \delta)$, and our claim follows from connectedness of $\leq^\gamma A$ (Proposition 26).

Applying this to $A = q^- b$, we get a function $\alpha_b : \mathrm{Cosh}\, q(\pi_0)(b) \to G(b)$. In fact, these α_bs form a homomorphism of cosheaves. To see this, suppose $b_1 \leq b_2$, so $q^- b_1 \subseteq q^- b_2$, and suppose $\gamma_i \in \pi_0(q^- b_i)$ with $\gamma_2 = (\pi_0)_{q^- b_1, q^- b_2}(\gamma_1)$. Let $\delta_i = \alpha_{b_i}(\gamma_i)$. We wish to show $\delta_2 = G_{b_1, b_2}(\delta_1)$. We have $(\leq^{\gamma_1} q^- b_1) \subseteq (\leq^{\gamma_2} q^- b_2)$ and it follows that $(\leq^{\gamma_1} q^- b_1) \lhd f^-(b_2, \delta_2)$. But also $(\leq^{\gamma_1} q^- b_1) \lhd f^-(b_1, \delta_1)$ and $(b_1, \delta_1) \leq (b_2, G_{b_1, b_2}(\delta_1))$, so $(\leq^{\gamma_1} q^- b_1) \lhd f^-(b_2, G_{b_1, b_2}(\delta_1))$. Hence by uniqueness (by the original remark using b_2, $q^- b_1$ and γ_1 for b, A and γ) we have $\delta_2 = G_{b_1, b_2}(\delta_1)$ as required.

Now let $\alpha : \mathrm{Cosh}\, q(\pi_0) \to G$ be an arbitrary homomorphism. We define a map f_α on points as follows. If y is a point of Y, then by Theorem 20 we have a unique homomorphism $! : \delta(y) \to \pi_0$, and hence a homomorphism

$$\mathrm{Cosh}\, q(!); \alpha : \delta(q(y)) = \mathrm{Cosh}\, q(\delta(y)) \to \mathrm{Cosh}\, q(\pi_0) \to G.$$

Then f_α is defined by

$$f_\alpha(y) = (q(y), \mathrm{Cosh}\, q(!); \alpha).$$

Let us calculate the inverse image. If $f_\alpha(y) \vDash (b, \delta)$ then $y \vDash q^- b$ so $\delta(q(y))(b) = \delta(y)(q^- b) = 1$. The image of its single element under $\mathrm{Cosh}\, q(!)_b$ is the component of $q^- b$ containing y, i.e. the unique $\gamma \in \pi_0(q^- b)$ such that $y \vDash (\leq^\gamma q^- b)$, and then $f_\alpha(y) \vDash (b, \delta)$ tells us that $\delta = \alpha_b(\gamma)$. Hence

$$a f_\alpha(b, \delta) \text{ iff } a \lhd \bigcup\{\leq^\gamma q^- b \mid \gamma \in \pi_0(q^- b), \alpha_b(\gamma) = \delta\}.$$

Referring to Proposition 28, the reader sceptical of these geometric pointwise methods can check that this is a map.

Clearly if $\gamma \in \pi_0(q^- b)$ and $\alpha_b(\gamma) = \delta$ then $(\leq^\gamma q^- b) \lhd f_\alpha^-(b, \delta)$, so it follows that α is recovered from f_α by the process described in the first part of the proof.

Finally, let us start from a map f, and define α as in the first part. By definition of α, if $\gamma \in \pi_0(q^- b)$ and $\alpha_b(\gamma) = \delta$ then $(\leq^\gamma q^- b) \lhd f^-(b, \delta)$, so if $a f_\alpha(b, \delta)$ then $a \lhd f^-(b, \delta)$, i.e. $a f(b, \delta)$. Conversely, suppose $a f(b, \delta)$; since f is over X, we also have aqb. If $\varepsilon \in \pi_0(a)$ then $(\leq^\varepsilon a) \lhd f^-(b, \delta)$, and by the argument at the start, δ is the unique such. Let $\gamma = (\pi_0)_{\{a\},q^- b}(\varepsilon)$. Then $(\leq^\varepsilon a) \subseteq (\leq^\gamma q^- b) \lhd f^-(b, \alpha_b(\gamma))$, and we deduce $\alpha_b(\gamma) = \delta$. Hence $a f_\alpha(b, \delta)$. ■

