Positivity relations on a locale

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Abstract

This paper analyses the notion of a *positivity relation* of Formal Topology from the point of view of the theory of Locales. It is shown that a positivity relation on a locale corresponds to a suitable class of points of its lower powerlocale. In particular, closed subtopologies associated to the positivity relation correspond to overt (that is, with open domain) weakly closed sublocales. Finally, some connection is revealed between positivity relations and localic suplattices (these are algebras for the powerlocale monad).

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Introduction

Much of the theory of locales can be developed in a fully predicative way provided that "bases" are assumed as given data. Of course, this makes no difference within an impredicative setting where any locale has a base. Also predicatively, however, requiring bases does not appear as a real restriction, for there seems to be no other way to define a locale but presenting it by generators (hence at least a subbase) and relations (in a suitable sense). In Formal Topology (that is, predicative pointfree Topology) a presentation of a locale usually takes the form of a "cover relation" on a set. In [7] the name *formal topology* was given to a cover relation with a unary positivity predicate: this corresponds to the case of overt (or open) locales. In [9] a new definition, called *positive topology*, is proposed in which a binary relation replaces the positivity predicate. This positivity relation is used to define *formal closed subsets*, which give a suitable notion of *closed* subtopologies.

The main aim of this paper is to characterize positivity relations in a base-independent way (at the cost of introducing some impredicativity). In other words, we find the unknown value x in the proportion: formal topology is to overt locale as positive topology is to x.

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We show that each formal closed subset is "splitting" (it has inhabited intersections with all covers of its elements) and that a positivity relation corresponds to a sub-suplattice of the suplattice of all splitting subsets. This leads to a number of characterizations of what a positivity relation on a locale is. In particular, it follows that each formal closed subset is a point of the lower powerlocale and thus corresponds to an overt weakly closed sublocale.

Further, we show some connections between positivity relations and *localic suplattices*, as introduce in [6], which are algebras for the powerlocale monad. Classes of points of localic suplattices give rise to positivity relations (and vice versa with classical logic).

A positivity relation on a locale $\mathcal{L}$ can also be read as a condition for selecting a class of points of $\mathcal{L}$. This idea becomes particularly clear when the positivity arises from localic sub-suplattice of the lower powerlocale $\mathcal{P}_L\mathcal{L}$.

To make the paper as general as possible, we begin from positivity relations on suplattices (with reversed morphisms). This is essentially the category of *basic topologies* [9].

1. Suplattices

We start by summarizing some of the impredicative facts about suplattices (complete join semilattices) and suplattice homomorphisms (join-preserving maps). Most of these are well known.

If L and M are suplattices, then so is $\mathbf{SupLat}(L, M)$, the set of suplattice homomorphisms $L \to M$, with joins calculated argumentwise: $\varphi \leq \psi$ when $\varphi(x) \leq \psi(y)$ for all $x \in L$.

For $1 = \{*\}$, let $\Omega \stackrel{def}{=} \mathsf{Pow}(1)$, the powerset of 1. In topos theory, this is the subobject classifier. Ω is the free suplattice over $\{*\}$, with injection of generators $* \mapsto \{*\}$ – in fact, for any set I, the powerset $\mathsf{Pow}(I)$ is the free suplattice over I. It follows that elements of a suplattice L are equivalent to suplattice homomorphisms $\Omega \to L$.

Because a suplattice L also has all meets – though we do not require homomorphisms to preserve them – it follows that L^{op} is also a suplattice. Moreover, a suplattice homomorphism $f\colon L \to M$ has a right adjoint g, which preserves all meets and hence is a suplattice homomorphism $M^{op} \to L^{op}$. This provides a self-duality $L \leftrightarrow L^{op}$ on the category of suplattices. It follows that elements of L^{op}, equivalent to suplattice homomorphisms $\Omega \to L^{op}$, are also equivalent to suplattice homomorphisms $L \to \Omega^{op}$.

However, we shall be particularly interested in suplattice homomorphisms $L \to \Omega$. *Classically,* with $\Omega^{op} \cong \Omega$, these are again equivalent to elements of L^{op}. More generally they are different.

We obtain two functors $\mathbf{SupLat}(-, \Omega), \Omega^{-}\colon \mathbf{SupLat}^{op} \to \mathbf{SupLat}$ acting on morphisms by composition, and with a natural transformation from the first to the second.

Since arbitrary maps $\varphi\colon L \to \Omega$ are equivalent to subsets $\varphi^{-1}(1)$ of L, we should identify which subsets correspond to the suplattice homomorphisms.

Definition 1.1. *Let* L *be a suplattice. A subset* $Z \subseteq L$ *is* splitting *if*

$$Z \ni x \leq \bigvee Y \Longrightarrow Z \between Y \text{ for every } x \in L \text{ and every } Y \subseteq L,$$

where, following Sambin, by $X \between Y$ *we mean that* $X \cap Y$ *is inhabited. We write* $\mathsf{Split}(L)$ *for the collection of all splitting subsets of* L *.*

Splitting subsets can be characterized also by the following two conditions

1. if $x \in Z$ and $x \leq y$, then $y \in Z$ (upward closed)
2. if $(\bigvee Y) \in Z$, then $y \in Z$ for some $y \in Y$ (completely prime)

and so they can as well be called *completely-prime upsets*. More succinctly, they can be characterized by a single condition that $(\bigvee Y) \in Z$ if and only if $Y \between Z$ – the "if" direction gives the upward closedness.

It is easy to check that any union of splitting subsets is splitting and so $\mathsf{Split}(L)$ is a suplattice – in fact, a sub-suplattice of $\mathsf{Pow}(L)$. We get two functors $\mathsf{Split}, \mathsf{Pow}\colon \mathbf{SupLat}^{op} \to \mathbf{SupLat}$, acting on morphisms by inverse image and with a natural transformation from the first to the second.

Proposition 1.2. *There exists a natural isomorphism between* $\mathbf{SupLat}(-, \Omega)$ *and* Split*.*

Proof For every set L, there is a bijection between functions $\varphi\colon L \to \Omega$ and subsets of L, with φ corresponding to $Z_\varphi = \varphi^{-1}(1)$, and then $\varphi(x) \in \Omega$ is the truth value of "$x \in Z_\varphi$". Now if L is a suplattice, then φ is a homomorphism iff for all $Y \subseteq L$ we have $(\bigvee Y) \in Z_\varphi$ iff $(\exists y \in Y) y \in Z_\varphi$, in other words iff $Y \between Z_\varphi$, from which we gain the isomorphism. Naturality follows from that between Pow and Ω^-. q.e.d.

1.1. Suplattice presentations

A presentation of a suplattice by generators and relations comprises a set S of generators, and a set $\lhd_0$ of cover relations, each of the form $a \lhd_0 U$ where $a \in S$ and $U \subseteq S$ and denoting the relation $a \leq \bigvee U$.

