# Surviving without function types:

## Life in an arithmetic universe

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Talk given
22 JUN 2010
CT2010 Genova

- Arithmetic universes as generalized spaces categories of sheaves
- An induction principle
- Classical logic — of some subspaces

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# Part I

## Arithmetic universes

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## Arithmetic universe

Originated: Joyal (1970s) unpublished

- constructed examples (inc. initial one)
- applied to Gödel's theorem

General definition? Which?

Pretopos + internal free algebras — Joyal, Wraith ... free categories & diagrams

Cockett & Jay: algebra in locoses

Maietti:

AU = list-arithmetic pretopos
= pretopos + parametrized list objects

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A category is a pretopos if

- has finite limits
- has finite coproducts
  - and they are stable and disjoint
- has stable, effective quotients of equivalence relations

cf. Giraud's theorem

Similar conditions + all small coproducts + size constraints

⇒ Grothendieck topos

| | predicative? | geometric? | pretopos? |
|---|---|---|---|
| • $\mathbb{N}$, hence free algebras | ✓ | ✓ | ✗ |
| • cartesian closed | ✓ | ✗ | ✗ |
| • powerobjects | ✗ | ✗ | ✗ |

## Arithmetic universe = list-arithmetic pretopos

enough for finitary algebra

For every $A$ : list$(A)$ has

$$1 \xrightarrow{[\,]} \mathrm{list}(A)$$

$$A \times \mathrm{List}(A) \xrightarrow{\mathrm{cons}} \mathrm{List}(A)$$

$\mathrm{cons}(a, [a_1, \dots, a_n]) = [a, a_1, \dots, a_n]$

$$\begin{tikzcd}
 & \mathrm{List}(A) \times B \arrow[d, "\mathrm{rec}(y_0, g)"] & & A \times \mathrm{List}(A) \times B \arrow[ll, "\mathrm{cons} \times B"'] \arrow[d, "A \times \mathrm{rec}(y_0, g)"] \\
B \arrow[ur, "\langle B, [\,] \circ ! \rangle"] \arrow[r, "y_0"'] & Y & & A \times Y \arrow[ll, "g"]
\end{tikzcd}$$

$\forall\, y_0, g\ \exists!\ \mathrm{rec}(y_0, g)$

write $g(a, y) = a \cdot y$. Then $\mathrm{rec}(y_0, g)([a_1, \dots, a_n], b) = a_1 \cdot \cdots \cdot a_n \cdot y_0(b)$

## Classifying AUs

Theory of AUs is cartesian

$\Rightarrow$ can present AUs by generators & relations

Geometric theory

$\downarrow$ IF can replace $\bigvee$ by internal types + $\exists$

AU presentation

e.g. $\bigvee_{q \in Q} L_q \;\downarrow\; \exists q : Q.\, L(q)$

AU stands in for classifying topos

AUs support limited fragment of sheaf theory

how limited?

## Arithmetic spaces

Generalized space = Grothendieck topos

cf frames, locales

Continuous Map = functor (backwards)

geometric morphism

preserving all colimits

has right adjoint

finite limits

also free algebras

Idea: Arithmetic space $X$ given by AU $\mathbb{A}X$

Map $f : X \to Y$ is functor $\mathbb{A}X \leftarrow \mathbb{A}Y$

preserving finite colimits

finite limits

List

strictly or non-strictly?

# Part II

# INDUCTION

## Induction I

$\phi \subseteq N$. Suppose:

$$\begin{tikzcd} 0^*\phi \arrow[r] \arrow[d, hook, "\cong"'] & \phi \arrow[d, hook] \\ 1 \arrow[r, "0"'] & N \end{tikzcd}$$

Base case $\phi(0)$

Induction step $\forall n\,(\phi(n) \to \phi(n+1))$

$$\underbrace{\{n \mid \phi(n)\}}_{\phi} \subseteq \underbrace{\{n \mid \phi(n+1)\}}_{s^*\phi}$$

$$\begin{tikzcd} \phi \arrow[r] \arrow[dr] & s^*\phi \arrow[r] \arrow[d] & \phi \arrow[d] \\ & N \arrow[r, "s"'] & N \end{tikzcd}$$