Corollary 33 *Let F, G be cosheaves on $X = (P, \leq, \lhd_0)$. Then there is a bijection between homomorphisms $F \to G$ and maps $CS(X; F) \to CS(X; G)$ over X.*

Proof. In Theorem 32, let Y be $CS(X; F)$, and let q be its p map. If $b \in P$, then

$$\mathrm{Cosh}\, p(\pi_0)(b) = \pi_0(p^- b) = \pi_0(\{(b, \delta) \mid \delta \in F(b)\}) \cong F(b)$$

and we see that $\mathrm{Cosh}\, p(\pi_0) \cong F$. ■

Proposition 34 *Let $X = (P, \leq, \lhd_0)$ be equipped with a cosheaf π_0. Then the following are equivalent.*

1. *X and π_0 have relations $\leq^\gamma$ to make X locally connected.*
2. *$p : CS(X; \pi_0) \to X$ is a homeomorphism.*

Proof. (1)⇒(2): This follows by combining Propositions 23 and 31.

(2)⇒(1): By Proposition 29 $CS(X; \pi_0)$ is formally locally connected. By Lemma 21 we deduce that X is locally connected with connected component cosheaf $\mathrm{Cosh}\, p(\pi_0')$, and so it suffices to prove that the unique homomorphism $\pi_0 \to \mathrm{Cosh}\, p(\pi_0')$ is an isomorphism. We have

$$\begin{aligned} \mathrm{Cosh}\, p(\pi_0')(a) &= \pi_0'(\{(a, \gamma) \mid \gamma \in \pi_0(a)\}) \\ &= \mathrm{colim}\{1 \mid (b, \delta) \leq (a, \gamma), \gamma \in \pi_0(a)\} \\ &\cong \pi_0(a). \end{aligned}$$

The identity function $\pi_0(a) \to \pi_0(a) \cong \mathrm{Cosh}\, p(\pi_0')(a)$ gives an isomorphism $\pi_0 \to \mathrm{Cosh}\, p(\pi_0')$. ■

7 "Formal toposes"

Grothendieck said that a topos is a generalized topological space, and in fact we can already see such a generalized space in our working so far: it is the space MX of cosheaves over X, and is called the symmetric topos over X. What makes this a real generalization is that the homomorphisms between cosheaves, the analogue of the specialization order on points of a space, do not form a preorder – there may be distinct homomorphisms between the same two cosheaves. What gives it its spatial qualities is the fact that cosheaves are the points of a site (or the models of a geometric theory) in the same way as points of a formal topology (models of a *propositional* geometric theory). Technically, the opens are not enough to define the topological structure, but sheaves can be used instead.

In this Section we continue our overall story of how one might find predicative content in the topos theory of [BF06]. However, we shall stray from the confines of established predicative formal topology, and instead sketch proofs of some known results in a style that we believe will be amenable to a predicative treatment. Our main purpose is to show toposes in action as generalized spaces, with the suggestion that there should be a treatment of them as "formal toposes". We shall also use "geometricity" arguments (see Section 1.1) to analyse constructions under change of base. This is seen in Lemmas 38 and 42.

We propose here the idea that general sites could be treated as "formal toposes" in a way that generalizes formal topologies; the generalized notion of continuous map will then correspond to geometric morphisms. Such a theory has not yet been formulated, but our aim in this Section is to show how features of topos theory would appear in it. Since the discussion is based on sites, in this section *all formal topologies are assumed to be inductively generated.*

Note that for generalized spaces, the specialization order (see Section 2) between maps – i.e. geometric morphisms – $f, g : X \to Y$ will become specialization *morphisms* $\alpha : f \Rightarrow g$. Pointwise, α is a natural transformation. For each point x of X, it gives a homomorphism $\alpha_x : f(x) \to g(x)$. (Since points of a topos are models of a geometric theory, there is a natural definition of homomorphism.) These specialization morphisms are the 2-cells in a 2-category of toposes, and have horizontal and vertical composition in the usual way.