The theory of suplattice presentations $(S, \lhd_0)$ is very similar to the theory of inductively generated formal topologies [5]. In fact, the latter is essentially about how to present frames as suplattices, but with added complexity to take care of the finite meets. From $\lhd_0$ we can inductively generate the full cover $\lhd$ by the rules

$$\frac{a \in U}{a \lhd U} \quad \text{and} \quad \frac{a \lhd_0 V \lhd U}{a \lhd U}$$

(where $V \lhd U$ means $v \lhd U$ for every $v \in V$). By induction on proofs (of $a \lhd U$) it is easy to see that $\lhd$ is transitive: if $a \lhd U \lhd V$ then $a \lhd V$. Now, impredicatively, we find that the partial order $\mathsf{Pow}(S)/\lhd$ is a suplattice, and indeed is the suplattice freely generated by S subject to the relations in $\lhd_0$. To see this, let L be a suplattice and $f\colon S \to L$ a function respecting the relations in $\lhd_0$. By induction on proofs we find that if $a \lhd U$ then $f(a) \leq \bigvee_{u \in U} f(u)$, and from this we deduce that the unique suplattice homomorphism $\mathsf{Pow}(S) \to L$ given by freeness of $\mathsf{Pow}(S)$ factors uniquely via $\mathsf{Pow}(S)/\lhd$.

Note that each element of $\mathsf{Pow}(S)/\lhd$ has a canonical representative, namely the greatest. In other words, if $U \subseteq S$ represents an element of $\mathsf{Pow}(S)/\lhd$, then $\lhd U = \{a \in S \mid a \lhd U\}$ is the greatest subset in the same equivalence class as U.

We now look at suplattice homomorphisms from $(S_2, \lhd_0)$ to $(S_1, \lhd_0)$. (Note the direction; note also that we use the same symbol for both cover relations. In practice it will always be clear which one is being referred to.) Let L_2 and L_1 be the suplattices presented. By the universal property of presentations, a homomorphism $f \colon L_2 \to L_1$ is equivalent to a function $f_0 \colon S_2 \to L_1 \cong \mathsf{Pow}(S_1)/\lhd$ such that if $b \lhd_0 V$ in S_2 then $f_0(b) \leq \bigvee_{v \in V} f_0(v)$. We define a relation $R \subseteq S_1 \times S_2$ by aRb if a is in the canonical (greatest) representative of $f_0(b)$. The functions f_0 are equivalent to relations R satisfying that

$$\text{if } a \lhd_0 U \subseteq R^- b \text{ then } aRb. \tag{1}$$

f_0 then respects covers if R also has the property that

$$\text{if } aRb \lhd_0 V \text{ then } a \lhd R^- V. \tag{2}$$

In summary, we define a category **SLP** (suplattice presentations) as follows. An object is a pair $(S, \lhd_0)$ where $\lhd_0 \subseteq S \times \mathsf{Pow}(S)$. A morphism from $(S_1, \lhd_0)$ to $(S_2, \lhd_0)$ is a relation $R \subseteq S_1 \times S_2$ satisfying the two conditions (1),(2) above. The identity morphism on $(S, \lhd_0)$ is the relation $a \,\mathrm{Id}\, b$ if $a \lhd \{b\}$. Composition is given by

$$a(R_2 \circ R_1)c \text{ if } a \lhd R_1^- R_2^- \{c\}.$$

Then the transformation of $(S, \lhd_0)$ into the presented suplattice gives a contravariant functor $\mathbf{SLP} \to \mathbf{SupLat}$ that is full, faithful and essentially surjective, thus making **SLP** dual equivalent to **SupLat**. (The contravariance is built in solely to match later applications with formal topologies.)

Applying this to the case where S_2 is S and S_1 is $\{*\}$ with the empty cover relation, we see that suplattice homomorphisms $L \to \Omega$, when described as relations from 1 to S, are equivalent to splitting subsets of S. Thus splitting subsets of L can equivalently be described as subsets Z of S that split $\lhd_0$, in the sense that if $a \lhd_0 U$ and $a \in Z$ then $U \between Z$. We shall write the set of such as $\mathsf{Split}(S, \lhd_0)$: thus

$$\mathsf{Split}(S, \lhd_0) \cong \mathbf{SupLat}(L, \Omega).$$

Because of the reversal of morphisms for presentations, we find that Split becomes a *co*variant functor from **SLP** to **SupLat**. If $R \colon (S_1, \lhd_0) \to (S_2, \lhd_0)$ in **SLP**, then

$$\mathsf{Split}(R) \colon \mathsf{Split}(S_1, \lhd_0) \to \mathsf{Split}(S_2, \lhd_0)$$

is defined by direct image of R,

$$\mathsf{Split}(R)(Z) = R^+(Z) = \{b \in S_2 \mid (\exists a \in Z) aRb\}.$$

The same formula allows us to define a covariant powerset functor $\mathsf{Pow}\colon \mathbf{SLP} \to \mathbf{SupLat}$, $(S, \lhd_0) \mapsto \mathsf{Pow}S$, with a natural transformation from Split to Pow.

The argument earlier shows that Split is naturally isomorphic to $\mathbf{SupLat}(-,\Omega)$ composed with the dual equivalence $\mathbf{SLP} \cong \mathbf{SupLat}^{op}$.

2. Positivity relations

We recall the basic theory of positivity relations [8], with the slight change that we assume a cover base $\lhd_0$ rather than the full cover $\lhd$. The details are little changed, and the essential point to note is that it is enough to check that basic covers are split.

If L is a suplattice, we say that $S \subseteq L$ is a *base* if every $x \in L$ can be written as a join of elements in S.

Definition 2.1 (Sambin [9]). *Let $(L, \leq, \bigvee)$ be a suplattice presented by $(S, \lhd_0)$. A* positivity relation *is a predicate $\ltimes \subseteq S \times \mathsf{Pow}(S)$ such that:*

1. *if $a \ltimes U$, then $a \in U$*
2. *if $a \ltimes U$ and $U \subseteq V$, then $a \ltimes V$*
3. *if $a \ltimes U$, then $a \ltimes \{x \in S \mid x \ltimes U\}$*
4. *if $a \lhd_0 U$ and $a \ltimes V$, then $u \ltimes V$ for some $u \in U$ (compatibility)*

for all $a \in S$ and $U, V \subseteq S$. The triple $(S, \lhd_0, \ltimes)$ is called a basic topology. *When L is a frame, then $(S, \lhd_0, \ltimes)$ is called a* positive topology.

For every $U \subseteq S$, we put $\ltimes U \stackrel{def}{=} \{x \in S \mid x \ltimes U\}$. Rules (1)-(3) say precisely that this makes $\ltimes$ an *interior operator* on $\mathsf{Pow}(S)$, that is, a monotone, idempotent operator such that $\ltimes U \subseteq U$ for any U. The compatibility rule (4) says that the subsets of S of the form $\ltimes U$ for some $U \subseteq S$ are all splitting subsets for $(S, \lhd_0)$. We call the sets $\ltimes U$ *formal closed sets*; equivalently, they are the fixed-points of $\ltimes$ as operator on $\mathsf{Pow}(S)$.

Let us write $FC(S, \lhd_0, \ltimes)$ for the collection of all formal closed sets of the basic topology $(S, \lhd_0, \ltimes)$.

Lemma 2.2. *For every basic topology $(S, \lhd_0, \ltimes)$, the collection $FC(S, \lhd_0, \ltimes)$ is a sub-suplattice of $\mathsf{Split}(S, \lhd_0)$.*

Proof Let $\{V_i\}_{i\in I}$ be any family of formal closed sets. For any $x \in S$, assume $x \in \bigcup_{i\in I} V_i$. Then $x \in V_i$ for some $i \in I$ and so $x \ltimes V_i$ because V_i is formal closed. Therefore $x \ltimes \bigcup_{i\in I} V_i$ and hence $\bigcup_{i\in I} V_i$ is formal closed as well. q.e.d.

