pp. 288-314 in M.P. Fourman, P.T. Johnstone and A.M. Pitts (eds) *Applications of Categories in Computer Science* (Proceedings of the LMS Symposium, Durham 1991), London Mathematical Society Lecture Note Series **177**, Cambridge University press 1992.

Geometric Theories and Databases

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Abstract

Domain theoretic understanding of databases as elements of powerdomains is modified to allow multisets of records instead of sets. This is related to geometric theories and classifying toposes, and it is shown that algebraic base domains lead to algebraic categories of models in two cases analogous to the lower (Hoare) powerdomain and Gunter's mixed powerdomain.

Terminology

Throughout this paper, "domain" means algebraic poset – not necessarily with bottom, nor second countable. The information system theoretic account of algebraic posets fits very neatly with powerdomain constructions. Following Vickers [90], it may be that essentially the same methods work for continuous posets; but we defer treating those until we have a better understanding of the necessary generalizations to topos theory.

More concretely, a domain is a preorder (information system) (D, ⊑) of *tokens,* and associated with it are an algebraic poset pt D of *points* (ideals of D; one would normally think of pt D as the domain), and a frame ΩD of *opens* (upper closed subsets of D; ΩD is isomorphic to the Scott topology on pt D).

"Topos" always means "Grothendieck topos", and not "elementary topos"; morphisms between toposes are understood to be geometric morphisms.

S, italicized, denotes the category of sets.

We shall follow, usually without comment, the notation of Vickers [89], which can be taken as our standard reference for the topological and localic notions used here.

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Acknowledgements
It has taken me a long time to understand toposes well enough to be able to handle them at all mathematically. I think others will share my experience, for even now there is no introductory text that treats in depth the relation between toposes and *geometric* logic (as opposed to intuitionistic logic), this being the connection that is relevant to Grothendieck's original intuition of a topos as a generalized topological space. My thanks are therefore due to all the people who have ever told me anything about toposes. I should like in particular to mention: Martin Hyland, for his unfailing helpfulness; Mike Fourman, for his clear talk (reported as Fourman and Vickers [85]) on classifying toposes; Samson Abramsky, Axel Poigné and Mike Smyth, whose participation in a reading group at Imperial in 1985 helped me gain a toehold of understanding of Johnstone [77]; and Peter Johnstone and other participants at the Durham Symposium for their advice on technical matters.

I should also like to acknowledge the influence of Carl Gunter, whose elegant database account of powerdomains gave me both the intuitions (databases) and the mathematical context (generalized powerdomain theory) that I needed for a natural application of topos theory.

The work was made possible by the financial support of the British Science and Engineering Research Council for two research projects at Imperial: Formal Methods for Declarative Languages, and Foundational Structures for Computer Science.

1. Introduction

Gunter [89] gives a powerdomain theoretic account of some aspects of databases (or, perhaps more fairly, a database account of powerdomains). Let us first summarize the intuitions in the case of the lower (or Hoare) powerdomain.

The account starts with a domain D in which each real-world object considered by the database has a semantic value as a point. A *record* is then a token of the domain. It can be thought of in two ways:

(i) The token represents a compact point, and hence a possible finite approximation to the semantic values of objects (which may be more infinite in some sense).

(ii) The token represents a completely coprime open, and hence an observable property of points (namely, that they are approximated by the token's corresponding compact point; so if t is the token, we shall write $\uparrow$t for this property). Every other open is then a disjunction of those represented by tokens.

In other words, the token lives both in a straightforward domain theoretic world as a point, and in a logical world as a property. The logic is an observational logic, and it is *geometric* – that is to say, in this propositional case, its connectives are finite meets and arbitrary joins. It has been argued enough elsewhere (see Vickers [89]) that these connectives, unlike $\neg$ and $\rightarrow$, have a direct observational content.

The real world contains more than one object to be described, and in Gunter's account a database has a finite *set* of records to represent a set of objects. One can refine a database in two ways, either by refining the records (how to do this is described by the base domain), or by adding new records. These are encapsulated in the *lower* preorder on finite sets of tokens,

$$X \sqsubseteq_L Y \quad \Leftrightarrow \quad \forall s \in X.\ \exists t \in Y.\ s \sqsubseteq t$$

These finite sets of tokens are themselves the tokens for a new domain, the lower powerdomain P_L D. Such a set, X, can be seen either as a finite approximation to a reality comprising possibly infinitely many, possibly non-compact points, or as a property of that reality – that for each token s in X, there is a point in reality satisfying it (in the modal logic of powerdomains, X represents $\bigwedge_{s \in X} \Box\Diamond \uparrow s$).

There are curiosities here that arise from the use of sets. For instance, suppose a database has two records that are only partial:

{forename = "John"}
{surname = "Smith"}

In other words, the database has records of – apparently – two people, John and Smith. If it now turns out that Smith's forename is John, that record gets refined to

{forename = "John", surname = "Smith"}

But this record now subsumes the old record for John. The powerdomain semantics implies that the two databases

{forename = "John"}
{forename = "John", surname = "Smith"}

and

{forename = "John", surname = "Smith"}

are equivalent, so the computer might as well save space by dropping the record about John. But that's odd, because the original John was in fact John *Smythe,* a completely different person.

The proposal to be developed here is that the world is full of Johns, all different, and the database should be prepared to contain distinct copies of the record {forename = "John"}. In other words, the database is not a *set* of records, but a *bag*. (Many databases do indeed work this way.)

Definition 1.1 Let S be a set. A *bag* in S is a set X (the *base* set) equipped with a function from X to S, written $x \mapsto |x|$, the *value* of x.

It is also possible to define a bag by stating, for each element of S, what its multiplicity (possibly infinite) is in the bag. But the set X in the definition here makes concrete the "underlying distinct identities" of the elements of the bag and makes it easier to define bag morphisms.

Having accepted that a database should be a finite bag of records (tokens from the base domain), we should now like to extend the domain theory by defining a "bagdomain"; moreover, this will be a category rather than a domain. One possible definition of "bagdomain" ("categorical powerdomain") has been given by Lehmann [76] and developed by Abramsky [83], but we define a different notion. The relation between Lehmann's bagdomain and ours is roughly that between the upper and lower powerdomains.

Let us look carefully at how one bag, Y, can refine another, X. As before, we want the addition of extra elements to refine the bag. We also want to be able to refine a bag by refining its elements, so for each x in X we want some y in Y with $|x| \sqsubseteq |y|$. But remember also that the distinct elements of the bag were supposed to represent distinct objects in reality; so in making the refinement perhaps we should keep track of which element is which.

Consider –

X =	{surname = "Smith"}	(object x)
Y =	{surname = "Smith", age = 0}	(object y_1)
	{surname = "Smith"}	(object y_2)

Obviously Y can refine X, and it is very reasonable to suppose that it is by mapping x to y_2, with y_1 new. But it could equally well be by mapping x to y_1 with y_2 new, and these are really two different ways of making the refinement.

Definition 1.2 Let D be a domain, and let X and Y be bags in D (i.e. bags of *tokens*, representing *compact* points). A *refinement* from X to Y is a function $f: X \to Y$ such that $\forall x \in X.\ |x| \sqsubseteq |f(x)|$.

Note a subtle observational implication of this definition. If f is not onto, then we have refined X by adding new elements. This has already been mentioned. But if f is not 1-1, then the refinement says that two elements that were thought to be distinct have now been found to be the same. The idea of making this kind of observation is quite plausible, but it must be understood that it is a physical assumption about the systems we are trying to model. Another way of understanding it is that there must be an observational, intensional meaning to equality (physical identity of objects, not equality of their values). We shall see later that this helps to extend the observational content of propositional geometric logic to the predicate case.

Definition 1.3 Let D be a domain. Then the *lower bagdomain* over D, $B_L(D)$, is the category for which –

- objects are finite bags of tokens from D
- morphisms are refinements.

It is not hard to show that this actually is a category. It is also essentially small, for it is equivalent to its full subcategory in which the base sets of the bags are restricted to be finite subsets of the natural numbers, or indeed of any standard countable alphabet. We shall tacitly replace $B_L(D)$ by this small category equivalent to it, and use as the "standard countable alphabet" the same one as supplies the stock of logical variables; this will ease some technical proofs later on.

Our aim now is to show that this is a good categorical extension of powerdomain theory. The analogy is as follows.

