Is geometric logic constructive?*

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* Hard to say. Probably not exactly.

Geometric logic (or type theory)

Logic of *sets* highly restricted

- finite limits
- colimits
- includes natural numbers, free algebras

No exponentials or power objects!

Pervasive need for *spaces*.

Spaces are point-free, and include toposes as generalized spaces.

- Space = geometric theory (of the points)
- Map = geometric construction of points from points
- Bundle = geometric construction of spaces from points

Everything is topologized and continuous.

Further discussion and details in [Vic22].

Example: Point-free real analysis

Real line $\mathbb{R}$ = geometric theory of Dedekind sections.

[NV22]: real exponentiation and logarithms geometrically

Typical technique: Analyse Dedekind real as a pair, lower real and upper real, with disjointness and locatedness axioms.
Apply constructions in one-sided cases, then put results together to get Dedekind result.

[Vic23] calculates their integrals and derivatives

Technique: First prove Fundamental Theorem of Calculus, after which the calculations are more or less standard.
Point-free accounts of integrals and derivatives are already available [Vic08, Vic09].
Lower and upper integrals are lower and upper halves of a Dedekind integral.

Are these constructive?

Some aspects

Constructive taboos

Some are valid geometrically, but that's not a problem, because they are interpreted *topologically.*

Ontology

Constructivism: "there exists" = "we can construct".
That's an ontological assertion.

Compare with "serendipitous" ontology for geometric logic.

Point-free surjections

For discrete computations, must be working with *disconnected* version of $\mathbb{R}$, with notational redundancy.

Using point-free surjections, reason *as if* every Dedekind real can be equipped with disconnecting data.

Constructive taboos – eg

- Double negation rule $\neg\neg\phi \equiv \phi$
- Law of excluded middle LEM $\phi \vee \neg\phi \equiv \top$

Considered inimical to constructive maths: any logic that validates them is not constructive.

Geometric maths validates both!
– Because the weakness of geometric logic forces us to reinterpret the rules in a topological way.

That's not inconsistent with the different interpretation in constructive maths.

Double negation $\neg\neg\phi \equiv \phi$ – geometrically

- ϕ a subset of 1, then $\neg\phi$ is exponential 0^ϕ.
- But exponentiation is not a geometric construction *of sets.*
- ϕ still exponentiable (locally compact) *as space*.
- 0^ϕ is not discrete (a set), but a *Stone space.*
- Its Boolean algebra of clopens is presented by no generators, and set of relations $\{\top \leq \bot \mid \phi\}$.
- Stone spaces are also exponentiable. 0^{0^ϕ} turns out to be discrete, and isomorphic to ϕ. (Use Stone duality.)

Moral: arrow types (and Π-types) distort the logic

– when postulated as constructions of *sets.*

They conflict with the topology. Problem lies in taking "*set* of points" of a function *space.*

LEM $\phi \vee \neg\phi \equiv \top$ – geometrically

We need to think of subspaces. Given a space, described by a geometric theory, a subspace is described by additional geometric axioms.

- $\neg\phi$ is not a set, but it is a subspace of 1, described by axiom $\phi \vdash \bot$.
- ϕ is an open subspace of 1, $\neg\phi$ is its closed complement.
- $\vee$ cannot be the usual logical disjunction (of subsets of 1). Instead it is join in the lattice of subspaces.
- There we find the closed complement is a Boolean complement.

Geometric case splitting: Suppose X a space, Y a subspace

To show $X = Y$ want: every $x{:}X$ is in Y

- Find some open subspace U of X.
- Show (case 1) every $x{:}U$ is in Y, and ...
- ... (case 2) every $x{:}\neg U$ is in Y.
- That's enough! This is frequently used in [NV22].

Limitations

- It's not true that every $x{:}X$ is either in U or in $\neg U$. (No map from X to $U + \neg U$.)
- Y must be a *subspace*, defined by geometric axioms.
- It doesn't work for properties defined by unique structure. For example, property of a lower real of being Dedekind. That relies on having the (uniquely defined) *structure* of the corresponding upper real.
- These are monics into X, subspaces are regular monics.