Note that a positivity relation can be reconstructed from the suplattice of its formal closed sets. In fact, it is easy to check that *$a \ltimes U$ if and only if $a \in V \subseteq U$ for some formal closed set V.*

Theorem 2.3. *Let L be a suplattice, presented by $(S, \lhd_0)$. Then there is a bijection between positivity relations on L with respect to S and sub-suplattices of $\mathsf{Split}(S, \lhd_0)$.*

Proof Every $\ltimes$ determines, and is determined by, the suplattice of its formal closed subsets $FC(S, \lhd_0, \ltimes)$ as sub-suplattice of $\mathsf{Split}(S, \lhd_0)$. It therefore remains only to show that each sub-suplattice $\mathcal{F}$ of $\mathsf{Split}(S, \lhd_0)$ comes from a unique positivity in this way.

We consider the relation $\ltimes_{\mathcal{F}}$ defined by

$$a \ltimes_{\mathcal{F}} U \quad \stackrel{def}{\Longleftrightarrow} \quad (\exists Z \in \mathcal{F})\big(a \in Z \subseteq U\big)$$

for $a \in S$ and $U \subseteq S$. We claim that $\ltimes_{\mathcal{F}}$ is a positivity relation – actually, this happens for $\mathcal{F}$ *any* subfamily of $\mathsf{Split}(S, \lhd_0)$. Trivially, $a \ltimes_{\mathcal{F}} U$ yields $a \in U$. Also $a \ltimes_{\mathcal{F}} V$ follows from $a \ltimes_{\mathcal{F}} U$ and $U \subseteq V$. The proof of $a \ltimes_{\mathcal{F}} U \Rightarrow a \ltimes_{\mathcal{F}} \{x \in S \mid x \ltimes_{\mathcal{F}} U\}$ is as follows. Take a splitting subset Z such that $a \in Z \subseteq U$. It is sufficient to show that Z is contained in $\{x \in S \mid x \ltimes_{\mathcal{F}} U\}$, that is, that $x \in Z \Rightarrow x \ltimes_{\mathcal{F}} U$ holds; this follows from $Z \subseteq U$. To prove compatibility, assume $a \lhd_0 U$ and $a \ltimes_{\mathcal{F}} V$, that is, $a \in Z \subseteq V$ for some $Z \in \mathcal{F}$; then $U \between Z$ because Z is splitting; hence there exists $u \in U$ such that $u \in Z \subseteq V$, that is, $u \ltimes_{\mathcal{F}} V$.

We now check that the correspondence just described is in fact a bijection. We have already noted that any positivity $\ltimes$ is equal to $\ltimes_{\mathcal{F}}$ where $\mathcal{F} = FC(S, \lhd_0, \ltimes)$. It remains to prove that, for any sub-suplattice $\mathcal{F}$, we have $\mathcal{F} = FC(S, \lhd_0, \ltimes_{\mathcal{F}})$. First of all, the definition of $\ltimes_{\mathcal{F}}$ yields that Z is formal closed for every $Z \in \mathcal{F}$. Vice versa, let V be a formal closed set with respect to $\ltimes_{\mathcal{F}}$. Thus

$$V = \ltimes_{\mathcal{F}} V = \bigcup\{Z \mid Z \in \mathcal{F} \ \&\ Z \subseteq V\}.$$

Since $\mathcal{F}$ is closed under joins, we have $V \in \mathcal{F}$. q.e.d.

This result has a couple of interesting consequences. First of all, from now on we can imagine a basic topology $(S, \lhd_0, \ltimes)$ as just a presentation of the pair $(L, \mathcal{F})$ where $\mathcal{F}$ is the sub-suplattice of $\mathsf{Split}(L)$ that corresponds to $FC(S, \lhd_0, \ltimes)$ as sub-suplattice of $\mathsf{Split}(S, \lhd_0)$. Alternatively, using the isomorphism $\mathsf{Split}(L) \cong \mathbf{SupLat}(L, \Omega)$, we can view it as a pair (L, Φ) where Φ is a sup-suplattice of $\mathbf{SupLat}(L, \Omega)$.

A second corollary is that *the collection of all possible positivity relations on L (with respect to S) is a suplattice.* In particular, it must have both a bottom $\ltimes_{\perp}$ and a top $\ltimes_{max}$. The relation $\ltimes_{\perp}$, which corresponds to the trivial sub-suplattice of $\mathsf{Split}(L)$, is such that $a \ltimes_{\perp} U$ never holds. It has one and only one formal closed set, namely $\emptyset$.

More interestingly, the *greatest* positivity on L (with respect to S) is characterized by

$$a \ltimes_{max} U \quad \Longleftrightarrow \quad (\exists Z \in \mathsf{Split}(L))\,(a \in Z \subseteq U)$$

for all $a \in S$ and $U \subseteq S$. Also one has $FC(S, \lhd_0, \ltimes_{max}) = \mathsf{Split}(S, \lhd_0)$. Following [4], we give the following

Definition 2.4. *Basic topologies of the form $(S, \lhd_0, \ltimes_{max})$ are called* saturated*.*

2.1. Positivity relations on a topological space

We now look at situations where the suplattice under consideration is a frame, initially the frame τ of opens for a topological space (X, τ). We shall assume it also has a given base $S \subseteq \tau$.

As discussed earlier, the suplattice homomorphisms $\tau \to \Omega^{op}$ are equivalent to the elements of τ^{op}, and hence – taking order into account – to the closed subsets of X. This is taking "closed" in the sense of complement of open. Even constructively, that notion of closed is still important, since an open sublocale for a (got by imposing the cover $1 \lhd a$) has a closed complement *as sublocale,* got by imposing the cover $a \lhd \emptyset$.

However, that notion of closed diverges from another, classically equivalent: closed as containing all its closure (or adherent) points. We shall see now that that other notion is related to the splitting subsets of τ, the suplattice homomorphisms $\tau \to \Omega$. Thus the constructive difference between the two notions of closed is related to the constructive failure of the isomorphism between Ω and Ω^{op}. From now on, "closed" will be taken in this second sense.

Recall that a point is in the *closure* of a set if all its basic neighbourhoods, hence all its open neighbourhoods, have inhabited intersection with the given set, and a set is *closed* if it equals its closure. To analyse this notion, let us define, following [9], two maps

$$D \mapsto \{a \in S \mid a \between D\} \stackrel{def}{=} \Diamond D \quad \text{and} \quad U \mapsto \{x \in X \mid \Diamond\{x\} \subseteq U\} \stackrel{def}{=} \mathsf{rest}U$$

for $D \subseteq X$ and $U \subseteq S$.

First, note that

$$D \subseteq \mathsf{rest}U \Leftrightarrow \Diamond D \subseteq U.$$

This is because both conditions are equivalent to

$$(\forall x \in X, a \in S)\,(x \in a \cap D \to a \in U)\,.$$

Categorically, this means that $\Diamond$ is left adjoint to rest, and this restricts to an equivalence between the fixed points of $\Diamond\mathsf{rest}$ and those of $\mathsf{rest}\Diamond$.

Second, note that $\mathsf{rest}\Diamond D$ is the closure of D, so the fixed points of $\mathsf{rest}\Diamond$ are the closed sets, and these are equivalent to the fixed points of $\Diamond\mathsf{rest}$.

Third, note the obvious fact that every set $\Diamond D$ is splitting.

Classically, every splitting set Z is equal to $\Diamond\mathsf{rest}Z$ – in other words, we have $Z \subseteq \Diamond\mathsf{rest}Z$. For suppose $a \notin \Diamond\mathsf{rest}Z$. Then for every $x \in a$ we have $x \notin \mathsf{rest}Z$, and so there is some $b_x \in S$ with $x \in b_x$ but $b_x \notin Z$. Now $a \subseteq \bigcup_{x \in a} b_x$, and it follows from the fact that Z is splitting that $a \notin Z$. Thus, classically, the splitting sets are the fixed points of $\Diamond\mathsf{rest}$ and so are equivalent to closed sets.