We shall illustrate the idea with a discussion of a converse of Theorem 20, which said that if X is locally connected then its connected components cosheaf π_0 is a terminal cosheaf – i.e., a terminal point of MX. The converse is not actually true. In fact (see [Fun99]), for every Grothendieck topos X there is a terminal cosheaf, whether or not X is locally connected. The corresponding complete spread, a locally connected coreflection of X, is called the *Gleason core* of X (see [BF06, Section 6.2]), and it need not be homeomorphic to X. However, we can find a converse by reinterpreting "point" in the generalized way – recall that a *generalized* point of X (at *stage* W) is a map $W \to X$. Let us say that a (global) point $t : 1 \to X$ is *strongly terminal* if for every generalized point $x : W \to X$ there is a unique homomorphism from x to $t\circ !$. This is equivalent to t being right adjoint to $! : X \to 1$, and X is called *totally connected* if it has such a point (see [Joh02b, C3.6.16]). The geometricity of Theorem 20 in fact shows that if X is locally connected then the connected components cosheaf π_0 is a strongly terminal cosheaf, so that MX is totally connected. We shall give a new proof of the converse.

To motivate our presentation of results for local connectedness and MX, we shall first give an analogous presentation of known results regarding overtness and the lower powerlocale. These do not involve generalized spaces.

7.1 The lower powerlocale

A junior version of cosheaf is the *lower powerpoint* (or [Val05] *up-complete* set of basics). Whereas a cosheaf is a colimit-preserving functor from sheaves to sets, a lower powerpoint is a join-preserving function from opens to propositions. If

$X = (P, \leq, \lhd_0)$ is a flat site, then a lower powerpoint over it is a subset $F \subseteq P$, up-closed with respect to $\leq$, such that if $a \lhd_0 U$ and $a \in F$ then U meets F. This then gives a sublocale $\mathsf{Rest}F$ of X [Vic07] by adding covers $a \lhd \{a\} \cap F$. The fundamental result of the lower powerlocale ([BF96]; see also [Vic97], [Vic07]) is that the assignment $F \mapsto \mathsf{Rest}F$ gives a bijection between lower powerpoints and overt, weakly closed sublocales (classically, these are the same as the closed sublocales). In the reverse direction, starting from the sublocale Y, F can be recovered as the set of basic opens that are positive modulo Y.

From the techniques of [Vic04] (see also [Vic06]) it can be deduced that the lower powerpoints are the points of another flat site, for the *lower powerlocale* $P_L X$ of X. As a flat site it can be presented as $(\mathcal{F}P, \leq_U, \lhd_0)$, $\mathcal{F}P$ being the (Kuratowski) finite powerset of P, where (i) $S \leq_U T$ if for each $t \in T$ there is some $s \in S$ with $s \leq t$, and (ii) for each basic cover $a \lhd_0 U$ in X, and for each $S \in \mathcal{F}P$, we have a basic cover $\{a\} \cup S \lhd_0 \{\{u\} \cup S \mid u \in U\}$ in $P_L X$. There is a map $\downarrow : X \to P_L X$, corresponding to the fact that every point is already a lower powerpoint. Each lower powerpoint F has a map $1 \to P_L X$, and the corresponding sublocale is obtained as a comma object

$$\begin{array}{ccc}
\mathsf{Rest}F & \longrightarrow & 1 \\
\downarrow & \sqsubseteq & \downarrow F \\
X & \underset{\downarrow}{\longrightarrow} & P_L X
\end{array}$$

By definition, this means that the (generalized) points of $\mathsf{Rest}F$ are those points x of X such that $\downarrow x \sqsubseteq F$.

In fact, F can be replaced by a generalized point $W \to P_L X$. The geometricity results of [Vic04] then show that, working internally in $\mathcal{S}W$ (sheaves over W), this corresponds to a lower powerpoint of the locale $W \times X \to W$ over W, and the comma object (which we shall write as $\mathsf{Rest}_W F$) is the corresponding sublocale of $W \times X$.

In general, a comma object is like a pullback, but universal with respect to having an inequality in the square instead of equality.

Lemma 35 *Suppose we have maps $f : X \to Z$ and $g : Y \to Z$. Consider the diagram*

$$\begin{array}{ccc}
P & \longrightarrow & Y \\
\downarrow & & \downarrow g \\
C & \longrightarrow & Z \\
\downarrow & \sqsubseteq & \downarrow \mathrm{Id} \\
X & \underset{f}{\longrightarrow} & Z
\end{array}$$

where the lower square is a comma square, and the upper is a pullback. Then the two squares together form a comma square.

Proof. Straightforward checking of universal property. ■

A lower powerpoint Pos is a *positivity predicate* if $a \lhd \{a\} \cap \mathrm{Pos}$ for every a. As mentioned in Definition 2, X is overt iff it has a positivity predicate.

