INFORMATION SYSTEMS FOR CONTINUOUS POSETS

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Abstract

The method of information systems is extended from algebraic posets to continuous posets by taking a set of tokens with an ordering that is transitive and interpolative but not necessarily reflexive. This develops results of Raney on completely distributive lattices and of Hoofman on continuous Scott domains, and also generalizes Smyth's "R-structures". Various constructions on continuous posets have neat descriptions in terms of these continuous information systems; here we describe Hoffmann-Lawson duality (which could not be done easily with R-structures) and Vietoris power locales.

We also use the method to give a partial answer to a question of Johnstone's: in the context of continuous posets, Vietoris algebras are the same as localic semilattices.

1. Introduction

When in denotational semantics one chooses the structures to be used as the semantic domains, various requirements conspire to narrow the choice quite a lot: one wants, for instance, to be able to model recursive definitions, construct function spaces and solve domain equations. A frequent choice is to use Scott domains (bounded complete algebraic cpos).

One aspect of Scott domains [20] is that they can be presented using *information systems:* you don't describe the set of points directly, but present it using an *information system* comprising *tokens,* a selection of points from which the others can be derived as joins. Moreover, Scott continuous maps between Scott domains can be described as relations (*approximable mappings*) involving tokens. The information systems are crucial when one comes to solving domain equations (Scott [20] and Larsen and Winskel [14]), essentially because a colimit of a chain of domains (obtained by iterating the constructor in the domain equation) is constructed by taking the set-theoretic union of the corresponding information systems.

It is well known that algebraic posets (otherwise known as algebraic dcpos) can be treated in roughly the same way, using the compact points as tokens; general points are then ideals (directed, downward closed sets) of tokens. The algebraic poset itself is just the ideal completion of the poset of compact points, and (Scott) continuous maps between algebraic posets can be described as "approximable mappings", certain sets of pairs of compact points. It is not hard to extend these methods by using a preorder instead of a poset (this is particularly useful when dealing with powerdomains, for the orderings on sets of tokens are only preorders).

This is not a direct generalization of Scott's information systems, because in the absense of bounded completeness we must give more tokens – enough so that every point is a *directed* join of tokens. However, we shall follow Scott's original

terminology of *tokens, information systems, ideals* and *approximable mappings*. What distinguishes an information system from an ordinary preorder is, of course, the implied use of approximable mappings as morphisms instead of order preserving functions.

Let us note at this point a fact that is known but less commonly seen: algebraic posets are easy to treat locally, because the frame of Scott opens of an algebraic poset Idl(P) (which is isomorphic to the frame of Alexandroff opens of P) can be presented as (in the notation of Vickers [22])

$$\text{Fr} \langle \uparrow\{t\} \ (t \in P) \mid \bigwedge_{t \in S} \uparrow\{t\} = \bigvee_{s \in \text{ub}(S)} \uparrow\{s\} \qquad (S \subseteq_{\text{fin}} P) \rangle$$

(Here, ub(S) is the set of upper bounds of S.)

An immediate benefit of the information system approach is that power locales can be treated very easily: from an original information system P, just construct $\wp_{\text{fin}}(P)$ with the lower, upper or Egli-Milner preorder; this new information system represents an algebraic poset which is homeomorphic to the lower, upper or Vietoris power locale of Idl(P).

Mathematically speaking, it is only a short step from algebraic to continuous posets (or continuous dcpos; see Gierz et al. [5], or, more conveniently for our purposes, Johnstone [8]) and these also cover important examples based on the real line. It is the purpose of this paper to generalize the technique of information systems and use it to prove results about general continuous posets. An approach to this that has been taken by, for instance, Smyth [21], is to exploit the fact that every continuous poset is a retract of an algebraic poset. Hence one might describe an ordinary information system (giving the algebraic poset) together with extra structure representing the retraction. More generally, one might try to describe a basis of a continuous poset, i.e. a set of points such that every other point is a directed join of some of them, and Smyth's [21] *R-structures* (see note 4 at the end of Section 4) use this idea. Our information systems, which generalize R-structures, go beyond this: the tokens do not form a basis of points, but are closer to a basis of opens. In Theorem 4.13 it is actually easier to use R-structures; the main advantage of our information systems seems to come when one works with Hoffmann-Lawson duality (Section 3.2).

Hoofman [7] has already studied the case of *continuous Scott domains,* generalizing Scott's information systems. Though his applications are somewhat different from ours, his method of generalization is essentially the same. In a Scott information system there is an entailment relation ⊢ between finite sets of tokens, and in the passage to continuity one replaces reflexivity (X ⊢ X) with a weaker *interpolation* property: if X ⊢ Z then X ⊢ Y ⊢ Z for some Y.

Beware!

In ordinary topology, opens are concretely sets of points, while in pure locale theory points are concretely sets (actually completely prime filters) of opens. In the presentation we shall give here, *both points and opens are concretely sets of tokens.* Tokens are not faithfully represented either as points or as opens, so both the topological and the localic standards are inappropriate, even though the continuous posets constructed are sober spaces or spatial locales. We get round this by using the language of *topological systems* as in Vickers [22], in which points x "satisfy" opens a (x ⊨ a). There are traps for the unwary here. For instance, the intersection of two opens is their intersection *as sets of tokens* and is not necessarily itself open. The point set topological "intersection" is the *meet* of the opens, and is contained in their intersection.

2. Continuous information systems

Definition 2.1 A *continuous information system* (or *infosys;* but I'd happily change this name) is a set D equipped with a relation < that is –

- transitive,　　　　　i.e. if s < t and t < u then s < u

- interpolative,　　　　i.e. if s < u then for some t, s < t and t < u

In short, $< \circ < \; = \; <$.

The elements of D are known as *tokens*.

Note that reflexivity implies interpolativity, so preorders are infosyses. This corresponds to the algebraic case, where < appears as $\sqsubseteq$ on compact points. Some authors use the symbol $\vdash$ to relate tokens. This corresponds to our >, i.e. $s \vdash t$ iff $t < s$.

Our aim is to show how infosyses can be used to present continuous posets. We shall first give a quick localic account that exploits the standard result (see Johnstone [8]) that continuous posets are Stone dual to completely distributive frames: any continuous poset (with its Scott topology) is homeomorphic to a locale whose frame is completely distributive, and vice versa. The localic treatment is very natural, because – as we shall see – the tokens represent opens rather than points.

Recall that a frame A is *completely distributive* iff whenever $X_\lambda \subseteq A$ ($\lambda \in \Lambda$, which may be infinite), we have

$$\bigwedge_\lambda \bigvee X_\lambda \leq \bigvee_f \bigwedge_\lambda f(\lambda)$$

where f ranges over the functions from Λ to $\bigcup_\lambda X_\lambda$ such that $f(\lambda) \in X_\lambda$.

Definition 2.2 Let D be an infosys.

If $S \subseteq D$, then the *upper closure,* ↑*S,* of S is $\{t \in D : \exists s \in S.\ s < t\}$.

If $S \subseteq D$, then S is *upper closed* iff $S = {\uparrow}S$.

The upper closed subsets of D are called the *opens* of D.

Beware!

- S "upper closed" means not only that if $t > s \in S$ then $t \in S$, but also that if $t \in S$ then $s < t$ for some $s \in S$. Note that if S satisfies just the first of these conditions, then it *contains* its upper closure.
- It is convenient to use the abbreviation "↑s" for "↑{s}". But we shan't do this, because it introduces ambiguities when we come to the power locales.
- The opens of D do not form a topology *on the set D*. They are not open subsets of D in any sense.

Some useful properties of ↑ are –

- if $S \subseteq T$ then $\uparrow S \subseteq \uparrow T$

- $\uparrow(\bigvee_i S_i) = \bigvee_i \uparrow S_i$

- $\uparrow\uparrow S = \uparrow S$

- $\uparrow S$ is the least upper closed set containing S

Proposition 2.3 Let D be an infosys.

(i)   (Raney [16]) The opens of D form a completely distributive frame ΩD. Join is just union (as sets of tokens) and the meet of a family of opens is the upper closure of the intersection.

(ii) ΩD can be presented as

Fr $\langle$ $\uparrow\{t\}$ $(t \in D)$ | $\bigwedge_{t \in S} \uparrow\{t\} = \bigvee_{s \in ub(S)} \uparrow\{s\}$ $(S \subseteq_{fin} D)$ $\rangle$ (†)

where $ub(S) = \{s \in D : \forall t \in S. \ t < s\}$.

Recall that the notation (†) means the frame generated by formal symbols $\uparrow\{t\}$, subject to the relations (equations) given. In effect, the result shows a universal property of ΩD: defining a frame homomorphism from ΩD to another frame B is equivalent to interpreting the formal generators $\uparrow\{t\}$ in B and showing that this "respects the relations", i.e. transforms the formal relations to actual equalities in B. (For further details, see Vickers [22].)

Proof

(i) Raney proves this for D^{op} instead of D. (But note that this requires the axiom of choice – see the remarks round Definition 2.16.) Raney's results are considered in more detail in 2.12 to 2.15.

(ii) Let us write A for the frame presented as in the statement. Note that by taking S to be a singleton, we have $\uparrow\{t\} = \bigvee_{s>t} \uparrow\{s\}$, so if $s > t$ then $\uparrow\{s\} \leq \uparrow\{t\}$. Also in A,

$\bigwedge_{t \in S} \uparrow\{t\} = \bigvee_{s' \in ub(S)} \bigvee_{s>s'} \uparrow\{s\} = \bigvee_{s \in \uparrow ub(S)} \uparrow\{s\}$ (*)

There is a frame homomorphism $\theta_1 : A \to \Omega D$ defined by mapping the symbol $\uparrow\{t\}$ to the open $\uparrow\{t\}$. Conversely, we can define $\theta_2 : \Omega D \to A$ by taking a to $\bigvee\{\uparrow\{t\} : t \in$

a}. This clearly preserves joins; as for finite meets, first we have **true** = $\uparrow$D = $\uparrow$ub(Ø)

maps to **true** in A. Next, if a and b are opens of D then

$$\theta_2(a) \wedge \theta_2(b) = \bigvee\{\uparrow\{s\} \wedge \uparrow\{t\}: s \in a, t \in b\} \qquad \text{(by frame distributivity)}$$

$$= \bigvee\{\uparrow\{u\}: \exists v. \exists s\in a, t\in b. (u >v, v>s \text{ and } v>t)\}$$

$$\text{(by (*) with S = \{a, b\})}$$

$$= \bigvee\{\uparrow\{u\}: u \in \uparrow(a\cap b)\} = \theta_2(a\wedge b)$$

θ_1 and θ_2 are mutually inverse.   ⟧

The theory of completely distributive frames (see, e.g., Johnstone [8]) tells us immediately that ΩD is isomorphic to the frame of Scott opens for a continuous poset; we shall use 2.3 (ii) to determine what its points are.