- The category $B_L(D)$ is a categorical information system. Its objects are tokens and its morphisms are refinements.
- The categorical "ideal completion" (the analogy of pt D) is the *ind-completion* (see Johnstone [82]), whose objects are filtered diagrams in the base category. We shall show that for $B_L(D)$, the objects of the ind-completion are the arbitrary bags of possibly non-compact points of D, with refinements as morphisms.
- The observational theory for $P_L(D)$ is a *propositional* geometric theory, generated by propositions (nullary predicates) $\Diamond a$ ($a \in \Omega D$). For $B_L(D)$ this is replaced by a *predicate* geometric theory with unary predicates a(x) ($a \in \Omega D$). Abstracting away from its presentation, a propositional geometric theory can be identified with a frame, its geometric Lindenbaum algebra; but for a predicate theory one must instead use the more complicated *classifying topos*. (For a propositional theory, the classifying topos is the topos of sheaves over the corresponding locale.) We shall

show that the classifying topos for the theory here is equivalent to the functor category $S^{BL(D)}$ (where we write S for the category of sets). This is analogous to the localic proof with $P_L(D)$ that the locale whose frame is presented with generators $\Diamond a$ (i.e. given by a theory presentation with those as primitive propositions) is homeomorphic to the algebraic poset whose compact points are represented by finite sets of compact points of D under the lower preorder.

2. Geometric logic

We summarize here the principal ideas of geometric logic and topos theory that we shall use. For fuller references, see Johnstone [77] and Makkai and Reyes [77]; for another introduction for computer scientists see Fourman and Vickers [85].

A geometric theory is presented by –

- sorts
- primitive predicate and function symbols, each with an arity specifying the number and sorts of its arguments and (for a function) the sort of its result
- axioms of the form $\phi \vdash \psi$, where ϕ and ψ are geometric formulas, constructed using the primitive symbols, sorted variables and the geometric connectives $\wedge$ (and **true**), $\bigvee$ (including infinite disjunctions), = (sorted) and $\exists$.

Note the special case of a propositional theory, i.e. one with no sorts. The only primitive symbols are the nullary predicates (propositions), there can be no variables, and = and $\exists$ have no role to play. Such a presentation is exactly a presentation of a frame.

Formulas as sets

The crucial intuition is that a *formula* is a parametrized *set*, the parameter being a model. In other words, for a given theory there is a pairing Models × Formulas → Sets. It pairs a model M with a formula ϕ to give the *extent* of ϕ in M, the set of ways in which the free variables of ϕ can be instantiated in M to make ϕ true.

We write this extent as $\{M|\phi\}$ (the notation was suggested by Dirac's [47] bras and kets). This can be thought of as $\{x \in M \mid \phi(x)\}$, though x is really a vector of all the free variables in ϕ, so that $\{M|\phi\}$ is a set of tuples of elements of the appropriate carriers of M. (Remember that the theory may be many-sorted.)

A formula with *no* free variables has an extent that is a subset of M^0, which is a singleton set {*}. Classically, either M satisfies the formula, in which case the extent is all of {*}, or it doesn't, and the extent is Ø. Also, a sort can be represented by the formula x=x; {M|x=x} is the carrier of M for the sort of x.

For both models and formulas, we have a notion of morphism and the pairing {M|ϕ} is functorial in both arguments.

For models, the morphisms are homomorphisms in an obvious sense (with functions mapping carriers to carriers, and preserving the operations and predicates). Then for a homomorphism from M to N we get for each ϕ a corresponding function from {M|ϕ} to {N|ϕ} *because ϕ is geometric.*[*] (To see an example where this breaks down for non-geometric formulas, consider $\phi \equiv \forall y.(x \cdot y = y \cdot x)$ in the theory of monoids. This states that x is central (commutes with all other elements), so {M|ϕ} is the centre of M; but homomorphisms of monoids do not necessarily map centres to centres.)

For formulas, once we have agreed that they are sets, the morphisms should be functions; the idea is to use their graphs. If ϕ(x) and ψ(y) are formulas (x and y here may be vectors of variables), then a function from ϕ to ψ is a formula θ(x,y) satisfying –

- $\theta(x, y) \vdash \phi(x) \wedge \psi(y)$ (ϕ and ψ are the source and target of θ)
- $\theta(x, y) \wedge \theta(x, y') \vdash y = y'$ (single-valuedness)
- $\phi(x) \vdash \exists y.\ \theta(x, y)$ (totality)

When we parametrize by (take extents in) a model M, these conditions ensure that {M|θ} is the graph of a function from {M|ϕ} to {M|ψ}.

The category of formulas, the *syntactic category* of the theory, has as objects the formulas (actually, formulas modulo relabelling of variables) and as morphisms the function formulas (actually modulo provable equivalence).

Because the formulas are parametrized sets, one can imagine applying set constructions to them, such as products, equalizers, unions, etc. Some of these can be done just using the logical connectives. For instance, Cartesian product and intersection can both be constructed using conjunction:

$$\{M|\phi(x)\} \cap \{M|\phi(x)\} = \{M|\phi(x) \wedge \phi(x)\}$$
$$\{M|\phi(x)\} \times \{M|\phi(x)\} = \{M|\phi(x) \wedge \phi(y)\}$$

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[*] This also holds when infinite conjunctions are used in the construction of ϕ, though we shan't use this.

Others can not. Some, such as complements and function spaces, are essentially non-geometric. But some, notably disjoint union, are geometric in flavour and can be added. In the next section, we shall describe more precisely, though in categorical terms, what are "geometric set constructions".

Although the syntactic category does not contain all the geometric set theory, its logical structure is very convenient for reasoning with. This is illustrated in our main Theorems, 3.1 and 4.2, where in order to interpret certain geometric theories in terms of other ones, it suffices to define functors into the syntactic category.

"Giraud frames"

Makkai and Reyes [77] use the phrase "Giraud toposes" temporarily as a description of categories *E* satisfying the following conditions (see Johnstone [77] p. 17 for more details) –

- *E* has all finite limits
- *E* has all small colimits, and they are universal (preserved under pullback)
- Coproducts in *E* are disjoint, i.e. the injections pull back pairwise to the initial object Ø
- Epimorphisms out of an object are equivalent to "equivalence relations" on that object
- *E* is locally small, i.e. each "hom-set" is indeed a set
- *E* has a set of generators, i.e. a set *G* of objects such that if f and g are distinct morphisms from X to Y, then there is some h: G → X with G in *G* such that h;f ≠ h;g

The last two conditions are size conditions, needed to construct a small site for which the Giraud topos can be the category of sheaves. The first four are the categorical embodiment of "geometric set theory": so the constructions wanted are finite limits (which can already be done by logic, in the syntactic category), and all small colimits (which cannot).

Makkai and Reyes use their term "Giraud topos" only while they are proving Giraud's theorem, that Giraud toposes and (Grothendieck) toposes are the same – the corollary to this, that every Giraud topos is an elementary topos, is somewhat remarkable, for nowhere in the definition are mentioned Cartesian closedness or subobject classifiers.

Let us try to give a more permanent usefulness to the notion of Giraud topos by making the same distinction as there is between frames and locales: the morphisms go in opposite directions. We therefore rename Giraud toposes as *Giraud frames:* categories *E* satisfying the conditions given above. A homomorphism of Giraud frames will then be a functor that preserves this geometric structure of finite limits and all colimits; or, rather, let us express the preservation of colimits by the possession of a right adjoint, so that a homomorphism of Giraud frames from E to F is an adjoint pair (f^*, f_*), $f^*: E \to F$ and $f_*: F \to E$ such that f^* preserves finite limits. f^* (the *inverse image*) is the primary part, the structure-preserving functor, and in fact we shall often talk as though f^* *is* the Giraud frame homomorphism.

Note that these homomorphisms do not necessarily preserve other structure of elementary toposes, such as exponentials and subobject classifiers. An exact analogue (the propositional case) is with frames: frames are, as it happens, complete Heyting algebras, but frame homomorphisms are only required to preserve joins and finite meets and do not necessarily preserve the Heyting arrow.

Now a locale is a frame "pretending to be a topological space", and this pretence is maintained in part by the morphism reversal. The same can be done with toposes. Grothendieck says that a topos is a generalized topological space. More precisely, a topos is a generalized *locale,* and in a directly generalized way it too pretends to be a space, its "space" of models – to aid the pretence, the set-theoretic models are called *points* of the topos. The reason that this can only be a pretence is that we do not identify toposes on the mere grounds that they have the same space of points – for instance, some non-trivial toposes don't have any points at all. Again, to maintain the pretence, the appropriate morphisms between toposes, the *geometric morphisms,* are defined to be homomorphisms of Giraud frames, but in the opposite direction. We can now say with precision that toposes (and geometric morphisms) are to locales as Giraud frames (and homomorphisms) are to frames.

The classifying topos

Let us first recall the definition of the classifying topos, and then try to elucidate it.

