Geometricity

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- As in geometric logic
- (With Bertfried Fauser, Guillaume Raynaud)

Applying to topos approach to quantum theory
(Isham, Butterfield, Döring; Landsman, Spitters, Heunen)

Part 1:

TOPOLOGY VIA LOGIC

- Old idea (long before my book!)
- logical theory $\mapsto$ topological space of models

Boolean ..... Stone
coherent .... spectral
geometric ..... sober

- use theories instead of spaces ... Why?

Geometric theory First-order, many sorted

Signature Σ – sorts, predicates, functions
Context $\vec{x}$ – finite list of variables, with sorts
Terms in context $(\vec{x}.t)$ – built using variables & function symbols
Formulae in context $(\vec{x}.\phi)$ – use $=, \top, \wedge, \bigvee, \exists$

- free variables must be in the context
- Disjunction $\bigvee$ may be infinitary
- Write $\perp$ for $\bigvee\emptyset$ (false)

Main interest: propositional case –
no sorts, ∴ no variables or terms, ∴ no = or $\exists$

Axioms

(Sequents, sentences)

$$\phi \vdash_{\vec{x}} \psi$$

- ϕ, ψ formulae in context $\vec{x}$

Read as $(\forall \vec{x})(\phi \to \psi)$ — $\to, \neg, \forall$ at just one level – no nesting

Geometric theory $(\Sigma, \mathbb{T})$ – $\mathbb{T}$ a set of axioms
(don't require $\mathbb{T}$ closed under consequence)

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Rules of geometric logic

Sequent calculus

Identity, cut $\quad \phi \vdash \phi \qquad \dfrac{\phi \vdash \psi \quad \psi \vdash \chi}{\phi \vdash \chi}$

Conjunction $\quad \phi \vdash \top \qquad \dfrac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$

$\phi \wedge \psi \vdash \phi \qquad \phi \wedge \psi \vdash \psi$

Disjunction $\quad \phi \vdash \bigvee S \ (\phi \in S) \qquad \dfrac{\phi \vdash \psi \ \text{(all } \phi \in S)}{\bigvee S \vdash \psi}$

Distributivity $\quad \phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$

Predicate rules

Need to be explicit about context $\vec{x}$:

$\phi \vdash_{\vec{x}} \psi \quad ((\forall \vec{x})(\phi \to \psi))$

Substitution $\quad \dfrac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$ ($\vec{s}$ terms in context $\vec{y}$)

Equality $\quad \top \vdash_{x} x = x \qquad (\vec{x} = \vec{y}) \wedge \phi \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]$

Existential $\quad \dfrac{\phi \vdash_{\vec{x}y} \psi}{(\exists y)\phi \vdash_{\vec{x}} \psi}$

Frobenius $\quad \phi \wedge (\exists y)\psi \vdash_{\vec{x}} (\exists y)(\phi \wedge \psi)$

Example: real numbers

Propositional theory - no sorts

Propositional symbols: $L_q, R_q \quad (q \in \mathbb{Q})$

Axioms: $L_q \vdash L_{q'} \qquad R_{q'} \vdash R_q \qquad (q' \leq q)$

$L_{q'} \vdash \bigvee_{q'<q} L_q \qquad R_q \vdash \bigvee_{q'<q} R_{q'}$

$\top \vdash \bigvee_q L_q \qquad \top \vdash \bigvee_q R_q$

$L_q \wedge R_q \vdash \bot$

$\top \vdash L_{q'} \vee R_q \quad (q' < q)$

Models?

Example: real numbers

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$\top \vdash \bigvee_q L_q \qquad \top \vdash \bigvee_q R_q$

$L_q \wedge R_q \vdash \bot$

$\top \vdash L_{q'} \vee R_q \quad (q' < q)$

Determined by:

$L = \{q \in \mathbb{Q} \mid L_q$ interpreted as tt$\}$

$R = \{q \in \mathbb{Q} \mid R_q$ " " "$\}$

Axioms assert:

L down-closed $\qquad R$ up-closed

L, R rounded, inhabited

$L \cap R = \emptyset$

If $q' < q$ then $q' \in L$ or $q \in R$

$\Rightarrow L, R$ come arbitrarily close

Model = (variant of) Dedekind section of rationals
= real number x

$L = \{q \in \mathbb{Q} \mid q < x\}$
$R = \{q \in \mathbb{Q} \mid x < q\}$

Topology: subbase $L_q = (q, \infty)$
$R_q = (-\infty, q)$

$\therefore$ usual topology on reals

Opens ~ formulae ($\bigcup$ ~ $\bigvee$)

In general $(\Sigma, \mathbb{T})$ a propositional geometric theory

Predicate theories? Get toposes.