Definition 2.4 Let D be an infosys.

An *ideal* in D is a subset x $\subseteq$ D satisfying –

- if s < t $\in$ x then s $\in$ x
- if S $\subseteq_{\text{fin}}$ x, then S has an upper bound in x

Note that if S is a singleton, this says that whenever t $\in$ x then we can find t' $\in$ x with t < t'.

An ideal of D is called a *point* of D; we write pt D for the set of points of D.

Note that although each token t gives rise to an open $\uparrow\{t\}$, it need not – by contrast with the algebraic case – give an ideal $\downarrow\{t\}$.

Proposition 2.5 Let D be an infosys. Then the points of D correspond bijectively with the locale points of ΩD, with x ⊨ a (the point x "satisfies" the open a) iff the ideal x and the upper closed set a have a token in common.

The specialization ordering on pt D is inclusion on ideals of D.

Proof

Proposition 2.3 (ii) tells us that the locale points of ΩD (that is to say the points of the locale whose frame of opens is ΩD, which are in effect frame homomorphisms from ΩD to the two-element frame $\mathbb{2}$) can be considered to be functions from the generators

$\uparrow\{t\}$ to $\mathbb{2}$ that respect the relations in the presentation. We can describe such a function by the set

$x = \{t \in D: \uparrow\{t\} \mapsto \textbf{true}\}$

and then "respecting the relations" says exactly that x is an ideal (the two clauses in Definition 2.4 correspond to the two directions $\geq$ and $\leq$ in the frame presentation).

$x \vDash a = \bigvee\{\uparrow\{t\}: t \in a\}$ iff $x \vDash \uparrow\{t\}$ for some $t \in a$, i.e. some t is in $x \cap a$.

$x \sqsubseteq y$ means that for all tokens t, if $x \vDash \uparrow\{t\}$, i.e. $t \in x$, then $y \vDash \uparrow\{t\}$, $t \in y$; i.e. $x \subseteq y$.   ⟧

Definition 2.6 Let D be an infosys. Then we also use the symbol D to denote the topological system (Vickers [22]) whose points and opens are the ideals and upper closed sets as in Definitions 2.2 and 2.4, with $x \vDash a$ iff $x \cap a \neq \emptyset$. (Note Proposition 2.3 describing the frame structure of ΩD.)

We shall say that the infosys *presents* the topological system.

The next Proposition now follows from Proposition 2.5 and the standard theory of completely distributive frames.

Proposition 2.7 Let D be an infosys. Then D (as a topological system) is homeomorphic to a continuous poset (with its Scott topology).   ⟧

Examples

(i)   If D is a preorder, then it is an infosys. Thus infosyses subsume information systems for algebraic posets. Moreover, for any infosys there is a bijection between compact points and equivalence classes of tokens under the partial equivalence relation $s \sim t$ iff $s < t$ and $t < s$ – as we shall see from 2.9 (ii), a point is compact iff it has the form $\downarrow\{s\}$ for some $s < s$, and $\downarrow\{s\} = \downarrow\{t\}$ iff $s \sim t$. Therefore, an infosys representing an algebraic poset must contain a copy of the poset of compact points, the standard information system.

(ii)   Let I be the set of dyadic rationals (i.e. of the form $m/2^n$) in the closed interval [0, 1], with $<$ as the strict numerical comparison except that $0 < 0$.

Then as topological system, I is the interval [0, 1] with its numerical ordering and Scott topology.

Completely distributive frames

The theory above relies on the established theory of completely distributive frames. Since this is somewhat involved in the standard presentation, let us sketch a direct account of two facts: first, the topological systems described in Definition 2.6 are continuous posets with their Scott topologies; and, second, every completely distributive frame arises as the frame of opens for such a system.

First, let us fix an infosys D. The key result is the following:

Lemma 2.8 Let $s < t$ be tokens. Then

(i)   There is a point x such that $s \in x$ and t is an upper bound for x.

(ii)   There is an open a such that $t \in a$ and s is a lower bound for a.

Proof

(i)   Define a sequence of tokens (s_i) as follows: $s_0 = s$, and $s_i < s_{i+1} < t$. Let x be $\{u:\ \exists u < s_i\}$. Then x is an ideal.

(ii) is similar.   ⟧

Lemma 2.9

(i)   For opens, we have $a \ll b$ iff there is some $S \subseteq_{fin} b$ such that $a \subseteq \uparrow S$.

(ii)   pt D is a continuous poset, with $x \ll y$ iff there is some token $t \in y \cap ub(x)$.

Proof

(i) ⟸: Suppose $b \leq \bigvee^{\uparrow} W = \bigcup^{\uparrow} W$ (where the $\uparrow$ is to indicate that W is directed). Every element of S is contained in some element of W, so by directedness all of S is contained in some $c \in W$. Then $a \leq c$.

⟹: Let $W = \{\uparrow S: S \subseteq_{fin} b\}$. Then $b = \bigvee^{\uparrow} W$, so $a \leq \uparrow S$ for some $S \subseteq_{fin} b$.

(ii) ⟸: Much as in part (i), and using the fact that directed unions of ideals are still ideals.

⟹: Let $W = \{z \in pt\ D: \exists s \in y.\ s$ is an upper bound for $z\}$, which is directed. We show that $y = \bigsqcup^{\uparrow} W$. Let $s' \in y$, and let $s \in y$ with $s' < s$. By Lemma 2.8 (i) there is a point z

such that s' ∈ z and s is an upper bound for z, so z ∈ W. The rest is now as in part (i).

⟧

Lemma 2.10 As a topological system, D is spatial and localic.

Proof Proposition 2.5 tells us that D is localic. For spatiality, suppose extent(a) ⊆ extent(b) (where "extent(a)" means {x ∈ pt D: x ⊨ a}). If s ∈ a, then we can find t such that s > t ∈ a. Let x be a point containing t, for which s is an upper bound. Then x ⊨ a, so x ⊨ b. If u ∈ x∩b then s > u, so s ∈ b. Hence a ≤ b.   ⟧

Lemma 2.11 ΩD is the Scott topology on pt D.

Proof

It is standard, and straightforward, that a base for the Scott topology on a continuous poset is given by the sets ⇑x = {y: x « y}. But by Lemma 2.9 (ii), y » x ⇔ y ⊨ ↑ub(x), which is open.   ⟧

This completes the proof that as topological system, D is the Scott topology on a continuous poset.

We next present Raney's results [16] (adapted to our purposes) as 2.12 to 2.15. These show how to construct an infosys directly from a completely distributive frame so that every completely distributive frame is isomorphic to the Scott topology on a continuous poset. Some of the proofs are sketched in Gierz et al. [5], Exercise I.2.22.

Definition 2.12 (This is Raney's anonymous relation "ρ".) Let A be a frame, and let a, b ∈ A. Then a is *completely below* b, a «$_c$ b, iff for every W ⊆ A if b ≤ ⋁W then a ≤ c for some c ∈ W.

Proposition 2.13 Let A be a frame. Then A is completely distributive iff every element is the join of the elements completely below it.   ⟧

Lemma 2.14 Let A be a completely distributive frame. Then «$_c$ on A is transitive and interpolative.⟧

Proposition 2.15 Let A be a completely distributive frame, and let D be the infosys whose tokens are the elements of A, ordered by s < t iff t «$_c$ s in A (note the order reversal). Then A is isomorphic to ΩD.

Proof

(Let us write a, b, etc. for opens of D, and s, t, etc. for tokens, i.e. elements of A.)

An open for D is a subset of A *lower* closed under $\ll_c$. Map ΩD to A by $a \mapsto \bigvee a$. This is monotone, and onto by Proposition 2.13. Now suppose $\bigvee a \le \bigvee b$. If $s \in a$, then we can find s' and s" such that $s \ll_c s' \ll_c s" \in a$. Then $s" \le \bigvee a \le \bigvee b$, so $s' \le t$ for some $t \in b$, so $s \ll_c t$ and $s \in b$. Hence $a \le b$. It follows that this map is an order isomorphism.

⟧

Let us briefly mention a defect in this treatment from a constructivist point of view. To prove complete distributivity according to the standard definition (before 2.2) we need a sufficiency of the choice functions f, and this usually requires the axiom of choice. In particular, choice is needed for Proposition 2.13.

Fawcett and Wood [3] have given a definition of *constructive* complete distributivity, classically equivalent to the standard definition but constructively weaker than it.

Definition 2.16 Let A be a complete lattice, and let $\bigvee: \mathrm{D}A \to A$ be the join map, where $\mathrm{D}A$ is the set of lower closed (D for "down closed") subsets of A. Then A is *constructively completely distributive* (CCD) iff $\bigvee$ has a left adjoint.

(Compare this with the definition of continuous poset in Johnstone [8], according to which a dcpo D is continuous iff $\bigvee: \mathrm{Idl}(D) \to D$ has a left adjoint.)

Proposition 2.17 A complete lattice A is constructively completely distributive iff it is isomorphic to ΩD for some infosys D.

Proof The straightforward proof is constructively valid. If A is a CCD lattice and $\bigvee$ has left adjoint g: A $\to \mathrm{D}A$, then A is made into an infosys by defining $a < b$ iff $b \in g(a)$. ⟧

Further results on constructive complete distributivity can be found in Rosebrugh and Wood [18].

Approximable mappings

It remains to consider the morphisms between infosyses. What we want to do is show how Scott continuous maps between continuous posets can be described solely in terms of tokens. The idea is seen most clearly in the algebraic case (when the infosyses are posets and the tokens represent compact points).

