Continuity and Geometric Logic

Steven Vickers[1]

School of Computer Science, University of Birmingham,
Birmingham, B15 2TT.

Abstract

This paper is largely a review of known results about various aspects of geometric logic. Following Grothendieck's view of toposes as generalized spaces, one can take geometric morphisms as generalized continuous maps. The constructivist constraints of geometric logic guarantee the continuity of maps constructed, and can do so from two different points of view: for maps as point transformers and maps as bundles.

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1. Introduction

Geometric logic has arisen in topos theory out of the fact that toposes (by "topos" unqualified we shall always mean Grothendieck topos) may be described as classifying toposes for geometric theories – that is to say, any topos may be presented as being generated by a generic model of some geometric theory.

The historical roots of this idea must surely go back to Grothendieck's dictum that "A topos is a generalized topological space" [10], but there is a specific technical understanding that underlies this: that for the the purposes of sheaf cohomology, what was important was particular categorical structure and properties of categories of sheaves over spaces; and that it was fruitful to generalize to other categories (the toposes) with the same structure and properties.

I have not been able to trace in detail how this developed over the 1970s into the idea of toposes as geometric theories as mentioned above. Some of the difficulties are described in the 1986 paper *Theories as Categories* [9], which grew out of notes I made on a talk given by Mike Fourman to computer scientists and gave in outline form some of the ideas and results on which the present paper is based. Fourman said,

> This theory and its applications developed initially without the benefit of widespread publication. Many ideas were spread among a

Email address: `s.j.vickers@cs.bham.ac.uk` (Steven Vickers)
URL: `http://www.cs.bham.ac.uk/~sjv` (Steven Vickers)

> relatively small group, largely by word of mouth. The result of this is that the literature does not provide an accessible introduction to the subject. ... To apportion credit for the ideas presented here is difficult so long after the event. Lawvere and Joyal have a special position in this subject. Many others ... contributed also. Their contributions are, in general, better reflected in their published work.

The difficulty is compounded by the fact that for a long time the terminology was not settled. In both [13] and [20], standard references for topos theory, "geometric" is used for the coherent fragment, without infinitary disjunctions, and classifying toposes are constructed for coherent theories. By contrast [23], which also sets out the results on classifying toposes, uses "coherent" (or $L^g_{\infty\omega}$) for the full geometric logic, and "finitary coherent" for what we call coherent. [17, D1.1.6] uses the terms as we have them here.

I would also refer the reader to [31], which shows in more detail how results in the standard texts [20, 16, 17] justify the view of toposes described here.

While geometric logic can be treated as just another logic, it is an unusual one. Much of this arises from its infinitary disjunctions, which make it possible to characterize a number of constructions up to isomorphism by geometric structure and axioms. This gives rise to a geometric *mathematics*, going beyond the merely logical – technically it is the mathematics that can be conducted in the internal mathematics of toposes, and, in addition to that topos-validity, is moreover preserved by the inverse image functors of geometric morphisms. To put it another way, the geometric mathematics has an intrinsic continuity (since geometric morphisms are the continuous maps between toposes).

In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *"continuity is geometricity"*. In other words, to "do mathematics continuously" is to work within the geometricity constraints. In the rest of this section I shall set out four slogans of the manifesto, and the remaining sections will give the technical elaboration.

As one might expect, discussing continuity requires one first to discuss topological spaces, and the first slogan of the manifesto sets this out. It includes a rephrasing of Grothendieck's dictum that toposes are generalized topological spaces.

1. Spaces are geometric theories

To put this more carefully, a space is going to be described as the space of models for a geometric theory, with its topological nature arising naturally from that theory. This is in essence the approach of *point-free topology*, as adopted in locale theory and in formal topology, though we also generalize from propositional geometric theories to predicate ones, and thereby see Grothendieck's generalization from (point-free) topological spaces to toposes. There is ample evidence that it is the correct approach in a number of constructivist settings, including topos theory: point-free topology retains important results of classical topology that fail in a constructivist point-set approach.

If one stays with propositional geometric theories, the new spaces are equivalent to *locales* (see, e.g., [14, 25]). One might say that they are *frames* (complete Heyting algebras) pretending to be topological spaces.

It is well known that there is an adjunction between topological spaces and locales, but to reach common ground (the *Stone* equivalence between sober spaces and spatial locales), concessions have to be made on both sides.

On the point-free side, the concession is to assume *spatiality*: that the frame can be embedded in a powerset (of a set of points). This is often thought of as harmless, since in classical mathematics enough important locales are spatial that the non-spatial ones can be regarded as pathological. This will not work constructively, however. Even for the real line, to validate standard results of analysis one needs a version (Examples 2 and 6) that is non-spatial in general. Thus in general it is is essential to forgo spatiality and work outside the Stone equivalence.

On the point-set side, the concession is to assume that spaces are *sober*: that the assignment of open neighbourhood filters to points gives a bijection between points and completely prime filters of opens (or, which is classically equivalent, irreducible closed sets). This tells us that the points are not arbitrarily decreed as a set, but depend on some prior structure, the frame, and in fact the special features of sober topology carry over to point-free topology where the points are determined by the logical theory.

Any sober space is T_0, in other words each point is uniquely determined by its open neighbourhood filter. It also has the important property of being a *dcpo* (*directed complete poset*) with respect to the specialization order. It has all directed joins, found by taking directed unions of completely prime filters.

Accepting, as is inevitable in point-free topology, that the core of topology is sober, then the additional layer of non-sobriety in the usual theory can be understood as providing a set of labels (the arbitrarily decreed points) for some or all of the "abstract points" derived as a sober space from the topology. The labelling may have repetition, in other words the T_0 property may fail. The labelling can be described as a map from a discrete space to a sober space (or locale more generally), and as such is equivalent to a "topological system" as defined in [25]. However, we shall not be interested in such structures here. We are looking at the sober core of topology.

We can now discuss continuity. Note that, for us, the word *map* will always assume continuity.

2. Maps are point transformers, defined geometrically

In other words, a map $f : X \to Y$ is described by a geometric transformation $x \mapsto f(x)$.

There are two surprises here. The first is that geometric logic is incomplete, which means there may be an insufficiency of models to discriminate between logically inequivalent formulae. Traditionally one might see this as a deficiency in the logical rules, but in topos theory it is better seen as a deficiency in any individual set-theory's ability to supply models: this is the phenomenon of non-spatiality referred to above. For example, there are non-trivial locales with no points at all. Hence it is surprising that a map can be satisfactorily described as a point transformer. However, geometricity entails that the description can be applied not only to *global* points, maps $1 \to X$, of which there may be insufficient, but also to *generalized* points, maps $W \to X$ for arbitrary W, including the generic point $\mathrm{Id} : X \to X$. The global points are the models in the default category of sets (or base topos), while the generalized points allow the set theory to vary.

The second surprise is that no explicit continuity proof is required. Effectively, by adhering to geometricity constraints we forgo the ability to define discontinuous maps.

3. Bundles are indexed spaces defined geometrically

Here, by a *bundle over* Y we simply mean a map $p : X \to Y$ for some X. We have already called these generalized points of Y, but now there is a change of point of view. The generalized point was a "point of Y parametrized by points of X". As a bundle, we view it as a space (the fibre $X_y = p^{-1}\{y\}$) parametrized by points of Y. I shall explain how bundles can be understood as geometric constructions $y \mapsto X_y$.

4. Geometricity is preservation under pullback of bundles

In this setting we become interested in constructions on bundles, and geometricity comes out as a simple criterion: that the constructions are preserved under pullback. We previously defined geometricity of constructions on sets as preservation by inverse image functors. This revised view generalizes that, since the bundles corresponding to discrete spaces ("indexed *sets* defined geometrically") are local homeomorphisms, and pullback of them is the action of inverse image functors. Geometricity has the important consequence that the constructions work fibrewise, since fibres are pullbacks along points. The construction $X \mapsto F(X)$ on individual spaces can be extended to bundles just by sprinkling it with base-point indexes, $(X_y)_{y \in Y} \mapsto (F(X_y))_{y \in Y}$. To put it another way, point-free topology done geometrically automatically gives fibrewise results for bundles. This has significant promise as a tool even for classical topologists.