Constructive analysis

The preface to [BIRS23] says –

> *In constructive mathematics, 'there exists' is interpreted strictly as 'we can find/construct/compute'.*

Is geometric mathematics compatible with this?

It's a question of *ontology*, of how the formalism is meant to represent what it is supposed to be talking about.

Serendipitous ontology of observations

> *Serendipity* – The faculty of making happy chance finds.

For propositional geometric logic (*Topology via Logic* [Vic89]):

Open (proposition) = *observable* property of the things you want to talk about.

$\wedge$ and $\bigvee$ can be explained in observational terms, $\neg$ and $\rightarrow$ can't.

Just because a property is true (of a particular thing), doesn't mean you will observe it. That might take hard work, and in the end still come down to luck.

Serendipitous ontology of observations

Propositional geometric logic:

Axiom (sequent) $\phi \vdash \psi$ is not observable. It's an assumption about how one property entails another.

Propositions $+$ axioms $=$ geometric theory $=$ space.

The assumption is that the things you want to talk about can be abstractly represented as points of the space, defined by which observations are true for them.

Axioms of the form $\phi \vdash \bot$ can act as *Popperian falsification*. If you do observe ϕ, then the theory, or its interpretation, is wrong – they don't accurately describe the thing you are observing.

Can suggest testable questions. eg, does theory of Dedekind reals accurately represent how physical quantities are observed?

Serendipitous ontology of observations

Denotational semantics (Abramsky [Abr91] developing Scott):
Domain D, point-free, is observational account of *how user observes program*, by watching it run.

Syntax is how coder writes program. For each syntactic type, get a set (discrete space) P of program fragments of that type.

Semantics $[\![-]\!] \colon P \to D$ represents each program (fragment) as a user-observable object. It relates *what coder writes* to *what user sees.*

Important clue to describing constructive content of geometric structures?

Serendipitous ontology of observations

For predicate geometric logic [Vic10]:

To describe a set, must prescribe

1. – how to "ascertain that you have apprehended" an element of the set,
2. – how to ascertain that two apprehended elements are equal.

(cf. Bishop and Bridges [BB85, Chapter 1.1], how to *construct* an element and how to *prove* two elements equal.)

Then existence has clear observational meaning (unlike universals). To observe an existence $\exists x.\phi(x)$, we must apprehend an element and observe ϕ for it.

Prescription doesn't work for spaces in general!

eg for Dedekind reals it is *in*equality that is observable. (Inequality is open subspace of $\mathbb{R} \times \mathbb{R}$.)

Serendipitous ontology of observations

For predicate geometric logic [Vic10]:

What is the meaning of axiom $\phi \vdash \exists x.\psi(x)$?

[Vic10] discusses three possibilities.

1. "Already done"

Whatever was done to observe ϕ has already apprehended the x needed for $\exists x.\psi(x)$.

This is extremely strong, since it means that validity of such sequents follows directly from knowing how formulae are interpreted.
This makes for less flexibility when using theory axioms as background assumptions.

Serendipitous ontology of observations

For predicate geometric logic [Vic10]:

What is the meaning of axiom $\phi \vdash \exists x.\psi$? Three possibilities [Vic10].

2. “Nearly done”

A well defined program of extra work will yield a suitable x.

This is essentially the constructivist interpretation – “we can construct”.

3. “Can be done”

There is some suitable x “out there”, though we don't necessarily know how to find it.

This is the serendipitous interpretation.

$\exists$ in proofs

$$\frac{\phi \vdash \exists x.\psi(x) \quad \frac{\psi(x) \vdash_x \chi}{\exists x.\psi(x) \vdash \chi}}{\phi \vdash \chi}$$

From ϕ deduce $\exists x.\psi(x)$, then –

Constructively

1. construct x with $\psi(x)$,
2. use $\psi(x) \vdash_x \chi$ to deduce χ.

Seredipitously

χ doesn't involve x, so knowing that x is "out there somewhere" is enough to reason *as if* we already have it.

Predictive mathematics[1]

Geometric mathematics makes *predictions.*

eg: Your theory says your algorithm will terminate (provided your computer is fast enough and you have the patience),

and when it does you will be able to ...

cf. Scientific theories

They make predictions – which may be experimentally falsifiable.