A continuous map from Idl(D) to Idl(E) is equivalent to a monotone map from D to Idl(E) and hence can be described by a relation f from D to E, s f t iff t ∈ pt f($\downarrow${s}). The constraints on f are that pt f($\downarrow${s}) should be an ideal and that it should be monotone as a function in s:

- s f t ⊒ t' ⟹ s f t'   (ideals are lower closed)

- s f t_i (1 ≤ i ≤ n) ⟹ ∃t. (s f t and t ⊒ t_i (1 ≤ i ≤ n))   (ideals are directed)

- s' ⊒ s f t ⟹ s' f t   (monotonicity)

These three conditions have their analogues as (3), (4) and (1) in Definition 2.18; however, direct application of the idea breaks down in the continuous case, because $\downarrow${s} isn't necessarily an ideal. Instead, we define s f t iff s ∈ Ωf($\uparrow${t}) ($\uparrow${t} *is* open), which in the algebraic case is equivalent to the definition given above. Proposition 2.20 will show that the following definition characterizes the relations that arise in this way from continuous maps.

Definition 2.18 Let D and E be two infosyses. An *approximable mapping* from D to E is a relation f from D to E such that –

(1)   if s' > s f t then s' f t

(2)   if s' f t then s' > s f t for some s

(3)   if s f t' > t then s f t

(4)   suppose s' > s f t_i (1 ≤ i ≤ n). Then there is some t' ∈ E such that s' f t' and t' > t_i for all i.

Note the nullary case (n = 0) in (4) (which corresponds to ideals having to be non-empty): if s' > s then ∃t' ∈ E. s' f t'. Of course, n = 0 and n = 2 are the significant cases; all the others are corollaries.

Note also that a dual of (2) follows:

if s' f t then s' f t' > t for some t'

For by (2) we can find s such that s' > s f t, and then by the unary case of (4) we can find t' as required.

The *identity* approximable mapping from D to itself is the relation >.

If f: D → E and g: E → F are two approximable mappings, then their *composition* f;g (or g ∘ f) is the relational composition: s (f;g) u iff for some t we have s f t g u.

Proposition 2.19 Infosyses under approximable mappings form a category **Infosys**.

Proof The hardest part is to show that compositions still satisfy 2.18 (4). Suppose s' > s f t_i g u_i. Take s" such that s' > s" > s, then t" such that s" f t" > t_i, then t' such that s' f t' > t" (use the unary case of clause (4)). Then t" g u_i for all i, so we can find u' such that t' g u' > u_i. Then s' (f;g) u' > u_i.

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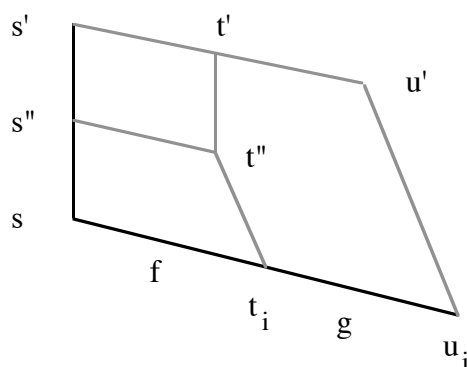
Proposition 2.20 The construction of topological systems out of infosyses extends to a full and faithful functor from **Infosys** to **Topological Systems**.

If f: D → E is an approximable mapping, then pt f: pt D → pt E and Ωf: ΩE → ΩD are defined by

pt f (x) = {t ∈ E: ∃s ∈ x. s f t}

Ωf (b) = {s ∈ D: ∃t ∈ b. s f t}

Proof

First, pt f(x) is a point of E and Ωf(b) is an open of D, and Ωf obviously preserves joins. As for finite meets, suppose s" ∈ $\bigwedge_{b \in X}$ Ωf(b), so s" > s' > s with s f t_b ∈ b for each b ∈

X. Then we can find t' with s' f t' > t_b, so t' $\in \bigcap$ X, and t" with s" f t" > t', so t" $\in \bigwedge$ X and s" $\in \Omega$f($\bigwedge$ X).

Now it is easy to see that

pt f(x) $\vDash$ b $\Leftrightarrow$ $\exists$s, t. (s $\in$ x, t $\in$ b and s f t) $\Leftrightarrow$ x $\vDash$ Ωf(b)

so that pt f and Ωf are the two parts of a continuous map between the topological systems.

It is easy to check that this constructs a functor.

For faithfulness, note that s f t iff s $\in \Omega$f($\uparrow$\{t\}) (the $\Rightarrow$ direction uses the remark after Definition 2.8) so that f is uniquely determined by Ωf.

For fullness, note that the topological systems involved are continuous posets, and hence localic. Therefore it is only necessary to consider Ωf. Given Ωf, define f by s f t iff s $\in \Omega$f($\uparrow$\{t\}). This is an approximable mapping from which Ωf is reconstructed. ⟧

Theorem 2.21 Infosys is equivalent to the category **CtsPO** of continuous posets under Scott continuous maps.

Proof

CtsPO can be considered a full subcategory of **Topological Systems**, so Proposition 2.20 constructs a full and faithful functor from **Infosys** to **CtsPO**. It remains to show that every continuous poset D can be presented by an infosys. Take as tokens, the points of D; and define s < t iff s « t (the way below relation, which is well-known to be transitive and interpolative). Now to each element s of the poset, there corresponds a set of tokens $\Downarrow$s = \{t: t « s\}, and by definition of continuous poset this is an ideal of tokens. Conversely, an ideal of tokens is directed, and hence has a join in the original poset. Moreover, s = $\bigsqcup^{\uparrow}$ $\Downarrow$s. Now let x be an ideal of tokens. If t « $\bigsqcup^{\uparrow}$ x, then we can find t « t' « $\bigsqcup^{\uparrow}$ x, and by definition of «, t' $\leq$ s for some s in x and t « s. Hence t $\in$ x, and it follows that x = $\Downarrow$($\bigsqcup^{\uparrow}$ x). This shows that pt D is order-isomorphic to the original continuous poset D.   ⟧

3. Examples of constructions using information systems

3.1 Finitary products

We don't consider infinite products, because they don't preserve continuity of posets.

Definition 3.1.1 Let D_λ ($\lambda \in \Lambda$) be a finite indexed family of infosyses. Their *product,* $\prod_\lambda D_\lambda$, is the set-theoretic product with the product ordering.

Theorem 3.1.2 Let D_λ ($\lambda \in \Lambda$) be a finite indexed family of infosyses.

(i)   The infosys product D presents the localic product.

(ii)   The projection p_μ: D → D_μ is the approximable mapping

   t p_μ s   iff $\exists$t'. t > t' and t'_μ > s

(iii)   If for each μ, f_μ: E → D_μ is an approximable mapping, then the tupling map

   f: E → D is defined by

   u f (t_λ)   iff $\exists$u'. (u > u' and $\forall\lambda$. u' f_λ t_λ)

(iv)   Let E be an infosys. Then the diagonal map δ: E → E^n is defined by

   s δ (t_λ) iff $\exists$s'. (s > s' and $\forall\lambda$. s' > t_λ), i.e. s $\in \bigwedge_\lambda \uparrow\{t_\lambda\}$

(v)   Let f_λ: D_λ → E_λ ($\lambda \in \Lambda$) be approximable mappings. Then the product $\prod_\lambda f_\lambda$

   = f (say) is defined by

   (s_λ) f (t_λ) iff s_λ f_λ t_λ for all λ

Proof There are various ways of proving this, but in preparation for Section 4 let us give a localic proof.

(i)   Consider these two frames:

   $A_1 = $ Fr $\langle \uparrow\{t\}$ (t $\in$ D) | $\bigwedge_{t\in S} \uparrow\{t\} = \bigvee_{s\in ub(S)} \uparrow\{s\}$ (S $\subseteq_{fin}$ D) $\rangle$

   $A_2 = $ Fr $\langle \uparrow\{t\}$ (t $\in D_\lambda, \lambda \in \Lambda$) |

               $\bigwedge_{t\in S} \uparrow\{t\} = \bigvee_{s\in ub(S)} \uparrow\{s\}$ (S $\subseteq_{fin}$ $D_\lambda, \lambda \in \Lambda$) $\rangle$

A_2 is the coproduct of the frames ΩD_λ. Then $A_1 \cong A_2$, the isomorphisms being given by

   θ_{12}: A_1 → A_2, $\uparrow\{t\} \mapsto \bigwedge_\lambda \uparrow\{t_\lambda\}$

$\theta_{21}: A_2 \to A_1,$ $\qquad \uparrow\{t\} \mapsto \bigvee \{ \uparrow\{u\}: u_\mu > t\}$ $\quad (t \in D_\mu)$

To show that θ_{12} respects the relations, suppose $S \subseteq_{fin} D$. Then
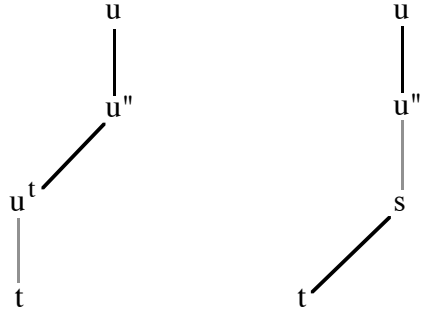
$$\bigwedge_{t \in S} \bigwedge_\lambda \uparrow\{t_\lambda\} = \bigwedge_\lambda \bigwedge_{t \in S} \uparrow\{t_\lambda\} = \bigwedge_\lambda \bigvee\{\uparrow\{s\}: s \in ub\{t_\lambda: t \in S\}\}$$

$$= \bigvee\{\bigwedge_\lambda \uparrow\{s_\lambda\}: \forall\lambda.\ s_\lambda \in ub\{t_\lambda: t \in S\}\} = \bigvee_{s \in ub(S)} \bigwedge_\lambda \uparrow\{s_\lambda\}$$

Now for θ_{21}, suppose $S \subseteq_{fin} D_\mu$ for some μ. We wish to show that

$$\bigwedge_{t \in S} \theta_{21}(\uparrow\{t\}) = \bigvee_{s \in ub(S)} \theta_{21}(\uparrow\{s\})$$

Note that $u \in \theta_{21}(\uparrow\{t\})$ iff there is some u' such that $u > u'$ and $u'_\mu > t$. Hence

$u \in \bigwedge_{t \in S} \theta_{21}(\uparrow\{t\}) \Leftrightarrow \exists u'', u^t\ (t \in S).\ (u > u''$ and $\forall t.\ (u'' > u^t$ and $u^t_\mu > t))$

$u \in \bigvee_{s \in ub(S)} \theta_{21}(\uparrow\{s\}) \Leftrightarrow \exists u''', s.\ (u > u'''$ and $u'''_\mu > s$ and $\forall t \in S.\ s > t)$

To get from the first to the second, interpolate u''' between u'' and u, and take $s = u''_\mu$.