2. Geometric logic and theories

We start by outlining the basic definitions of geometric logic and its rules and semantics. Note that because it is a positive logic, lacking implication amongst its connectives, it is given as a sequent style presentation. We follow the account of [17, Section D1].

Definition 1. *Let Σ be a first order signature: it comprises sorts, function symbols (including constants) and predicates, each with an arity describing the number and sorts of the arguments and (for function symbols) the sort of the result. Then, over Σ, we define the following.*

1. *A* context *is a finite list* $\overrightarrow{x}$ *of distinct symbols (not already in* Σ*), called* variables*, each with a stipulated sort* $\sigma(x_i)$*. Note that free variables are provided not in a global way, but context by context.*
2. *A* term in context $(\overrightarrow{x}.t)$ *is a term* t *build up in the usual way from the variables in* $\overrightarrow{x}$ *and the function symbols. It has a sort* $\sigma(t)$*.*
3. *A* geometric formula in context $(\overrightarrow{x}.\phi)$ *is a formula* ϕ *built up in the usual way from the variables in* $\overrightarrow{x}$ *and the functions and predicates in* Σ*, using connectives* $\top$ *(true),* $\wedge$ *(binary conjunction),* $\bigvee$ *(arbitrary disjunction; we can also define* $\bot$*, false, as the nullary disjunction),* $=$ *(for each sort) and* $\exists$*.*
   Note that only the free variables have to be in the context $\overrightarrow{x}$*. The precise rule for bound variables is that if* $(\overrightarrow{x}y.\phi)$ *is a formula in context, then so too is* $(\overrightarrow{x}.(\exists y)\phi)$*. All our geometric formulae will be in context, and it follows that each formula can have only finitely many free variables, although it may have infinitely many bound variables – an example appears in Proposition 5.*
4. *A* geometric sequent *is an expression* $\phi \vdash^{\vec{x}} \psi$ *where* ϕ *and* ψ *are formulae in context* $\vec{x}$*. (The sequent has the same meaning as the sentence* $(\forall x_1 \cdots \forall x_n)(\phi \to \psi)$*, but that is not a geometric formula because it uses* $\to$ *and* $\forall$*.)*
5. *A* geometric theory *is a set* $\mathbb{T}$ *of geometric sequents, the* axioms *of the theory.*

We say a theory is *propositional* if its signature has no sorts: so all predicates are propositional symbols and there can be no function symbols, no variables, no terms, and no use of $=$ or $\exists$. In this case we can see the connection with topology, since the remaining connectives, $\wedge$ and $\bigvee$, correspond to the set theoretic operations, $\cap$ and $\bigcup$, that preserve openness. Indeed, point-free approaches to topology such as locale theory and formal topology may be understood as describing the points of a space as the models of a propositional geometric theory. Then there is a topology in which each formula describes an open, comprising those models for which that formula is assigned the value true. There is an intrinsic sobriety in this approach – the points are exactly the completely prime filters of opens.

A theory is *coherent* if all the disjunctions appearing in it are finite. Topologically this corresponds to *spectral* spaces, those sober spaces for which the compact opens form a base closed under intersection. Many important geometric theories are coherent, and indeed [20] define geometric to mean coherent. However, as we shall see, some vital properties of geometric logic depend on infinitary disjunctions.

As a major example of how a propositional geometric theory can capture topology, we look at the reals: a theory for which each model is a real number. A standard presentation is that in [14], but we give a slightly different version from [31, Section 2.5].

Example 2 (The real line $\mathbb{R}$). *Take a signature with no sorts (it's propositional) and an infinite family of propositional symbols P_{qr} indexed by $q, r \in \mathbb{Q}$. The axioms are*

$$P_{qr} \wedge P_{q'r'} \vdash\dashv \bigvee\{P_{st} \mid \max(q,q') < s < t < \min(r,r')\}$$

$$\top \vdash \bigvee\{P_{q-\varepsilon,q+\varepsilon} \mid q \in \mathbb{Q}\} \text{ for each } 0 < \varepsilon \in \mathbb{Q}.$$

Note that we do not *assume $q < r$ in P_{qr}. But if $r \leq q$ then the first axiom (with $q' = q$ and $r' = r$) tells us that P_{qr} is equivalent to the empty disjunction $\bot$.*

There is a bijection between models of this theory and Dedekind sections of $\mathbb{Q}$. (We use a definition of Dedekind section in which both the lower and upper cuts are rounded, so the section for a rational q omits q on both sides. See Example 6.) If x is a model, then we obtain a Dedekind section $(\underline{x}, \overline{x})$, where $\underline{x} = \{q \mid \text{some } P_{qr} \text{ is true in the model } x\}$ and similarly for $\overline{x}$. In the other direction, if (L,R) is a Dedekind section, then we define a model x in which P_{qr} is true if $q \in L$ and $r \in R$. The proposition P_{qr} corresponds to the open interval (q,r), and so geometric formulae correspond to the opens in the usual topology. Then $\bigvee$, $\wedge$ and $\vdash$ correspond to $\bigcup$, $\cap$ and $\subseteq$.

When we move to predicate theories, an important and quite different family of examples is given by finitary algebraic theories. Logically these are very special, since the only connective they use is $=$. (An interesting generalization is that of cartesian, or essentially algebraic, theories. For these it can be natural to use partial terms, such as in Coste's *limit theories* [4]. [24] describes a simple adaptation of geometric logic to partial terms, in which cartesian theories correspond to the fragment with only $=$ and $\wedge$ as connectives.)

Example 3 (Commutative rings). *Take a signature with a single sort R, and function symbols*

$$\begin{aligned} 0, 1 &: 1 \to R \\ - &: R \to R \\ +, \cdot &: R^2 \to R. \end{aligned}$$

(Apologies for the overloading of 1. In the arity $1 \to R$, 1 denotes R^0 and so is the arity of a constant, with no arguments.)

All the algebraic laws of commutative rings can be expressed as geometric sequents of the form $\top \vdash^{\vec{x}} t_1 = t_2$. For example, distributivity is

$$\top \vdash^{xyz} x \cdot (y+z) = (x \cdot y) + (x \cdot z).$$

The next example needs $\vee$ and $\exists$ and so is neither purely topological nor purely algebraic. However, it does not need the infinitary disjunctions – it is a *coherent* theory.

Example 4 (Commutative local rings). *The signature is the same as for commutative rings, and the axioms are the same with, in addition,*

$$(\exists z)\ (x+y)\cdot z = 1 \vdash^{xy} (\exists z)\ x\cdot z = 1 \vee (\exists z)\ y\cdot z = 1$$
$$0 = 1 \vdash \bot.$$

These may be read as saying the invertible elements form the complement of a proper ideal. However, that would be a classical reading because it relies on having a classical notion of complement.

2.1. Inference rules

The inference rules of geometric logic are ones that derive sequents from sequents. We summarize them here, as presented in [17], but stress that there are few surprises.