Let $\mathbb{T}$ be a geometric theory. A topos $S[\mathbb{T}]$ is a *classifying topos* for $\mathbb{T}$ iff –

- $S[\mathbb{T}]$ has a specified model of $\mathbb{T}$, the *generic* model, and
- if E is any other topos with a model of $\mathbb{T}$, then there is a unique (up to equivalence) geometric morphism f from E to $S[\mathbb{T}]$ such that f^* maps the generic model to the given model in E.

Fact *Every geometric theory has a classifying topos (unique up to equivalence).*
Proof See Makkai and Reyes [77]. (Johnstone [77] gives a proof for coherent theories, i.e. geometric theories in which no infinite disjunctions are used in the axioms). ⟧

$S[\mathbb{T}]$ is to be the Giraud frame freely generated by the primitives of $\mathbb{T}$ (as sets and functions) subject to the relations expressed in the axioms in $\mathbb{T}$. To say that we have interpreted the primitives in $S[\mathbb{T}]$ (sorts as objects, predicates as subobjects of products of sort objects, functions as morphisms) and satisfied the axioms is precisely to say that we have a model of $\mathbb{T}$ in $S[\mathbb{T}]$. To say that this is done freely is to imply a universal property, that if we have a model of $\mathbb{T}$ in any other Giraud frame *E,* then there is a unique (up to equivalence here) homomorphism from $S[\mathbb{T}]$ to *E* that preserves the generators, and in the language of toposes and geometric morphisms this is as stated in the Definition.

2.1 Flat functors and Diaconescu's Theorem

Consider a domain D. A point is an ideal of D, but it can be viewed another way: it is a function f from D to $\mathbb{2} = \wp(\{*\})$, mapping s to $\{*\}$ iff s is in the ideal, and this function is antitone and satisfies

$$\{*\} = \bigcup\{f(u) \colon u \in D\}$$
$$f(s) \cap f(t) = \bigcup\{f(u) \colon s \sqsubseteq u, t \sqsubseteq u\} \qquad (s, t \in D)$$

The first of these says that there is some u such that $* \in f(u)$, i.e. the ideal is non-empty. The second says that if $* \in f(s)$ and $* \in f(t)$, i.e. s and t are both in the ideal, then they have some upper bound in the ideal: in other words, the ideal is directed.

These translate directly into the localic presentation of ΩD (see Vickers [90]; ub(S) is the set of upper bounds of S):

$$\Omega D = \mathrm{Fr}\,\langle\, \uparrow\{s\}\ (s \in D) \mid \textstyle\bigwedge_{s\in S} \uparrow\{s\} = \bigvee_{u\in ub(S)} \uparrow\{u\}\ \ (S \subseteq_{fin} D)\,\rangle$$

so that the points of the locale are exactly the ideals of D.

Our aim now is very quickly to present the categorical version of this idea. Throughout this discussion, let D be a small category.

First, the *ind-completion,* Ind-D, is the analogue of the ideal completion of a poset. A very good account is given in Johnstone [82]. Its objects are filtered diagrams in D, considered as formal representatives of their colimits. The morphisms are defined between the diagrams in such a way as to represent morphisms between colimits.

Now the Yoneda embedding from D into $S^{D^{op}}$ (where *S* is the category of sets) extends to Ind-D and in fact gives an equivalence between Ind-D and a full subcategory of $S^{D^{op}}$. Johnstone describes this as the category of "filtered colimits of representable functors", but that is for the sake of technical simplicity. There is an alternative characterization (see Johnstone [77]) of these functors as *flat.*

Definition 2.1

(i) Let G be a category. G is *filtered* iff every finite diagram in G has a cocone over it, i.e.

- G is nonempty.
- Given any two objects X and Y in G, there can be found a third object Z and morphisms f: Y → Z, g: Y → Z.
- Given two morphisms f, g: X → Y, there can be found a third morphism h: Y → Z with f;h = g;h.

(ii) Let D be a category, and let F: $D^{op} \to \mathcal{S}$ be a functor.
The *Grothendieck construction* on F, Groth F, is the category in which

- an object is a pair (x, X) where X is an object of D and x ∈ F(X)
- a morphism from (x, X) to (y, Y) is a morphism f: X → Y in D such that x = F(f)(y)

F is *flat* iff Groth F is filtered.

Example 2.2 Let D be a poset. Then flat functors F: $D^{op} \to \mathcal{S}$ are equivalent to ideals of D.
Proof Let F be flat. First, note that F(X) is a singleton for all X in D, for suppose x, y ∈ F(X). Consider the objects (x, X) and (y, X) in Groth F. By flatness, there is an object (z, Z) and morphisms f and g from (x, X) and (y, X) respectively to (z, Z). But we must have f = g in D, so then x = F(f)(z) = F(g)(z) = y. Let I be the set of X in D such that F(X) ≠ Ø. Again by flatness, it follows that I is an ideal. ⟧

The next step is to observe that *flatness is geometric*.

Definition 2.3 The geometric theory Flat(D) has –

sorts: for each object X of D, a corresponding sort
functions: for each morphism f: X → Y in D, a corresponding function f: Y → X
axioms:
$\vdash \mathrm{Id}(x) = x$
$\vdash (f;g)(z) = f(g(z)) \qquad$ (f: X → Y, g: Y → Z in D)
$\vdash \bigvee_{X \in D} \exists x{:}X.\ x = x$
$\vdash \bigvee_{Z \in D} \bigvee\{\exists z{:}Z.\ (x = f(z) \wedge y = g(z)) : f: X \to Z,\ g: Y \to Z \text{ in } D\}$
$f(y) = g(y) \vdash \bigvee_{Z \in D} \bigvee\{\exists z{:}Z.\ (y = h(z)) : h: Y \to Z,\ f;h = g;h\}$
(f, g: X → Y in D)

A model of Flat(D) in $\mathcal{S}$ is precisely a flat functor from D^{op} to $\mathcal{S}$. The object part is described by the carriers, and the morphism part by the interpretation of the functions. The first two axioms say that it is a (contravariant) functor, and the remaining three that it is flat.

Theorem 2.4 (Diaconescu's Theorem)

Let D be a small category. Then the classifying topos for Flat(D) is S^D.

The generic model is the Yoneda embedding of D contravariantly into S^D.

Proof In other words, for any topos F there is an equivalence between models of Flat(D) in F and geometric morphisms from F to S^D. This is Theorem 4.34 in Johnstone [77]. To see why, first note that Johnstone covers not just Grothendieck toposes but more generally toposes "defined over" an elementary topos E. We are interested in the case where E = S, the category of sets. The geometric morphism f: F → S has for its inverse image part the functor f*: S → E that maps a set X to the coproduct in F of |X| copies of 1; f* maps the small category D, an internal category in S, to an internal category f*D in F. Now Johnstone shows that geometric morphisms from F to S^D are equivalent to "flat internal presheaves on f*D", so there is a gap: we ought to show that flat internal presheaves on f*D, which are defined categorically, are equivalent to the models in F of Flat(D), which are defined more in logical terms.

This is a gap of exposition rather than of real mathematics. A development of topos theory using the Mitchell-Bénabou language would naturally define the flat internal presheaves in terms similar to Definition 2.3. Johnstone took a deliberately categorical approach, and presented the "obvious" categorical formalization of the logical ideas. Nonetheless, once the treatments – logical and categorical – have diverged it seems a non-trivial exercise to show that corresponding points on them are genuinely equivalent.

The basic trick here is that because coproducts are preserved by pullback in a topos, and because we are interested in flat internal presheaves over the category f*D whose object and morphism families are copowers of 1, the objects used in the course of defining "flat internal presheaves" can be decomposed as coproducts indexed by structures from the external category D. (Recall also that in a Giraud frame colimits are universal, i.e. preserved by pullback.) ⟧

2.2 Geometric logic as observational logic

In Abramsky [87] and Vickers [89], it is argued that disjunction and finitary conjunction, the connectives of propositional geometric logic, have observational content in that observability of properties is preserved by these connectives. The notion of "observability" is not formally defined, but it is intended to capture these two ideas:

- *positivity:* if an observable property holds, then it is possible to observe it; but if it fails, there need not be any way of ever discovering that.
- *serendipity:* one is told how to know in retrospect when one has observed something, but not any method that's guaranteed to result in the observation whenever possible.

(Serendipity perhaps represents the distinction between this notion of observability and semi-decidability.) We wish here to extend these intuitions to full predicate geometric logic. Although this section is necessarily informal, we hope that it helps to answer the question "What use is geometric logic in the real world?"