Also, a positivity predicate Pos is greatest amongst the lower powerpoints F: for if $a \in F$ then, since $a \lhd \{a\} \cap \mathrm{Pos}$, we see that $\{a\} \cap \mathrm{Pos}$ meets F and so $a \in \mathrm{Pos}$. However, the converse is not true. From [MV04] we know that *every* X, overt or not, has a greatest lower powerpoint F, and $\mathsf{Rest}F$ is an overt coreflection of X. Nonetheless, we can find a valid converse by strengthening the idea of "greatest lower powerpoint". The greatest lower powerpoint F of [MV04] is greatest amongst all the *global* points $1 \to P_L X$. However, we can also ask whether F is greatest amongst the *generalized* points $G : W \to P_L X$: do we always have $G \sqsubseteq\ !; F$, where $! : W \to 1$ is the unique map?

Proposition 36 *Let X be a space, and $t : 1 \to X$. Then t is greatest amongst all generalized points iff t is right adjoint to $!_X : X \to 1$.*

Proof. $\Rightarrow$: Considering the generic point $\mathrm{Id}_X : X \to X$, we see $\mathrm{Id}_X \sqsubseteq\ !_X; t$. Since in any case $\mathrm{Id}_1 = t; !_X$, it follows that t is right adjoint to $!_X$.

$\Leftarrow$: Let $x : W \to X$ be a generalized point. Since $\mathrm{Id}_X \sqsubseteq\ !_X; t$, we have $x \sqsubseteq x; !_X; t = !_W; t$. ∎

Let us call such a point t *strongly top*. We shall prove (Theorem 39) that a lower powerpoint is a positivity predicate iff it is strongly top in the lower powerlocale. This result is already known [Vic95], but we shall give a proof that is more categorical as a pattern for the corresponding discussion of cosheaves in Section 7.2.

Lemma 37 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and let F be a lower powerpoint. Then F is a positivity predicate iff $\mathsf{Rest}F = X$.*

Proof. $\Rightarrow$: The covers $a \lhd \{a\} \cap F$ presenting $\mathsf{Rest}F$ already hold in X.

$\Leftarrow$: For any F, the sublocale $\mathsf{Rest}F$ is overt, so X is overt. The positivity predicate for $\mathsf{Rest}F$ is F, which is thus also the positivity predicate for X. ∎

Lemma 38 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and let $F, G : W \to P_L X$ be two generalized lower powerpoints. Then $F \sqsubseteq G$ iff $\mathsf{Rest}_W F \leq \mathsf{Rest}_W G$ (as subspaces of $X \times W$).*

Proof. Note that $\mathsf{Rest}_W F \leq \mathsf{Rest}_W G$ iff there is a (unique) map $\mathsf{Rest}_W F \to \mathsf{Rest}_W G$ over $X \times W$, and by definition of $\mathsf{Rest}_W G$ as comma object this holds if $p_F; \downarrow\ \sqsubseteq q_F; G : \mathsf{Rest}_W F \to P_L X$. ($p_F$ and q_F are the projections from $\mathsf{Rest}_W F$ to X and W.)

$\Rightarrow$: This is the easy way round. If $F \sqsubseteq G$ then $p_F; \downarrow\ \sqsubseteq q_F; F\ \sqsubseteq q_F; G$.

$\Leftarrow$: We first prove this in the case $W = 1$. Since $\mathsf{Rest}_W F \leq \mathsf{Rest}_W G$, the positivity predicate for $\mathsf{Rest}_W F$ is included in that for $\mathsf{Rest}_W G$: in other words, $F \sqsubseteq G$. But that argument is constructive and hence applies also for internal locales in $\mathcal{S}W$, i.e. maps with codomain W. Let us apply it to $X \times W$, with its projection down to W. The geometricity of P_L [Vic04] tells us that, constructing P_L internally in $\mathcal{S}W$, we have $(P_L)_W(X \times W) \cong P_L X \times W$. In this context we have *global* lower powerpoints of $X \times W$, namely $\langle F, W\rangle, \langle G, W\rangle : W \to P_L X \times W$. Let us calculate the Rest sublocale for $\langle F, W\rangle$, the comma object

$$\begin{array}{ccc} C & \xrightarrow{q} & W \\ \downarrow p & \sqsubseteq_W & \downarrow \langle F, W\rangle \\ X \times W & \underset{\downarrow \times W}{\longrightarrow} & P_L X \times W \end{array}$$

