The interval object [-1, 1]

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* Escardo and Simpson: universal characterization of real interval [-1, 1], using *midpoint algebra* structure.

* Localic [-1, 1] has that property. Should also work in formal topology.

* Part of proof: *localic surjection* $2^\omega \to [-1, 1]$

"The localic compact interval is an Escardo-Simpson interval object"
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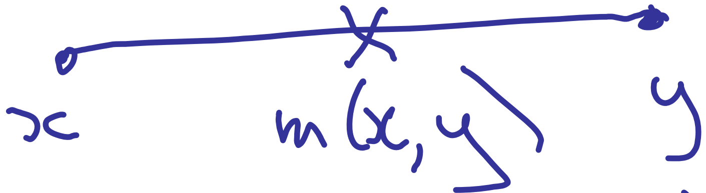
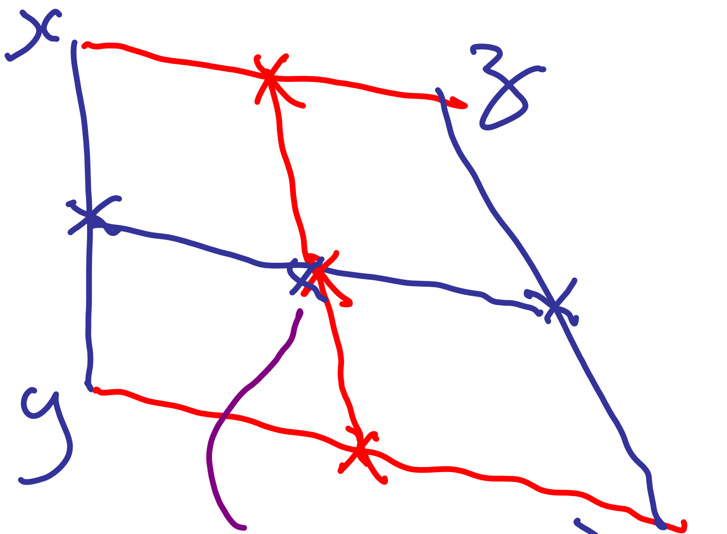
Midpoint algebras

Paradigm: points in space



$= \frac{1}{2}(x+y)$



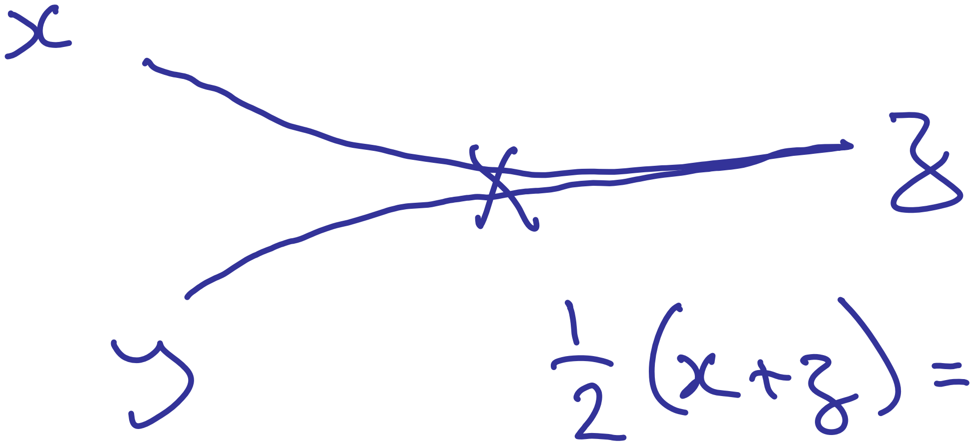
$\frac{1}{4}(x+y+z+w)$

A with binary operation m satisfying

m(x,x) = x
m(x,y) = m(y,x)
m(m(x,y), m(z,w)) = m(m(x,z), m(y,w))

It's *cancellative* if

m(x,z) = m(y,z) => x = y



$\frac{1}{2}(x+z) = \frac{1}{2}(y+z) \Rightarrow x = y$

Dyadic rational convex combinations - are derived operations

e.g.

$$\frac{1}{8}x + \frac{5}{8}y + \frac{1}{4}z$$

$$= \frac{1}{2}\left\{ \frac{1}{4}x + \frac{3}{4}y + \frac{1}{2}y + \frac{1}{2}z \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}y\right) + \frac{1}{2}y + \frac{1}{2}y + \frac{1}{2}z \right\}$$

$$= m\left(m\left(m(x,y), y\right), m(y,z)\right)$$

More general convex cominations

- can use recursion

e.g. $\frac{1}{3}x + \frac{2}{3}y$

Define recursively:
  u = m(y, v)
  v = m(x,u)

$$u = \frac{1}{2}(y+v) = \frac{1}{2}\left(y + \frac{1}{2}(x+u)\right)$$

$$\frac{3}{4}u = \frac{1}{2}y + \frac{1}{4}x$$

$$u = \frac{2}{3}y + \frac{1}{3}x$$

x v u y

Iterative midpoint algebras: ability to do the recursion

A midpoint algebra A is *iterative* if:

for every - object X
- morphism h: X -> A   head
- morphism t: X -> X   tail

there is a unique M: X -> A such that

M(x) = m(h(x), M(t(x)))

e.g. X = {0, 1}, t(x) = 1-x

M(0) = m(h(0), M(1))
M(1) = m(h(1), M(0))

u = m(y, v)
v = m(x,u)

$$
\begin{tikzcd}
A\times X \arrow[r, dotted, "A\times M"] & A\times A \arrow[d, "m"] \\
X \arrow[u, "\langle h,t\rangle"] \arrow[r, dotted, "M"'] & A
\end{tikzcd}
$$