In the framework outlined above, $\{M|\phi\}$ (when ϕ is a proposition) is a subset of the singleton $\{*\}$. Hence observing ϕ is equivalent to discovering an element of this subset. We should like to extend this to predicate logic by the notion of "apprehending" elements of the sets $\{M|\phi\}$ (for general ϕ), i.e. observing the existence of elements and moreover getting some kind of grasp on them so that they can be related to others. We also want to be able to observe equality between elements, though this is an *intensional* equality – two twin elements may be indistinguishable in all respects, but still not equal.

We therefore propose the idea of "observability set", specified by two methods (both interpreted positively and serendipitously):

- how to apprehend elements of it
- how to observe equality between two elements of it

Equality (i.e. possibility of observing equality) must be an equivalence relation.

This idea is actually not very different from Bishop's (Bishop and Bridges [85]) account of sets as "the totality of all mathematical objects constructed in accordance with certain requirements … endowed with a binary relation = of *equality,*" though in generalizing "construction" to "apprehension" (one can plausibly apprehend an element by constructing it, though not necessarily the other way round) we are trying to be more neutral about the physical reality.

Similarly, we drop Bishop's constructional or operational import from the notion of function.* A function $f: X \to Y$ is simply its graph, a subset of $X\times Y$, a retrospective way of observing that for two elements x and y we have "y=f(x)". (To apprehend an element of f you apprehend a pair (x,y) and observe that "y=f(x)"; to observe that (x,y) = (x',y') you observe that x = x' and y = y'. We shall preserve the quotes round "y=f(x)" to make it plain that this is not an instance of the the equality observation.) Of course, this method must be extensional (with respect to =), single-valued and total, though these are matters for proof rather than observation.

It is now possible to justify the first four conditions in the definition of Giraud frame in terms of observability sets, to reason informally that observability

* For this reason, we believe that geometric logic has a role to play in *specification* of computer programs.

sets are preserved under finite limits and small colimits and that they satisfy the properties relating these.

For instance, consider how one could construct the equalizer of two functions f, g: X → Y. We must describe the equalizer E, the function h: E → X, and the mediating functions as observability sets, and we must also prove that these satisfy the right properties.

To apprehend an element of E, apprehend elements x and y of X and Y, and observe that "y=f(x)" and "y=g(x)". To observe that (x,y) = (x',y'), just observe that x = x' (though it will follow from single-valuedness of f and g that it is then possible to observe that y = y').

To observe (having apprehended x', x and y) that "x'=h(x, y)", observe that x' = x. One can then prove that h is a function. Moreover, h;f = h;g: for suppose it is possible to observe "y'=(h;f)(x,y)". The construction of composite functions is that to observe "c=(φ;ψ)(a)" you must apprehend some b and observe that "c=ψ(b)" and "b=φ(a)". In our case, therefore, it is possible to apprehend some x' and observe that "y'=f(x')" and x' = x. But (x,y) ∈ E, so we already know that "y=f(x)" and "y=g(x)". By extensionality and single-valuedness of f, it is possible to observe that y' = y and hence by extensionality of g that "y'=g(x')". It follows that it is possible to observe that "y'=(h;g)(x,y)". By symmetry it follows that it's possible to observe "y'=(h;f)(x,y)" iff it's possible to observe "y'=(h;g)(x,y)", i.e. h;f = h;g.

Now suppose we have a function k: Z → X such that k;f = k;g. We want to construct the mediating function k': Z → E such that k = k';h. (Note that these equalities between functions are not themselves observable, since they relate sets, not elements.) To observe that "(x,y)=k'(z)", observe that "x=k(z)". Again, one can prove that k' is a function, that k = k';h, and that k' is the unique such.

We shall not give the details of the rest of the proofs needed, but restrict ourselves to describing the constructions of products, coproducts and coequalizers.

Products: To apprehend an element of X×Y, apprehend a pair (x,y) of elements from X and Y. To observe that (x,y) = (x',y'), observe that x = x' and y = y'. The projections and mediating functions are all obvious. (Note also the nullary product 1: to apprehend an element of it or to observe two elements equal you need do nothing. Hence it has a single element that exists of its own accord.)

Coproducts, $X = \coprod_{\lambda \in \Lambda} X_\lambda$: Note that Λ is not an observability set, but an ordinary discrete set in which equality and inequality are perfectly known. To apprehend an element (x,λ) of X, apprehend an element x of X_λ for some λ. To observe $(x,\lambda) = (y,\lambda)$, observe that $x = y$ in X_λ; if $\lambda \neq \mu$ then to observe that $(x,\lambda) = (y,\mu)$ is impossible.

Coequalizers: Let f, g: $X \to Y$ be two functions. We wish to construct the coequalizer h: $Y \to C$. To apprehend an element of C, just apprehend an element of Y. To observe that $y = y'$ in C, apprehend elements $y_0, y_1, \ldots, y_n$ of Y and elements $x_1, \ldots, x_n$ of X, observe that $y = y_0$ and $y' = y_n$, and for each i ($1 \leq i \leq n$) observe either that "$y_{i-1}=f(x_i)$" and "$y_i=g(x_i)$" or that "$y_{i-1}=g(x_i)$" and "$y_i=f(x_i)$".

The thesis, then, is that geometric theories can be satisfactorily used to formalize informal (or real-world) structures that seem to contain the ingredients of observability sets. Each sort will then represent (or be interpreted in a real-world model as) an observability set, each predicate symbol an observability subset of a product of sort sets, and each function symbol a function as described above. The axioms – including axioms to specify functionhood of functions – will be constraints on the admissible models, justifiable on physical grounds for the physical models that we have in mind. They are not in themselves observable.

The classifying topos will contain not only the observability sets specified in the theory presentation (the generic model), but also all others that can be derived using limits and colimits as described above.

The remainder of this paper attempts to apply these intuitions to databases. The idea is that *people* in the real world can be considered an observability set. To apprehend a person, you take a firm grasp of his or her collar and say "'Allo, 'allo, 'allo, what's going on 'ere, then?" To show that two apprehended people are equal, you try to knock their heads together and discover that you can't. In the examples dealt with, the only other primitive observability sets are subsets of this, and propositional observations. There are no observations other than equality to relate different people. This is rather a simple case, but we hope that the same intuitions can be used in other contexts, and in particular for specifying software systems.

3. The lower bagdomain

Theorem 3.1 *Let D be a domain. Then the theory Flat ($B_L(D)$) is equivalent to the single-sorted theory T presented by:*

predicates:	$a(x)$	$(a \in \Omega D)$
axioms:	$a(x) \vdash b(x)$	$(a \leq b \text{ in } \Omega D)$
	$\bigwedge_{a \in S} a(x) \vdash (\bigwedge_{a \in S} a)(x)$	$(S \subseteq_{fin} \Omega D)$

$$(\bigvee_{a \in S} a)(x) \vdash \bigvee_{a \in S} a(x) \qquad (S \subseteq \Omega D)$$

Proof We must show that $\mathcal{S}^{B_L(D)}$ (the classifying topos of Flat ($B_L(D)$), by Diaconescu) is equivalent to $\mathcal{S}[T]$, finding a model of each theory in the other's classifying topos in order to describe the geometric morphisms between them.

First, we describe a flat functor F from $B_L(D)^{op}$ to $\mathcal{S}[T]$. This automatically extends to a Giraud frame homomorphism F: $\mathcal{S}^{B_L(D)} \to \mathcal{S}[T]$ (i.e. a geometric morphism from $\mathcal{S}[T]$ to $\mathcal{S}^{B_L(D)}$; F is the inverse image part).

Let X be an object of $B_L(D)$, a finite bag of tokens of D. Then

$$F(X) = \bigwedge_{x \in X} (\uparrow |x|)(x)$$

We are assuming here that the stock of elements from which the base sets of the bags are formed is actually *the same as* the stock of variables used in the logic. If $x \in X$ then $|x|$ is a token of D, and $\uparrow|x|$ is the corresponding open. Note that because the objects of the syntactic category (formulas) are defined only up to renaming of variables, we may always rename the base elements of bags to ensure that base sets are disjoint.

Let f: X → Y be a morphism in $B_L(D)$, i.e. a refinement; by renaming, we can assume that X and Y are disjoint. Then F(f) is the function formula

$$F(X) \wedge F(Y) \wedge \bigwedge_{x \in X} (x = f(x))$$

Note that "f(x)" here is not a composite term, but an actual variable. The refinement f maps variables to variables. Also, because $|x| \sqsubseteq |f(x)|$, this formula is equivalent to $F(Y) \wedge \bigwedge_{x \in X} (x = f(x))$.