Most of the propositional rules are standard ones for identity, cut, conjunction and disjunction:

$$\phi \vdash^{\vec{x}} \phi,\ \frac{\phi \vdash^{\vec{x}} \psi \qquad \psi \vdash^{\vec{x}} \chi}{\phi \vdash^{\vec{x}} \chi},$$

$$\phi \vdash^{\vec{x}} \top, \quad \phi \wedge \psi \vdash^{\vec{x}} \phi, \quad \phi \wedge \psi \vdash^{\vec{x}} \psi, \quad \frac{\phi \vdash^{\vec{x}} \psi \quad \phi \vdash^{\vec{x}} \chi}{\phi \vdash^{\vec{x}} \psi \wedge \chi},$$

$$\phi \vdash^{\vec{x}} \bigvee S \quad (\phi \in S), \quad \frac{\phi \vdash^{\vec{x}} \psi \quad (\text{all } \phi \in S)}{\bigvee S \vdash^{\vec{x}} \psi}.$$

We also need *frame distributivity* – which would be derivable from other rules if we had implication as a connective:

$$\phi \wedge \bigvee S \vdash^{\vec{x}} \bigvee \{\phi \wedge \psi \mid \psi \in S\}.$$

Turning to the predicate rules, the *substitution* rule is

$$\frac{\phi \vdash^{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash^{\vec{y}} \psi[\vec{s}/\vec{x}]}.$$

Here, $\vec{s}$ is a sequence of terms in context $\vec{y}$, matching the variables in $\vec{x}$ in number and in sorts. From the substitution rule we can also deduce *context weakening,*

$$\frac{\phi \vdash^{\vec{x}} \psi}{\phi \vdash^{\vec{x},y} \psi}.$$

The *equality* and *existential* rules are

$$\top \vdash^{x} x = x, \qquad (\vec{x} = \vec{y}) \wedge \phi \vdash^{\vec{z}} \phi[\vec{y}/\vec{x}],$$

$$\frac{\phi \vdash^{\vec{x},y} \psi}{(\exists y)\phi \vdash^{\vec{x}} \psi}, \qquad \frac{(\exists y)\phi \vdash^{\vec{x}} \psi}{\phi \vdash^{\vec{x},y} \psi}.$$

In the second equality rule, $\vec{z}$ has to include all the variables in $\vec{x}$ and $\vec{y}$, as well as those free in ϕ, and the variables in $\vec{x}$ have to be distinct.

Finally, again we need an unexpected *Frobenius* rule that would be derivable if we had implication as connective.

$$\phi \wedge (\exists y)\psi \vdash^{\vec{x}} (\exists y)(\phi \wedge \psi).$$

One point to note is that although we have context *weakening* (a sequent that holds in a smaller context will still hold in a bigger one), we do not have context "strengthening". We cannot drop variables from a context even if they are unused. The explicit listing in the sequent of a context of free variables, whether used in the formulae or not, enables the logic to have a satisfactory treatment of empty carriers. As an example, suppose in a theory we have $\top \vdash^{x} \phi$ as axiom. This asserts $(\forall x)\ \phi$ and is unproblematic for an empty carrier – it holds vacuously, in fact. Then we can derive

$$\frac{\top \vdash^{x} \phi \qquad \frac{\overline{(\exists x)\phi \vdash (\exists x)\phi}}{\phi \vdash^{x} (\exists x)\phi}}{\top \vdash^{x} (\exists x)\phi}$$

Again, this is unproblematic for the empty carrier. It says *for every element of the carrier* the proposition $(\exists x)\ \phi$ holds. But, even though neither formula $\top$ nor $(\exists x)\ \phi$ has free variables, we cannot derive $\top \vdash (\exists x)\ \phi$. That is just as well, for this sequent *would* be problematic with an empty carrier – it asserts that $(\exists x)\ \phi$ holds unconditionally. To summarize, many standard accounts of logic would have a valid inference $\frac{(\forall x)\ \phi}{(\exists x)\ \phi}$, which is incompatible with empty carriers, but we do not have the corresponding $\frac{\top \vdash^{x} \phi}{\top \vdash (\exists x)\ \phi}$.

2.2. Categorical semantics

The categorical semantics is standard, and is described in [17]. It allows us to talk about not only ordinary models, carried by sets, but also models in suitable categories, categories with enough structure for the logical connectives to be interpreted in a uniform way. We summarize it in this table.

Syntax	**Interpretation**
sort	object (carrier)
sequence of sorts, context	product of carriers
term in context	morphism
formula in context	subobject
$\wedge$	pullback
$=$	equalizer
$\exists$	image
$\bigvee$	image of coproduct
sequent	truth value (order relation between subobjects)

Note that the interpretation of a sequent is an external truth value, not something internal in the category. To interpret $\phi \vdash^{\vec{x}} \psi$, we interpret ϕ and ψ as subobjects of the carrier product for $\vec{x}$, and ask whether the subobject for ϕ is less than that for ψ.

We should also define the notion of *homomorphism* between two models, from M to N. It consists of a family of carrier morphisms $f_\sigma : M(\sigma) \to N(\sigma)$, one for each sort σ. This then extends to morphisms between finite products of carriers, and we also require that the appropriate product morphisms should commute with the interpretations of the function symbols and restrict to the interpretations of the predicates. In concrete set-theoretic terms, the condition on function symbols is the one familiar from algebra homomorphisms, while on predicates ϕ it is that if $\phi(a_1, \ldots, a_n)$ holds in M then $\phi(f_{\sigma_1}(a_1), \ldots, f_{\sigma_n}(a_n))$ holds in N (where σ_i is the sort of a_i). Note that if a function symbol is replaced by a predicate for its graph, together with axioms of totality and single-valuedness, then the homomorphisms are unchanged.

Clearly, for each topos its $\mathbb{T}$-models and homomorphisms form a category.

For a propositional theory there are no sorts and hence if there is a homomorphism, then it is unique: it is the empty family of carrier morphisms. Thus the category of models is a preorder. The question of *whether* there is a morphism from M to N is the question of whether each propositional symbol ϕ that is true in M is also true in N. In topological terms, where propositions are opens, this is the *specialization order* on points. Thus more generally for predicate theories, one can think of the homomorphisms as "specialization morphisms" between points.

It was mentioned earlier that a sober topological space has all directed joins of points (with respect to the specialization order), calculated as unions of the completely prime filters. The same goes for models of a propositional geometric theory. More generally, for a predicate theory we have all filtered colimits of points (with respect to the specialization morphisms).

We now briefly discuss the categorical structure needed in the semantic category where the models are sought. Some of this is already apparent: finite limits, arbitrary coproducts and images. Taking into account the need for the inference rules to be valid, the exact categorical structure needed is that of a *geometric category* [16]. (These do not even demand coproducts – disjunction is just join of subobjects.) However, in practice we use a more restricted class of categories, the Grothendieck toposes. These are cocomplete and also have the advantage of embodying a non-logical principle, of *unique choice:* every total, single-valued relation is the graph of a morphism. Categorically, it says that the category is *balanced*, i.e. that every morphism that is both monic and epi is an isomorphism (because monic and epi imply that the relational converse of the graph is total and single-valued). We shall see that principle in use in Proposition 5.

Actually, in discussing Grothendieck toposes we shall not be assuming an ambient logic of classical sets. The shift from finitary to infinitary logic begs the question categorically of what infinities are allowed, and the general answer in topos theory is to take them as being provided abstractly as the objects of a given base topos, which for us will need to be an elementary topos with natural numbers object. Then, relative to that, a Grothendieck topos is an elementary topos equipped with a bounded geometric morphism to the base. (*Bounded* is a condition that implies the Grothendieck topos can be got from an internal site in the base – see [16, B3.1.7].)

We shall use the phrase *topos-valid*, ostensibly for constructions that can be carried out in Grothendieck toposes; but in the light of the discussion above that would be parametrized by the choice of base topos and in practice our topos-valid constructions are those that can be carried out in elementary toposes with nno. The existence of an nno is sufficient to support other free algebra constructions [18], and this is enough to characterize many countable disjunctions internally, without reference to a base topos.

These ideas, taken with those of the next section, suggest a more radical choice of semantic category: that of Joyal's *arithmetic universes* [22]. These are not fit for arbitrary geometric theories, and there remain significant technical questions regarding their use, so we defer their discussion until Section 6. Nonetheless, they seem to cover geometric theories found in practice, including all our examples except for Example 2 (Example 6 must be used instead). They would also have the foundational advantage of not needing to explain "arbitrary" (i.e. set-indexed) in arbitrary disjunctions.

