On the trail of the Formal Topos

Steve Vickers
School of Computer Science
University of Birmingham

- predicative content of topos theory
- open directions of research

Talk given at: Advances in Constructive Topology & Logical Foundations, Padua 8-11 Oct 08

History (rationalized!)

Grothendieck Generalized space	Site	Topos
Ungeneralized space Point-free topology	Site on poset	Frame

Grothendieck topos

Formal topology

Frame presentation

Locale theory

Predicative

Topos-valid

Elementary topos

Elementary topos

finite limits + powerobjects

impredicative

$\Downarrow$ monadic

contravariant P self adjoint

cf. Taylor ASD

$\Downarrow$

colimits, exponentials, ..., internal frames

$\Rightarrow$ impredicative locale theory

$\Downarrow$

predicative content?

Bundles as variable spaces

no other assumptions in general

Bundle = continuous map $p: Y \to X$

bundle space (Y), base space (X)

Point of view: variable space

fibre $p^{-1}(\{x\})$ varies with $x \in X$

Sheaf ≈ local homeomorphism

= variable set (fibres are discrete)

Sheaves over X

= set theory "parametrized by $x \in X$"

dependent sets

"Geometricity"

- named after geometric logic
- cf. Bunge/Funk "equivariant monad"

Geometric constructions — preserved under pullback of bundles

$$\begin{tikzcd} f^*Y \arrow[r] \arrow[d, "f^*p"'] & Y \arrow[d, "p"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

= substitution of base point

$(x:X)\ p^{-1}(\{x\}) \mapsto (x':X')\ p^{-1}(\{f(x')\})$

Geometric constructions work fibrewise

Predicativity: powerset construction is not geometric. (on local homeomorphisms)

(Neither is function set.)

Idea: Fibrewise topology over X

- "Do topology" using variable sets — internal mathematics of Sh(X)
- = study of bundles over X ?

YES! But ...

- Can't use "point-set" topology
  - local homeomorphisms don't approximate arbitrary bundles well enough.
- Use internal frames
  - "doing topology" point-free

Joyal & Tierney

earlier – Fourman & Scott

$\mathcal{E}$ a topos: then duality

internal frames in $\mathcal{E}$

$\sim$ localic geometric morphisms $\mathcal{F} \to \mathcal{E}$

Special case $\mathcal{E} = Sh(X)$ — X a locale

internal frames in $Sh(X)$

$\sim$ locale maps $Y \to X$

internal frame $\sim$ localic bundle

topos-valid locale theory $\Rightarrow$ results about maps

Geometricity

$$\begin{tikzcd} \text{internal formal topology} \arrow[rr, mapsto, "\text{geometric!}"] \arrow[dr, mapsto, "\text{not geometric}"'] & & \text{bundle} \\ & \text{internal frame} \arrow[ur, mapsto, "\text{not geometric}"'] & \end{tikzcd}$$

because frame structure not geometric

Geometricity restoring points

Y, Z two formal topologies

Φ a geometric construction pts $Y \to$ pts Z

Then have corresponding map $Y \xrightarrow[f]{} Z$

s.t. for any pt $1 \xrightarrow{y} Y$, $\Phi(y) = f \circ y$

Note $Sh(Y)$ is classifying topos for points of Y.

Hence maps $W \xrightarrow{y} Y$ (generalized points of Y)

$\sim$ "points of Y in $Sh(W)$"

Again, $\Phi(y) = f \circ y$

Lemma

Suppose W an internal formal topology in $Sh(X)$.

Then internal points of W

$\sim$ global sections of bundle $\begin{array}{c} W \\ \downarrow \\ X \end{array}$

Example

X a formal topology — bundle $\begin{array}{c} X \\ \downarrow \\ 1 \end{array}$

Pullback to X — $\begin{array}{c} X \times X \\ \downarrow \\ X \end{array}$

Generic point corresponds to section $\begin{array}{c} X \times X \\ \Delta \uparrow \downarrow \\ X \end{array}$

Φ : pts of $Y \longrightarrow$ pts of Z

Pull back to Y:

$$\begin{tikzcd}
Y \times Y \arrow[dr, shift left] & & Z \times Y \arrow[dl, shift left] & Y \arrow[d] & Z \arrow[dl] \\
& Y \arrow[ul, shift left, "\Delta"] \arrow[ur, shift left, "\Phi(\Delta)"'] \arrow[rr] & & 1 &
\end{tikzcd}$$