The identity morphism is mapped to the identity function. As for composition, suppose also that g: Y → Z in $B_L(D)$. Then $F(f) \circ F(g)$ is $\exists \mathbf{y}.(F(f) \wedge F(g))$; when this is expanded as a logical formula, it is easily shown to be equivalent to F(f;g).

Next, we must prove flatness. The first flatness axiom, follows from the easily verified $\vdash \exists \mathbf{x}\colon F(\emptyset).\ \mathbf{x} = \mathbf{x}$ where Ø is the empty bag. (The bold face **x** represents the – empty – vector of free variables in F(Ø).)

For the second flatness axiom, let X and Y be bags of tokens with disjoint base sets and let Z be a disjoint union of X and Y with injections f: X → Z, g:□Y□→ Z. We can assume Z is disjoint from X and Y and write x' = f(x), y' = g(y). Then it suffices to show that in $\mathcal{S}[T]$,

$$F(X) \wedge F(Y) \vdash \exists \mathbf{z}\colon F(Z).\ (\mathbf{x} = f(\mathbf{z}) \wedge \mathbf{y} = g(\mathbf{z}))$$

For the third flatness axiom, let X and Y be disjoint bags of tokens, with refinements f, g: X → Y. The left hand side of the axiom is interpreted as

$\exists \mathbf{x}.(F(f) \wedge F(g))$, which is equivalent to $F(Y) \wedge \bigwedge_{x \in X} f(x) = g(x)$. Let ~ be the equivalence relation on Y generated by the pairs $f(x) \sim g(x)$ ($x \in X$), and let Z_0 be Y/~: as a set, this is the coequalizer of f and g. Choose a representative element of each equivalence class, and let Y_0 be the set of chosen representatives. Then our formula is equivalent to

$$\begin{aligned}
&\bigwedge_{y \in Y_0} [(\bigwedge_{y' \sim y} \uparrow |y'|)(y) \wedge \bigwedge_{y' \sim y} (y' = y)] \\
&\quad \dashv\vdash \quad \bigwedge_{y \in Y_0} [(\bigvee\{\uparrow t: \forall y'. (y' \sim y \to |y'| \sqsubseteq t)\})(y) \wedge \bigwedge_{y' \sim y} (y' = y)] \\
&\quad \dashv\vdash \quad \bigvee \{\bigwedge_{y \in Y_0} [(\uparrow t_y)(y) \wedge \bigwedge_{y' \sim y} (y' = y)]: \\
&\qquad\qquad\qquad\qquad \forall y \in Y_0. \forall y' \in Y. (y' \sim y \to |y'| \sqsubseteq t_y)\} \\
&\quad \dashv\vdash \quad \bigvee \{\bigwedge_{y \in Y} (\uparrow t_{[y]})(y) \wedge \bigwedge_{y' \sim y} (y' = y): \forall y \in Y. |y| \sqsubseteq t_{[y]})\}
\end{aligned}$$

where we are writing [y] for the equivalence class of y, an element of Z_0. Given a family of tokens t_z ($z \in Z_0$) with all $|y| \sqsubseteq t_{[y]}$, we can define a bag Z whose underlying set is Z_0 and with $|z| = t_z$, and then the projection $h: Y \to Z$ is a refinement with f;h = g;h. Therefore, the LHS of the axiom is a disjunction of formulas of the form

$$\bigwedge_{y \in Y} (\uparrow |h(y)|)(y) \wedge \bigwedge_{y' \sim y} (y' = y)$$

which is equivalent to $\exists \mathbf{z}: F(Z).\ \mathbf{y} = h(\mathbf{z})$.

Next, we describe a model for T in $\mathcal{S}^{B_L(D)}$, giving a Giraud frame homomorphism G from $\mathcal{S}$[T] to $\mathcal{S}^{B_L(D)}$. The single sort is carried by the functor (object of $\mathcal{S}^{B_L(D)}$) that maps each bag X to its base set X, and each refinement to its underlying function. If $a \in \Omega D$, then G(a(x)) is the subfunctor that takes X to $\{x \in X: |x| \models a\}$. If $a \leq b$, then clearly a(x) gives a subfunctor of b(x).

Limits and colimits in $\mathcal{S}^{B_L(D)}$ are computed argumentwise, so it is clear that the subfunctor corresponding to $\bigwedge_{a \in S} a(x)$ (the intersection) is $\{x \in X: \forall a \in S. |x| \models a\}$, i.e. $\{x \in X: |x| \models \bigwedge_{a \in S} a\}$, i.e. the subfunctor corresponding to $(\bigwedge_{a \in S} a)(x)$. The other axiom is similar.

We must now show that the two geometric morphisms thus constructed are the two parts of an equivalence.

First, let X be an object of $B_L(D)$. Then $GF(X) \cong X$, for

$$\begin{aligned}
GF(X)(Z) &= G(\bigwedge_{x \in X} (\uparrow |x|)(x))(Z) = \prod_{x \in X} G((\uparrow |x|)(x))(Z) \\
&= \prod_{x \in X} \{z \in Z: |x| \sqsubseteq |z|\} \cong B_L(D)(X, Z) = X(Z)
\end{aligned}$$

Next, let $f: X \to Y$ be a morphism in $B_L(D)$. GF(f)(Z) is a subset of $GF(Y)(Z) \times GF(X)(Z) \cong B_L(D)(Y, Z) \times B_L(D)(X, Z)$, and this subset is the graph of a function. We must show that the function is the same as f(Z), i.e. that $(g, h) \in GF(f)(Z)$ iff h = f;g. (Some of the notation is a bit condensed here. When we write X(Z) to mean $B_L(D)(X, Z)$, we are identifying X with its image under the Yoneda embedding; f(Z) is similar.)

$$\begin{aligned} GF(f)(Z) &= G(F(X) \wedge F(Y) \wedge \textstyle\bigwedge_{x\in X} (x = f(x)))(Z) \\ &= \{\mathbf{z}\in Z^Y\times Z^X\colon |x| \sqsubseteq |z_x|, |y| \sqsubseteq |z_y|, z_x = z_{f(x)}\} \\ &\cong \{(g, h) \in Y(Z)\times X(Z)\colon \forall x.\ h(x) = g(f(x))\} \\ &= \{(g, h) \in Y(Z)\times X(Z)\colon h = f;g\} \end{aligned}$$

Hence, F;G is equivalent to the identity on $\mathcal{S}^{B_L(D)}$.

Let us now consider G;F. First, note that G(a(x)) is the *colimit* (not just the union) of the subfunctors $G((\uparrow s)(x))$ for tokens $s \models a$. For suppose we have a functor $U \in \mathcal{S}^{B_L(D)}$, and natural transformations $f^s\colon G((\uparrow s)(x)) \to U$ ($s \models a$). If $z \in G(a(x))(Z)$, i.e. $z \in Z$ and $|z| \models a = \bigvee_{s\models a}\Box\uparrow s$, then $|z| \sqsupseteq s$ for some $s \models a$ and $z \in G((\uparrow s)(x))(Z)$. We wish to define $f_Z(z) = f^s_Z(z)$, and in fact this is unambiguous. For if also $|z| \sqsupseteq s' \models a$ then

$$|z| \models \uparrow s \wedge \uparrow s' = \bigvee \{\uparrow t\colon t \sqsupseteq s, t \sqsupseteq s'\}$$

so $|z| \sqsupseteq$ some such t, and $f^s_Z(z) = f^t_Z(z) = f^{s'}_Z(z)$.

Next, $G((\uparrow s)(x))$ is isomorphic to the (image in $\mathcal{S}^{B_L(D)}$ of the) singleton bag $X_s = \{x\}$ with $|x| = s$. F preserves colimits, so F(G(a(x))) = A (say) is the colimit of the objects $F(X_s) = (\uparrow s)(x)$, $s \models a$, and it remains to show that this colimit is isomorphic to a(x). The inclusions $(\uparrow s)(x) \to a(x)$ ($s \models a$) form a cocone over the diagram; let $f\colon A \to a(x)$ be the mediating morphism. f is epi, because a(x) is the union of its subobjects $(\uparrow s)(x)$. Let R, a subobject of $A\times A$, be the corresponding equivalence relation. $A\times A$ is the colimit of objects $(\uparrow s)(x) \times (\uparrow t)(x)$, and pulling back to R we get that R is the colimit of objects $(\uparrow s)(x) \cap (\uparrow t)(x) \cong (\uparrow s \wedge \uparrow t)(x)$, the intersection being taken in a(x). Hence R is a colimit of objects $(\uparrow u)(x)$. It follows from this that we can find an inverse to the reflexivity monic $A \to R$, and hence that the equivalence relation R is equality and f is an isomorphism.