3. Geometric types and constructions

Unlike the case with finitary logic, the infinitary disjunctions allow some important set-theoretic constructions to be characterized up to isomorphism by geometric structure and axioms. These include some, but not all, of the topos-valid constructions, and we are led to a notion of geometric *mathematics*, a fragment of the internal mathematics of toposes. The next result shows how this works for one particular construction, that of list objects. If A is a set, then we write $\mathrm{List}(A)$ for its list object, the set of finite lists of elements from A. The categorical characterization can be found in [16, A2.5.15], but in fact we shall use the more general characterization of the *parametrized* list object – see [21] –, which is equivalent in cartesian closed categories such as toposes.

Proposition 5. *Let A and L be sorts in some geometric theory. Then L can be constrained to be isomorphic to the list type $\mathrm{List}(A)$ by functions $\mathsf{nil} : 1 \to L$ and $\mathsf{cons} : A \times L \to L$, together with axioms as follows:*

$$\mathsf{cons}(a, l) = \mathsf{nil} \vdash^{al} \bot$$
$$\mathsf{cons}(a, l) = \mathsf{cons}(a', l') \vdash^{ala'l'} a = a' \wedge l = l'$$
$$\top \vdash^{l} \bigvee_{n \in \mathbb{N}} (\exists a_0 a_1 \cdots a_{n-1})\ l = [a_0, a_1, \ldots, a_{n-1}]$$

where $[a_0, a_1, \ldots, a_{n-1}]$ is an abbreviation for $\mathsf{cons}(a_0, \mathsf{cons}(a_1, \ldots, \mathsf{cons}(a_{n-1}, \mathsf{nil}) \cdots))$. Obviously the formula on the right of the final axiom is not written in the strict syntax of geometric logic, but it is intended to suggest the recursive definition of a countable family of disjuncts. Note that it is an example of a geometric formula with infinitely many bound variables.

Proof. (Sketch) We show the universal property of the parametrized list object. Suppose we are given functions $f : B \to Y$ and $g : A \times Y \to Y$. We want there to be a unique $r = \mathsf{rec}(f,g) : L \times B \to Y$ making these diagrams commute.

$$\begin{tikzcd} B \arrow[r, "\langle \mathsf{nil} \circ !, B\rangle"] \arrow[dr, "f"'] & L \times B \arrow[d, "r"] & A \times L \times B \arrow[l, "\mathsf{cons} \times B"'] \arrow[d, "A \times r"] \\ & Y & A \times Y \arrow[l, "g"] \end{tikzcd}$$

Logically, we define the graph of r, a relation $\gamma \subseteq L \times B \times Y$, by

$$\gamma(l,b,y) \stackrel{def}{=} \bigvee_{n \in \mathbb{N}} (\exists a_0 a_1 \cdots a_{n-1}) \\ (l = [a_0, a_1, \ldots, a_{n-1}] \wedge y = g(a_0, g(a_1, \ldots, g(a_{n-1}, f(b)) \cdots))).$$

It is clear that if r exists at all, its graph has to be equivalent to γ. One next proves that γ is total and single-valued, and then appeals to unique choice to get the morphism r. ■

The *geometric constructions* on sets (or on objects of Grothendieck toposes) are the ones that can be characterized geometrically in this way. They include finite limits and arbitrary colimits, and in a sense that covers them all because of the way Giraud's Theorem characterizes Grothendieck toposes in terms of finite limits and arbitrary colimits. However, they also include free algebras – such as the list construction just described. This enables us to get $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, with their arithmetic and (decidable) order, and also the (Kuratowski) finite powerset $\mathcal{F}X$ along with finitely bounded universal quantification. (Examples of geometric theories that use $\mathcal{F}$ and finitely bounded $\forall$ can be found in [28] and [29].)

However, there are also topos-valid constructions that are *non-geometric*. These include exponentials (function types), powersets, and the reals (of various kinds) and complex numbers. Their non-geometricity may be seen concretely in the fact that in general they are not preserved by the inverse image functors of geometric morphisms. The problem lies not so much in the constructions themselves, but in viewing them as sets. In fact they can all be described as point-free spaces – there are geometric theories whose models are functions from X to Y, or subsets of X, or real numbers, but they naturally give a non-discrete topology. The non-geometric step – topos-valid, but not preserved by inverse image functors – is that of taking the *set* of points, i.e. imposing the discrete topology.

There are two different ways to view this notion of geometric types.

The first is as *syntactic sugar*. Knowing that these types can be characterized geometrically, it is legitimate to include them in presenting geometric theories. That is to say, when we declare a sort, we can also require it to be isomorphic to a geometrically constructed type; but we think of that as an abbreviation for some geometric structure and axioms so that at base it is all presented in pure geometric logic.

On the other hand one might also say that in essence geometric logic is a *type theory:* the type constructions are an intrinsic part of it. This is the idea behind an alternative definition of geometric theory given in [16, B4.2.7]. It also points a way towards *foundational simplification.* When we characterize a geometric type such as $\mathbb{N}$ in terms of geometric logic, we are in effect using the arbitrary (set-indexed) disjunctions to explain internal infinities – in the case of $\mathbb{N}$, the natural number object in the topos – in terms of external infinities, infinities in our ambient mathematics of sets. But we could take the geometric types, or a suitable selection of them, as a given part of geometric logic, characterized semantically by universal properties such as that used for parametrized list objects in Proposition 5. Once that is done, arbitrary disjunctions become less essential.

As an illustration of the use of geometric types, here is an alternative presentation of the real line, this time as a predicate theory.

Example 6 (The reals $\mathbb{R}$ – again). *Take a signature with one sort, the rationals $\mathbb{Q}$ (together with its strict order $<$), and two unary predicates $L, R \subseteq \mathbb{Q}$. The axioms are as follows.*

$$\begin{array}{ll}
\top \vdash (\exists q : \mathbb{Q})\ L(q) & \top \vdash (\exists q : \mathbb{Q})\ R(q) \\
L(q) \vdash\dashv^{q:\mathbb{Q}} (\exists q' : \mathbb{Q})\ (q < q' \wedge L(q')) & R(r) \vdash\dashv^{r:\mathbb{Q}} (\exists r' : \mathbb{Q})\ (r' < r \wedge R(r')) \\
L(q) \wedge R(q) \vdash^{q:\mathbb{Q}} \bot & q < r \vdash^{q,r:\mathbb{Q}} L(q) \vee R(r)
\end{array}$$

The models of this are the Dedekind sections, with lower and upper cuts L and R. The the top two axioms on the left or right say that L or R is an inhabited, rounded downset or upset respectively. The bottom left axiom says L and R are disjoint, and the bottom right ("locatedness") is used in proving that the cuts come arbitrarily close together. In Proposition 7 we shall see how the argument sketched in Example 2 can be used to show that this predicate theory is equivalent to the previous propositional one.

Note that specifying $\mathbb{Q}$ geometrically requires infinitary disjunctions, so the theory is not coherent. However, apart from that, all the disjunctions are finitary.

3.1. Ontology

By "ontology" I mean how you match the logic to whatever it is you are talking about, and in computer science the ideas of Samson Abramsky [1] and Mike Smyth in effect provided an ontology for propositional geometric logic in terms of observability. [25] uses this as the basis for its treatment of topology. Although it plays no role in the mathematical development of geometric logic, it has proved fruitful in motivating applications. The idea is that in a model of a propositional geometric theory, a formula is to be interpreted as a finitely observable property – let us say a finitely *ascertainable* property, meaning that if it holds then there is some possibility of ascertaining it in a finite way. (For a countable disjunction it will even be semidecidable, since there is a systematic way to try out all the disjuncts in parallel. For other infinities one should rather think of the process as serendipitous[1] since there may be no systematic way of

[1]Serendipity is "the faculty of making happy chance finds".

seeking out the situation in which the property is finitely ascertained.) The idea is that ascertainability is closed under finite conjunctions and arbitrary disjunctions, but not negation or implication. A sequent is not an ascertainable property, but a background assumption, or scientific hypothesis, about how observations interact with each other.