Constructively, the fixed points of $\Diamond\mathsf{rest}$ form a sub-suplattice of τ and so we can describe them using a positivity relation. It is obtained by putting $a \ltimes_X V$ if $a \in \Diamond\mathsf{rest}V$, i.e. *whenever there exists a point $x \in a$ such that all basic neighbourhoods of x belong to V* (see [9]). From our preliminary remarks it is now clear that $\ltimes_X$ is a positivity on τ, and that $FC(S, \lhd, \ltimes_X)$ is the lattice of fixed points of $\Diamond\mathsf{rest}$ and hence *is isomorphic to the lattice of closed sets.* (Of course, the join of a family of closed sets is given by the *closure* of their union.)

Note that $FC(S, \lhd, \ltimes_X)$, being isomorphic to the lattice of closed sets of τ, depends neither on the base S, nor on the positivity $\ltimes_X$, but on the topology only.

For every point x, the subset $\Diamond\{x\} \subseteq S$ is formed by all basic neighbourhoods of x. Note that one can rewrite $a \ltimes_X U$ as $(\exists x \in X)(a \in \Diamond\{x\} \subseteq U)$. An argument similar to that above proves the following, more general result.

Proposition 2.5. *Let (X, τ) be a topological space with a base S. For every subset $Y \subseteq X$, the relation*

$$a \ltimes_Y U \quad \stackrel{def}{\Longleftrightarrow} \quad (\exists y \in Y)(a \in \Diamond\{y\} \subseteq U)$$

is a positivity relation on ΩX (with respect to S). Moreover

$$FC(S, \lhd, \ltimes_Y) \;=\; \{\Diamond(D \cap Y) \mid D \subseteq X \text{ closed}\}$$

and hence $FC(S, \lhd, \ltimes_Y)$ is isomorphic to the lattice of closed sets in the subspace topology on Y.

Proof We see that $a \ltimes_Y U$ iff $Y \cap a \cap \mathsf{rest}U$ is inhabited, and so $\ltimes_Y U = \Diamond(Y \cap \mathsf{rest}U)$. From this, the proof that $\ltimes_Y$ is a positivity relation is straightforward.

It follows that the formal closed sets, of the form $\ltimes_Y U$, are those of the form $\Diamond(Y \cap D)$ where D is closed. q.e.d.

2.2. Maps that respect positivity

Definition 2.6 (Sambin). *Let $(S_i, \lhd_0, \ltimes_i)$, $i = 1, 2$, be two basic topologies, and let the relation $R \subseteq S_1 \times S_2$ be a morphism from $(S_1, \lhd_0)$ to $(S_2, \lhd_0)$. R* respects positivity *when the following holds*

if aRb and $a \ltimes_1 U$, then $b \ltimes_2 R^+U$

for all $a \in S_1$, $b \in S_2$ and $U \subseteq S_1$.

This condition says precisely that $b \in R^+ \ltimes_1 U$ implies $b \in \ltimes_2 R^+ U$ for all $b \in S_2$ and $U \subseteq S_1$. In other words, R respects positivity if and only if $R^+ \ltimes_1 U \subseteq \ltimes_2 R^+ U$ for all $U \subseteq S_1$. We shall refer to this last condition as

$$R^+ \ltimes_1 \;\subseteq\; \ltimes_2 R^+. \tag{3}$$

Proposition 2.7 (Sambin). *In the notation of the previous definition, R respects positivity if and only if $\mathsf{Split}(R)$ maps $FC(S_1, \lhd_0, \ltimes_1)$ to $FC(S_2, \lhd_0, \ltimes_2)$.*

Proof Assume that R respects positivity. If $U \subseteq S_1$ is a formal closed set, then $R^+U = R^+ \ltimes_1 U \subseteq \ltimes_2 R^+ U$. Thus $\mathsf{Split}(R)(U) = R^+U$ is formal closed, as claimed.

Conversely, if $\mathsf{Split}(R)$ maps formal closed sets to formal closed sets, then $R^+ \ltimes_1 U = \ltimes_2 R^+ \ltimes_1 U$ holds for every $U \subseteq S_1$, because $\ltimes_1 U$ is formal closed. In particular, $R^+ \ltimes_1 U \subseteq \ltimes_2 R^+ \ltimes_1 U$ and hence $R^+ \ltimes_1 U \subseteq \ltimes_2 R^+ U$ because $\ltimes_1 U \subseteq U$ and both R^+ and $\ltimes_2$ are monotone. q.e.d.

Recall (from section 1.1) that $\mathsf{Split}(R)$ is a suplattice homomorphism from $\mathsf{Split}(S_1, \lhd_0)$ to $\mathsf{Split}(S_2, \lhd_0)$. Consequently, if R respects positivity, then $\mathsf{Split}(R)$ automatically restricts to a suplattice morphism between $FC(S_1, \lhd_0, \ltimes_1)$ and $FC(S_2, \lhd_0, \ltimes_2)$. In other words, "$R$ respects positivity" expresses the fact that the suplattice morphism $\mathsf{Split}(R)$ can be restricted to the sub-suplattices of formal closed sets.

$$\begin{array}{ccc} \mathsf{Split}(S_1, \lhd_0) & \xrightarrow{\mathsf{Split}(\mathrm{R})} & \mathsf{Split}(S_2, \lhd_0) \\ \uparrow & & \uparrow \\ FC(S_1, \lhd_0, \ltimes_1) & \dashrightarrow & FC(S_2, \lhd_0, \ltimes_2) \end{array}$$

In the light of Proposition 2.3, we thus see the following interpretation of respecting positivity.

Proposition 2.8. *Let $(S_i, \lhd_0, \ltimes_i)$ $(i = 1, 2)$ be two basic topologies presenting pairs $(L_i, \mathcal{F}_i)$ and let $R \subseteq S_1 \times S_2$ be a* **SLP** *morphism from $(S_1, \lhd_0)$ to $(S_2, \lhd_0)$ corresponding to a suplattice homomorphism $f\colon L_1 \leftarrow L_2$. Then R respects positivity if and only if $f^{-1}\colon \mathsf{Split}(L_1) \to \mathsf{Split}(L_2)$ restricts to a map from $\mathcal{F}_1$ to $\mathcal{F}_2$.*

In this way, the following result from [9] comes to have a very straightforward proof.

Proposition 2.9. *Basic topologies and presentation morphisms which respect positivity form a category, called* **BTop***, with respect to the usual identities and composition of functions.*

It is equivalent to the category whose objects are pairs $(L, \mathcal{F})$, with L a suplattice and $\mathcal{F}$ a sub-suplattice of $\mathsf{Split}(L)$, and whose arrows $(L_1, \mathcal{F}_1) \to (L_2, \mathcal{F}_2)$ are join-preserving maps $f\colon L_1 \leftarrow L_2$ such that f^{-1} maps $\mathcal{F}_1$ to $\mathcal{F}_2$.

It follows that, impredicatively, we can dispense with bases, in the following sense.