To get from the second to the first, interpolate u'' between u''' and u, and take u^t equal to u''' for all t.

By rather similar methods, one shows that θ_{12} and θ_{21} are mutually inverse.

(ii) This is now immediate from the characterization

$u\ p_\mu\ s \qquad \Leftrightarrow \qquad u \in \Omega p_\mu(\uparrow\{s\})$

(iii) $u\ f\ (t_\lambda) \Leftrightarrow u \in \Omega f(\uparrow\{(t_\lambda)\}) = \Omega f(\bigwedge_\lambda \Omega p_\lambda(\uparrow\{t_\lambda\})) = \bigwedge_\lambda \Omega f_\lambda(\uparrow\{t_\lambda\})$. The result is now immediate.

(iv) and (v) These follow from (iii). $\rrbracket$

Note that the nullary product is the one-element poset $\{*\}$, with $u\ f\ *$ iff $u > u'$ for some u'.

3.2 Hoffmann-Lawson duality

A good account of Hoffmann-Lawson duality for continuous posets is given in Johnstone [8]. In domain theory, it is unusual to apply the duality directly to the domains themselves, but an indirect application is that the Smyth power domain $P_U D$ is the Hoffmann-Lawson dual of the frame ΩD of Scott open sets.

Recall the basic definition and results:

Definition 3.2.1 Let D be a continuous poset. Then the *Hoffmann-Lawson dual* of D, written $\hat{D}$, has as its points the Scott open filters in D, i.e. the Scott open, downward directed sets of points of D.

Theorem 3.2.2 Let D be a continuous poset.

(i) $\hat{D}$ is also a continuous poset.

(ii) $\Omega\hat{D} \cong (\Omega D)^{op}$ (Recall that ΩD is a completely distributive frame, from which it follows that so is its opposite lattice.)   ⟧

The duality is very easily representable by infosyses; just turn the token set upside down.

Theorem 3.2.3 Let D be an infosys. We write $\hat{D}$ for the infosys with the same tokens but the opposite order. Then $\hat{D}$ presents the Hoffmann-Lawson dual of the continuous poset presented by D.

Proof

We show that $\Omega\hat{D} \cong (\Omega D)^{op}$. If u is an open for $\hat{D}$, in other words a *lower closed* subset of D, write u* for $\uparrow(u^c)$, the upper closure of its complement. Similarly, if a is an open for D, write a* for $\downarrow(a^c)$. We show that u** = u.

$s \in u^{**} \Leftrightarrow \exists s'. \ s < s' \notin u^* \Leftrightarrow \exists s'. \ (s < s' \text{ and } \neg\exists t. \ (s' > t \text{ and } t \notin u))$

$\Leftrightarrow \exists s'. \ (s < s' \text{ and } \forall t. \ (s' > t \Rightarrow t \in u)) \Leftrightarrow s \in u$

(For the last $\Leftarrow$, choose s' such that $s < s' \in u$.)

By symmetry, a** = a and so * is a bijection from ΩD to $\Omega\hat{D}$; it is clearly order reversing.   ⟧

Note that the argument given here is not intuitionistically valid. Rosebrugh and Wood [18] show that if you conduct your set theory internally in a topos, then the property that the opposite of any constructively completely distributive lattice is still constructively completely distributive characterizes Boolean toposes. Hence in general, Hoffmann-Lawson duality of continuous posets has to be defined by taking the opposite of the infosys, not of the lattice.

The Hoffmann-Lawson dual is not functorial with respect to continuous maps (approximable mappings), but there are two other kinds of morphism between infosyses that give categories of infosyses for which the Hoffmann-Lawson dual is a genuine duality, contravariantly functorial. In fact on **Infosys** it is a very good duality, for the composition of the duality functor with itself is *equal* to the identity functor.

Let us reconsider the definition (2.18) of approximable mappings. The idea is to symmetrize the definition so that it can be dualized by simultaneously reversing the orders and the relation. Under this dualization, clauses (1) and (3) are interchanged, but (2) and (4) are not. It was remarked after the definition that an approximable mapping satisfies the dual of (2).

We make one symmetrization (lower semicontinuity) by weakening (4) to the dual of (2), giving a non-deterministic generalization of approximable mappings, and the other (Lawson maps) by strengthening (2) to the dual of (4), giving a specialization of approximable mappings.

Definition 3.2.4 Let D and E be two infosyses.

(i)   A *lower approximable semi-mapping* from D to E is a relation f from D to E such that–

(1)   if $s' > s \, f \, t$ then $s' \, f \, t$

(2)   if $s' \, f \, t$ then $s' > s \, f \, t$ for some s

(3)   if $s \, f \, t' > t$ then $s \, f \, t$

(4)   if $s \, f \, t'$ then $s \, f \, t > t'$ for some t.

        (In short, $>;f = f = f;>$. (1), (2) and (3) are exactly as for approximable mappings.)

(ii)   An approximable mapping f from D to E is a *Lawson approximable mapping*

iff it satisfies

(5)   suppose s_i f t ($1 \leq i \leq n$) and t > t'. Then there is some $s \in D$ such that $s_i > s$

for all i and s f t'.

It is straightforward to verify that there are two more categories whose objects are

infosyses:

- **Infosys$_L$** has as morphisms the lower approximable semimappings
- **Infosys$_\Lambda$** has as morphisms the Lawson approximable mappings

In each case, the identity is the relation >, and composition is relational composition.

Moreover, each of these categories is antiisomorphic to itself by a functor extending the

Hoffmann-Lawson dual.

Recall that if D and E are locales, then a *lower semicontinuous map* from D to E is

a continuous map from D to $P_L E$ and that these amount to join preserving functions

from ΩE to ΩD.

Also, if D and E are continuous posets, then a continuous map g: D $\rightarrow$ E is a

Lawson map iff Ωg preserves open filters of points (as referred to in 3.2.1).

Theorem 3.2.5 Let D and E be infosyses.

(i)   There is a bijection between lower approximable semimappings from D to E,

and lower semicontinuous maps from D to E under which θ: $\Omega E \rightarrow \Omega D$

corresponds to the relation $\{(s,t): s \in \theta(\uparrow\{t\})\}$.

(ii)   The bijection of (i) restricts to a bijection between Lawson approximable

mappings from D to E, and Lawson maps from D to E (treated as continuous

posets).

Proof

(i)   Given a lower semicontinuous map θ, define the relation f by s f t iff $s \in \theta(\uparrow\{t\})$.

This is a lower approximable semimapping, and it determines θ. Conversely, given a

lower approximable semimapping f, define θ by $\theta(a) = \{s: \exists t \in a.\ s\ f\ t\} \in \Omega D$; θ

preserves joins. Now $s \in \theta(\uparrow\{t\}) \Leftrightarrow s$ f t' $> t$ for some t', i.e. s f t. Hence we regain f from θ.

(ii) We first show that $a \in \Omega D$ is a filter (of points) iff it is a filter as a set of tokens, in other words an ideal of $\hat{D}$.

Suppose a is a filter in the standard sense, and let S be a finite set of tokens in a. For each $s \in S$, we can find s' $< s$ with s' $\in a$, and then by Lemma 2.8 a point x_s such that s' $\in x_s$ and s is an upper bound for x_s. We have $x_s \vDash a$, so there is a point $y \vDash a$ such that $y \sqsubseteq x_s$ for all $s \in S$. Take $t \in y \cap a$. Then for each s, $t \in x_s$, so t $< s$ and t is a lower bound for S in a.

Now suppose a is an ideal for $\hat{D}$, and let X be a finite set of points of D all satisfying a. For each $x \in X$, we can find a token $s_x \in x \cap a$. Let t be a lower bound in a for $\{s_x : x \in X\}$, let t' $< t$ with t' $\in a$, and let (by Lemma 2.8) y be a point with t' $\in y$ and t an upper bound for y. Then $y \vDash a$; and if $u \in y$ then u $< t < s_x \in x$, so $u \in x$. Hence y is a lower bound for X.

Having cleared that hurdle, let's return to the main result. We must show that if f is an approximable mapping from D to E, then f is Lawson iff whenever $b \in \Omega E$ is a filter of tokens, then so is $\Omega f(b)$.

Suppose f is Lawson, b is a filter and $S \subseteq_{fin} \Omega f(b)$. For each $s \in S$ we can find $t_s \in b$ such that s f t_s; let t be a lower bound in b for the tokens t_s, and let t' $< t$ also be in b. By the Lawson property in 3.2.4, we can find s', a lower bound for S, with s' f t' so that s' $\in \Omega f(b)$.

Now suppose Ωf preserves filters, and suppose s_i f t $(1 \leq i \leq n)$ and t $> t'$. By Lemma 2.8 applied to $\hat{D}$, there is an open filter b' $\in \Omega E$ such that $t \in b'$ and t' is a lower bound for b'. $s_i \in \Omega f(b')$ for every i, and $\Omega f(b')$ is an open filter, so we can find a lower bound s in $\Omega f(b')$ for the tokens s_i. Then s f t'' for some t'' $\in b'$, so t'' $> t'$ and s f t'.

⟧

Note that the bijection of 3.2.5 (i) also restricts to the bijection (2.20) between approximable mappings and continuous maps.

4. The Vietoris power locale

In this section we treat the power domains of continuous posets (though we shall usually call them power *locales,* because of our localic viewpoint). The three main power domains (lower, or Hoare; upper, or Smyth; and convex, Plotkin, or Vietoris) are used in computer science in the semantics of non-determinism – see Plotkin [15] for the standard results. We shall also assume some of the technical results in Johnstone [9]. Note that our power locales will always include the empty set as a point, and that for the Vietoris power locale it is isolated, incomparable with any other point.

There are four main methods of describing power locales, which we describe in relation to the Vietoris power locale (on which we shall mainly concentrate). For fuller details on most of these, see Plotkin [15].