In fact one can see a Popperian idea of refutation here. Suppose a geometric theory $\mathbb{T}$ includes some axioms $\phi \vdash \bot$, making it refutable, and experimental observations over the same signature are expressed as a set $\mathbb{E}$ of sequents of the form $\top \vdash \psi$, because that is the general form of observations. If in $\mathbb{T} \cup \mathbb{E}$ we can infer $\top \vdash \bot$, then the theory $\mathbb{T}$ is refuted by the experiments $\mathbb{E}$. More carefully, either the experimental reality does not obey the axioms of $\mathbb{T}$, or there is a mismatch between the way the signature is interpreted for $\mathbb{E}$ and what was envisaged for $\mathbb{T}$.

The ontology extends to predicate logic, and this is discussed in some detail in [36]. The idea is that for an "observable set" you need two kinds of information about existence and equality: (1) how to ascertain when you have "apprehended" an element of the set, and (2) how to ascertain when two apprehended elements are equal.

For example, for a finitely presented group, to apprehend an element you write down a word in the generators, and to find equality you find a proof of equality from the relations. Note that if the word problem is undecidable, then *in*equality will not be ascertainable in the same sense.

As another example, if A is an observable set, then $\mathrm{List}(A)$ is observable in the following way. To apprehend an element, you get a natural number n, and, for each i with $0 \leq i < n$, apprehend an element a_i of $\dot{A}$. To ascertain $\langle n, (a_i)_{i=0}^{n-1} \rangle = \langle n', (a'_i)_{i=0}^{n'-1} \rangle$, you find $n = n'$ and ascertain $\bigwedge_{i=0}^{n} a_i = a'_i$.

In these terms we can often see clear reasons why non-geometric constructions are problematic. For example in a function space, equality is not in general finitely ascertainable: to ascertain $f = g$ we may have to ascertain $f(x) = g(x)$ for infinitely many x. Similarly, for real numbers x and y, apprehended as convergent processes of rational approximation, *in*equality (or apartness) is finitely ascertainable, but equality is not.

The ontology of $\exists$ is interesting. To apprehend an element of $(\exists y)\ \phi(x, y)$ you apprehend a and b, and ascertain $\phi(a, b)$ – in other words, the same as to apprehend an element of ϕ. But equality is different. To ascertain $(a, b) = (a', b')$ in $(\exists y)\ \phi(x, y)$, you just ascertain $a = a'$.

We can now see three different ontologies for the sequent $\psi \vdash^x (\exists y)\ \phi$, and in fact the principle of unique choice implies three corresponding ontologies for function symbols. Each starts with the assumption that you have apprehended some a (for the variable x) and ascertained ψ for it. Somehow that must entail the possibility of apprehending some b and ascertaining ϕ for (a, b). The strongest interpretation, generally too strong to be useful, is that apprehending a already involves apprehending b somehow. The constructivist interpretation is that there is some finite procedure for finding b from a. The observational interpretation, closer to scientific hypotheses, is that b is merely "out there somewhere".

4. Toposes as spaces

There's a very general idea in categorical logic, by which a theory gives rise to a "classifying category" that may somehow be thought of as the "space of models" of the theory. I must stress that the classifying category is *not* the category of models. In fact, the classifying category is a useful tool in situations where the logic is incomplete and the category of models (standard models in ordinary sets) is insufficient. This is important for geometric logic, which is incomplete, and in this case the classifying categories are the classifying toposes. There are some features of this approach that are very general, and apply for rather mundane categorical reasons. However, there are also some specific features in geometric logic that set it apart and support the slogan "continuity is geometricity".

In general, the technique is for a logic $\mathcal{L}$, using some specific categorical structure to interpret the logic and its rules, and with particular importance given to the functors that preserve that structure. We shall describe it for geometric logic and toposes, with the important functors being the ones that preserve finite limits and arbitrary (set-indexed) colimits – that is to say, the structure using which Grothendieck toposes are characterized in Giraud's theorem (see, e.g., [20]). Just temporarily (before moving to geometric morphisms) we shall call such functors *geometric*, and write $\mathrm{gfun}(C, D)$ for the category of geometric functors from C to D. (The morphisms are the natural transformations.)

Suppose we are given some geometric theory $\mathbb{T}$.

- For each topos C, there is a category $\mathrm{Mod}_{\mathbb{T}}(C)$ of models of $\mathbb{T}$ in C.
- For each geometric functor $F : C \to D$, there is a functor $\mathrm{Mod}_{\mathbb{T}}(F) : \mathrm{Mod}_{\mathbb{T}}(C) \to \mathrm{Mod}_{\mathbb{T}}(D)$.
- The *classifying topos* $\mathcal{S}[\mathbb{T}]$ is a topos equipped with a *generic* $\mathbb{T}$-model M_g. They are characterized by the property that for every topos C, the functor $\mathrm{gfun}(\mathcal{S}[\mathbb{T}], C) \to \mathrm{Mod}_{\mathbb{T}}(C)$, defined by $F \mapsto \mathrm{Mod}_{\mathbb{T}}(F)(M_g)$, is an equivalence of categories. $\mathcal{S}[\mathbb{T}]$ may be thought of as freely generated, as a Gothendieck topos (using finite limits and arbitrary colimits), by the generic model M_g.
- The trivial theory $\mathbb{T}_\emptyset$ (no signature, no axioms), which has a unique, vacuous model in any topos, is classified by **Set**. This is because **Set** is essentially initial with respect to geometric functors: for every topos C there is a geometric functor, unique up to unique isomorphism, from **Set** to C.
- Immediately from the characterization of classifying topos, we see that a geometric functor $\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$ is equivalent to a model of $\mathbb{T}_1$ in $\mathcal{S}[\mathbb{T}_2]$.

The same pattern applies to a range of other logics – see, e.g., [17, Section D1] –, and is most evident in propositional logics, where the categories playing the same role as toposes can be taken as posets. For example, for propositional classical, intuitionistic and geometric logic, the corresponding categories are Boolean algebras, Heyting algebras and frames respectively. Then the classifying category is the *Lindenbaum algebra* of formulae modulo logical equivalence.

Looking at propositional geometric logic in more detail, those categories, the frames, are complete lattices in which binary meet distributes over arbitrary joins (frame distributivity) and the corresponding functors are frame homomorphisms, functions preserving finite meets and arbitrary joins. Then a propositional geometric theory $\mathbb{T}$ is the same as a frame presentation by generators and relations – the generators are the propositional symbols in the signature, and the relations are the axioms. It presents a frame $\Omega[\mathbb{T}]$, which is the geometric Lindenbaum algebra for $\mathbb{T}$.

In predicate logic, we need categories. For example, for finitary algebraic, finitary cartesian and geometric theories, the corresponding categories are finite product categories, finite limit categories and toposes respectively.

Now we look at how to understand the classifying toposes as spaces of models. The trick is to work in the *opposite* of the category of toposes and geometric functors. Since a geometric functor F, preserving arbitrary colimits, has a right adjoint, we see that it corresponds to an adjoint pair $(f^* = F \dashv f_*)$ for which the left adjoint preserves finite limits, in other words a geometric morphism f in the opposite direction to F. Thus to use the (2-)category **Top** of toposes and geometric morphisms is to play this trick of taking the opposite category. Let us write $[\mathbb{T}]$ for $\mathcal{S}[\mathbb{T}]$ considered as an object of the opposite category: we wish to foster an illusion that it is "the space of models of $\mathbb{T}$". Thus we make a notational distinction between toposes as generalized spaces ($[\mathbb{T}]$) and toposes as generalized universes of sets ($\mathcal{S}[\mathbb{T}]$).