(The subscript W in $\sqsubseteq_W$ denotes that the order between maps to $P_L X \times W$ has to restrict to equality when projected down to W.) A point of C is a triple $\langle x, w_1, w_2\rangle$ such that $\downarrow x \sqsubseteq F(w_2)$ and $w_1 = w_2$, so $C \cong \mathsf{Rest}_W F$ as defined above. Hence our working in $\mathcal{S}W$ shows that if $\mathsf{Rest}_W F \leq \mathsf{Rest}_W G$ then $\langle F, W\rangle \sqsubseteq \langle G, W\rangle$ over W, i.e. $F \sqsubseteq G$. ∎

Theorem 39 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and F a lower powerpoint. Then the following are equivalent.*

1. *X is overt, with F its positivity predicate.*
2. *$\mathsf{Rest} F = X$.*
3. *F is strongly top in $P_L X$.*

Proof. (1)⇔(2): This is Lemma 37.

For (2)⇔(3) we shall use purely categorical methods together with Lemma 38.

(3)⇒(2): Condition (2) is equivalent to having a map $X \to \mathsf{Rest} F$ over X, and by the comma property of $\mathsf{Rest} F$ this is equivalent to $\downarrow \sqsubseteq !_X ; F$ – in other words, $\downarrow : X \to P_L X$ as generalized lower powerpoint is less than F, a particular consequence of (3).

(2)⇒(3): This is harder, and we shall use Lemma 38 with $W = P_L X$ to get $\mathrm{Id}_{P_L X} \sqsubseteq !; F$. We must calculate the two comma squares for $\mathrm{Id}_{P_L X}$ and $!; F$, and Lemma 35 allows us to fit them together.

$$\begin{array}{ccc} \mathsf{Rest}_{P_L X}(!; F) & \xrightarrow{q'} & P_L X \\ \downarrow p' & & \downarrow ! \\ \mathsf{Rest} F & \xrightarrow{!} & 1 \\ \downarrow q^* F & & \downarrow F \\ \mathsf{Rest}_{P_L X}\,\mathrm{Id} & \xrightarrow{q} & P_L X \\ \downarrow r & \sqsubseteq & \downarrow \mathrm{Id} \\ X & \underset{\downarrow}{\longrightarrow} & P_L X \end{array}$$

Here the bottom square is a comma, and the two upper ones are pullbacks – in fact the top one shows $\mathsf{Rest}_{P_L X}(!; F) \cong \mathsf{Rest} F \times P_L X$.

By hypothesis (2), the composite $p = (q^* F); r$ has an inverse $p^{-1} : X \to \mathsf{Rest} F$. Define

$$\theta = \langle r; p^{-1}, q\rangle : \mathsf{Rest}_{P_L X}\,\mathrm{Id} \to \mathsf{Rest} F \times P_L X \cong \mathsf{Rest}_{P_L X}(!; F)$$

Then $\theta; p'; (q^* F); r = r; p^{-1}; (q^* F); r = r$ and it follows that θ is over $X \times P_L X$. Consequently, we can apply Lemma 38 to deduce that F is strongly top. ∎

7.2 The symmetric topos

Analogous to the lower powerlocale monad on **Loc**, there is a symmetric topos monad M on **Top**, the 2-category of Grothendieck toposes and geometric morphisms [BF06, Section 4.2]. The points of MX are the cosheaves over X. (Our account here in the context of formal topologies addressed the case where X is a locale, thought of as a localic topos. However, the notion of cosheaf over X also makes sense when X is a more general topos.)