$X = pt[\Sigma, \mathbb{T}]$ = set of models of $(\Sigma, \mathbb{T})$

extent
$ext : \Sigma \longrightarrow \mathcal{P}X$
$ext(P) = \{x \mid P \mapsto \text{true in model } x\}$ $x \Vdash P$

If ϕ a geometric formula:
$ext(\phi) \subseteq X$ - evaluate ϕ using $\bigcup$ for $\bigvee$

Topology on X: opens are all sets $ext(\phi)$

Sets $ext(P)$ ($P \in \Sigma$) a subbase of opens.

Theory defines both points and topology.

Maps

continuous

f: models x of $(\Sigma_1, \mathbb{T}_1) \rightarrow$ models $f(x)$ of $(\Sigma_2, \mathbb{T}_2)$

Continuity $\Rightarrow \forall Q \in \Sigma_2 . f^{-1}(Q)$ geometric ($\bigvee \wedge$) in $\mathbb{T}_1$

f^{-1} preserves $\bigvee \wedge$, hence extends to all geom. formulae

Also, if $\phi \vdash \psi$ in $\mathbb{T}_2$ then $f^{-1}(\phi) \vdash f^{-1}(\psi)$ in $\mathbb{T}_1$

$(x \Vdash f^{-1}(\phi) \Leftrightarrow f(x) \Vdash \phi \Rightarrow f(x) \Vdash \psi \Leftrightarrow x \Vdash f^{-1}(\psi))$

Inverse image gives model transformation

$f(x)$ defined by $f(x) \Vdash Q$ if $x \Vdash f^{-1}(Q)$.

Model transformation gives inverse image?

Logic is incomplete!

Declare truth values $[x \Vdash P]$ ($P \in \Sigma_1$) satisfying $\mathbb{T}_1$

> Let x be a model of $(\Sigma_1, \mathbb{T}_1)$
>
> Some construction that delivers a point y of $(\Sigma_2, \mathbb{T}_2)$
>
> Define $f(x)$ to be y.

Construction needs to be geometric ($\bigvee \wedge$)

Have got $f^{-1}(Q)$ ($Q \in \Sigma_2$) geometrically in $\mathbb{T}_1$.

$f(x)$ a model $\Rightarrow f^{-1}$ respects $\mathbb{T}_2$ axioms

Continuity is geometricity

Point-free topology

Treat theory $(\Sigma, \mathbb{T})$ as "point-free" space $[\Sigma, \mathbb{T}]$ (or $[\mathbb{T}]$)

Map $f : [\mathbb{T}_1] \longrightarrow [\mathbb{T}_2]$

defined using

- geometric transformation of models

or

- inverse image function

Point-free topology: various approaches

Locales: Lindenbaum algebra $\Omega[\Sigma, \mathbb{T}]$

(formulae modulo equivalence)

is a frame – $\bigvee, \wedge$, $\wedge$ distributes over $\bigvee$

Inverse image function = frame homomorphism

Locale = frame pretending to be space

Categorically: $\mathbf{Loc} = \mathbf{Fr}^{op}$

Formal topology: Base + cover relation

One way to describe a $(\Sigma, \mathbb{T})$

$a \lhd U$ (U covers a) means $a \leq \bigvee U$

Ordinary models

& a unique model

$1 = [\emptyset, \emptyset]$ – no symbols or axioms

Formula = ordinary truthvalue

Map $1 \longrightarrow [\Sigma, \mathbb{T}]$ is model of $(\Sigma, \mathbb{T})$

Inside the box

Let x be a model of $(\Sigma_1, \mathbb{T}_1)$

Some construction that delivers a point y of $(\Sigma_2, \mathbb{T}_2)$

Define $f(x)$ to be y.

1. There is a generic point, x
2. The truth values $[x \Vdash \phi]$ aren't either true or false (they're parametrized by x)
3. Mathematics is geometric

It's world of sheaves over $[\Sigma_1, \mathbb{T}_1]$.

(Sets parametrized by x)

Part 2:

SHEAVES

Higgs (unpublished); Loullis; Fourman & Scott; ... Höhle; Vickers

= $\mathbb{T}$-sets

= Sets parametrized by point x

Parametrized sets

$\mathbb{T}$-sets $(\Sigma, \mathbb{T})$ a prop^l geom. theory

? Set S_y parametrized by model y?