1 –   The *localic* method is the localic Vietoris constuction of Johnstone [9], in which subbasic (generating) opens of the power locale are of the form $\Box a$ and $\Diamond a$, where a is an open of the original locale:

$\Omega(VD) \quad \cong Fr \langle \Box a, \Diamond a \ (a \in \Omega D) \mid$

$\Diamond$ preserves joins

$\Box$ preserves finite meets

$\Box$ preserves directed joins

$\Box a \wedge \Diamond b \le \Diamond(a \wedge b)$

$\Box(a \vee b) \le \Box a \vee \Diamond b \quad \rangle$

One should think of this as a way of building properties of sets out of properties of points: $X \vDash \Box a$ iff $\forall x \in X. \ x \vDash a$, and $X \vDash \Diamond a$ iff $\exists x \in X. \ x \vDash a$. But the method is purely localic, and does not depend on points at all. It applies to arbitrary locales, and we take it as our reference point.

2 –   The *topological* method describes the points of the power space as sets of points of the original space. For instance, for a spectral locale D (these include SFP domains), the Vietoris power locale VD is spatial and its points can be identified with the convex, patch closed sets of points of D.

3 –   The *information system theoretic* method shows how, given an information system representing the original locale, to construct one for the power locale. In standard domain theory, one takes finite sets of tokens (compact points), preorders them appropriately (for the Vietoris locale this is the Egli-Milner preorder), and takes equivalence classes with respect to the preorder to make a poset. This is the basis of the power locale.

4 –   The *algebraic* method, in which one specifies the power locale as a free localic semilattice. This method is standard in domain theory (see, e.g., Plotkin [15]), but it is apparently not known for general locales whether the Vietoris power locale is a free localic semilattice. Johnstone [9] addresses the problem in the following form. Every algebra for the Vietoris monad is a localic semilattice; moreover, a localic semilattice structure extends to a Vietoris algebra structure in at most one way. He asks what conditions on localic semilattices are sufficient for them to be also Vietoris algebras. When this happens, the Vietoris power locale is the free Vietoris algebra and hence the free localic semilattice.

In our treatment here, we describe methods 1, 3 and 4 for infosyses (i.e. general continuous posets) and prove that they are equivalent. The crux is the proof (Theorem 4.3) that methods 1 and 3 yield homeomorphic locales. We then provide part of an answer to Johnstone's question by showing that for any continuous poset, a localic semilattice structure will always extend to a Vietoris algebra structure; hence method 4 works. It seems that 4 and 3 are closely related, 4 exploiting the fact that the information system construction uses *finite* sets of tokens. On the other hand, I do not know of a nice treatment even for algebraic posets along the lines of the spatial method 2, and I conjecture that the method is really appropriate for stably locally compact locales. (Johnstone [9] showed that in general the points of a Vietoris power locale VD can be identified with certain sub*locales* of D, but that if D is stably locally compact then these sublocales are spatial.)

Definition 4.1 Let D be an infosys. The *lower, upper and Vietoris power systems* $P_L D$, $P_U D$ and VD are defined as follows. For each, the tokens are all finite sets of tokens of D. The corresponding orderings $<_L$, $<_U$ and $<_{EM}$ (all easily shown to be transitive and interpolative) are defined by –

- $S <_L T$ iff $\forall s \in S.\ \exists t \in T.\ s < t$

- $S <_U T$ iff $\forall t \in T.\ \exists s \in S.\ s < t$

- $S <_{EM} T$ iff $S <_L T$ and $S <_U T$

Definition 4.2 We have already defined the Vietoris power locale. If D is a locale, then the lower (Hoare) and upper (Smyth) power locales are defined by

$$\Omega P_L D = Fr \langle \diamond a\ (a \in \Omega D) \mid \diamond \text{ preserves joins} \rangle$$

$$\Omega P_U D = Fr \langle \square a\ (a \in \Omega D) \mid \square \text{ preserves finite meets and directed joins} \rangle$$

Theorem 4.3 Let D be an infosys. Then –

(i)   VD is homeomorphic to the Vietoris power locale on D.

(ii) $P_L D$ is homeomorphic to the lower power locale on D.

(iii) $P_U D$ is homeomorphic to the upper power locale on D.

Proof

(i) The method of proof is essentially that outlined in Robinson [17] to prove the corresponding result for algebraic posets with bottom. We first prove a general result about the Vietoris construction.

Proposition 4.4 Let D be a locale, and $X \subseteq \Omega D$. Then in the Vietoris locale on D, we have

$$\square(\bigvee X) = \bigvee \{\square(\bigvee Y) \wedge \bigwedge_{a \in Y} \diamond a: Y \subseteq_{fin} X\}$$

(The spatial intuition is this. If K is a compact set of points contained in $\bigvee X$, then we can first (by compactness) replace X by a finite subset Y, and then throw away the elements of Y from which K is disjoint until we get $K \subseteq \bigvee Y$ and K meets every a in Y.)

Proof

$\Box$ preserves $\bigvee^{\uparrow}$ (where again the arrow is used to indicate that the join is of a *directed* set), so $\Box(\bigvee X) = \bigvee^{\uparrow}\{\Box(\bigvee Y): Y \subseteq_{fin} X\}$. Hence it suffices to prove the result for finite X, which we do by induction on $|X|$. In the base case, $X = \emptyset$, both sides reduce to $\Box$**false**.

Now assume $|X| \geq 1$. The difficult direction is $\leq$.

Lemma 4.4.1 Let $X' \subseteq X$. Then

$$\Box(\bigvee X) = \bigvee_{b \in X'} \bigvee_{b \notin Y \subseteq X} (\Box(\bigvee Y) \wedge \bigwedge_{a \in Y} \Diamond a) \quad \vee \quad (\Box(\bigvee X) \wedge \bigwedge_{b \in X'} \Diamond b)$$

Proof Again, the difficult direction is $\leq$. We use induction on $|X'|$. The base case, $X' = \emptyset$, is trivial.

Now suppose $|X'| \geq 1$. Pick $c \in X'$, and let $X'' = X' \backslash \{c\}$. By induction on X',

$$\Box(\bigvee X) = \bigvee_{b \in X''} \bigvee_{b \notin Y \subseteq X} (\Box(\bigvee Y) \wedge \bigwedge_{a \in Y} \Diamond a)$$

$$\vee (\Box(\bigvee(X \backslash \{c\}) \vee c) \wedge \bigwedge_{b \in X''} \Diamond b)$$

Now for any u and v in ΩD, $\Box(u \vee v) = \Box(u \vee v) \wedge (\Box u \vee \Diamond v) = \Box u \vee (\Box(u \vee v) \wedge \Diamond v)$. Hence,

$$\Box(\bigvee(X \backslash \{c\}) \vee c) \wedge \bigwedge_{b \in X''} \Diamond b \leq \Box(\bigvee(X \backslash \{c\})) \vee (\Box(\bigvee X) \wedge \Diamond c \wedge \bigwedge_{b \in X''} \Diamond b)$$

$$= \bigvee_{c \notin Y \subseteq X} (\Box(\bigvee Y) \wedge \bigwedge_{a \in Y} \Diamond a) \vee (\Box(\bigvee X) \wedge \bigwedge_{b \in X'} \Diamond b)$$

using induction on X. Hence we get the required result. ⟧

We return to the proof of Proposition 4.4. Put $X' = X$ in the Lemma:

$$\Box(\bigvee X) = \bigvee_{b \in X} \bigvee_{b \notin Y \subseteq X} \{\Box(\bigvee Y) \wedge \bigwedge_{a \in Y} \Diamond a\} \vee (\Box(\bigvee X) \wedge \bigwedge_{b \in X} \Diamond b)$$

$$= \bigvee_{Y \subseteq X} (\Box(\bigvee Y) \wedge \bigwedge_{a \in Y} \Diamond a) ⟧$$

Let us now embark on Theorem 4.3 (i). For the sake of the proof, let us write $V_i D$ for the locale defined using the infosys called VD in Definition 4.1, and VD for the Vietoris locale on (the continuous poset presented by) D. We show that the frame $\Omega(VD)$ (as defined at the start of this section) is isomorphic to –

$$\Omega(V_i D) \qquad \cong Fr \langle \uparrow\{S\} (S \subseteq_{fin} D) |$$

$$\bigwedge_{S \in X} \uparrow\{S\} = \bigvee_{T \in ub(X)} \uparrow\{T\} \qquad (X \subseteq_{fin} VD) \rangle$$

If $S \subseteq_{fin} D$, write

$$\alpha(S) = \square\uparrow S \wedge \bigwedge_{s\in S} \diamond\uparrow\{s\}$$

If $S <_{EM} T$, then $\alpha(S) \geq \alpha(T)$. Also, Proposition 4.4 says that for any $a \in \Omega D$,

$$\square\, a = \bigvee\{\alpha(T)\colon T \subseteq_{fin} a\}$$

We define a homomorphism $\theta_1\colon \Omega(V_iD) \to \Omega(VD)$ by $\uparrow\{S\} \mapsto \alpha(S)$. To show that this respects the relations, suppose that $X \subseteq_{fin} VD$. Then the difficult direction is to show that

$$\bigwedge_{S\in X} \alpha(S) \leq \bigvee_{T\in ub(X)} \alpha(T)$$

Now,

$$\bigwedge_{S\in X} \alpha(S) = \square\, \bigwedge_{S\in X} \uparrow S \wedge \bigwedge_{s\in\cup X} \diamond\uparrow\{s\}$$

$$= \bigvee\{\alpha(R)\colon R \subseteq_{fin} \bigwedge_{S\in X}\uparrow S\} \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{s\}$$

$$\leq \bigvee\{\alpha(R) \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{s\}\colon S <_U R \text{ for all } S\in X\}$$

We can consider such sets R individually.

$$\alpha(R) \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{s\}$$

$$\leq \alpha(R) \wedge \bigwedge_{s\in\cup X}\diamond(\uparrow\{s\}\wedge\uparrow R) \qquad (\text{because } \alpha(R) \leq \square\uparrow R)$$

$$= \bigvee\{\alpha(R) \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{t_s\}\colon t_s \in \uparrow\{s\} \wedge \uparrow R\ (s\in\cup X)\}$$

Again, we can consider such families $\{t_s\}$ individually. Let $T = R \cup \{t_s\colon s\in\cup X\}$. Since $t_s \in \uparrow R$, we have $\uparrow T = \uparrow R$. Hence

$$\alpha(R) \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{t_s\} = \square\uparrow R \wedge \bigwedge_{s\in R}\diamond\uparrow\{s\} \wedge \bigwedge_{s\in\cup X}\diamond\uparrow\{t_s\}$$

$$= \square\uparrow T \wedge \bigwedge_{t\in T}\diamond\uparrow\{t\} = \alpha(T)$$

Also, $S <_{EM} T$ for all $S \in X$, i.e. $T \in ub(X)$.