- The topos $\mathbf{Set} = \mathcal{S}[\mathbb{T}_\emptyset]$ becomes essentially terminal as $[\mathbb{T}_\emptyset]$ in **Top**. Let us denote it by 1 when we consider it in this opposite category.

- A *point* of $[\mathbb{T}]$ is defined to be a morphism $1 \to [\mathbb{T}]$, and that is equivalent to a model of $\mathbb{T}$ in $\mathcal{S}1 = \mathbf{Set}$.

- More generally, let us call a *generalized* point of $[\mathbb{T}]$ any morphism $C \to [\mathbb{T}]$. This is equivalent to a model of $\mathbb{T}$ in C.

- Hence (generalized) points of $[\mathbb{T}]$ are equivalent to models of $\mathbb{T}$ (in arbitrary toposes).

- A geometric morphism $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$ transforms points of $[\mathbb{T}_1]$ to points of $[\mathbb{T}_2]$ by $M \mapsto f \circ M$ $(M : C \to [\mathbb{T}_1])$.

- It also transforms models M of $\mathbb{T}_1$ (in C, say) into models of $\mathbb{T}_2$. f is a model $f(M_{g_1})$ of $\mathbb{T}_2$ in $\mathcal{S}[\mathbb{T}_1]$. But everything in $\mathcal{S}[\mathbb{T}_1]$ is constructed by colimits and finite limits out of the generic model M_{g_1} and those constructions are preserved by M as geometric functor, and it follows that the construction that in $\mathcal{S}[\mathbb{T}_1]$ constructs $f(M_{g_1})$ out of M_{g_1} also, in C, constructs the model for $f \circ M$ out of that for M. Hence the point transformer matches the model transformer.

Thus a map f, though formally a functor from $\mathcal{S}[\mathbb{T}_2]$ to $\mathcal{S}[\mathbb{T}_1]$, can be understood as a model transformer: it transforms the generic model M_{g_1} of $\mathbb{T}_1$ into a model $f(M_{g_1})$ of $\mathbb{T}_2$, and then the same construction works for all other models. In general logics the way in which $f(M_{g_1})$ is constructed out of M_{g_1} is closely bound to the syntax of the logic and is little real advance on thinking of a logical interpretation of $\mathbb{T}_2$ into formulae of $\mathbb{T}_1$. However, for geometric logic, if we make good use of the geometric types, the model transformer can look just like ordinary mathematics – albeit with constructivist restrictions.

For propositional geometric logic, the trick of using the opposite category is well known as locale theory. We are writing $[\mathbb{T}]$ for the locale whose frame of opens is $\Omega[\mathbb{T}]$. The maps $[\mathbb{T}_1] \to [\mathbb{T}_2]$ are the frame homomorphisms $\Omega[\mathbb{T}_2] \to \Omega[\mathbb{T}_1]$.

We can now exploit the argument above about model transformers to define geometric morphisms in a way that really makes them look like maps transforming models into models. (This was explained in detail in [28].) Suppose we want to define $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$. We can say:

Let x be a model of $\mathbb{T}_1$. Then $f(x) = \cdots$ is a model of $\mathbb{T}_2$.

As long as the $\cdots$, and the proof that it defines a model, are all geometric, then this defines a map f. First of all, this is because we can apply it to the generic model of $\mathbb{T}_1$ in the topos $\mathcal{S}[\mathbb{T}_1]$ to get a model of $\mathbb{T}_2$ in $\mathcal{S}[\mathbb{T}_1]$ and hence a map $[\mathbb{T}_1] \to [\mathbb{T}_2]$. But the geometricity of the construction also tells us that it is preserved by inverse image functors, and so the same construction in other toposes agrees with the point transformer got by composing with f.

Here is a sample application of the method.

Proposition 7. *The two theories of the reals presented in Examples 2 and 6 are equivalent.*

Proof. We outline the salient points. Fuller details are in [31].

Let us write $\mathbb{T}_1$ for the propositional theory of Example 2, and $\mathbb{T}_2$ for the predicate theory of Example 6. For clarity we shall distinguish here between the external rationals $\mathbb{Q}$ and the internal object Q of rationals in a topos, though in practice it is not necessary to be so fussy.

To define a map $\alpha : [\mathbb{T}_1] \to [\mathbb{T}_2]$, let x be a model of $\mathbb{T}_1$ (in any topos). For each pair of external rationals q, r, the model x provides us with a subobject of 1 (a truthvalue). From this we obtain, as outlined in Example 2, two subobjects $(\underline{x}, \overline{x})$ of Q. This relies on the isomorphism $Q \cong \coprod_{q \in \mathbb{Q}} 1$; this is shown in [31, Theorem 1.46] to be a general phenomenon relating internal and external infinities. After that, it is straightforward to prove that $(\underline{x}, \overline{x})$ is a model of $\mathbb{T}_2$.

In the other direction we define a map $\beta : [\mathbb{T}_2] \to [\mathbb{T}_1]$. Let (L, R) be a model of $\mathbb{T}_2$. Since each external rational corresponds to an internal morphism

$1 \to Q$, we obtain the $(\mathbb{Q}\times\mathbb{Q})$-indexed family of truthvalues for the propositions P_{qr}. It is reasonably straightforward to prove that these satisfy the axioms for Example 2, except for one: that

$$\top \vdash \bigvee\{P_{q-\varepsilon,q+\varepsilon} \mid q \in \mathbb{Q}\}$$

For this an induction is needed on natural numbers n such that $\varepsilon < 2^{-n}$. This arises from the particular choice of axioms used for the predicate theory $\mathbb{T}_2$, and is discussed further in Section 6.

Now that α and β have been defined, it is not hard to show that there are isomorphisms $\beta(\alpha(x)) \cong x$ and $\alpha(\beta(L,R)) \cong (L,R)$. We conclude that $[\mathbb{T}_1]$ and $[\mathbb{T}_2]$ are equivalent toposes. (We should not expect equality or isomorphism, since the universal property of a classifying topos characterizes it only up to equivalence.) ■

Thus, although the theory of Example 6 is not itself propositional, it is equivalent to one that is and so we say it is *essentially propositional.* A sufficient condition for this is that it does not declare any sorts other than ones that can be constructed geometrically out of the rest of the theory.

We now turn to a simpler example that illustrates the way continuity proofs are unnecessary.

Example 8 (Addition of reals). *Let x_1, x_2 be reals. To define $x_1 + x_2$ (as a Dedekind section) we must say which rationals q and r have $q < x_1 + x_2 < r$. We have $q < x_1 + x_2$ if $q = q_1 + q_2$ for rationals $q_i < x_i$. To express this as a geometric formula for a Dedekind section (L, R), in terms of Dedekind sections (L_i, R_i) for x_i,*

$$L(q) \stackrel{def}{=} (\exists q_1 q_2)(q = q_1 + q_2 \wedge L_1(q_1) \wedge L_2(q_2)).$$

The upper cut R is similar, and then the proof that (L, R) is Dedekind is straightforward.

Notice how frame theory does not enter into this description, nor is a continuity proof required. The sceptical reader may reconstruct the inverse image by

$$+^*(q, \infty) = \bigvee_{q=q_1+q_2} (q_1, \infty) \times (q_2, \infty).$$

As mentioned before, the technique works despite the fact that geometric logic is incomplete, so that a geometric theory may have an insufficiency of models in the conventional sense. Some non-trivial locales have no points at all. Thus global points (maps from 1) are inadequate for defining a map, but the geometricity means that the construction also applies to the generalized points, and there are enough of them – in fact, for what we just did, the generic point was enough in itself.