Idea: S a set of "tokens"

For each y, $\sim_y$ a partial equivalence relation (p.e.r.) on S (symmetric, transitive)

$S_y = S/\sim_y = \{\text{eq. classes } [s]_y \ (s \sim_y s)\}$

For $(s, s') \in S \times S$, $[\![s = s']\!] = \{y \mid s \sim_y s'\}$

Assume open, for "continuity": geo. formula in $(\Sigma, \mathbb{T})$

Symmetry: $[\![s = s']\!] \vdash [\![s' = s]\!]$

Transitivity: $[\![s = s']\!] \wedge [\![s' = s'']\!] \vdash [\![s = s'']\!]$

Parametrized functions

$\mathbb{T}$-functions $(S, [\![\cdot = \cdot]\!]) \xrightarrow{f} (T, [\![\cdot = \cdot]\!])$

For each y: $f_y : S/\sim_y \longrightarrow T/\sim_y$

Graph $\Phi_y \subseteq S \times T$: closed under $\sim_y$ on both sides

Single valued: $s \Phi_y t$, $s \Phi_y t' \Rightarrow t \sim_y t'$

Total : $s \sim_y s \Rightarrow \exists t.\ s \Phi_y t$

$[\![fs = t]\!] = \{y \mid s \Phi_y t\}$ – assume open again

strict: $[\![fs = t]\!] \vdash [\![s = s]\!] \wedge [\![t = t]\!]$

extensional: $[\![s' = s]\!] \wedge [\![fs = t]\!] \wedge [\![t = t']\!] \vdash [\![fs' = t']\!]$

single valued: $[\![fs = t]\!] \wedge [\![fs = t']\!] \vdash [\![t = t']\!]$

total: $[\![s = s]\!] \vdash \bigvee_t [\![fs = t]\!]$

Category of $\mathbb{T}$-sets & $\mathbb{T}$-functions

$\simeq$ $Sh[\Sigma, \mathbb{T}]$ (sheaves over $[\Sigma, \mathbb{T}]$)

A topos – similar to category of sets

- can do mathematics "internally" in it
- but logic is intuitionistic
- and no axiom of choice

except axiom of unique choice

Examples

A a set: constant Π-set on A

$[\![a=a']\!] = \bigvee\{\top \mid a=a'\}$ crisp equality

$\sim_x$ is equality for all x

$A/\sim_x = A$

Fuzzy set on A: $E(a)$ geo. formula in (Σ,Π)

Π-subset of constant Π-set

Same A, different equality

$[\![a=a']\!] = \bigvee\{E(a) \mid a=a'\}$

Generic point of $[\Sigma,\Pi]$

Ordinary model is subset of Σ (satisfying axioms)

In world of Π-sets:

model is fuzzy set on Σ

Use identity $\Sigma \to \Sigma$! $E(P)$ is formula P

satisfies axioms by construction

Fibrewise constructions

e.g. finite powerset $\mathcal{F}$

Given $(S, [\![\cdot=\cdot]\!])$: finite powerset fibrewise

$$\mathcal{F}S,\ [\![U=V]\!] = \underbrace{\bigwedge_{u\in U}}\ \bigvee_{v\in V} [\![u=v]\!] \wedge \underbrace{\bigwedge_{v\in V}}\ \bigvee_{u\in U} [\![u=v]\!]$$

NB wouldn't work for infinite U,V

$(\mathcal{F}S)/\sim_y \cong \mathcal{F}(S/\sim_y)$

$[U]_y \mapsto \{[u]_y \mid u\in U\}$

Other examples: finite limits, arbitrary colimits

Inverse image of Π-sets

$f : [\Sigma_1, \Pi_1] \longrightarrow [\Sigma_2, \Pi_2]$

$(B, [\![\cdot=\cdot]\!]_B)$ a Π_2-set

Inverse image $f^*(B, [\![\cdot=\cdot]\!]_B) = (B, [\![\cdot=\cdot]\!]_A)$

$[\![b=b']\!]_A = f^{-1}[\![b=b']\!]_B$

Geometricity

for arbitrary $\mathbb{T}$

General construction on $\mathbb{T}$-sets is geometric if preserved by inverse image

$\Rightarrow$ calculated fibrewise

- if $x : 1 \longrightarrow [\Sigma, \mathbb{T}]$

  then $x^* A \cong A/\!\sim_x$

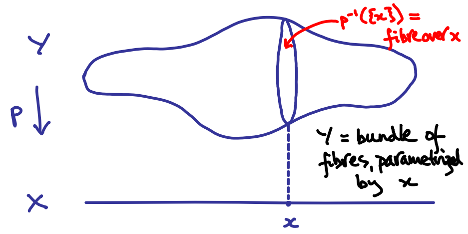
e.g. colimits, finite limits, $\mathcal{F}$, constants

non e.g. $\mathcal{P}$

Part 3 :