Bunge and Funk [BF06] describe the site for MX in their Lemma 4.2.4. If X has site $\langle \mathbb{C}, J \rangle$ then MX (appearing as $\Sigma(\mathcal{E})$, where $\mathcal{E} = \mathcal{S}X$) has site $\langle \mathbb{C}^*, K \rangle$. Here $\mathbb{C}^*$ is the lex (finite limit) completion of $\mathbb{C}$, and the construction of K out of J, makes K pullback stable. Their construction is in the generality of X being a topos, but it can be adapted to the situation where X is an inductively generated formal topology $(P, \leq, \lhd_0)$. The preorder $(P, \leq)$ can be considered to be a category, and the basic covers in $\lhd_0$ become a coverage J. Of course, the lex completion $(P, \leq)^*$ is no longer a preorder, so we have left the realm of formal topology as we know it. It is important to note that the construction of $\langle \mathbb{C}^*, K \rangle$ from $\langle \mathbb{C}, J \rangle$ is geometric – preserved under inverse image functors of maps (geometric morphisms). In fact, they are predicative. The construction of $\mathbb{C}^*$ from $\mathbb{C}$ is geometric because it is a free construction with respect to cartesian theories. (For a predicative proof, see [PV07].) After that, the construction of K out of J is also geometric. The geometricity enables us to calculate how M changes when we change the base topos: specifically, it shows that M is *equivariant* [BF06, Definition 6.4.4]. This enables us to use arguments about global points to derive properties of generalized points. See Lemma 42 for an application here.

The point cosheaves $\delta(x)$ give us a map (i.e. geometric morphism) $\delta : X \to MX$, and this is the unit of the monad. Then [BF06, Proposition 5.2.1] for a cosheaf F, a map $1 \to MX$, the corresponding complete spread is a localic map p obtained as a bicomma object

$$\begin{array}{lcl} CS(X;F) & \longrightarrow & 1 \\ \downarrow p & \Rightarrow & \downarrow F \\ X & \underset{\delta}{\longrightarrow} & MX \end{array}$$

in **Top**. (A "bicomma" square is a bicategorical analogue of the comma square and has essentially the same universal property, but up to equivalence instead of isomorphism. We shall not dwell on the details.) Its points, already described in Proposition 28 for the localic case, can be described as pairs (x, α) such that x is a point of X and $\alpha : \delta(x) \to F$ is a homomorphism of cosheaves.

Again, F can be replaced by a generalized point $W \to MX$, and the bicomma square (we shall write its vertex as $CS_W(X;F)$) gives a complete spread over $W \times X$.

Our aim now (Theorem 43, essentially a rephrasing of [BF06, Proposition 6.2.5]) is to sketch a proof of a topos analogue of Theorem 39, with a cosheaf F being a connected components cosheaf iff it is "strongly terminal".

Lemma 40 *The analogue of Lemma 35 holds, with comma squares replaced by bicomma squares, and the pullback square replaced by a pseudopullback (i.e. it commutes only up to an isomorphism).*

Proposition 41 *Let X be a topos, and $t : 1 \to X$. Then t is terminal amongst all generalized points iff t is right adjoint to $!_X : X \to 1$.*

Proof. As is already known, the general 2-categorical definition of adjunction can be applied to toposes as follows. Let $f : X \to Y$ and $g : Y \to X$ be two maps. Then an adjunction $f \dashv g$ (f left adjoint to g) is a pair of 2-cells $\eta : \mathrm{Id}_X \Rightarrow f;g$ and $\varepsilon : g;f \Rightarrow \mathrm{Id}_Y$ satisfying the two triangle equations $\begin{bmatrix} \eta & f \\ f & \varepsilon \end{bmatrix} = f$ and $\begin{bmatrix} g & \eta \\ \varepsilon & g \end{bmatrix} = g$. (This is a straightforward generalization of one of the characterizations of adjunction given in [ML71].) Within the square brackets we are writing horizontal composition of 2-cells horizontally, from left to write in diagrammatic order, and vertical composition vertically, from top to bottom. We also abbreviate Id_f (the identity 2-cell on the 1-cell f) to f, etc.

In our current situation we are considering an adjunction $!_X \dashv t$ and the triangle equations become

$$\begin{bmatrix} \eta & !_X \\ !_X & \varepsilon \end{bmatrix} = !_X$$

$$\begin{bmatrix} t & \eta \\ \varepsilon & t \end{bmatrix} = t$$

The topos 1 is pseudo-terminal: in other words, for any topos W there is a map $!_W : W \to 1$, and although it is not necessarily unique, any two maps $W \to 1$ have a unique isomorphism between them. Hence the existence of ε is trivial, and the first equation is automatic. The 2-cell η itself amounts to a homomorphism from the generic point of X to t, and by horizontally composing it with a generalized point $x : W \to X$ we get $\eta_x = \begin{bmatrix} x\eta \\ \cong t \end{bmatrix} : x \Rightarrow x;!_X;t \cong !_W;t$. The second equation then says that, modulo the isomorphism εt between maps to 1, η_t is the identity on t.