We have seen how *propositional* geometric theories $\mathbb{T}$ can be dealt with as theories for propositional geometric logic, with frames as the classifying categories and frame homomorphisms as the localic analogue of continuous map.

However, an important result of topos theory is that the classifying topos $\mathcal{S}[\mathbb{T}]$ is the category of sheaves over the frame $\Omega[\mathbb{T}]$ and geometric morphisms between the classifying toposes are equivalent to frame homomorphisms between the frames of opens. Hence it doesn't matter whether we think of the space $[\mathbb{T}]$ in the locale way, embodied by the frame $\Omega[\mathbb{T}]$ presented by $\mathbb{T}$, or in the topos way, embodied by the classifying topos $\mathcal{S}[\mathbb{T}]$. The space $[\mathbb{T}]$ *has* a frame of opens $\Omega[\mathbb{T}]$ and *has* a topos of sheaves $\mathcal{S}[\mathbb{T}]$, but we do not insist that it *is* either one of them. (Indeed, even a predicate theory has a frame of opens $\Omega[\mathbb{T}]$, the frame of subsheaves of 1, though in general the frame is not enough to determine the topos. For generalized spaces the opens are not enough, and we must use sheaves, i.e. objects of $\mathcal{S}[\mathbb{T}]$ – we think of $\mathcal{S}$ as standing for "sheaf".) Likewise, it doesn't matter whether we think of maps $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$ as embodied by frame homomorphisms or by geometric morphisms. We shall extend this notation to locales X and write ΩX and $\mathcal{S}X$ for the frame of opens and the category of sheaves.

Here is an example where the geometric constructions involve a non-propositional theory. The geometric point of view here is expanded in [35].

Example 9 (Sheaves). *Let the theory $\mathbb{T}_{ob}$ have one sort and no functions, predicates or axioms. A model of $\mathbb{T}_{ob}$ in a topos is simply an object of that topos, so $[\mathbb{T}_{ob}]$ is the space of sets – if we understand sets in a generalized way as objects of whichever topos we wish to work in. A map $[\mathbb{T}] \to [\mathbb{T}_{ob}]$ can then be understood either as an object of $\mathcal{S}[\mathbb{T}]$ or as a geometric construction of sets out of models of $\mathbb{T}$. In other words, a sheaf (object of the topos) is a "continuous set-valued map" – a map from $[\mathbb{T}]$ to the space of sets.*

The topos $[\mathbb{T}_{ob}]$ is usually called the object classifier.

5. Bundles

For simplicity we shall work now with propositional theories and locales, although the results apply more generally. A locale X will be a space $[\mathbb{T}]$ for some propositional (or essentially propositional) geometric theory, so the points of X are the models of $\mathbb{T}$. We have discussed geometricity of constructions on *sets:* constructions that are preserved by inverse image functors f^*. However, the notion generalizes to constructions on locales. We shall see that this has an important relativization effect, allowing us to deal with continuously indexed families of spaces (i.e. bundles). The theory of individual topological spaces easily gives results about bundles, as long as one adheres to geometricity constraints.

Definition 10. *Let Y be a locale. Then a* bundle *over Y is a map $p : X \to Y$ for some X. Morphisms of bundles are the usual commutative triangles.*

That looks too trivial to be useful, but it embodies a particular point of view. When we say that a map $p : X \to Y$ is a bundle, we are thinking of it as being an indexed family of spaces: for each point y of Y we have a fibre $p^{-1}(\{y\})$. It is given by the pullback y^*X in

$$\begin{array}{ccc} y^*X & \xrightarrow{p^*y} & X \\ {\scriptstyle y^*p}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow[y]{} & Y \end{array}$$

Actually, this is a generalized fibre for a generalized point, but the usual fibres arise the same way when $W = 1$.

The following result is of fundamental importance.

Theorem 11 (Localic Bundle Theorem). *Let Y be a locale.*[2] *Then there is an equivalence between –*

1. *the category of bundles over Y (the morphisms being the commutative triangles), and*
2. *the category of internal locales in $\mathcal{S}Y$.*

Proof. (Sketch) This has been proved by Fourman and Scott in [8] and by Joyal and Tierney in [19]. It relies on the fact that the theory of frames works satisfactorily in any elementary topos, with the arbitrary disjunctions in a frame A being given by a function $\bigvee : \mathcal{P}A \to A$.

It also relies on the fact that for any geometric morphism p, the right adjoint p_* preserves frames. Given $p : X \to Y$, the corresponding internal frame in $\mathcal{S}Y$ is got by applying p_* to the subobject classifier in $\mathcal{S}X$. Conversely, given an internal frame A in $\mathcal{S}Y$, ΩX is got as the external frame of its global elements. ■

This gives us an important principle for constructing bundles. Suppose we have a construction on frames that is topos-valid. Then it also gives a construction on bundles. For starting from a bundle $p : X \to Y$ we get a frame A in $\mathcal{S}Y$. We can apply our construction to that, giving another frame A' in $\mathcal{S}Y$, and hence another bundle $p' : X' \to Y$.

Now since we think of bundles as indexed families of spaces, we would really like such a construction to work fibrewise: in other words, we want to be able to see the indexes as just indexing the whole construction. Since the fibres are got as pullbacks along points, we should like the construction to be preserved at least by pullbacks along global points; but, actually, it is much more satisfactory if they are preserved by all pullbacks.

Definition 12. *A construction on localic bundles is* geometric *if it is preserved (up to isomorphism) by pullback.*

[2] It works quite generally for toposes Y, once one has the notion of localic bundle as localic geometric morphism p.

Actually there are some definite questions of coherence here – how the different pullbacks must fit together. Work is in progress to understand this better.

An important special case is for bundles that are local homeomorphisms (see [19] for the localic definition and proofs of the results; see also [35] for a development from a specifically geometric point of view). Under the correspondence of Theorem 11 these correspond to internal frames that are powerobjects $\mathcal{P}X$, i.e. discrete locales, and the correspondence between the local homeomorphisms and the objects X is essentially that well known for spaces between local homeomorphisms and presheaves with the sheaf pasting condition. Hence the local homeomorphisms are the bundle form of internal locales that are discrete. Such discreteness is geometric: local homeomorphisms are preserved under pullback. Hence local homeomorphisms are fibrewise discrete.

Now the inverse image functors f^*, when reinterpreted as acting on local homeomorphisms, act by pullback. Hence in the special case of bundle constructions for local homeomorphisms, geometricity under the new localic definition restricts to the old definition of preservation by inverse image functors.

Unfortunately, frames are not geometric objects. As we saw in the proof of Theorem 11, the structure of a frame A includes a join map $\bigvee : \mathcal{P}A \to A$, and the powerset construction $\mathcal{P}$ is not geometric.[3] Although p_* preserves frames, p^* does not. Hence geometricity of locale constructions (viewed through the localic bundles) cannot be deduced from geometricity of a corresponding frame construction. However, there is a useful way round this. Suppose an internal frame in $\mathcal{S}Y_2$ has an internal presentation as $\Omega[\mathbb{T}]$, where $\mathbb{T}$ is an internal propositional geometric theory. A simple general form is the *GRD-system* of [30] described by a (non-commutative) diagram

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ \mathcal{F}G & R \arrow[l, "\lambda"] \end{tikzcd}$$

and presenting a theory with propositional symbols in G and, for each $r \in R$, an axiom

$$\bigwedge \lambda(r) \vdash \bigvee_{\pi(d)=r} \bigwedge \rho(d).$$

This presents a frame $\Omega[\mathbb{T}]$ in $\mathcal{S}Y_2$ and hence gives a bundle $p_2 : X_2 \to Y_2$. Now suppose we have a map $f : Y_1 \to Y_2$. This gives an internal theory $f^*\mathbb{T}$ in $\mathcal{S}Y_1$ (using f^*G, f^*R and f^*D, and using the fact that the Kuratowski finite powerset $\mathcal{F}$, as free semilattice, is geometric) and hence an internal frame $\Omega[f^*\mathbb{T}]$ and a bundle $p_1 : X_1 \to Y_1$. What is the relation between p_1 and f^*p_2? On the face of it, we don't know, because the passage in $\mathcal{S}Y_2$ from $\mathbb{T}$ to $\Omega[\mathbb{T}]$ to p_2 is not all geometric and so is not preserved by pullback. However, it is proved

[3] This also creates a problem in predicative type theory, since $\mathcal{P}$ is impredicative.

in [30] that the connection between $\mathbb{T}$ and p_2 *is* geometric: p_1 is isomorphic to $f^* p_2$.