FIBRED SPACES

Maps understood as bundles



$\sim$ point-free space in $\mathbb{T}$-sets

Fourman & Scott

Joyal & Tierney

Geometric theory as GRD-system

Generators
Relations
Disjuncts

$(\Sigma, \mathbb{T})$ $\qquad \Sigma \mapsto G$

Axioms in $\mathbb{T}$ of form $\phi \vdash \psi$

Formula $\phi \approx \bigvee$ of $\wedge$ of symbols from Σ

$\bigvee_i \quad \bigwedge S_i \qquad S_i \in \mathcal{F}\Sigma$

Replace $\phi \vdash \psi$ by set of axioms $\bigwedge S_i \vdash \psi$

$\therefore$ Can assume each axiom r of form

$r: \quad \bigwedge S \vdash \bigvee_j \bigwedge T_j$

$R = \{\text{axioms}\}$ $\quad \lambda : R \longrightarrow \mathcal{F}G \quad \lambda(r) = S$

GRD-systems

Vickers

Disjuncts: pairs (r, j), j an index on RHS of

$r: \quad \bigwedge S \vdash \bigvee_j \bigwedge T_j$

D = set of all disjuncts (r,j)

$\pi : D \longrightarrow R \qquad \pi(r,j) = r$

$\rho : D \longrightarrow \mathcal{F}G \qquad \rho(r,j) = T_j$

GRD system

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ \mathcal{F}G & R \arrow[l, "\lambda"] \end{tikzcd}$$

Theory: signature G

Axioms

$\bigwedge \lambda(r) \vdash \bigvee_{\pi(d)=r} \bigwedge \rho(d)$

Parametrized theories

$(\Sigma, \mathbb{T})$ a propositional geometric theory (e.g. given by (G,R,D))

Let
$$\begin{tikzcd} & D' \arrow[dl, "e'"'] \arrow[d, "\pi'"] \\ \mathcal{F}G' & R' \arrow[l, "\lambda'"] \end{tikzcd}$$
be a GRD system in $\mathbb{T}$-sets

G', R', D' are $\mathbb{T}$-sets, λ', π', ρ' $\mathbb{T}$-functions

For each point x of $[\Sigma, \mathbb{T}]$:
$$\begin{tikzcd} & D'/\sim_x \arrow[dl, "e'_x"'] \arrow[d, "\pi'_x"] \\ (\mathcal{F}G')/\sim_x & R'/\sim_x \arrow[l, "\lambda'_x"] \end{tikzcd}$$
– an ordinary GRD-system $(G',R',D')_x$

Fibre space $[\mathbb{T}][\mathbb{T}']$

$(\Sigma, \mathbb{T})$ given by (G,R,D)
$$\begin{tikzcd} & D' \arrow[dl, "e'"'] \arrow[d, "\pi'"] \\ \mathcal{F}G' & R' \arrow[l, "\lambda'"] \end{tikzcd}$$
be a GRD system in $\mathbb{T}$-sets

Theory whose models are pairs (x,y), x model of (G,R,D), y model of $(G',R',D')_x$

Signature: $G + G'$

Axioms:
- from $\begin{tikzcd} & D \arrow[dl] \arrow[d] \\ \mathcal{F}G & R \arrow[l] \end{tikzcd}$
- $g' \vdash [\![g' = g']\!]$
- $g'_1 \wedge [\![g'_1 = g'_2]\!] \vdash g'_2$
- $[\![\lambda' r' = S']\!] \wedge \bigwedge S' \vdash \bigvee_{d' \in D',\, T' \in \mathcal{F}G'} [\![\pi' d' = r']\!] \wedge [\![\rho' d' = T']\!] \wedge \bigwedge T'$

Bundle

$$\begin{tikzcd} {[\mathbb{T}][\mathbb{T}']} \arrow[d] \\ {[\mathbb{T}]} \end{tikzcd}$$

Signature: $G + G'$

Axioms:
- from $\begin{tikzcd} & D \arrow[dl] \arrow[d] \\ \mathcal{F}G & R \arrow[l] \end{tikzcd}$
- $g' \vdash [\![g' = g']\!]$
- $g'_1 \wedge [\![g'_1 = g'_2]\!] \vdash g'_2$
- $[\![\lambda' r' = S']\!] \wedge \bigwedge S' \vdash \bigvee_{d' \in D',\, T' \in \mathcal{F}G'} [\![\pi' d' = r']\!] \wedge [\![\rho' d' = T']\!] \wedge \bigwedge T'$

Models: $(x,y) \longmapsto x$

Inverse image: $g \longmapsto g$

Fibre over x given by $(G',R',D')_x$

Joyal & Tierney

$[\Sigma, \mathbb{T}]$ a point-free space

Equivalent:
- Bundles over $[\Sigma, \mathbb{T}]$ ∘∘ Maps $[\Sigma_0, \mathbb{T}_0] \to [\Sigma, \mathbb{T}]$, some $[\Sigma_0, \mathbb{T}_0]$
- Point-free spaces in $\mathbb{T}$-sets ∘∘ GRD-systems

(Must get right notion of maps.)