Using iterativity: example

$$\begin{tikzcd} A \times N \arrow[r, "A \times M"] & A \times A \arrow[d, "m"] \\ N \arrow[u, "\langle h, s \rangle"] \arrow[r, "M"'] & A \end{tikzcd}$$

$$M(n) = \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} h(i)$$

infinitary convex combination: $\displaystyle\sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} = 1$

Using iterativity: example

Fix two elements of A $\quad a_-, a_+$

Take X = Cantor space $\quad 2^\omega$

Think: 2 = {-, +}
Point of Cantor space = sequence of signs $\quad s = (s_i)_{i=1}^{\infty}$
t: X -> X is tail map - omit first entry
h: X -> A $\quad h(s) = a_{s_1} \in \{a_-, a_+\}$

$$\begin{tikzcd}
A \times 2^\omega \arrow[r, "A \times M_{a_- a_+}"] & A \times A \arrow[d, "m"] \\
2^\omega \arrow[u, "{\langle h, t \rangle}"] \arrow[r, "M_{a_- a_+}"'] & A
\end{tikzcd}$$

$$M_{a_- a_+}(s) = \sum_{i=1}^{\infty} \frac{1}{2^i} a_{s_i}$$

Interval object I

= free iterative midpoint algebra on two *endpoints* x_-, x_+

For every other iterative midpoint algebra A equipped with two points

a_-, a_+

there is a unique midpoint homomorphism I -> A mapping

$x_\pm \longmapsto a_\pm$

Intuition: I should be like convex hull of two points,
i.e. real interval [-1, 1]

Is that exactly what it is?
In point-set topology: Yes (Escardo and Simpson)
Point-free (locales): Yes (Vickers)

$\mathbb{I} =$ point-free space of Dedekind reals [-1, 1]

Steps of proof

1. I is iterative. (It's also cancellative.)

2. Consider c: 2^ω -> I, defined as $c = M_{-1,+1}$

   It evaluates infinite binary expansions,

$$c(s) = \sum_{i=1}^{\infty} \frac{s_i}{2^i}$$

3. c is a localic surjection,

4. ... and in fact it is a coequalizer of two maps 2* -> 2^ω, expressing

$$c(-+^{\omega}) = c(+-^{\omega}) \quad -\frac{1}{2} + \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{2} - \sum_{i=2}^{\infty} \frac{1}{2^i} = 0$$

5. We can now define I -> A using 2^ω -> A

6. Finally we prove it preserves midpoints.

I is iterative

Use metric space theory and ball domains.

I is localic completion of D = {dyadic rationals in (-1,1)}

Formal ball = (x, δ), x in D, δ positive rational

$$(x, \delta) \subset (y, \varepsilon) \text{ if } d(y,x) + \delta < \varepsilon$$