Can show $\phi \hookrightarrow N$ is iso

i.e. $\forall n\, \phi(n)$

$$\begin{tikzcd} & N \arrow[d, shift left, "{rec(0,s)}"] & N \arrow[l, "s"'] \arrow[d, shift left, "{rec(0,s)}"] \\ 1 \arrow[ur, "0"] \arrow[r, "0"'] & \phi \arrow[u, shift left] & \phi \arrow[l, "s"] \arrow[u, shift left] \end{tikzcd}$$

$\forall n\, \phi(n)$

## Induction for $\phi \to \psi$

Two predicates $\phi, \psi$ on $N$ $\qquad$ $\phi, \psi \subseteq N$

Want $\forall n.\ \phi(n) \to \psi(n)$ $\qquad$ $\phi \subseteq \psi$

Function types give predicate $\phi \to \psi$

Use same induction:

Base case $\phi(0) \to \psi(0)$

Induction step

$$\frac{\forall n\,\big((\phi(n) \to \psi(n)) \to (\phi(n+1) \to \psi(n+1))\big)}{\forall n\,\big(\phi(n) \to \psi(n)\big)}$$

## Without function types?

IF classical logic —

$\phi(n) \to \psi(n) \equiv \neg\phi(n) \vee \psi(n)$

$$\begin{aligned} & (\phi(n) \to \psi(n)) \to (\phi(n+1) \to \psi(n+1)) \\ \equiv\ & (\neg\phi(n) \vee \psi(n)) \to (\neg\phi(n+1) \vee \psi(n+1)) \\ \equiv\ & \neg\phi(n) \to (\neg\phi(n+1) \vee \psi(n+1)) \\ & \wedge\ \psi(n) \to (\neg\phi(n+1) \vee \psi(n+1)) \\ \equiv\ & \phi(n+1) \to \phi(n) \vee \psi(n+1) \\ \wedge\ & \phi(n+1) \wedge \psi(n) \to \psi(n+1) \end{aligned}$$

Know how to interpret these

## Theorem (Maietti/Vickers)

In any arithmetic universe: if have $\phi, \psi \subseteq N$

$\phi(0) \to \psi(0)$

$\forall n\,\big(\phi(n+1) \to \phi(n) \vee \psi(n+1)\big)$

$\forall n\,\big(\phi(n+1) \wedge \psi(n) \to \psi(n+1)\big)$

Then $\forall n\,\big(\phi(n) \to \psi(n)\big)$

## Proof

In an arithmetic universe: if have $\phi, \psi \subseteq N$

$\phi(0) \to \psi(0)$ — base case

$\forall n\,(\phi(n+1) \to \phi(n) \vee \psi(n+1))$ IS1

$\forall n\,(\phi(n+1) \wedge \psi(n) \to \psi(n+1))$ IS2

Then $\forall n\,(\phi(n) \to \psi(n))$

Define

$(k:N)\quad A(k) \subseteq N$

$A(k) = \{ j \in N \mid j \le k,\ \phi(j), \dots, \phi(k) \}$

If $\phi(k)$ then $k \in A(k)$ – use $f_k(k)$

Define $f_k : A(k) \to \psi(k)$ recursively so $f_k(j) = k$

recursion variant $= k + j$

$j = k = 0$: $\phi(0)$, $\therefore \psi(0)$ by base case

$j = k > 0$: $\phi(j)$. By IS1, $\phi(j-1) \vee \psi(j)$

$$f_k(j) = \begin{cases} f_k(j-1) & \text{if } \phi(j-1) \\ k & \text{if } \psi(j) \end{cases}$$

$j < k$: From $f_{k-1}(j)$ get $\psi(k-1)$

$\therefore \phi(k) \wedge \psi(k-1)$, so $\psi(k)$ by IS2

## Summary

IF classical logic in AU then induction proof for $\phi(n) \to \psi(n)$ reduces to new induction principle

In an arithmetic universe: if have $\phi, \psi \subseteq N$

$\phi(0) \to \psi(0)$

$\forall n\,(\phi(n+1) \to \phi(n) \vee \psi(n+1))$

$\forall n\,(\phi(n+1) \wedge \psi(n) \to \psi(n+1))$

Then $\forall n\,(\phi(n) \to \psi(n))$

New principle valid in any AU

# Part III

for any AU

Classical logic of subspaces

- Replace subsets of N by subspaces
  - $\parallel$ open subspaces
  - closed subspaces $\Rightarrow$ Boolean complements
- Boolean calculations on subspaces
  $\Rightarrow$ valid properties of subsets