Hence! "Fibrewise topology" of bundles
= topology "in a topos" constructive

Is this just "parametrizing topology by point of $[\Sigma, \mathbb{T}]$"? geometricity

Discrete bundles = local homeomorphisms

Special case – equivalent:

- Fibrewise discrete bundles over $[\Sigma, \Pi]$

  $[\Sigma_0, \Pi_0] \to [\Sigma, \Pi]$ a local homeomorphism

- Π-sets

  give "discrete spaces" internally

Fibred product / pullback

$$\begin{tikzcd} & Y_1 \times_X Y_2 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$= \{(y_1, y_2) \mid p_1(y_1) = p_2(y_2)\}$

with subspace topology from $Y_1 \times Y_2$

Fibre of $Y_1 \times_X Y_2$ = product of fibres of Y_1 and Y_2

Fibres are pullbacks

pullback

fibre

$$\begin{tikzcd} 1 \times_X Y = p^{-1}(\{x\}) \arrow[r] \arrow[d, "!"'] & Y \arrow[d, "p"] \\ 1 \arrow[r, "x"'] & X \end{tikzcd}$$

All works point-free

Inverse image of Π-sets is pullback

Inverse image of Π-sets

$f: [\Sigma_1, \Pi_1] \longrightarrow [\Sigma_2, \Pi_2]$

$(B, [\![\cdot = \cdot]\!]_B)$ a Π_2-set

Inverse image $f^*(B, [\![\cdot = \cdot]\!]_B) = (B, [\![\cdot = \cdot]\!]_A)$

$[\![b = b']\!]_A = f^{-1} [\![b = b']\!]_B$

$$\begin{tikzcd} {[\Sigma_1, \Pi_1] \times_{[\Sigma_2, \Pi_2]} Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ {[\Sigma_1, \Pi_1]} \arrow[r, "f"'] & {[\Sigma_2, \Pi_2]} \end{tikzcd}$$

local homeomorphism for $f^*(B, [\![\cdot = \cdot]\!]_B)$

local homeomorphism for $(B, [\![\cdot = \cdot]\!]_B)$

Geometricity

General construction on bundles is geometric if preserved by pullback.

$\Rightarrow$ Calculated fibrewise

because fibres are pullbacks

Generalizes geometricity for Π-sets

because inverse image is pullback

Geometricity

for arbitrary Π

General construction on Π-sets is geometric if preserved by inverse image

$\Rightarrow$ calculated fibrewise

- if $x: 1 \to [\Sigma, \Pi]$ then $x^* A \cong A/\sim_x$

Examples of geometric bundle constructions

Finite limits, coproducts

Fibre space: (G', R', D') a GRD-system in Π_2-sets, $f: [\Pi_1] \to [\Pi_2]$.

Then $f^*(G', R', D')$ gives pullback

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {[\Pi_2][\Pi']} \arrow[d] \\ {[\Pi_1]} \arrow[r, "f"] & {[\Pi_2]} \end{tikzcd}$$

Hyperspaces (powerlocales), spaces of measures.

cf. power set not geometric

Geometricity: cost / benefit

Costs:
- stringent constraints on logic
- limitations on maths available (e.g. $\mathcal{P}X$, $\mathbb{R}$ not sets)
- pervasive use of point-free spaces
- new techniques of topology (e.g. hyperspaces)

Benefits:
- Rely on points of point-free spaces (e.g. generic points)
- fibrewise reasoning for point-free bundles

Geometrization programme

How much mathematics can be treated geometrically?

examples
- analysis – integration, differentiation, IVT, Rolle's Thm (suitably formulated)
- metric completion
- domain theory
- ? Gelfand duality

developing Banaschewski–Mulvey

Topos approach to quantum theory

Isham Döring
Heunen Spitters
Landsman

Roughly: for quantum system given by A

- $\mathcal{C}(A)$ = space of "classical points of view"
  commutative subalgebras of A
- Spectral bundle over $\mathcal{C}(A)$ – each fibre a classical spectrum
- Internal working in $Sh(\mathcal{C}(A))$ is "neoclassical" but deals with whole system
- If geometric, works fibrewise

References (selective!)

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