What this means is that one can define a bundle over Y as indexed space in the following style:

> Let y be a point of Y. Then $\mathbb{T}(y) = \cdots$ is a propositional geometric theory.

Here the $\cdots$ must be geometric. Then for each y the fibre over y is the space of models $[\mathbb{T}(y)]$.

Consequently, suppose we have a locale construction that can be described by a geometric construction on theories $\mathbb{T}$. Then it follows that the locale construction is geometric. (However, there are still questions to ask about whether the construction is presentation independent, since there may be quite different theories giving isomorphic locales.)

5.1. Example: powerlocales

Important examples of geometric constructions on locales are the Vietoris, upper and lower powerlocales, three kinds of localic hyperspaces. They all arise out of the Vietoris powerlocale described in [15]. See [27] for more information and history.

We shall focus here on the upper powerlocale and its relation to compactness, which, frame theoretically, can be defined as a natural reformulation of the finite subcover property: that if the top open $\top$ is a directed join, then it must already be one of the opens in the join.

A hyperspace is a space whose points are subspaces of some other space, and the localic notion of subspace is the sublocale. There are various technical ways to formulate these, but from a geometric point of view the most natural is generally as extra axioms added to the geometric theory, putting extra constraints on the points and hence restricting to a subspace. See [32]. It is then rather obvious that we have arbitrary meets of sublocales, by taking unions of sets of extra axioms.

An open U corresponds to an open sublocale, with a single added axiom $\top \vdash U$, and a meet of open sublocales is called *fitted*.

Now suppose X is a locale. Given a fitted sublocale Y, let F be the filter of opens whose corresponding sublocales are greater than Y, so clearly Y can be recovered as the meet of all the open sublocales corresponding to opens in F. It is also clear that if Y is compact, then F is Scott open. Johnstone's localic form of the Hofmann-Mislove Theorem (see [27] for a proof valid in elementary toposes) shows that this sets up a bijection between the Scott open filters of ΩX and the compact fitted sublocales of X.

If X is a locale then its upper powerlocale $P_U X$ has for its frame the free frame over ΩX "qua preframe", i.e. preserving finite meets and directed joins. Its global points are easily seen to be equivalent to the Scott open filters of ΩX, and hence to the compact fitted sublocales of X, and so $P_U X$ is a localic hyperspace. (There is an order reversal – high in the specialization order on $P_U X$ means a large Scott open filter but a small compact fitted sublocale.)

In these terms, another way of expressing compactness of X is that $\{\top\}$ is a Scott open filter, and so corresponds to a compact, fitted sublocale – it is X as a sublocale of itself, and therefore compact. Because of the order reversal, it is a bottom point in $P_U X$.

This treatment of compactness is closely bound to the frame, and therefore not geometric. However, it can be expressed in a geometric way using the fact that the upper powerlocale itself is a geometric construction of locales. This follows from results in [30]. The central point is that if X is presented as $[\mathbb{T}]$, then $P_U X$ is presented by a theory that can be constructed geometrically from $\mathbb{T}$. Now compactness can be expressed geometrically. As mentioned above, a compact X corresponds to a bottom point $\perp : 1 \to P_U X$, so the question now is when a bottom point $\perp$ corresponds to X as sublocale of itself. One can show [26] that this holds iff $\perp$ is "strongly bottom" in the sense of being less than every *generalized point* – alternatively, iff $\perp : 1 \to P_U X$ is left adjoint to the unique map $! : P_U X \to 1$. This condition is stable under pullback (now that we know P_U is geometric) and so gives a geometric criterion for compactness.

The lower and Vietoris powerlocales are also geometric, and the lower powerlocale gives a geometric account of the constructively important property of overtness (or openness) of locales. Further examples are the double powerlocale (see [30], which also sets out the general geometricity arguments), the connected Vietoris powerlocale [34] and the valuation locales [33], [3]. These latter two have been used in localic accounts of differentiation and integration.

See also [37], which uses the geometricity of the symmetric topos (see [2]; they call geometricity "equivariance") and arguments similar to those of Section 5.1 to give a geometric criterion for local connectedness.

6. Conclusions

My main take-home message is that geometric reasoning, when it can be done, is a powerful tool. By accessing the generalized points, it restores the points to point-free spaces, thus making localic reasoning much more pleasant; and by its applicability in toposes of sheaves, where point-set spaces have grave disadvantages, it provides a natural treatment of fibrewise topology of bundles.

Whether geometric reasoning can be applied in any given situation is, however, a non-trivial question, and work is in progress on case studies that have included domain theory, differentiation and integration. A recent project of my own at Birmingham (see [6]) is to test its applicability to the topos approaches to quantum foundations of [5] and [12] (see also [11], which explicitly identifies a desire for geometricity). Here bundles seem to enter naturally through the notion of states "in context". The base points of the bundle are contexts, or classical points of view, sets of observables that commute and so – by Gelfand-Naimark duality – are compatible with states in the sense of classical physics.

In making the geometric type constructors an intrinsic part of geometric logic, one might wonder whether one can dispense with the infinitary disjunctions. Except for Example 2, the examples in this paper make do with the free algebra constructions such as the list object and an otherwise finitary version of geometric logic. In fact, a start has been made in investigating such a logic, with a categorical semantics using Joyal's arithmetic universes – so the logic may be thought of as *arithmetic logic,* a fragment of geometric logic. Arithmetic universes are defined more precisely in [21] as *list arithmetic pretoposes,* i.e. pretoposes with parametrized list objects, and it is shown how the list objects enable the construction of other free algebras. (In an elementary topos, whose starting structure includes the non-geometric constructions of function types and powersets, a natural number object is enough.) I believe also that the techniques of [24], constructing free algebras for cartesian theories, will work in arithmetic universes. Then theories such as Example 6 may be considered as arithmetic theories, with models taken in arithmetic universes. This is by contrast with Example 2, with its explicit infinitary disjunctions, even though the two theories are equivalent for Grothendieck toposes. This radically simplifies the foundations needed, since arithmetic universes can be treated by finitary algebra: they are the models of a finitary, essentially algebraic theory.

Now the fact that Grothendieck toposes are elementary toposes, and have the non-geometric constructions of function types and powersets, makes it very much easier to reason geometrically with them: for it is often permissible to use the non-geometric (but still topos-valid) reasoning as long as the result being proved can be stated geometrically. This is an obstacle to transferring the techniques to arithmetic universes, which are not cartesian closed and do not have powerobjects. There the geometric reasoning has to be very pure. For example, frame theory does not work in arithmetic universes and so Theorem 11 does not hold in the way it is proved. Nonetheless, a start has been made in [22] in showing how to live in such a restrictive mathematics and still benefit from the geometricity ideas. For example, there is another version of Example 6, with the "locatedness" axiom in a different form

$$\varepsilon > 0 \vdash^{\varepsilon:\mathbb{Q}} (\exists q : \mathbb{Q})(L(q-\varepsilon) \wedge R(q+\varepsilon))$$

that corresponds more closely to the second axiom in Example 2. The proof that they are equivalent uses induction to prove a geometric sequent, which on the face of it requires cartesian closedness so that the sequent can be treated as a formula with implication. [22] shows how the same proof can also be justified in arithmetic universes.

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