Corollary 2.10. *Let L be a suplattice with two presentations $(S_i, \lhd_0)$. Let $\ltimes_i$ be a positivity relation on each $(S_i, \lhd_0)$, giving equal sub-suplattices of $\mathsf{Split}(L)$. Then the basic topologies $(S_i, \lhd_0, \ltimes_i)$ are isomorphic.*

Given two basic topologies (L_1, Φ_1) and (L_2, Φ_2), it is immediate from the definition of $\mathbf{SupLat}(-, \Omega)$ on morphisms that a map $f \in \mathbf{SupLat}(L_2, L_1)$ respects positivity if and only if whenever $\varphi \in \Phi_1$ then $\varphi \circ f \in \Phi_2$. The following synoptic table summarizes three different ways of conceiving of positivity relations and, consequently, of the category of basic topologies.

Objects

$(S, \lhd_0, \ltimes)$	$(L, \mathcal{F})$	(L, Φ)
$(S, \lhd_0)$ *cover*	L *suplattice*	
$\ltimes$ positivity relation	$\mathcal{F} \hookrightarrow \mathsf{Split}(L)$	$\Phi \hookrightarrow \mathbf{SupLat}(L, \Omega)$
Formal closed subset	$Z \in \mathcal{F}$	$\varphi \in \Phi$

Morphisms

R between covers s.t.	$f\colon L_1 \leftarrow L_2$ join-preserving s.t.	
$R^+ \ltimes_1 \subseteq \ltimes_2 R^+$	$f^{-1}(\mathcal{F}_1) \subseteq \mathcal{F}_2$	$\Phi_1 \circ f \subseteq \Phi_2$

And one gets from one notion to the others as follows:

$$\{a \in S \mid a \ltimes U\} \mapsto \{x \in L \mid a \ltimes U \text{ for some } a \in S \text{ with } a \leq x\} \in \mathsf{Split}(L),$$

$$\mathcal{F} \ni Z \mapsto \big(x \mapsto \{* \mid x \in \mathcal{F}\}\big) \in \mathbf{SupLat}(L, \Omega) \qquad \text{and}$$

$$\Phi \mapsto \big\{(a, U) \in S \times \mathsf{Pow}(S) \mid a \in \big(S \cap \varphi^{-1}(1)\big) \subseteq U \text{ for some } \varphi \in \Phi\big\}.$$

From a purely categorical point of view, a positivity on L is just a suplattice monomorphism to $\mathbf{SupLat}(L, \Omega)$. Accordingly a morphism from (L_1, m_1) to (L_2, m_2), with $m_1\colon \Phi_1 \hookrightarrow \mathbf{SupLat}(L_1, \Omega)$ and $m_2\colon \Phi_2 \hookrightarrow \mathbf{SupLat}(L_2, \Omega)$, is an $f\colon L_1 \leftarrow L_2$ such that there exists a (necessarily unique) morphism from Φ_1 to Φ_2 which makes the following diagram commute.

$$\begin{array}{ccc} \mathbf{SupLat}(L_1, \Omega) & \xrightarrow{\_\circ f} & \mathbf{SupLat}(L_2, \Omega) \\ {\scriptstyle m_1}\uparrow & & \uparrow{\scriptstyle m_2} \\ \Phi_1 & \dashrightarrow & \Phi_2 \end{array}$$

We end this section with a quick look to what happens to the category of basic topologies when classical logic is assumed (this is what the label CLASS stands for).

Proposition 2.11 (CLASS). *Let (L, Φ) be a basic topology. Then there exists a suplattice quotient $L \twoheadrightarrow M$ such that $\Phi \cong \mathbf{SupLat}(M, \Omega)$.*

Proof Classically $\Omega^{op} \cong \Omega$ and so $\mathbf{SupLat}(X, \Omega) \cong \mathbf{SupLat}(\Omega^{op}, X^{op}) \cong \mathbf{SupLat}(\Omega, X^{op}) \cong X^{op}$ for any suplattice X. Now put $M = \Phi^{op}$. Since Φ is a sub-suplattice of $\mathbf{SupLat}(L, \Omega) \cong L^{op}$, it follows that M is a quotient of L. Moreover, $\mathbf{SupLat}(M, \Omega) \cong M^{op} = \Phi$. q.e.d.

This result gives another possible description of **BTop** in the classical case. Objects are suplattice quotients $p \colon L \twoheadrightarrow M$ and the corresponding positivity can be recovered as $\mathbf{SupLat}(M, \Omega) \circ p$. A morphism from $p_1 \colon L_1 \twoheadrightarrow M_1$ to $p_2 \colon L_2 \twoheadrightarrow M_2$ is now a suplattice homomorphism $f \colon L_2 \to L_1$ such that $p_1 \circ f$ can be (uniquely) factorized through p_2.

2.3. On saturated basic topologies

Some results on saturated basic topologies (definition 2.4) have been obtained in [4]. In terms of the suplattices, a saturated basic topology is one of the form $(L, \mathbf{SupLat}(L, \Omega))$, and it is easy to see that:

> every suplattice homomorphism $f \colon L_1 \leftarrow L_2$ is automatically a morphism of basic topologies $(L_1, \Phi) \to (L_2, \mathbf{SupLat}(L_2, \Omega))$ for every positivity Φ on L_1.

So the full subcategory of saturated basic topologies can be identified with $\mathbf{SupLat}^{op}$, the opposite of the category of suplattices. It is easy to check that the assignments $(L, \Phi) \mapsto (L, \mathbf{SupLat}(L, \Omega))$ and $f \mapsto f$ define a functor which is left adjoint to the inclusion functor. In other words $\mathbf{SupLat}^{op}$ embeds in **BTop** as a reflective subcategory.

The category **BTop** has a terminal object $\mathbf{1}$ given by the saturated basic topology on Ω, namely $(\Omega, \mathbf{SupLat}(\Omega, \Omega)) \cong (\Omega, \Omega)$. So it makes sense to consider (*global*) *elements/points* of a basic topology.

Proposition 2.12. *Let (L, Φ) be a basic topology and let $f \colon L \to \Omega$ be a suplattice homomorphism. Then $\mathbf{SupLat}(\Omega, \Omega) \circ f \subseteq \Phi$ (that is, f is a morphism $(\Omega, \mathbf{SupLat}(\Omega, \Omega)) \to (L, \Phi)$ of basic topologies) precisely when $f \in \Phi$. In symbols, $\mathbf{BTop}(\mathbf{1}, (L, \Phi)) \cong \Phi$.*

Proof One direction is trivial. For the other, we have to check that $f \in \Phi$ implies $\alpha \circ f \in \Phi$ for all $\alpha \in \mathbf{SupLat}(\Omega, \Omega)$. For each $a \in \Omega$ we have

$$a = \{* \in 1 \mid * \in a\} = \bigvee \{b \in \Omega \mid b = a = 1\}.$$

Translating this to $\mathbf{SupLat}(\Omega, \Omega) \cong \Omega$ (where $1 = \mathrm{Id}$, and $\alpha = 1$ iff $\alpha(1) = 1$) we have $\alpha = \bigvee \{\beta \in \mathbf{SupLat}(\Omega, \Omega) \mid \beta = \mathrm{Id},\, \alpha(1) = 1\}$, and so

$$\alpha \circ f = \bigvee \{\beta \circ f \mid \beta = \mathrm{Id},\, \alpha(1) = 1\} = \bigvee \{\varphi \in \Phi \mid \varphi = f,\, \alpha(1) = 1\} \in \Phi$$

q.e.d.