This completes the proof that θ_1 is well-defined.

Next, we define a homomorphism $\theta_2\colon \Omega(VD) \to \Omega(V_iD)$ mapping, for each $a \in \Omega D$,

$$\diamond a \qquad \mapsto \bigvee\{\uparrow\{S\}\colon S \subseteq_{fin} D, S\cap a \neq \varnothing\} \quad = \{S \subseteq_{fin} \uparrow D\colon S\cap a \neq \varnothing\}$$

$$\square a \qquad \mapsto \bigvee\{\uparrow\{S\}\colon S \subseteq_{fin} a\} \qquad\qquad = \{S\colon S \subseteq_{fin} a\}$$

These are both monotone in a.

For the first three relations in the presentation of $\Omega(VD)$, we must show that if $X \subseteq \Omega D$, then

- $\{S \subseteq_{fin} \uparrow D, S \cap \bigcup X \neq \emptyset\} \leq \{S \subseteq_{fin} \uparrow D, S \cap a \neq \emptyset \text{ for some } a \in X\}$

- if X is directed, then

  $\{S: S \subseteq_{fin} \bigcup \uparrow X\} \leq \{S: S \subseteq_{fin} a \text{ for some } a \in X\}$

- if X is finite, then

  $\bigwedge_{a \in X} \{S: S \subseteq_{fin} a\} \leq \{S: S \subseteq_{fin} \bigwedge X\}$

The first two of these are obvious. For the third, if S' is in the left hand side then $S' > S$ for some $S \subseteq_{fin} \bigcap X$, and it easily follows that $S' \subseteq_{fin} \bigwedge X$.

Next, we must show that if $a, b \in \Omega D$, then

- $\{S: S \subseteq_{fin} a \vee b\} \leq \{T: T \subseteq_{fin} a\} \vee \{R \subseteq_{fin} \uparrow D: R \cap b \neq \emptyset\}$

- $\{S: S \subseteq_{fin} a\} \wedge \{T \subseteq_{fin} \uparrow D: T \cap b \neq \emptyset\}$

  $\leq \{R \subseteq_{fin} \uparrow D: R \cap a \cap b \neq \emptyset\}$

The first is obvious. The second is also true, and easier to see, if we replace $\wedge$ by $\cap$; and this suffices to prove what we want.

This completes the proof that θ_2 is well-defined.

Next, we show that θ_1 and θ_2 are mutually inverse.

$\theta_2(\theta_1(\uparrow\{S\})) = \{R: R \subseteq_{fin} \uparrow S\} \wedge \bigwedge_{s \in S} \{T \subseteq_{fin} \uparrow D: T \cap \uparrow\{s\} \neq \emptyset\}$

$= \{R: R >_U S\} \wedge \{T \subseteq_{fin} \uparrow D: T >_L S\}$

$= \{R: R >_{EM} S\} = \uparrow\{S\}$

$\theta_1(\theta_2(\square a)) = \bigvee\{\alpha(S): S \subseteq_{fin} a\} = \square a$ \qquad (by Proposition 4.4)

$\theta_1(\theta_2(\diamond a)) = \bigvee\{\alpha(S): S \subseteq_{fin} \uparrow D, S \cap a \neq \emptyset\} \leq \diamond a$

$\qquad\qquad\qquad$ (if $s \in S \cap a$ then $\alpha(S) \leq \diamond \uparrow\{s\} \leq \diamond a$)

$\diamond a = \diamond a \wedge \square\mathbf{true} = \bigvee_{t \in a} \diamond \uparrow\{t\} \wedge \bigvee\{\alpha(S): S \subseteq_{fin} \uparrow D\}$ \qquad (by Proposition 4.4)

$\leq \bigvee\{\alpha(S \cup \{t\}): t \in a, S \subseteq_{fin} \uparrow D\} \leq \theta_1(\theta_2(\diamond a))$

This completes the proof of Theorem 4.3 (i). (ii) and (iii) are left to the reader; the method is the same as for (i), but much easier. $\qquad$]]

Note a corollary of (iii): the points of P_UD are the Scott open filters of ΩD, i.e. the points of the dual $(\Omega D)^\wedge$; in fact P_UD is homeomorphic to $(\Omega D)^\wedge$. It follows that we have an infosys presenting ΩD: the tokens are finite sets of tokens of D ordered by $>_U$.

We next describe certain constructions associated with the Vietoris monad.

Proposition 4.5

(i)   Let D be an infosys, and let $\eta: D \to VD$ and $\mu: V^2D \to VD$ be the unit and multiplication for the Vietoris monad. Then

$$s \, \eta \, T \iff T \neq \varnothing \text{ and } s \in \uparrow ub(T), \text{ i.e. } \exists s_1 < s \text{ with } \{s_1\} > T$$
$$G \, \mu \, T \iff \bigcup G > T$$

(ii)   Let f: D $\to$ E be an approximable mapping. Then

$$S \, (Vf) \, T \quad \iff \forall s \in S. \exists t \in T. \, s \, f \, t \quad \text{and} \quad \forall t \in T. \exists s \in S. \, s \, f \, t$$

(iii)   The semilattice structure of VD (see Johnstone [9]) has unit $p_0: 1 \to VD$ and binary operation n: VD$\times$VD $\to$ VD, where

$$* \, p_0 \, T \qquad \text{iff } T = \varnothing$$
$$(R, S) \, n \, T \qquad \text{iff } R \cup S > T$$

(We take 1 to be presented by the partial order {*}.)

Proof The proof methods are the same in each case. First we show (or claim) that the approximable mappings described are indeed approximable mappings, and then we show that the corresponding inverse image maps agree with those given by the standard locale theory of Johnstone [9].

(i) Define η' and μ' as in the statement: for instance, $G \, \mu' \, T$ iff $\bigcup G > T$. These are approximable mappings.

η': The only mildly difficult part of Definition 2.18 is (4). If $s' > s > s_i$ with $\{s_i\} > T_i$ ($1 \le i \le n$), then $s' \, \eta' \, \{s\} > T_i$.

μ': (1) and (3) in 2.18 are easy (note that if $G' > G$ then $\bigcup G' > \bigcup G$). For (4), if $G' > G$ with $G \, \mu' \, T_i$ ($1 \le i \le n$), then $G' \, \mu' \, \bigcup G > T_i$.

For (2), we must show that if $\cup G' > T$, then there exists $G < G'$ such that $\cup G > T$.
Let $P = \{(s,t): s \in \cup G', t \in T, s > t\}$ and for each $p = (s,t) \in P$, choose r_p with $s > r_p > t$. Now, for each $S \in G'$, let $R_S = \{r_p: p = (s,t) \text{ with } s \in S\}$, and let $G = \{R_S: S \in G'\}$.
Then $S > R_S$, so $G' > G$, and $\cup G > T$.

Now $\eta' = \eta$ and $\mu' = \mu$, for

$$\Omega\eta'(\Box a) = \{s: \exists s_1, T. \; s > s_1, T \subseteq_{fin} a, \{s_1\} > T\} = a = \Omega\eta(\Box a)$$

$$\Omega\eta'(\Diamond a) = \{s: \exists s_1, T. \; s > s_1, T \subseteq_{fin} \uparrow D, T \cap a \neq \emptyset, \{s_1\} > T\} = a = \Omega\eta(\Diamond a)$$

$$\Omega\mu'(\Box a) = \{G: \exists T. \; \cup G > T \subseteq_{fin} a\} = \{G: \cup G \subseteq_{fin} a\}$$

$$= \{G: \forall S. \; (S \in G \Rightarrow S \subseteq_{fin} a\} = \{G: G \subseteq_{fin} \Box a\} = \Box\Box a = \Omega\mu(\Box a)$$

$$\Omega\mu'(\Diamond a) = \{G: \exists T. \; \cup G > T \subseteq_{fin} \uparrow D \text{ and } T \cap a \neq \emptyset\}$$

$$= \{G \subseteq_{fin} \uparrow(VD): \cup G \cap a \neq \emptyset\}$$

$$= \{G \subseteq_{fin} \uparrow(VD): \exists S \in G. \; S \cap a \neq \emptyset\}$$

$$= \{G \subseteq_{fin} \uparrow(VD): G \cap \Diamond a \neq \emptyset\} = \Diamond\Diamond a = \Omega\mu(\Diamond a)$$

(ii) Let us define $f' \subseteq VD \times VE$ by the condition on the right-hand side of the statement. f' is an approximable mapping. For (2) of 2.18, suppose $S' \; f' \; T$ and define $P = \{(s,t) \in S' \times T: s \; f \; t\}$. For each $p = (s,t) \in P$, choose s_p such that $s > s_p \; f \; t$. Then

$$S' > \{s_p: p \in P\} \; f' \; T$$

For (4), suppose $S' > S \; f' \; T_i$ ($1 \leq i \leq n$), and let

$$P = \{(s',s,(t_i)_{1 \leq i \leq n}) \in S' \times S \times \prod_i T_i: s' > s \; f \; t_i \; (1 \leq i \leq n)\}$$

For each $p = (s',s,(t_i)) \in P$, choose r_p such that $\forall i. \; s' \; f \; r_p > t_i$, and let $T' = \{r_p: p \in P\}$.
Then $S' \; f' \; T' > T_i$. (In showing $T' >_L T_i$, note that if $t_i \in T_i$ for a particular i then we can find $s \in S$ with $s \; f \; t_i$, and then for each $j \neq i$ we can find $t_j \in T_j$ with $s \; f \; t_j$.)