X an AS: Subspace triple $= U \hookrightarrow I \hookleftarrow V$

two subsheaves

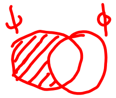
In topos: would use local operators

sheaf = object of AX

Corresponding subspace is $X[U \le_I V] \hookrightarrow X$

Preorder $(U \hookrightarrow I \hookleftarrow V) \le (U' \hookrightarrow I' \hookleftarrow V')$ if AX$[U \le_I V]$ also has $U' \le_{I'} V'$

Meets $(U_1 \hookrightarrow I_1 \hookleftarrow V_1) \wedge (U_2 \hookrightarrow I_2 \hookleftarrow V_2) = (U_1 + U_2 \hookrightarrow I_1 + I_2 \hookleftarrow V_1 + V_2)$




Special cases for $\phi, \psi \hookrightarrow 1$

Open $\psi = (1 \hookrightarrow 1 \hookleftarrow \psi)$

Crescent $(X - \phi) \wedge \psi$

Cocrescent $(\phi \hookrightarrow 1 \hookleftarrow \psi)$

Closed $X - \phi = (\phi \hookrightarrow 1 \hookleftarrow 0)$

$(X - \phi) \vee \psi$

# Representation theorems

Maietti: type theory

## ① Local homeomorphisms

$$\mathbb{A}X[a:1\to A] \simeq \mathbb{A}X/A$$

Adjoin generic element of A

$$A \xrightarrow{\Delta} A\times A \to A$$

Special case: opens $\quad A=\phi \hookrightarrow 1$

$$\mathbb{A}\phi \simeq \mathbb{A}X/\phi$$

Corollary

$$\phi \le (U \to I \leftarrow V) \iff U\times\phi \le_I V\times\phi$$

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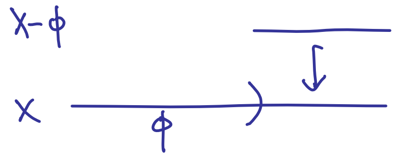
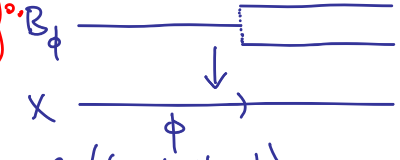
# Representation theorems ② Closeds $X-\phi$

Closed subspace of 1 is Stone



BA of clopens



$$B_\phi = 2/(0=1 \text{ if } \phi)$$

Sheaves F over $B_\phi$ $\quad F(0)=1$

$Sh(B_\phi)$ $\quad$ restriction $F(1)\to 1$ iso if $\phi$

$T_\phi: \mathbb{A}X \to \mathbb{A}X$ $\quad U\times\phi \rightrightarrows U, \phi \to U+\phi \twoheadrightarrow T_\phi(U)$

monad on $\mathbb{A}X$ $\quad$ coequalizer

Thm: $sh(B_\phi) \simeq Alg(T_\phi) \simeq \mathbb{A}(X-\phi)$

reflective subcat of $\mathbb{A}X$

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# Corollaries

done already

open $\quad \phi \le (U \to I \leftarrow V)$ iff $U\times\phi \le_I V\times\phi$ in $\mathbb{A}X$

closed $\quad X-\phi \le (U \to I \leftarrow V)$ iff $U \le_I V \vee I\times\phi$ in $\mathbb{A}X$

meets of opens

subspace meet $\phi_1\wedge\phi_2$ = subspace for subobject meet $\phi_1\wedge\phi_2$

$\phi_1\wedge\phi_2 = \phi_1\wedge\phi_2$

meets of closeds

$$(X-\phi_1)\wedge(X-\phi_2) = X-(\phi_1\vee\phi_2)$$

crescent

$(X-\phi)\wedge\psi \le (U \to I \leftarrow V)$ iff $U\times\psi \le_I V\vee I\times\phi$ in $\mathbb{A}X$