$\Rightarrow$: Considering the generic point $\mathrm{Id}_X : X \to X$, we get a unique homomorphism $\eta : \mathrm{Id}_X \Rightarrow !_X;t$. The second equation above now follows from the uniqueness part of terminality of t.

$\Leftarrow$: If $x : W \to X$ is a generalized point, then we already know we have some $\eta_x : x \Rightarrow !_W;t$. Suppose we also have $\eta'_x : x \Rightarrow !_W;t$. Then, with some application of the interchange law, we obtain

$$\eta'_x = \eta'_x X = \begin{bmatrix} \eta'_x & & X \\ !_W & t & \eta \\ & \varepsilon & t \end{bmatrix} = \begin{bmatrix} x & & \eta \\ \eta'_x & !_X & t \\ !_W & \varepsilon & \end{bmatrix} = \eta_x$$

where the last equation uses the uniqueness part of terminality of t to show that $\begin{bmatrix} \eta'_x & !_X & t \\ !_W & \varepsilon & \end{bmatrix} = [\cong t]$. ∎

Let us call such a point t *strongly terminal*. We shall prove that a cosheaf is a connected components cosheaf iff it is strongly terminal in the symmetric topos.

At this point, let us note that Proposition 34 is the analogue for cosheaves of Lemma 37.

Lemma 42 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and let $F, G : W \to MX$ be two generalized cosheaves. Then there is a bijection between homomorphisms $F \Rightarrow G$ and maps $CS_W(X; F) \to CS_W(X; G)$ over $X \times W$.*

Proof. The case $W = 1$ is Corollary 33. The general case follows from a geometricity argument similar to that of Lemma 38. The geometricity of M has already been remarked on. ∎

Theorem 43 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and π_0 a cosheaf. Then the following are equivalent.*

1. *X and π_0 have relations $\leq^\gamma$ to make X locally connected.*
2. *$p : CS(X; \pi_0) \to X$ is a homeomorphism.*
3. *π_0 is strongly terminal in MX.*

Proof. The statement of the Theorem is in fact largely a rephrasing of [BF06, Proposition 6.2.5]. The statement there, after some notation changes, is as follows. For any B, the identity Id_B is a discrete M-fibration (i.e. a complete spread for some cosheaf) iff $B \to 1$ admits an M-adjoint (i.e. is an essential geometric morphism – see their Proposition 4.3.16; but in this case, where the codomain is 1, that is equivalent to B being locally connected). In this case the corresponding cosheaf $1 \to M(B)$ is $\delta_1; r_b$. To see how this fits our phrasing, first suppose (2). Then we might as well replace p by Id_X. Then the first part of their statement is our (1)⇔(2) (except for the part about the relations $\leq^\gamma$). The second part of their statement then tells us that $\pi_0 = \delta_1; r_b$. But their $r_b : M1 \to MX$ is by definition the right adjoint to $M! : MX \to M1$. Also, $\delta_1 : 1 \to M1$ is a singleton set (a cosheaf over 1 is just a set) and δ_1 is right adjoint to $! : M1 \to 1$ because a singleton set is a terminal set. It follows that π_0 is right adjoint to $! : MX \to 1$, i.e. π_0 is strongly terminal.

For the converse, if π_0 is a strongly terminal cosheaf, then the inverse image functor $\pi_0^* : \mathcal{S}MX \to \mathbf{Set}$ is left adjoint to $!^* : \mathbf{Set} \to \mathcal{S}MX$. But [BF06, Remark 4.3.3] $\delta^* : \mathcal{S}MX \to \mathcal{S}X$ also has a left adjoint, so it follows that $!^* : \mathbf{Set} \to \mathcal{S}X$ has a left adjoint and we recover the topos-theoretic definition of local connectedness.

We now give a direct proof.

(1)⇔(2): This is Proposition 34.