3. Positivity and locales

Recall that a frame is a suplattice in which binary meet distributes over arbitrary joins. A morphism between frames is a suplattice homomorphism which, in addition, preserves finite meets. The opposite of the category of frames is **Loc**, the category of *locales*. If $\mathcal{L}$ is a locale, then we write $\Omega\mathcal{L}$ for the corresponding frame. Similarly, for $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ a morphism of locales, $\Omega f\colon \Omega\mathcal{L}_2 \to \Omega\mathcal{L}_1$ is the corresponding morphism of frames. Also recall that a *sublocale* is a(n isomorphism class of) regular monomorphisms in **Loc**, and that f is regular mono if and only if Ωf is surjective.

In view of the discussion in the previous section, a *positive topology* (definition 2.1) can be seen as a pair $(\mathcal{L}, \Phi)$ where $\mathcal{L}$ is a locale and $\Phi \hookrightarrow \mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$ is a positivity predicate on $\Omega\mathcal{L}$. The following is trivial.

Proposition 3.1. *Positive topologies and morphisms f between locales such that Ωf respects positivity form a category, called* **PTop**.

More explicitly, a morphism $f\colon (\mathcal{L}_1, \Phi_1) \to (\mathcal{L}_2, \Phi_2)$ between positive topologies is a morphism $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ of locales such that $\varphi \circ \Omega f \in \Phi_2$ for all $\varphi \in \Phi_1$.

3.1. Formal closed = overt, weakly closed

The category **Loc** has a terminal object $\mathbf{1}$. Its corresponding frame is $\Omega = \mathsf{Pow}(1)$. As usual, we write ! for the unique morphism from $\mathcal{L}$ to $\mathbf{1}$. So, for $p \in \Omega$, we have $\Omega!(p) = \bigvee\{y \in \Omega\mathcal{L} \mid y = 1 \;\&\; p = 1\}$.

A locale is *overt* (or *open*) when $\Omega!$ has a left adjoint, say $\exists$. When the elements of Ω are regarded as propositions, $\exists$ becomes a predicate over $\Omega\mathcal{L}$. In this case, it is usually written Pos and called a *positivity predicate*. Recall also (see [12] and [14]) that $\mathcal{L}' \hookrightarrow \mathcal{L}$ is a *weakly closed sublocale* of $\mathcal{L}$ if it is obtained by imposing on $\mathcal{L}$ some "axioms" of the form $x \leq \Omega!(p)$ with $x \in \Omega\mathcal{L}$ and $p \in \Omega$.

It is well-known [3] that *overt, weakly closed sublocales of $\mathcal{L}$ can be identified with the elements of* $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$. The correspondence is as follows. To every morphism $\varphi\colon \Omega\mathcal{L} \to \Omega$ one can associate the weakly closed sublocale $i\colon \mathcal{L}_\varphi \hookrightarrow \mathcal{L}$ characterized by $x \leq \Omega!\big(\varphi(x)\big)$ for all $x \in \Omega\mathcal{L}$. This is overt with $\mathsf{Pos}(\Omega i(x)) = \varphi(x)$. In other words, $\mathcal{L}_\varphi$ is the largest sublocale of $\mathcal{L}$ for which φ acts as a positivity predicate. Vice versa, every overt (weakly closed) sublocale $i\colon \mathcal{L}' \hookrightarrow \mathcal{L}$ gives a suplattice morphism $\varphi_{\mathcal{L}'}\colon \Omega\mathcal{L} \to \Omega$ defined by $\varphi_{\mathcal{L}'}(x) = \mathsf{Pos}(\Omega i(x))$. So (see also [14]):

> each formal closed set in a positive topology can be identified with an overt, weakly closed sublocale (of the locale presented by the cover).

Every positivity on a locale $\mathcal{L}$ is a sub-suplattice of $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$. However, its join operation does not corresponds to the join of sublocales. Actually, it is not difficult to see that taking a union of elements in $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$ corresponds to defining the least overt, weakly closed sublocale which contains (as a sublocale) all the given ones.

3.2. Points of a positive topology

The category **PTop** has a terminal object given by the saturated positive topology $(\mathbf{1}, \mathbf{SupLat}(\Omega, \Omega))$ where $\mathbf{1}$ is the terminal locale. So it is natural to consider morphisms from $(\mathbf{1}, \mathbf{SupLat}(\Omega, \Omega))$ to $(\mathcal{L}, \Phi)$ as points for the positive topology $(\mathcal{L}, \Phi)$. This is in fact Sambin's definition [9].

Definition 3.2. *Let $(\mathcal{L}, \Phi)$ be a positive topology. A(n* ideal*)* point *of $(\mathcal{L}, \Phi)$ is a morphism from $(\mathbf{1}, \mathbf{SupLat}(\Omega, \Omega))$ to $(\mathcal{L}, \Phi)$ in* **PTop**.

In other words, a point of $(\mathcal{L}, \Phi)$ is given by a (global) point p of $\mathcal{L}$ such that $\alpha \circ \Omega p \in \Phi$ for all $\alpha \in \mathbf{SupLat}(\Omega, \Omega)$. In view of proposition 2.12, we have:

Proposition 3.3. *The (ideal) points of a positive topology $(\mathcal{L}, \Phi)$ are the points of $\mathcal{L}$ (as a locale) whose corresponding frame morphisms belong to Φ.*

$$\mathbf{PTop}\big(\mathbf{1_{PTop}}, (\mathcal{L}, \mathbf{\Phi})\big) = \{p \in \mathbf{Loc}(\mathbf{1_{Loc}}, \mathcal{L}) \mid \Omega p \in \Phi\}$$

We write $\mathsf{Pt}(\mathcal{L}, \Phi)$ for the collection of all (ideal) points of the positive topology $(\mathcal{L}, \Phi)$ and $\mathsf{Pt}(\mathcal{L})$ for the points of the locale $\mathcal{L}$. Therefore, we have $\mathsf{Pt}(\mathcal{L})$ $= \mathsf{Pt}(\mathcal{L}, \mathbf{SupLat}\big(\Omega\mathcal{L}, \Omega)\big)$ but also, by a little abuse of notation,

$$\mathsf{Pt}(\mathcal{L}, \Phi) = \Phi \cap \mathsf{Pt}(\mathcal{L}) \ .$$

This result suggests that a positivity relation on locale, besides being a way for selecting a suplattice of overt weakly closed sublocales, is also a tool for choosing some points of the locale, that is, a subspace. This idea has already been discussed in [10]. For every family of points of $\mathcal{L}$, one can define a positivity by taking the sub-suplattice of $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$ generated by those points. The ideal points of such a positive topology include all (and, under extra assumptions [10], only) the points in the given family. In particular, every sublocale $\mathcal{L}'$ of $\mathcal{L}$ gives a positive topology $\big(\mathcal{L}, \mathbf{SupLat}(\Omega\mathcal{L}', \Omega)\big)$ each of whose ideal points is a point of $\mathcal{L}'$ as well.