[1] Not my phrase, but I can't track it down

Categorically

$\sum_x \psi(x) \to \exists x.\psi(x)$ is epi, and coequalizer of its kernel pair.

χ is a subobject of 1, so every map $\sum_x \psi(x) \to \chi$ factors via $\exists x.\psi(x)$.

Constructively, we split the epi.

Serendipitously, we reason *as if* every element of $\exists x.\psi(x)$ is in the image of $\sum_x \psi(x)$.

That was for sets. We can do something similar for surjections of spaces, but more care is needed.

Localic surjections $p\colon X \to Y$

(Locale = "ungeneralized space", propositional theory)

Frame theoretically – inverse image p^* is mono

Not well behaved in that generality.

Open surjections – see Joyal and Tierney [JT84]

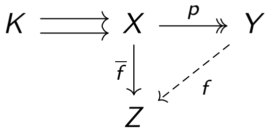
p^* has left adjoint $\exists_p$, and a Frobenius condition.

More localically: $\exists_p$ corresponds to a map $Y \to P_L X$, $y \mapsto$ fibre of p over y.

Can say [Vic95, Vic21] –
p open (and surjective) iff p fibrewise overt (and positive).
(Recall bundles, slide 2.)

Open surjection is coequalizer of kernel pair

– because it has effective descent [JT84].



To define $y \mapsto f(y)$:

1. Reason as if there is some x with $y = p(x)$. (Not true in general! And even when it is, can't make the choice depend continuously on y unless p splits.)
2. Define $\overline{f}(x)$.
3. Show that $\overline{f}(x)$ independent of choice of x.

Similarly for proper surjections [Ver86] and triquotients [Ple97].

Real analysis

$\mathbb{R}$ is connected

No non-constant maps from $\mathbb{R}$ to set of computer states or set of output symbols.

For realistic computation, can use triquotient cover of $\mathbb{R}$ to introduce some disconnectedness.

- Doesn't matter which.
- Don't have to *define* reals as other than Dedekind, just to fit computational needs.

Disconnectedness appears as redundancy in notation.

Example: Compact interval covered by Cantor space

$$p\colon 2^N \to [-1,1]$$
$$(s_i)_{i\geq 1} \mapsto \sum_i \frac{s_i}{2^i}$$

Think of 2 in 2^N as the 2-element set $\{+,-\}$.

p is a proper surjection – key lemma in [Vic17].

To define map on $[-1,1]$ –

Reason *as if* every x:$[-1,1]$ has a sign expansion.

Note redundancy: eg $p(+-^\omega) = 0 = p(-+^\omega)$.

For $\mathbb{R}$:

Have triquotient cover by a space of Cauchy sequences [Vic98, Section 7].

Locators (Auke Booij [Boo20])

Dedekind reals x have *locatedness* axiom:

$$q < r \vdash_{q,r:\mathbb{Q}} q < x \vee x < r$$

Located reals replace $\vee$ with BHK $+$ (constructive disjunction):

$$q < r \vdash_{q,r:\mathbb{Q}} (q < x) + (x < r)$$

Like an ordinary real, but, for each $q < r$, equipped with information to choose a case when we have both $q < x$ and $x < r$.

- Booij develops real analysis for located reals in univalent mathematics.
- Geometrically there is also a space $\mathbb{R}^{\mathfrak{L}}$ of located reals, and map $\mathbb{R}^{\mathfrak{L}} \to \mathbb{R}$ is an open surjection [Vic21].

Is the point-free real analysis of [NV22] etc. constructive?

What would we need in order to extract programs from them?

Suppose "programming language" for constructing reals can be embodied in a cover of $\mathbb{R}$ (such as $\mathbb{R}^{\mathfrak{L}}$). Then we require proofs at level of $\mathbb{R}$ (eg those in [NV22]) to lift to cover – proof steps lift to algorithm steps.

Geometric is relative to a base $\mathcal{S}$, elementary topos with nno
$\mathcal{S}$ describes the infinities allowed in disjunctions and coproducts.

Presumably constructive = algorithmic reasoning should be independent of $\mathcal{S}$ – which [NV22] is.

Conjecture – to guarantee lifting it suffices to restrict further and use the base-independent "arithmetic" reasoning of [MV12, Vic19].

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