For (2)⇔(3) we can use purely categorical methods analogous to those of Theorem 39, and using Lemma 42. Again, the difficult direction is (2)⇒(3). We calculate bicomma squares as –

$$\begin{array}{llcl}
\langle x, \beta : \delta(x) \Rightarrow \pi_0, G\rangle & CS_{MX}(X; (!; \pi_0)) & \xrightarrow{q'} & MX \\
 & \downarrow p' & & \downarrow ! \\
\langle x, \beta : \delta(x) \Rightarrow \pi_0\rangle & CS(X; \pi_0) & \xrightarrow{!} & 1 \\
 & \downarrow q^*\pi_0 & & \downarrow \pi_0 \\
\langle x, F, \alpha : \delta(x) \Rightarrow F\rangle & CS_{MX}(X; \mathrm{Id}) & \xrightarrow{q} & MX \\
 & \downarrow r & \Rightarrow & \downarrow \mathrm{Id} \\
 & X & \xrightarrow[\delta]{} & MX
\end{array}$$

The diagram itself is analogous to that in Theorem 39, but down the left hand side we have described the points of the corresponding complete spreads.

If $p = (q^*\pi_0); r$ has an inverse, then, just as in Theorem 39, we find a map $\theta = \langle r; p^{-1}, q\rangle : CS_{MX}(X; \mathrm{Id}) \to CS_{MX}(X; (!; \pi_0))$ over $X \times MX$. It maps $\langle x, F, \alpha\rangle$ to $\langle x, \beta, F\rangle$ where $\langle x, \beta\rangle = p^{-1}(x)$. By Lemma 42 this gives us the 2-cell $\eta : \mathrm{Id}_{MX} \Rightarrow !; \pi_0$.

It remains to show that $\eta_{\pi_0} : \pi_0 \Rightarrow \pi_0; !_{MX}; \pi_0 = \pi_0$ is the identity. (Note that since 1 is localic, the isomorphism $\pi_0; !_{MX} \cong \mathrm{Id}_1$ must be equality.) To calculate it we pull back θ along $\pi_0 : 1 \to MX$. Consider, on top of the column of squares in the diagram, a further pullback square got by pulling back along $\pi_0 : 1 \to MX$. Since $\pi_0; !_{MX} = \mathrm{Id}_1$, the vertex of the pullback square must be homeomorphic to $CS(X; \pi_0)$, so we can take the square to be of the form

$$\begin{array}{lcl}
CS(X; \pi_0) & \xrightarrow{!} & 1 \\
\downarrow p'' & & \downarrow \pi_0 \\
CS_{MX}(X; (!; \pi_0)) & \xrightarrow{q'} & MX
\end{array}$$

with $p''; p' = \mathrm{Id}_{CS(X;\pi_0)}$. Since $(q^*\pi_0); r$ is a homeomorphism, there is a unique map $CS(X; \pi_0) \to CS(X; \pi_0)$ over $X \times 1$, and so a unique homorphism $\pi_0 \to \pi_0$ which must be the identity. ■

8 Conclusions

As far as I am aware, connectedness and local connectedness have not been formalized in full generality for formal topologies (not necessarily overt). On the other hand, in topos theory they have been extensively studied in great generality, being applied not only to locales (as the ordinary topos-theoretic notion of topological space) but also to toposes (as generalized spaces) and to geometric morphisms (as spaces relative to other spaces). The topos theory also links these notions to that of cosheaf, and studies the space of cosheaves (the symmetric topos) and cosheaves as maps (the complete spreads) analogous to local homeomorphisms.

Much of the content of the present paper has been just to transfer a small part of this topos theory to formal topology. Where it deals with inductively generated formal topologies, it can be understood as translating the topos-theoretic use of sites. Nonetheless, we have shown that there is much predicative content in a body of topos theory that is generally conducted impredicatively.

Our main original contributions are as follows.

First, we have imported the technology of cosheaves into formal topology. In the inductively generated case the definition is predicative (no quantification over subsets) and we have also imported the associated technology of complete spreads.

Second, we have given general definitions of connectedness and local connectedness for formal topology. That of connectedness extends known definitions for the overt case, but is not predicative. For local connectedness we have used cosheaves to give a definition that (i) works with arbitrary bases, rather than requiring a special base of connected opens, and (ii) in the inductively generated case is predicative.

Third, we have presented a new approach to a particular result from topos theory, that X is locally connected iff MX is totally connected (i.e. has a strongly terminal point). Our new proof is aimed at being predicative, and suggests a notion of "formal topos", such as MX, as "generalized formal topology". Thus we have sketched a glimpse of "formal topos theory".

One important issue we have left untouched is that of when a *map* $f : X \to Y$ is locally connected as locale over Y. Geometricity will be a key technique here.

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