Hence G;F is equivalent to the identity on $\mathcal{S}[T]$. ⟧

In view of this Theorem, we can consider the presentations of T and $Flat\Box B_L(D)$ to be different presentations of the same theory, which we shall write $B_L(D)$ (the *lower bagdomain* theory over D).

Corollary 3.2 *The category of models of $B_L(D)$ is equivalent to*

(i) *The ind-completion of the category $B_L(D)$;*

(ii) *The category whose objects are bags (possibly infinite) of points (possibly non-compact) of D, and whose morphisms are the refinements.*

Proof

(i) follows because the theory is Flat $B_L(D)$.

(ii) A model X for the theory as presented in Theorem 3.1 has a carrier (let's call that X too) equipped with subsets $\{X|a\}$, and the axioms ensure that for each $\xi \in X$ the map $|\xi|: \Omega D \to \mathbb{2}$, $|\xi|(a) = \mathbf{true}$ iff $\xi \in \{X|a\}$, is a frame homomorphism. In other words, $|\xi|$ is a point of the locale D; so X is a bag of points of D.

If X and Y are two models, then a homomorphism from X to Y is a function $f: X \to Y$ that maps each $\{X|a\}$ to within $\{Y|a\}$. In other words, $\forall\xi\in X.\ \forall a\in\Omega D.\ (|\xi| \models a \to |f(\xi)| \models a)$, i.e. $\forall\xi\in X.\ |\xi| \sqsubseteq |f(\xi)|$, i.e. f is a refinement.
]]

The models are *topological systems* in the sense of Vickers [89], those systems E for which $\Omega E = \Omega D$. However, refinements are not necessarily the point parts of continuous maps.

Proposition 3.3 *B_L is functorial: if $f: D \to E$ is continuous, then there is a geometric morphism $B_L(f): B_L(D) \to B_L(E)$ (i.e. from the classifying topos of the theory $B_L(D)$ to that of $B_L(E)$).*

Proof It suffices to find a model for $B_L(E)$ in $\mathcal{S}^{BL(D)}$; we interpret b(x) ($b \in \Omega E$) as $(\Omega f(b))(x)$. Functoriality is clear.]]

Proposition 3.4 *There is a natural transformation from B_L to P_L, mapping – on models – the bag X to Cl $\{|\xi|: \xi \in X\}$.*

Proof To give a geometric morphism from $B_L(D)$ to $P_L(D)$, we find a model for $P_L(D)$ in $B_L(D)$. It maps $\Diamond a$ ($a \in \Omega D$) to $\exists x.\ a(x)$. Let F be its inverse image part.

Given a model X, let Y be the corresponding point of $P_L(D)$; it is defined by $\{Y|\phi\} = \{X|F(\phi)\}$ for $\phi \in \Omega P_L(D)$. (One could modify Dirac's notation and say $\{Y|\phi\} = \{X|F|\phi\}$, or $\{Y| = \{X|F$.) By the standard theory of the lower powerdomain, Y can be identified with a closed set of points of D, namely the complement of $\bigvee \{a \in \Omega D: Y \not\models \Diamond a\}$. But

$$Y \models \Diamond a \Leftrightarrow \{X|\exists x.\ a(x)\} = \{Y|\Diamond a\} = \{*\} \Leftrightarrow \text{extent}(a) \cap \{|x|: x \in X\} \neq \emptyset$$
$$Y \not\models \Diamond a \Leftrightarrow \{|x|: x \in X\} \subseteq \text{extent}(a)^c$$

and the result follows from this.]]

4. The mixed bagdomain

Gunter [89] presents a scheme in which databases have not only the records as we have described them above, representing actual objects, but also a different kind of record describing background assumptions about the objects of the form "all objects are such and such". The distinction represents the two modalities in the localic theory of the Vietoris (Plotkin) powerdomain: $\Diamond a$ says "there is some object satisfying a", whereas $\Box a$ says "all objects satisfy a".

This new kind of database has two sets of records, called the "flat" part* and the "sharp" part. If the flat records (tokens) are $r_1, \ldots, r_m$, and the sharp records are $s_1, \ldots, s_n$, then the database as a whole represents the property

$$\Diamond\uparrow r_1 \wedge \ldots \wedge \Diamond\uparrow r_m \wedge \Box(\uparrow s_1 \vee \ldots \vee \uparrow s_n\})$$

This says that for each r_i there is something in reality that it approximates, and everything in reality refines at least one of the s_j's.

Although these modalities are best known in connection with the Vietoris powerdomain, one of the axioms turns out to be inappropriate. It is:

$$\Box(a \vee b) \leq \Box a \vee \Diamond b$$

The reason is that it ties the sharp part of the database to the contingencies of what objects we have found, whereas we should actually prefer to have our background assumptions holding regardless.

Gunter therefore develops a new powerdomain, the "mixed" powerdomain, presented localically by

$$\begin{aligned} \Omega P_M(D) = \mathrm{Fr}\langle\ \Diamond a, \Box a\ (a \in \Omega D) \mid\ & \Diamond \text{ preserves all joins} \\ & \Box \text{ preserves finite meets and directed joins} \\ & \Diamond a \wedge \Box b \leq \Diamond(a \wedge b) \quad \rangle \end{aligned}$$

He shows that the usual methods of powerdomain theory work with this one, giving an information system theoretic construction (i.e. the tokens for the powerdomain are databases with flat and sharp parts, the flat parts having to conform with the sharp parts) and an algebraic characterization.

Our aim now is to revise this theory by replacing $\Diamond a$ by a(x) as before. We do not change $\Box a$; this remains as a proposition. The mixed axiom is replaced by $a(x) \wedge \Box b \vdash (a \wedge b)(x)$, which is slightly stronger. The old version says that if all objects satisfy b, and some object satisfies a, then some object satisfies $a \wedge b$. The new one says that if all objects satisfy b, and some object satisfies a, then *that very same object* satisfies $a \wedge b$.

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* This use of the word "flat" is by musical analogy, and is quite unconnected with the notion of flat functor.

Definition 4.1 Let D be a domain. (As usual, we think of D concretely as the set of *tokens*.) The *mixed bagdomain* over D, $B_M(D)$, is the category for which –

- objects are pairs $X = (X^b, X^\#)$ where X^b is a finite bag in D, $X^\#$ is a finite subset of D, and $X^\# \sqsubseteq_U \{|x|: x \in X^b\}$ (i.e. for every $x \in X^b$ there is some $t \in X^\#$ such that $|x| \sqsupseteq t$).
- morphisms from X to Y are refinements from X^b to Y^b, provided that $X^\# \square\sqsubseteq_U Y^\#$.

In Gunter's [89] construction, X^b is a *set*. Just as with B_L, we shall assume that the base sets for our bags X^b are finite subsets of some standard countable alphabet used also for logical variables.

Theorem 4.2 *Let D be a domain. Then the theory Flat ($B_M(D)$) is equivalent to the single-sorted theory T presented by:*

predicates:	$a(x)$, $\square a$	$(a \in \Omega D)$
axioms:	$a(x) \vdash b(x)$	$(a \leq b \text{ in } \Omega D)$
	$\bigwedge_{a\in S} a(x) \vdash (\bigwedge_{a\in S} a)(x)$	$(S \subseteq_{fin} \Omega D)$
	$(\bigvee_{a\in S} a)(x) \vdash \bigvee_{a\in S} a(x)$	$(S \subseteq \Omega D)$
	$\square a \vdash \square b$	$(a \leq b \text{ in } \Omega D)$
	$\bigwedge_{a\in S} \square a \vdash \square(\bigwedge_{a\in S} a)$	$(S \subseteq_{fin} \Omega D)$
	$\square(\bigvee^\uparrow_{a\in S} a) \vdash \bigvee^\uparrow_{a\in S} \square a$	$(S \subseteq \Omega D \text{ directed})$
	$a(x) \wedge \square b \vdash (a \wedge b)(x)$	$(a, b \in \Omega D)$

Proof First, we describe a flat functor F from $B_M(D)$ to $\mathcal{S}[T]$, extending to a Giraud frame homomorphism from $\mathcal{S}^{B_M(D)}$ to $\mathcal{S}[T]$.

Let X be an object of $B_M(D)$. Then

$$F(X) = \bigwedge_{x\in X^b} (\uparrow|x|)(x) \wedge \square(\uparrow X^\#)$$

Let $f: X \to Y$ be a morphism in $B_M(D)$; by renaming, we can assume that X^b and Y^b are disjoint. Then F(f) is the function formula

$$F(X) \wedge F(Y) \wedge \bigwedge_{x\in X^b} (x = f(x))$$

which is equivalent to $F(Y) \wedge \bigwedge_{x\in X^b} (x = f(x))$.