Now $f' = Vf$, for we have

$$S \in \Omega f'(\Box a) \Leftrightarrow \exists T \subseteq_{fin} a. \; S \; f' \; T \Leftrightarrow \forall s \in S. \; \exists t \in a. \; s \; f \; t$$

$$\Leftrightarrow S \subseteq_{fin} \Omega f(a) \Leftrightarrow S \in \Box\Omega f(a) = \Omega(Vf)(\Box a)$$

$$S \in \Omega f'(\Diamond a) \Leftrightarrow \exists T \subseteq_{fin} \uparrow E. \; S \; f' \; T \text{ and } T \cap a \neq \emptyset$$

$$\Leftrightarrow S \subseteq_{fin} \uparrow D \text{ and } \exists s \in S, t \in a. \; s \; f \; t$$

<div align="center">(for $\Leftarrow$, use the nullary case of Definition 2.18 (4))</div>

$\Leftrightarrow S \subseteq_{fin} \uparrow D$ and $S \cap \Omega f(a) \neq \emptyset \Leftrightarrow S \in \Diamond \Omega f(a) = \Omega(Vf)(\Diamond a)$

(iii) Define $p_0' \subseteq 1 \times VD$ and $n' \subseteq VD \times VD \times VD$ by the conditions on the right-hand sides in the statement. They are approximable mappings. Then $p_0' = p_0$ and $n' = n$, for

$* \in \Omega p_0'(\Box a) \Leftrightarrow \emptyset \subseteq a \Leftrightarrow \mathbf{true} \Leftrightarrow * \in \{*\} = \Omega p_0(\Box a)$

$* \in \Omega p_0'(\Diamond a) \Leftrightarrow \emptyset \cap a \neq \emptyset \Leftrightarrow \mathbf{false} \Leftrightarrow * \in \emptyset = \Omega p_0(\Diamond a)$

$(R,S) \in \Omega n'(\Box a) \Leftrightarrow \exists T \subseteq_{fin} a. \ R \cup S > T \Leftrightarrow R \cup S \subseteq_{fin} a$

$\quad \Leftrightarrow R \subseteq_{fin} a$ and $S \subseteq_{fin} a \Leftrightarrow (R,S) \in \Box a \otimes \Box a = \Omega n(\Box a)$

$(R,S) \in \Omega n'(\Diamond a) \Leftrightarrow \exists T \subseteq_{fin} \uparrow D. \ R \cup S > T$ and $T \cap a \neq \emptyset$

$\quad \Leftrightarrow R \cup S \subseteq_{fin} \uparrow D$ and $(R \cup S) \cap a \neq \emptyset$

$\quad \Leftrightarrow R \subseteq_{fin} \uparrow D$ and $S \subseteq_{fin} \uparrow D$ and either $R \cap a \neq \emptyset$ or $S \cap a \neq \emptyset$

$\quad \Leftrightarrow (R,S) \in \Diamond a \otimes \mathbf{true} \vee \mathbf{true} \otimes \Diamond a = \Omega n(\Diamond a)$

⟧

## *Vietoris power locales are free semilattices*

We now investigate continuous posets equipped with a localic semilattice structure, so throughout the section let D be an infosys so equipped. We write $\sigma_n: D^n \to D$ for the n-ary semilattice operation; so σ_0 is the unit, σ_1 is the identity map, σ_2 is the binary operation, and σ_n (n > 2) is σ_2 iterated (of course, we've used the associative law here).

If $(s_i)_{1 \leq i \leq n}$ is a sequence of tokens, then we also write $(s_i) \sigma_* t$ for $(s_i) \sigma_n t$.

The commutative law tells us that if $(s_i) \sigma_* t$, then $(s_{\pi(i)}) \sigma_* t$ for any permutation π of the indices. In other words, the validity of $(s_i) \sigma_* t$ depends only on the multiplicities of the tokens s_i, not on their order: so we can consider (s_i) to be not a sequence, but a *bag* (multiset) of tokens. We can think of a bag either as a sequence indexed over an unordered set or as a (finite) set in which the elements are assigned finite multiplicities. Note how the bags are ordered. B > C means that there is a way of arranging the elements of B and C so that as sequences, B > C. In particular, B and C must have the same size (length). We shall write Set(B) for the *set* of elements of a bag B.

It would be nice if the idempotent law enabled us to eliminate duplications and consider the bag to be a set: we could then define $\sigma: VD \to D$ by $S \ \sigma \ t$ iff $S \ \sigma_* t$

(treating S as a bag); but in general this is not possible: the converse of the following proposition does not hold. For a counterexample, consider the five-element infosys
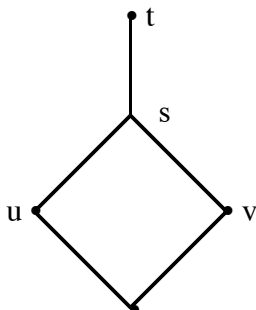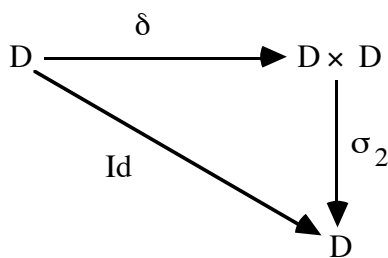

that is almost reflexive, except that $s \nless s$. Its points form the four-element Boolean algebra. If we consider its join operation, we find that $(u, v) \sigma_2 t$ and hence that $(s, s) \sigma_2 t$ – but we don't have $s > t$.

Proposition 4.6 $s > t \Rightarrow (s, s) \sigma_2 t$

Proof

The idempotent law says that the following diagram commutes (δ is the diagonal map).

If $s > t$, then there is a pair (u_1, u_2) such that $s\, \delta\, (u_1, u_2)\, \sigma_2\, t$. $s\, \delta\, (u_1, u_2)$ says there is some s' such that $s > s' > u_i$; then $(s, s) > (u_1, u_2)$, so $(s, s)\, \sigma_2\, t$. ⟧

In general, we must define $\sigma: VD \to D$ by $S\, \sigma\, t$ iff $B\, \sigma_*\, t$ for some bag B with $Set(B) = S$: i.e. B contains exactly the elements of S, but possibly with multiple copies. It can then be proved that this makes D into a Vietoris algebra. Let us introduce some language to ease the discussion.

Definition 4.7 Let D be a set and let $B = \{b_i: i \in I\}$, $C = \{c_j: j \in J\}$ be finite bags over D. Then B is an *inflation* of C iff there is a function f from I *onto* J such that $b_i = c_{f(i)}$

for every i. In other words, treating the bags as sets with multiplicities, B has exactly the same elements as C, all with multiplicities at least as great.

Lemma 4.8 Suppose S', S $\in$ VD with S' $>_{EM}$ S, and B_0 is an inflation of S. Then we can find bags B' > B such that B' is an inflation of S' and B is an inflation of B_0.

Proof Let P be the bag of pairs (s',s) such that s' $\in$ S', s $\in$ B_0 and s' > s. Then define B' and B as the bags

$$B' = \{s': (s',s) \in P\}, \qquad B = \{s: (s',s) \in P\} \qquad \rrbracket$$

Proposition 4.9 If B σ_* t and C is an inflation of B, then C σ_* t.

Proof C can be made from B by a series of duplications of elements, so without loss of generality we can assume that B and C are of the forms $(s, s_i)_{1 \le i \le n}$ and $(s,s, s_i)_{1 \le i \le n}$. By generalized associativity, σ_{n+2} can be expressed as $(\sigma_2 \times Id_{D^n}); \sigma_{n+1}$. Now B σ_{n+1} t, so we can find B' = $(s', s_i')_{1 \le i \le n}$ < B with B' σ_{n+1} t. By Proposition 4.6, (s,s) σ_2 s', so C $(\sigma_2 \times Id)$ B' and hence C σ_{n+2} t. $\qquad \rrbracket$

Definition 4.10 σ: VD $\rightarrow$ D is defined by S σ t iff B σ_* t for some inflation B of S (S treated as a bag with single multiplicities).

Proposition 4.11 Let σ: VD $\rightarrow$ D be defined as in 4.10. Then –

(i) σ is an approximable mapping.

(ii) σ makes D a Vietoris algebra.

(iii)   The Vietoris algebra structure on D extends the semilattice structure.

Proof

(i) If S' $>_{EM}$ S σ t, then S' σ t follows from Lemma 4.8.

Suppose S' σ t, with B' σ_* t for some inflation B' of S'. Then B' > B σ_* t for some bag B, and S' $>_{EM}$ Set(B) σ t.

Suppose S' $>_{EM}$ S σ t_i (1 $\le$ i $\le$ n). For each i we can find an inflation B_i of S such that B_i σ_* t_i; and by taking the maxima of the multiplicities we can assume there is a single inflation B_0 of S with B_0 σ_* t_i for all i. Take B' > B as in Lemma 4.8, so B σ_* t_i for all i. We can then find t such that B' σ_* t > t_i, so S' σ t > t_i as required.

(ii) We must show that the following two diagrams commute:
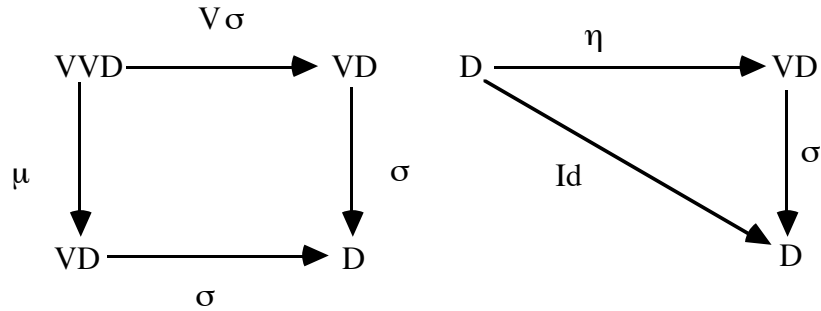
The right-hand diagram:

Suppose s η T σ u. Then we can find $s_1 < s$ such that $\{s_1\}$ $>_{EM}$ T, and an inflation B of T such that B σ_* u. Let B' be $\{s_1\}$ inflated to the same size as B: then s δ_* $B' > B$ σ_* u where δ_* is the appropriate diagonal map, and then by generalized idempotence we have $s > u$. Conversely, if $s > u$ then we can find s_1 and t with $s > s_1 > t > u$, and we can take $T = \{t\}$.