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# Joins

Subspace for subobject join $\phi_1\vee\phi_2$ is subspace join

$\phi_1\vee\phi_2 = \phi_1\vee\phi_2$

$X-(\phi_1\wedge\phi_2)$ is subspace join $(X-\phi_1)\vee(X-\phi_2)$

cocrescent

$(\phi \to 1 \leftarrow \psi) \le (U \to I \leftarrow V)$ iff $X-\phi$, $\psi$ both $\le (U \to I \leftarrow V)$

$\Rightarrow$ obvious

$\Leftarrow$ $U \le V\vee U\times\phi \le V\vee U\times\psi \le V$

$(\phi \to 1 \leftarrow \psi) = (X-\phi)\vee\psi$ $\quad$ cocrescents as joins

Also: $(X-\phi)\vee\phi = X$

$(X-\phi)\wedge\phi = 0$

open & closed are Boolean complements

## Lattice structure

$Y$ any subspace of $X$
$\phi_i, \psi_i$ opens

**Lemma** (A) $\bigvee_{i=1}^{n} (Y \wedge (X - \phi_i) \wedge \psi_i)$ exists, and equals

(B) $Y \wedge \bigwedge \{ X[\bigwedge_{i \in J} \phi_i \leq \bigvee_{i \in K} \psi_i] \mid \{1, \dots, n\} = J \cup K \}$

both finite

**Proof** (1) Conjuncts in (B) are upper bounds for disjuncts in (A)

(2) Over $Y$, suppose $\forall i \; Y \wedge (X - \phi_i) \wedge \psi_i \leq \begin{array}{ccc} U & & V \\ & \searrow \swarrow & \\ & I & \end{array}$ $\quad U \wedge \psi_i \leq V \vee I \times \phi_i$

& adjoin all $\bigwedge_{i \in J} \phi_i \leq \bigvee_{i \in K} \psi_i$. Prove by induction $n - |J|$

$U \times \bigwedge_{i \in J} \phi_i \leq V$

**Corollary** (1) Finite joins of crescents exist & are meets of cocrescents

(2) Any $Y \wedge$ distributes over those joins.

## Lattice structure

**Theorem**

Finite joins of crescents

= finite meets of cocrescents

They form a Boolean algebra.

Conjecture: Have all finite joins of subspaces?

$\begin{array}{ccc} U_1 & & V_1 \\ & \searrow \swarrow & \\ & I_1 & \end{array} \vee \begin{array}{ccc} U_2 & & V_2 \\ & \searrow \swarrow & \\ & I_2 & \end{array} = \begin{array}{ccc} U_1 \times U_2 & & V_1 \times I_2 \vee I_1 \times V_2 \\ & \searrow \swarrow & \\ & I_1 \times I_2 & \end{array}$ ?

## Induction

$\phi, \psi \hookrightarrow N$ in $\mathcal{A}X$

$\phi(n), \psi(n) \hookrightarrow 1$ in $\mathcal{A}X[n : 1 \to N]$

$\phi(n+1), \psi(n+1)$

**Theorem** Suppose $\phi(0) \leq \psi(0)$ in $\mathcal{A}X$

$\begin{array}{ccc} \phi(n) & & \psi(n) \\ & \searrow \swarrow & \\ & 1 & \end{array} \leq \begin{array}{ccc} \phi(n+1) & & \psi(n+1) \\ & \searrow \swarrow & \\ & 1 & \end{array}$ over $\mathcal{A}X[n]$

Then $\phi \leq \psi$ in $\mathcal{A}X$

**Proof** $(X[n] - \phi(n)) \vee \psi(n) \leq (X[n] - \phi(n+1)) \vee \psi(n+1)$

i.e. $\phi(n+1) \leq \phi(n) \vee \psi(n+1)$ over $X[n]$

$\phi(n+1) \wedge \psi(n) \leq \psi(n+1)$

From $\mathcal{A}X[n] \simeq \mathcal{A}X/N$ deduce corresponding relations in $\mathcal{A}X$ $\therefore$ can use previous induction thm

## Conclusions

Classical logic of (some) subspaces
even when logic of subobjects not classical

$\Rightarrow$ can work with implications
even when no internal exponentials

**General moral** Good properties of spaces spoiled when you discretize (take set of points)

e.g. Closed complement properly a Stone space
– don't expect a subobject of 1