The functorial axioms of Flat($B_M(D)$) are satisfied much as in Theorem 3.1.

The first flatness axiom follows from the fact that in ΩD,

$$\mathbf{true} = \uparrow D = \bigvee^\uparrow \{\uparrow X^\#: X^\# \subseteq_{fin} D\}$$

Hence in T,

$$\mathbf{true} \dashv\vdash \square\mathbf{true} \dashv\vdash \bigvee^\uparrow \{\square\uparrow X^\#: X^\# \subseteq_{fin} D\} \vdash \bigvee_{X\in B_M(D)} \exists \mathbf{x}: F(X).\ \mathbf{x} = \mathbf{x}$$

by considering those X for which $X^b = \emptyset$.

For the second flatness axiom, let X and Y be objects of $B_M(D)$, with X^b and Y^b are disjoint. Let Z_0 be their union, and $f: X^b \to Z_0$, $g: Y^b \to Z_0$ the injections, both refinements. Then in $\mathcal{S}[T]$,

$$F(X) \wedge F(Y) \dashv\vdash \bigwedge_{z\in Z_0} (\uparrow|z|)(z) \wedge \Box(\uparrow X^\# \wedge \uparrow Y^\#)$$
$$\dashv\vdash \bigvee^{\uparrow} \{\bigwedge_{z\in Z_0} (\uparrow|z|)(z) \wedge \Box(\uparrow Z^\#): Z^\# \sqsupseteq_U X^\#, Z^\# \sqsupseteq_U Y^\#\}$$
(because in ΩD $\uparrow X^\# \wedge \uparrow Y^\# = \bigvee^{\uparrow} \{\uparrow Z^\#: Z^\# \sqsupseteq_U X^\#, Z^\# \sqsupseteq_U Y^\#\}$)

Now

$$(\uparrow|z|)(z) \wedge \Box(\uparrow Z^\#) \dashv\vdash (\uparrow|z|\wedge\uparrow Z^\#)(z) \wedge \Box(\uparrow Z^\#)$$
$$\dashv\vdash \bigvee \{(\uparrow t_z)(z) \wedge \Box(\uparrow Z^\#): t_z \sqsupseteq |z|, t_z \in \uparrow Z^\#\}$$

so what we have is a join of formulas of the form $\bigwedge_{z\in Z_0} (\uparrow t_z)(z) \wedge \Box(\uparrow Z^\#)$, where $t_z \sqsupseteq |z|$, $t_z \in \uparrow Z^\#$, $Z^\# \sqsupseteq_U X^\#$ and $Z^\# \sqsupseteq_U Y^\#$. Let us define Z by taking $Z^b = Z_0$ with $|z|_Z = t_z$, and $Z^\#$ as we have it already. Then Z is an object in $B_M(D)$, and $f: X \to Z$, $g: Y \to Z$ are morphisms. This is enough to entail the right hand side of the axiom just as in Theorem 3.1.

For the third flatness axiom, let X and Y be objects of $B_M(D)$, with morphisms $f, g: X \to Y$. The argument is the same as in Theorem 3.1, the sharp parts $Z^\#$ all being $Y^\#$.

Next, we describe a model for T in $\mathcal{S}^{B_M(D)}$, giving a Giraud frame homomorphism G from $\mathcal{S}[T]$ to $\mathcal{S}^{B_M(D)}$. The single sort and the predicates are interpreted as functors that operate on objects Z by applying the method of Theorem 3.1 to Z^b. For instance, G(x=x)(Z) is the set Z^b. The axioms just involving those predicates are respected as in Theorem 3.1.

The proposition $\Box a$ is interpreted as the functor

$$G(\Box a)(Z) = \begin{cases} 1 & \text{if } \forall z\in Z^\#. |z| \models a \\ \emptyset & \text{otherwise} \end{cases}$$

$\Box$ preserves finite meets and directed joins for the same reason as in upper (Smyth) powerdomain theory, and using finiteness of $Z^\#$.

As for the mixed axiom,

$$G(a(x)\wedge\Box b)(Z) = \{z\in Z^b: |z| \models a \wedge \forall z'\in Z^\#.\ z' \models b\}$$
$$= \{z\in Z^b: |z| \models (a\wedge b) \wedge \forall z'\in Z^\#.\ z' \models b\}$$
(because if $z \in Z^b$ then $z \sqsupseteq$ some $z' \in Z^\#$)
$$\subseteq \{z\in Z^b: |z| \models (a\wedge b)\} = G((a\wedge b)(x))(Z)$$

We must now show that the two geometric morphisms thus constructed are the two parts of an equivalence.

First, let X be an object of $B_M(D)$. Then GF(X) ≅ X, for

$$GF(X)(Z) = G(\bigwedge_{x\in X^b} (\uparrow|x|)(x) \wedge \Box(\uparrow X^{\#}))(Z)$$
$$\cong \begin{cases}\{\text{refinements from } X^b \text{ to } Z^b\} & \text{if } X^{\#} \sqsubseteq_U Z^{\#}\\ \emptyset & \text{otherwise}\end{cases}$$
$$= B_M(D)(X, Z) = X(Z)$$

The argument for morphisms is somewhat as in Theorem 3.1, and so F;G is equivalent to the identity on $\mathcal{S}^{BL(D)}$.

Let us now consider G;F. Just as in 3.1, G(a(x)) is the colimit of the subfunctors G((↑s)(x)) for tokens s ⊨ a, and so F(G(a(x))) is the colimit of the F(G((↑s)(x)))'s. G((↑s)(x)) is the colimit in $\mathcal{S}^{B_M(D)}$ of the objects X from $B_M(D)$ for which X^b is the singleton bag $X_s = \{x\}$ with |x| = s, and $X^{\#} \sqsubseteq_U \{s\}$; for

$$G((\uparrow s)(x))(Z) \cong \{\text{refinements from } X_s \text{ to } Z^b\} \cong B_M(D)((X_s, Z^{\#}\cup\{s\}), Z)$$

Hence F(G((↑s)(x))) is the colimit of the objects $F(X) = (\uparrow s)(x)\wedge\Box(\uparrow X^{\#})$, and because

$$\bigvee \{\Box(\uparrow X^{\#})\text{: } X^{\#} \sqsubseteq_U \{s\}\} = \Box\ (\bigvee \{\uparrow X^{\#}\text{: } X^{\#} \sqsubseteq_U \{s\}\}) = \Box\,\mathbf{true} = \mathbf{true}$$

(the join is directed), it follows that this colimit is (↑s)(x). Hence F(G(a(x))) is the colimit of the objects (↑s)(x), and much as in Theorem 3.1 this is a(x). The proof that F(G(□a)) ≅ □a is somewhat similar. ⟧

Again, what we now have are two presentations of a single theory. Let us call it $B_M(D)$ (the *mixed bagdomain* theory over D).

Corollary 4.3 *The category of models of $B_M(D)$ is equivalent to*

(i) *The ind-completion of the category $B_M(D)$;*

(ii) *The category whose objects are pairs $X = (X^b, X^{\#})$ where $X^{\#}$ is a compact saturated set of points of D and X^b is a bag of points from $X^{\#}$; and whose morphisms from X to Y are refinements from X^b to Y^b provided that $Y^{\#} \subseteq X^{\#}$.*

Proof (i) follows because the theory is Flat $B_M(D)$. For (ii), the X^b part of the model arises as in 3.2, while $X^{\#}$ is a model for the propositions □a (by the standard theory for the upper, or Smyth powerdomain, using the Hofmann-Mislove theorem). The mixed axiom specifies that all the values of elements of X^b lie in $X^{\#}$. ⟧

Proposition 4.4 *B_M is functorial.*

Proof Given a continuous map f: D → E, we map b(x) to (Ωf(b))(x) and □b to □(Ωf(b)) (b ∈ ΩE). ⟧

Proposition 4.5 *There is a natural transformation from B_M to P_M. On models, it maps (X^b, $X^\#$) to (Cl {|x|: x ∈ X^b}, $X^\#$).*

Proof Much as for Proposition 3.4. ◇a ↦ ∃x. a(x), □a ↦ □a. ⟧

5. Further work

5.1 Categorical generalizations

The idea in B_L(D) of tacking arguments on the propositions generalizes to non-propositional theories, tacking an extra argument on all the predicates; this has now been done by Johnstone [91]. Moreover, rather than starting from this syntactic construction as a definition (which would leave open the question of presentation independence), he characterizes it universally as a "partial product". He also defines a monad on the category of toposes whose functor part is somewhat analogous to the construction given in Theorem 3.1, and shows – as in Theorem 3.1 –□that in the case of algebraic toposes (i.e. those of the form $\mathcal{S}^C$ for some small category C) it is equivalent to one analogous to the finite bagdomain construction.