$\Phi(\Delta) = \langle f, Id_Y \rangle$, some $f : Y \to Z$

Pull back along any generalized point:

$$\begin{tikzcd}
Y \times W \arrow[dr, shift left] & & Z \times W \arrow[dl, shift left] & Y \times Y \arrow[dr, shift left] & & Z \times Y \arrow[dl, shift left] \\
& W \arrow[ul, shift left, "\langle y, Id \rangle"] \arrow[ur, shift left, "{\langle f \circ y, Id \rangle = \Phi \langle y, Id \rangle}"'] \arrow[rrr, "y"'] & & & Y \arrow[ul, shift left, "\Delta"] \arrow[ur, shift left, "{\langle f, Id \rangle = \Phi(\Delta)}"'] &
\end{tikzcd}$$

General metatheorem

Continuous map = geometric transformation of points

Proof impredicative — internal frames

Result makes predicative sense.

Can check case by case:

geometric transformation

$\leftrightarrow$ inverse image function

$\leftrightarrow$ map between formal topologies

General predicative metatheorem?

Impredicative → Geometric

Powerlocales: free frames over suplattices/preframes → Achieved by geometric constructions on presentations

Compactness/overtness: properties of frames → Suitable points of powerlocales

↕

Properness/openness: properties of maps (bundles) → preserved under pullback

Predicative?

Categorical approach cf. Taylor ASD

FT – category of formal topologies

Slice FT/X – object = bundle over X {geometric}

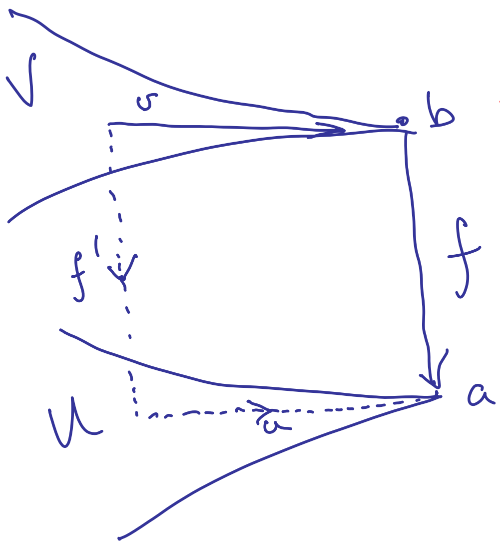
If enough structure on FT & slices

– replicate geometric results of topos-valid frames

cf. Townsend: abstract categorical structure of Loc in terms of double powerlocale monad

Sites ~ inductively generated case

"Propositional" = formal topology	"Predicate" = formal topos
Poset $(P, \leq)$ basic opens	Category $\mathcal{C}$ – basic sheaves
$\lhd_0 \subseteq P \times \mathcal{P}P$ basic covers	$\lhd_0 \subseteq \mathcal{C}_0 \times \mathcal{P}\mathcal{C}_1$
	$a \lhd_0 U \Rightarrow U \subseteq \mathcal{C}(-, a)$
$b \leq a \lhd_0 U$ $\Rightarrow \exists V \subseteq b \downarrow U . b \lhd_0 V$	$a \lhd_0 U,\ f: b \to a$ $\Rightarrow \exists V .\ b \lhd_0 V$ $\forall v \in V\ \exists f', u .\ u \in U$ $v;f = f';u$



$V \subseteq f^* U$

= set of v such that $v;f$ factors through something in U.

Formal points

$F \subseteq P$ $\qquad F : P \to \Omega$

F up-closed

F a filter $\qquad$ flatness

If $a \lhd_0 U$, $a \in F$

then $U \between F$

meets $\qquad$ point split

$F : \mathcal{C} \to \mathrm{Sets}$

- some $F(a)$ is inhabited
- $x \in F(a)$, $y \in F(b) \Rightarrow$
  $\exists$ $\begin{tikzcd} c \arrow[r, "f"] \arrow[r, shift right=2, "g"'] & a, b \end{tikzcd}$ $z \in F(c)$.
  $x = F(f)(z)$, $y = F(g)(z)$
- $\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$ $\quad F(f)(x) = F(g)(x)$
  $\Rightarrow \exists\, c \xrightarrow{h} a .\ h;f = h;g$
  $\exists\, z \in F(c).\ x = F(h)(z)$