The left-hand diagram:

Suppose G $(V\sigma)$ T σ u. Choose G' with $G > G'$ $(V\sigma)$ T. Then G μ $\cup G'$, and we show also that $\cup G'$ σ u. Choose an inflation C of T such that C σ_* u, and let P and C' be the bags

$$P = \{(S, t) \in G' \times C: S \ \sigma \ t\}, \qquad C' = \{t: \exists S. (S, t) \in P\}$$

(By this notation, C' is intended to have the same size as P.) C' is an inflation of C, so C' σ_* u. For each $p = (S, t) \in P$ let B_p be an inflation of S such that B_p σ_* t, and let B be the bag sum (disjoint union) of the B_p's. Then $\text{Set}(B) = \cup G'$ and by generalized associativity B σ_* u, so $\cup G'$ σ u.

Now suppose for some R that G μ R σ u. It follows that $\cup G$ σ u; let B be an inflation of $\cup G$ such that B σ_* u. Now let P, B' and B_S ($S \in G$) be the bags

$$P = \{(S, s) \in G \times B: s \in S\}, \qquad B' = \{s: \exists S. (S,s) \in P\}, \qquad B_S = \{s: (S,s) \in P\}$$

B' is an inflation of B, and so B' σ_* u. Moreover, B' is the bag sum of the B_S's, and it follows by generalized associativity that we can find a bag C of the form $\{t_S: S \in G\}$ such that B_S σ_* t_S (so, because B_S is an inflation of S, S σ t_S) and C σ u. It follows that G $(V\sigma)$ $\text{Set}(C)$ σ u.

(iii)   (cf. Proposition 4.5 (iii).) The nullary semilattice operation induced by the Vietoris algebra structure is $p_0;\sigma$, which we must prove equal to σ_0. We have

$$* \; p_0 \; T \; \sigma \; u \Leftrightarrow \emptyset = T \; \sigma \; u \Leftrightarrow \; * \; \sigma_0 \; u$$

The binary operation is $(\eta \times \eta);n;\sigma$, which should be σ_2. Suppose first that

$$(s_1, s_2) \; (\eta \times \eta) \; (T_1, T_2) \; n \; U \; \sigma \; v$$

We therefore have s_i' such that $s_i > s_i'$ and $\{s_1', s_2'\} >_{EM} T_1 \cup T_2 >_{EM} U \; \sigma \; v$. It follows (using Lemma 4.8) that there is an inflation B of $\{s_1', s_2'\}$ such that $B \; \sigma_* \; v$. Let us write B as the bag sum of B_1 and B_2, where each B_i is an inflation of s_i'. It follows from generalized associativity that we can find t_i with $B_i \; \sigma_* \; t_i$ and $(t_1, t_2) \; \sigma_2 \; v$. Also $s_i \; \delta_* \; B_i$, so by idempotence $s_i > t_i$ and so $(s_1, s_2) \; \sigma_2 \; v$.

Conversely, if $(s_1, s_2) \; \sigma_2 \; v$ then we can find

$$(s_1, s_2) > (s_1', s_2') > (s_1'', s_2'') \; \sigma_2 \; v$$
$$(s_1, s_2) \; (\eta \times \eta) \; (\{s_1'\}, \{s_2'\}) \; n \; \{s_1'', s_2''\} \; \sigma \; v \qquad \rbrack$$

We can also answer, in our present case, another question raised by Johnstone [9].

Proposition 4.12 Let D and E be two Vietoris algebras in **Infosys**, and let $f: D \to E$ be a semilattice homomorphism. Then f is a Vietoris algebra homomorphism.

Proof We must show that $(Vf);\sigma = \sigma;f$.

Suppose first that $S \; (Vf) \; T \; \sigma \; u$, let C be an inflation of T for which $C \; \sigma_* \; u$, and let P, B' and C' be the bags

$$P = \{(s,t) \in S \times C: s \; f \; t\}, \quad B' = \{s: \exists t. \; (s,t) \in P\}, \qquad C' = \{t: \exists s. \; (s,t) \in P\}$$

Then B' and C' are inflations of S and C, and $B' \; f^* \; C' \; \sigma_* \; u$. Hence we can find r such that $B' \; \sigma_* \; r \; f \; u$, so $S \; \sigma \; r \; f \; u$.

Conversely, suppose $S \; \sigma \; r \; f \; u$, and suppose $B \; \sigma_* \; r$ where B is an inflation of S. We can find C such that $B \; f^* \; C \; \sigma_* \; u$, and so $S = Set(B) \; (Vf) \; Set(C) \; \sigma \; u$. $\qquad \rbrack$

Let us summarize.

Theorem 4.13 When the carriers are restricted to be continuous posets, the categories of localic semilattices and Vietoris algebras are isomorphic.

Hence in **CtsPO**, the Vietoris power locale is a free semilattice.   ⟧

Notes

1.   For algebraic posets, these Theorems say that to construct a power locale you can take finite sets of compact points under an appropriate preorder, and then take the ideal completion. This is quite well-known, although to the best of my knowledge it has not been proved for general algebraic posets. Robinson [17] proved it for algebraic posets with bottom, while Vickers [22], following the methods used by Abramsky [1] for strongly algebraic (SFP) domains with bottom, proved it for spectral algebraic locales not necessarily with bottom.

2.   An immediate corollary of Theorem 4.3 is that the Vietoris locale of a continuous poset is again a continuous poset. As far as I know, this has not been shown before. Of course, once Theorem 4.3 has been proved for *algebraic* posets, this additional result follows cheaply from the fact that continuous posets are exactly the retracts of algebraic posets. But the proof of 4.3 is not actually made any easier by a restriction to the algebraic case.

3.   Readers interested in power locales with the empty set excluded should have no difficulties in applying the methods presented here.

4.   Smyth [21] defines an *R-structure* to be (in our terminology) an infosys for which, for every token s, the set {t: t < s} is an ideal.

      Since for every continuous poset the points form an R-structure under «, it follows that – classically at least – every infosys is isomorphic (in the category **Infosys**) to an R-structure. There is therefore no harm done by using R-structures instead of general infosyses, and in some ways they are much easier to handle. The tokens represent a basis of points, and ΩD is a topology on the tokens (an intersection of opens is still open). Of course, the Hoffmann-Lawson duality is much less trivially representable.

      In Section 4, a big advantage of using R-structures is that the converse of Proposition 4.6 becomes true, and as a corollary the V-algebra structure map σ (Definition 4.10) can be defined by S σ t if S σ_* t when S is treated as a bag. There

seem to be decidability issues here: the σ of 4.10 can only be semi-decidable, because there is (apparently) no bound on the size of the inflations of S that must be checked.

5. Conclusions and further directions

Other power locales

An immediate investigation is to apply the methods of Section 4 to the lower and upper power locales P_L and P_U. The same methods work, but much more simply, in showing that the information system theoretic and localic constructions are homeomorphic. There remain two main questions.

- What are the algebras for the P_U monad on **CtsPO**? Schalk [19] has shown that they are the continuous semilattices (continuous posets with finite meets, which are then automatically continuous) and homomorphisms that are continuous and preserve finite meets. Between continuous semilattices, the homomorphisms are precisely the Lawson maps (see Johnstone [8]); this should be very plain from an infosys account.

- What are the algebras for the P_L monad on **CtsPO**? I conjecture that they are the continuous lattices (continuous posets with finite joins; hence they are complete lattices) and homomorphisms that preserve all joins – perhaps, to make the morphisms implicit, these algebras should be called continuous sup-lattices.

A further direction is to investigate the composite $P_U \circ P_L$, which is isomorphic to $P_L \circ P_U$. (This has been known for Scott domains since Flannery and Martin [4]. It was proved for general locales by Johnstone and Vickers [10], and for general dcpos in Heckmann [7]. Heckmann's definitions are different form ours: his lower and upper powerdomains over D are, respectively (and using the terminology of Johnstone and Vickers [10]), the free suplattice and preframe over D qua dcpo. But the results of Schalk [19] show – at least for the upper powerdomain, which is the difficult one – that in the case of continuous posets these dcpo constructions are equivalent to the localic ones.) Heckmann shows that these composites give the free frame over a dcpo and

hence form a monad on the category of dcpos. Since P_U and P_L both preserve continuity, we therefore have a monad on **CtsPO**, for which I conjecture that the algebras are the continuous frames and the homomorphisms are the frame homomorphisms. There is thus the possibility of doing locale theory in **Infosys** to treat what, classically, are the locally compact locales.

Probabilistic power domains

Jones [11] (see also Jones and Plotkin [12]) has described how to construct, for an arbitrary dcpo P, a "probabilistic power domain" $E(P)$, also a dcpo. Its points are "evaluations", functions mapping ΩP to the unit closed real interval [0, 1] and satisfying certain conditions similar to those in measure theory. [0, 1] (with its Scott topology) is the prime example of a continuous poset that is not algebraic, and one could not expect $E(P)$ to be algebraic except in extraordinary cases (e.g. $P = \emptyset$). This rules out any conventional information system theoretic account of the probabilistic power domain. But there are excellent grounds for hoping that continuous information systems will work well.

I conjecture that if P is an infosys, then a token for $E(P)$ should be a finite bag S of pairs (s, r) where s is a token for P and r is a token for [0, 1] (concretely, a dyadic rational in the interval), subject to $\sum_{(s,r)\in S} r \leq 1$. The order < appears to be complicated. The basic idea is that we should like to say

$\{(s, r)\} < \{(t, r')\}$ if $s < t$ and $r < r'$

But we should also like to redistribute the numerical weights as in –

$\{(s_1, r_1), (s_2, r_2)\} < \{(t, r')\}$ if $s_1 < t$, $s_2 < t$ and $r_1+r_2 < r'$

$\{(s, r)\} < \{(t_1, r'_1), (t_2, r'_2)\}$ if $s < t_1$, $s < t_2$ and $r < r'_1+r'_2$

When these are put together, one apparently needs the Ford-Fulkerson Theorem on network flows (the Max-cut Min-flow Theorem mentioned by Jones and Plotkin) to make the order effective relative to the original order for P.

Function spaces

A crucial part of information system theory for domains is the treatment of function spaces. Of course, one cannot expect to give an information system for the function space of an arbitrary pair of information systems, because **CtsPO** is not cartesian closed; but Hoofman [7] has given the construction for continuous Scott domains. Jung [13] has now completed his very elegant account of maximal cartesian closed subcategories of **CtsPO**, so an obvious line of investigation is to construct information systems for functions spaces within each of these subcategories.

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