5.2 Continuous domains

In Vickers [90] it is shown that the method of information systems can also be applied to continuous posets, not just algebraic ones. A continuous information system as presented there has a token set with an order < that is transitive and *interpolative:* if s < u, then s < t < u for some t. This generalizes the reflexive axiom of posets. The construction of points and opens out of these tokens is then really very similar to that of the poset case, but now giving a continuous poset.

I conjecture that the bagdomains preserve continuity just as they do algebraicity. (The appropriate notion of continuous categories has been investigated by Johnstone and Joyal [82].)

5.3 More restricted information systems

The information systems used above are of the simplest kind, describing algebraic posets. By putting extra conditions on, one obtains more restricted kinds of information system that give more restricted kinds of domains: for instance, spectral algebraic, strongly algebraic (SFP), Scott domains.

Spectral algebraic domains can be rationalized as being governed by the desire to have a *coherent* theory, i.e. a geometric theory in which infinite disjunctions are not used in the presentation. This idea still makes sense in the predicate case, and in fact it is easy to see that if D is spectral algebraic then $B_L(D)$ and $B_M(D)$ are coherent. For instance, it is not hard to show that $B_M(D)$ can be presented using predicates a(x) and □a where a is a *compact* open.

Flat D is equivalent to a coherent theory if, for instance, D satisfies certain conditions analogous to the "2/3 SFP" conditions for spectral algebraic information systems, namely that for every finite diagram Δ in D there is a finite set S of cocones over Δ such that every cocone over Δ factors via one in S. Such D's would be a categorical generalization of spectral algebraic domains. There may be similar generalizations of SFP and Scott domains, expressible by conditions on information systems, and if so, one can ask whether they are closed under B_L and B_M.

5.4 Other uses of predicate geometric logic

Although the database ideas here are rather specialized, the methods may turn out to be useful models for other applications of predicate geometric logic, using the intuitions described in Section 2.

One interesting possibility that shows some promise is the use of geometric logic in specification languages. One reason for this may be the *ex post facto* nature of the intuition for observations of functions: the specifications tell you how to observe (or check) when you've got the right answer, but not how to calculate it. More concrete evidence is provided by some recent work of Hodges [90], who shows that specifications using the Z language can in practice be restricted to a logic (which he calls Σ_1^+) that is essentially geometric.

5.5 Dynamic predicate geometric logic

Another feature of the database ideas that may be intrinsic at this stage of development of the theory is that the databases are describing a *static universe*. The only process of change is that by which one refines one's knowledge of this universe by refining one's knowledge of the objects, by discovering new objects, or by observing equalities. The whole theory falls far short of real databases in its failure to account for change in the world.

It has been argued in Abramsky and Vickers [90] that in the propositional case, to take account of change one must introduce a dynamic observational logic in which the observations form a quantale instead of a frame. The essential difference is that conjunction $a \wedge b$ (observe a and b) must be replaced by a much more precise temporal composition $a \cdot b$ (observe a then b), with the understanding that there may have been a change in the object in connection with the observation a.

It therefore seems quite likely that in order to model dynamic databases we should use a theory that combines the dynamic features of quantale logic with the multiplicity of objects in the predicate logic: we need a predicate quantale logic. I do not yet understand what this theory would have to be.

References

S. Abramsky [83], "On semantic foundations of applicative multiprogramming", *Automata, Languages and Programming,* Lecture Notes in Computer Science **154**, (Springer-Verlag 1983), 1-14.

S. Abramsky [87], *Domain Theory and the Logic of Observable Properties,* PhD Thesis, Queen Mary College, University of London, 1987.

S. Abramsky [88], "Domain theory in logical form", *Annals of Pure and Applied Logic* **51** (1991), 1-77.

S. Abramsky and S.J. Vickers [90], *Quantales, Observational Logic and Process Semantics,* Report DOC 90/1, Department of Computing, Imperial College, London, 1990.

Errett Bishop and Douglas Bridges [85], *Constructive Analysis,* Grundlehren der mathematischen Wissenschaften **279**, Springer-Verlag (1985).

P. A. M. Dirac [47], *The Principles of Quantum Mechanics* (3rd edition), Oxford University Press (1947).

Michael P. Fourman and Steven Vickers [85], "Theories as categories", in David Pitt, Samson Abramsky, Axel Poigné and David Rydeheard (eds) *Category Theory and Computer Programming,* Lecture Notes in Computer Science **240** (Springer-Verlag 1986), 434-448.

Carl A. Gunter [89], "The mixed powerdomain", to appear in *Theoretical Computer Science*.

Wilfrid Hodges [90], "Another Semantics for Z", notes.

Peter Johnstone [77], *Topos theory,* Academic Press, London (1977).

Peter Johnstone [82], *Stone Spaces,* Cambridge University Press (1982).

Peter Johnstone [91], "Partial products, bagdomains and hyperlocal toposes", this volume.

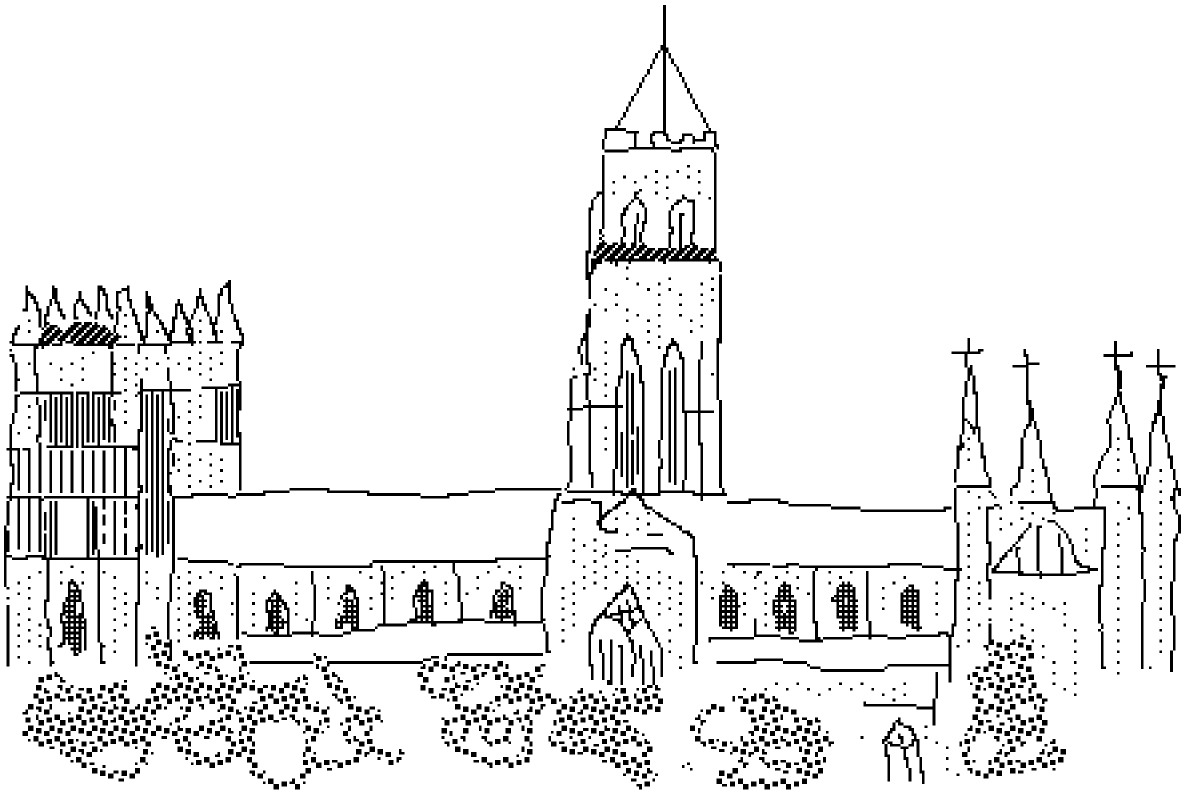
Peter Johnstone and André Joyal [82], “Continuous categories and exponentiable toposes”, *J. Pure and Appl. Algebra* **25**, 255-296.

D. Lehmann [76], “Categories for fixed point semantics”, FOCS 17.

Michael Makkai and Gonzalo E. Reyes [77], *First Order Categorical Logic,* Lecture Notes in Mathematics **611** (Springer-Verlag 1977).

Steven Vickers [89], *Topology via Logic,* Cambridge University Press (1989).

Steven Vickers [90], “Continuous information systems”, submitted for publication.



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