3.3. Relating **PTop** *to* **Loc** *and* **Top**

The notion of a saturated basic topology extends also to positive topologies. So a *saturated* positive topology is one of the form $\big(\mathcal{L}, \mathbf{SupLat}(\Omega\mathcal{L}, \Omega)\big)$. The facts discussed in section 2.3 remain valid when basic topologies are replaced by positive topologies (and, accordingly, sup-preserving maps are replaced by morphisms of frames). In particular we obtain an embedding of categories $\mathbf{Loc} \hookrightarrow \mathbf{PTop}$ which sends $\mathcal{L}$ to $\big(\mathcal{L}, \mathbf{SupLat}(\Omega\mathcal{L}, \Omega)\big)$ (and is the identity on morphisms). Moreover **Loc** can be identified with a reflective subcategory of **PTop**. Unfortunately this adjunction between **Loc** and **PTop** cannot be composed with the usual one between **Top** and **Loc** (which would be possible if **Loc** were a co-reflective subcategory of **PTop**). However, it is possible to obtain an adjunction between **Top** and **PTop** by mimicking the usual adjunction between **Top** and **Loc**, as we are now going to sketch. What follows is essentially a rephrasing of some results in [9]. As we saw in section 2.1, there exists a natural way to associate a positive topology to a topological space. Given a topological space (X, τ) and a base S for the topology, one defines a positivity $\ltimes_X$ on τ by requiring

$$a \ltimes_X U \iff a \between \mathsf{rest} U \iff (\exists x \in X)(a \in \Diamond\{x\} \subseteq U)$$

for $a \in S$ and $U \subseteq S$ (we use the same notation as in section 2.1). By following the arguments in section 2.2, one realizes that the suplattice morphisms $\tau \to \Omega$ corresponding to $\ltimes_X$ are precisely all those of the form

$$A \longmapsto A \between C$$

for C a closed set (provided that "closed" is defined via adherent points). In this view, a continuous map between topological spaces automatically gives a morphism of positive topologies because the composition $B \mapsto f^{-1}(B) \mapsto f^{-1}(B) \between C$ gives the same results as $B \mapsto B \between \mathsf{cl}(f(C))$ where cl is the topological closure in Y (in the sense of adherent points). Consequently one can define a functor from **Top** to **PTop**. This functor has a right adjoint whose object part sends each positive topology to its space of points. We refer to [9] for a detailed proof.

This adjunction between **Top** to **PTop** suggests a new, actually weaker, notion of *sobriety* for topological space. In our notation, a space (X, τ) is sober when it coincides with $\mathsf{Pt}(\tau, \ltimes_{max}) = \mathsf{Pt}(\tau)$, while it is *weakly sober* if $X = \mathsf{Pt}(\tau, \ltimes_X)$. The difference between the two is that $\ltimes_X$ corresponds to those suplattice homomorphisms $\tau \to \Omega$ that have the form $A \mapsto A \between C$ for C a closed set. With classical logic, every suplattice homomorphism $\varphi \colon \tau \to 2$ is seen to have this form by choosing C to be the complement of the largest open set U such that $\varphi(U) = 0$. In [2] it is shown that every Hausdorff space is weakly sober, though not sober, in general.

4. Positivities and the lower powerlocale

For any locale $\mathcal{L}$, the *lower powerlocale* $\mathcal{P}_L\mathcal{L}$ is constructed according to the following two steps: first, we make an isomorphic copy of the suplattice $\Omega\mathcal{L}$ in which a new symbol $\diamond x$ corresponds to each $x \in \Omega\mathcal{L}$ – so $\diamond \bigvee_{i \in I} x_i = \bigvee_{i \in I} \diamond x_i$; second, consider the frame freely generated by the $\diamond x$'s subject to preservation of all the joins, and define $\mathcal{P}_L\mathcal{L}$ as the corresponding locale. A base for $\Omega\mathcal{P}_L\mathcal{L}$ is therefore given by all objects of the form $\diamond x_1 \wedge \cdots \wedge \diamond x_n$ with n a non-negative integer. Our interest for lower powerlocales comes from the following well-known result [3]. Recall that $f \sqsubseteq g$ holds of two locale maps from X to Y if $\Omega f(y) \leq \Omega g(y)$ for all $y \in \Omega Y$.

Proposition 4.1. *Given a locale $\mathcal{L}$, let $\mathsf{Pt}(\mathcal{P}_L\mathcal{L})$ be the poset of all points of the lower powerlocale $\mathcal{P}_L\mathcal{L}$ equipped with the specialization order $\sqsubseteq$. Then $\mathsf{Pt}(\mathcal{P}_L\mathcal{L})$ is a suplattice isomorphic to* **SupLat**$(\Omega\mathcal{L}, \Omega)$.

Proof By the very definition of $\mathcal{P}_L\mathcal{L}$, the function $\diamond\colon \Omega\mathcal{L} \to \Omega\mathcal{P}_L\mathcal{L}$ that maps x to $\diamond x$ is a suplattice homomorphism. So, if $p\colon \mathbf{1} \to \mathcal{P}_L\mathcal{L}$ is a point of $\mathcal{P}_L\mathcal{L}$, then $\Omega p \circ \diamond\colon \Omega\mathcal{L} \to \Omega$ is a suplattice homomorphism. Conversely, given φ in $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$, the map $\diamond x \mapsto \varphi(x)$ extends uniquely to a frame homomorphism $\widetilde{\varphi}$ from $\Omega\mathcal{P}_L\mathcal{L}$ to Ω, that is, a point of $\mathcal{P}_L\mathcal{L}$. Clearly, $\widetilde{\varphi} \circ \diamond = \varphi$ and $\widetilde{\Omega p \circ \diamond} = \Omega p$, and this bijective correspondence between $\mathsf{Pt}(\mathcal{P}_L\mathcal{L})$ and $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$ is order-preserving. q.e.d.

4.1. Localic suplattices

One way to describe suplattice structure on a poset L is to consider the function $\downarrow\colon L \to \mathsf{Down}(L)$ where $\mathsf{Down}(L)$ is the poset (with respect to set-theoretic inclusion) of down-closed subsets of L and $\downarrow x$ is the principal downset of x. Then L is a suplattice if and only if $\downarrow$ has a left adjoint, which then calculates the joins of downsets. [6] proposes a localic form of this in which posets are replaced by locales (with the specialization order $\sqsubseteq$ on morphisms) and Down is replaced by $\mathcal{P}_L$. The function $\downarrow\colon \mathcal{L} \to \mathcal{P}_L\mathcal{L}$ then corresponds to the unique frame homomorphism which sends $\diamond x$ to x, for every $x \in \Omega\mathcal{L}$. (Note that such a frame morphism is surjective; so $\mathcal{L}$ is a sublocale of $\mathcal{P}_L\mathcal{L}$.) [6] defines a *localic suplattice* to be a locale $\mathcal{L}$ equipped with a morphism $\bigsqcup\colon \mathcal{P}_L\mathcal{L} \to \mathcal{L}$ left adjoint to $\downarrow$, that is, $\bigsqcup\downarrow \sqsubseteq \mathrm{Id}_{\mathcal{L}}$ and $\mathrm{Id}_{\mathcal{P}_L\mathcal{L}} \sqsubseteq \downarrow\bigsqcup$.

For instance, every lower powerlocale is a localic suplattice.

Localic suplattices are algebras for a monad whose functor part is just $\mathcal{P}_L$ [6]. As such, they are objects of a category (the Eilenberg-Moore category of the monad) in which morphisms are commutative diagrams of the following form.

$$\begin{array}{ccc} \mathcal{P}_L X & \xrightarrow{\ \mathcal{P}_L f\ } & \mathcal{P}_L Y \\ {\scriptstyle \bigsqcup}\downarrow & & \downarrow{\scriptstyle \bigsqcup} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

In this situation, we say that X is a *localic sub-suplattice* of Y if X is a sublocale of Y (that is, f is regular mono in the category of locales).

The fact that points of a lower powerlocale form a suplattice (proposition 4.1) extends to any localic suplattice.