$a \lhd_0 U$, $x \in F(a)$
$\Rightarrow \exists u \in U\ \exists y.\ x = F(u)(y)$

Flatness

F a filter $\qquad$ flatness

- some $F(a)$ is inhabited
- $x \in F(a)$, $y \in F(b) \Rightarrow$
  $\exists$ $\begin{tikzcd} c \arrow[r, "f"] \arrow[r, shift right=2, "g"'] & a, b \end{tikzcd}$ $z \in F(c)$.
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- $\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$ $\quad F(f)(x) = F(g)(x)$
  $\Rightarrow \exists\, c \xrightarrow{h} a .\ h;f = h;g$
  $\exists\, z \in F(c).\ x = F(h)(z)$

If $\mathcal{C}$ has finite limits:

$F : \mathcal{C} \to \mathrm{Set}$ flat

$\Longleftrightarrow$ F preserves finite limits

Formal toposes: some sightings

"Space of sets" ⇝ object classifier

Site is: $\mathcal{C} = (\mathrm{Set}_{\mathrm{fin}})^{op}$, no coverage

finite in strong sense. Object = natural number

$\mathrm{Set}_{\mathrm{fin}}$ has finite colimits, every object copower of 1

$\therefore$ flat functor $\mathcal{C} \to \mathrm{Set}$ determined by image of 1

Point = set

$\mathcal{C} = (\mathrm{Set}_{\mathrm{fin}})^{op}$, no coverage

Sheaves (objects of topos)

Contravariant $\mathcal{C} \longrightarrow \mathrm{Set}$

$=$ Covariant $\mathrm{Set}_{\mathrm{fin}} \longrightarrow \mathrm{Set}$

$\sim$ Covariant $\mathrm{Set} \longrightarrow \mathrm{Set}$ preserving filtered colimits

"functor points $\to$ sets with continuity condition"

("sheaf = continuous set-valued map")

Similarly: algebras

{Joyal (cf. Coquand's talk): Algebraic closure of k - f.p. k-algebras + coverage point = alg. closed ext^n of k

$\mathbb{T}$ - algebraic theory

$\mathcal{C} = (\mathbb{T}\text{-Alg}_{f.p.})^{op}$, no coverage

finitely presented $\mathbb{T}$-algebras

Point = $\mathbb{T}$-algebra

Sheaf = covariant $\mathbb{T}$-Alg $\to$ Set preserving filtered colimits

Formal topos is "space of $\mathbb{T}$-algebras"

Symmetric topos - analogue of lower powerlocale

Bunge - Funk

$X = (P, \leq, \lhd_0)$ a formal topology (inductively generated)

$\mathcal{C} = (P, \leq)^*$ = lex completion $\eta : P \to \mathcal{C}$

freely adjoin finite limits to category $(P, \leq)$

Coverage - if $a \lhd_0 U$ in P, $f : x \to \eta(a)$

$x \lhd_0 f^*(\mathcal{C}U)$ in $\mathcal{C}$ predicative

$\to$ site for symmetric topos MX

Points of MX

Cosheaves of $(P, \leq, \lhd_0)$

= covariant $(P, \leq) \to$ Set that "preserves colimits of sheaves"

$\simeq$ complete spreads over X (certain kind of bundle)

X locally connected $\Leftrightarrow$ MX has "strongly terminal" point (connected components cosheaf)

cf. lower powerlocale $P_L X$

- lower powerpoints
- $(P, \leq) \to \Omega$ preserves joins of opens
- overt, weakly closed subspaces of X
- X overt $\Leftrightarrow$ $P_L X$ has strongly top point (positivity predicate)

Where now?

- Geometrization: how much maths is geometrizable? On-going study
  $\Rightarrow$ points for point-free spaces, fibrewise reasoning for bundles
  cf Landsman, Spitters, Heunen "tovariance"
  should work predicatively
- Categorical account of spaces
  - ASD (locally compact)
  - Townsend

Formal toposes

- "Formal toposes" – predicative study of sites as generalized spaces.
- Structure of category of generalized spaces

? Lower powerlocale $\mapsto$ symmetric topos

Upper " $\mapsto$??