Proposition 4.2. *Let X be a localic suplattice. Then $\mathsf{Pt}(X)$ is a suplattice, ordered by specialization order $\sqsubseteq$.*

If $f\colon X \to Y$ is a morphism of localic suplattices, then $\mathsf{Pt}(f)\colon \mathsf{Pt}(X) \to \mathsf{Pt}(Y)$ is a suplattice homomorphism.

Proof Let p_i $(i \in I)$ be a family of global points of X and consider $\bigvee_i \Omega p_i \in \mathbf{SupLat}(\Omega X, \Omega)$. This corresponds to a point of $\mathcal{P}_L X$, say P. We claim that $\bigsqcup \circ P$ is the least upper bound of the p_i's in $\mathsf{Pt}(X)$. Suppose q is a global point of X. By the adjunction, $\bigsqcup \circ P \sqsubseteq q$ is equivalent to $P \sqsubseteq \downarrow \circ\, q$. This means that $\Omega P(y) \leq \Omega q(\Omega{\downarrow}(y))$ for all $y \in \Omega\mathcal{P}_L X$, which holds precisely when

$\Omega P(\diamond x) \leq \Omega q(\Omega{\downarrow}(\diamond x))$ for all $x \in \Omega X$. By unfolding definitions, this becomes $\bigvee_i \Omega p_i(x) \leq \Omega q(x)$ for all x, that is, $(\forall i \in I)(p_i \sqsubseteq q)$. This shows that $\bigsqcup \circ P$ is the desired least upper bound.

The second part is an immediate consequence of how the joins are calculated, bearing in mind that composition with Ωf preserves joins in $\mathbf{SupLat}(\Omega X, \Omega)$. q.e.d.

Note that, in the special case $X = \mathcal{P}_L\mathcal{L}$ and in view of proposition 4.1, the least upper bound constructed above has to coincide with the usual point-wise join in $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$ because joins are unique.

Corollary 4.3. *Let $\mathcal{L}$ be a locale. If X is a localic sub-suplattice of $\mathcal{P}_L\mathcal{L}$, then $\mathsf{Pt}(X)$ is a positivity on $\mathcal{L}$.*

Proof By proposition 4.2, $\mathsf{Pt}(X)$ is a sub-suplattice of $\mathsf{Pt}(\mathcal{P}_L\mathcal{L})$, which is isomorphic to $\mathbf{SupLat}(\Omega\mathcal{L}, \Omega)$. q.e.d.

As we already know, a positivity on a locale $\mathcal{L}$ is a way to select some of its points. This fact can be clearly seen now, at least when the positivity arises as the collection of points $\mathsf{Pt}(X)$ of localic sub-suplattice $m\colon X \to \mathcal{P}_L\mathcal{L}$. For, in that situation, we can consider the *intersection* of the two sublocales $\mathcal{L}$ and X of $\mathcal{P}_L\mathcal{L}$, namely the pullback $\mathcal{L} \times_{\mathcal{P}_L\mathcal{L}} X$ of m along ${\downarrow}\colon \mathcal{L} \to \mathcal{P}_L\mathcal{L}$. So the points of the positive topology $\big(\mathcal{L}, \mathsf{Pt}(X)\big)$, that, as we know, are essentially given by $\mathsf{Pt}(\mathcal{L}) \cap \mathsf{Pt}(X)$, are nothing but the points of $\mathcal{L} \times_{\mathcal{P}_L\mathcal{L}} X$. This says, in particular, that every positive topology of the kind $\big(\mathcal{L}, \mathsf{Pt}(X)\big)$, with X a localic sub-suplattice of $\mathcal{P}_L\mathcal{L}$, has the same ideal points of $\big(\mathcal{L}, \mathbf{SupLat}(\Omega\mathcal{L}', \Omega)\big)$ for a suitable sublocale $\mathcal{L}'$ of $\mathcal{L}$.

Now we want to investigate the converse to the previous corollary, namely whether any positivity on a locale $\mathcal{L}$ arises as the family of points of a localic sub-suplattice of $\mathcal{P}_L\mathcal{L}$. We are going to show that the answer is positive provided that classical logic is assumed.

In [6], also a contravariant functor $M \mapsto \widehat{M}$ is constructed from suplattices to localic suplattices, where $\Omega\widehat{M}$ is the free frame over M (qua suplattice). Thus for a locale $\mathcal{L}$, we have $\widehat{\Omega\mathcal{L}} = \mathcal{P}_L\mathcal{L}$. The points of $\widehat{M}$ can be identified with suplattice homomorphisms from M to Ω; in symbols: $\mathsf{Pt}\big(\widehat{M}\big) \cong \mathbf{SupLat}(M, \Omega)$.

Proposition 4.4 (CLASS). *Let $(\mathcal{L}, \Phi)$ be a positive topology. Then there exists a localic sub-suplattice X of $\mathcal{P}_L\mathcal{L}$ such that $\Phi = \mathsf{Pt}(X)$.*

Proof Let M be the quotient of $\Omega\mathcal{L}$ obtained as in proposition 2.11. By [6], $\widehat{M}$ is a subobject of $\mathcal{P}_L(\mathcal{L})$ in the category of localic suplattices. Moreover, $\mathsf{Pt}(\widehat{M}) = \mathbf{SupLat}(M, \Omega) = \Phi$. q.e.d.

In particular, within a classical setting, *every positivity Φ on $\mathcal{L}$ is the set of points of a sublocale of $\mathcal{P}_L\mathcal{L}$*, that is, *a sober subspace of* $\mathsf{Pt}(\mathcal{P}_L\mathcal{L})$.

Apparently, these last results do not hold intuitionistically. In particular, we suspect that the positivity $\ltimes_X$ associated to a spatial locale X (section 2.1) cannot be presented constructively as $\mathsf{Pt}(Y)$ for some localic sub-suplattice Y of $\mathcal{P}_L X$. (Recall that $\ltimes_X$ is classically – but not intuitionistically – equal to $\ltimes_{max}$ which, on the contrary, can be presented as $\mathsf{Pt}(\mathcal{P}_L X)$.)

5. Conclusions

The basic topologies of [9] have much appeal in their formal symmetries, but there remains the question of how to exploit them mathematically. Our impredicative version as pairs (L, Φ) involving suplattices, as a start, provides an alternative mathematical view of what the basic topologies are.

If one considers that many suplattices are constructed impredicatively, there are at least two approaches to seeking a predicative account. The first is to work instead with suplattice presentations instead of the concrete suplattices, and – as we have seen – that essentially was Sambin's starting point. Note, however, that the positivity relation may be constructed impredicatively, using coinduction principles. The maximum positivity relation $\ltimes_{max}$ is a typical example.

The second approach is to use a locale whose *points* are the elements of the suplattice under consideration. Formal topology then provides ways to make the locales predicative, using presentations of frames. In section 4.1 we developed ideas from [6] to deal with suplattices of the form $\mathbf{SupLat}(L, \Omega)$, where L is given by a presentation. This uses the lower powerlocale, which itself should be considered part of the toolkit of predicative formal topology in that it can be constructed predicatively on frame presentations. (More precisely, the lower powerlocale is *geometric* [13].)

We therefore propose a predicative (and geometric) framework in which a basic topology is a pair (L, Φ) where L is a presented suplattice, and Φ is a localic sub-suplattice of $\widehat{L}$ (represented as a formal topology).

However interesting this may be as a technical reworking, it still leaves open the question of how such pairs (L, Φ) may be be used mathematically. Perhaps some clues are already present in [6] in its use of both kinds (presented and localic) of suplattice. A particular achievement there was to give a geometric account, in terms of localic surjections, of the highly classical completeness theorems of [1].

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