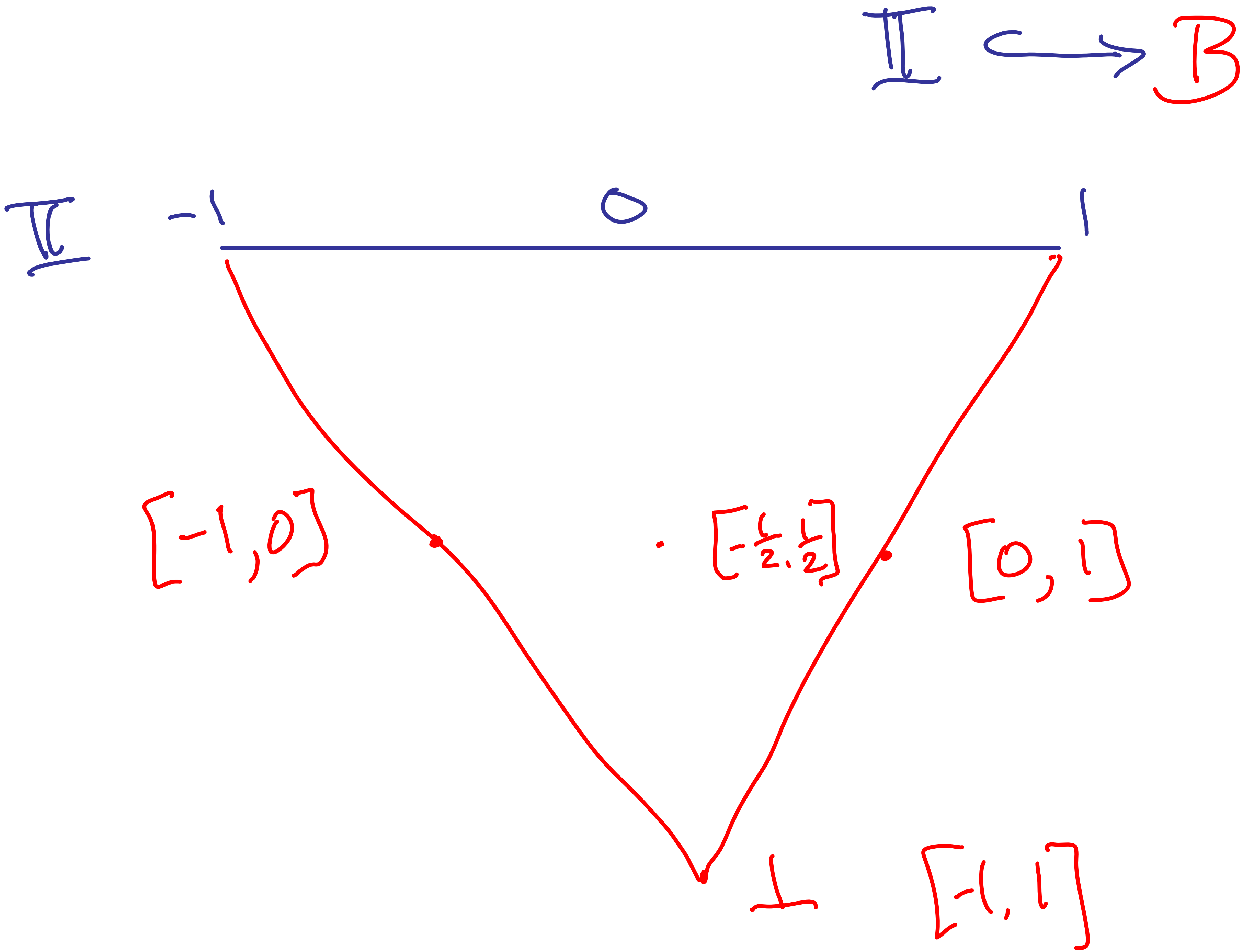
Ball domain Ball(D) = space of rounded filters of formal balls
radius of ball = inf { δ | (x, δ) in filter} (an upper real)

Localic completion I = space of Cauchy filters,
i.e. filters of radius 0

Let ι be rounded upset of (0, 1), and B be upper closure in Ball(D).

(High up = big filter = small ball.)

B



$$\begin{tikzcd}
\mathbb{I} \times X \arrow[rr, dashed, "\mathbb{I} \times M"] & & \mathbb{I} \times \mathbb{I} \arrow[d, "m"] \\
X \arrow[u, "\langle h,t \rangle"] \arrow[rr, dashed, "M"'] & & \mathbb{I}
\end{tikzcd}$$

1. m extends to m': I x B -> B

   m'(x,U) = {m(x,u) | u in U}

2. radius of m'(x,U) = 1/2 * radius of U.

3. Given f: X -> B, define
   T(f) = <h,t> ; (Ixf) ; m'

$$\begin{tikzcd}
\mathbb{I} \times X \arrow[rr, "\mathbb{I} \times f"] & & \mathbb{I} \times B \arrow[d, "m'"] \\
X \arrow[u, "\langle h,t \rangle"] \arrow[rr, dashed, "T(f)"'] & & B
\end{tikzcd}$$

4. T is monotone (w.r.t. specialization order)

5. Define f_0 = constant bottom, calculate directed join of

$$f_0 \sqsubseteq T(f_0) \sqsubseteq T^2(f_0) \sqsubseteq T^3(f_0) \sqsubseteq \dots$$

6. Radii halved at each stage, so join factors via I (radius 0).
This gives M: X -> I.

$$c = M_{-1,1} : 2^{\omega} \twoheadrightarrow \mathbb{I}$$ localic surjection

$$c(s) = \sum_{i=1}^{\infty} \frac{s_i}{2^i}$$

not surjective on points (constructively)

frame homomorphism is injection

conservativity:
  to infer inclusion between opens of I
- suffices to do it for inverse images in Cantor space

Proof - intricate, uses coverage theorems.
See paper

Remarks on localic surjections

f: X -> Y surjection ΩX <- ΩY: Ωf injection

In general: hard to prove predicatively

frames are impredicative constructions

In practical cases:

- show Ωf a *split* injection, with 1-sided inverse θ: ΩX -> ΩY
- such that θ is Scott continuous (preserves directed joins)

To define θ:

* present ΩX by generators and relations
  (e.g. inductively generated formal topology)
* manipulate presentation into form suitable for coverage theorem
* define θ by its action on generators

Then show Ωf;θ = Id

dcpo coverage theorem see, e.g., Jung-Moshier-Vickers

Suppose ΩX presented by generators and relations in form

Fr < generators L (qua DL) | relations a <= \/ U >

where
- L is a distributive lattice (DL)
- the joins in the relations are all directed
- the set of relations is meet- and join-stable

dcpo =
directed complete poset

Then ΩX is isomorphic to

dcpo < generators L (qua poset) | same relations a <= \/ U >

Similar, and easier, results for preframes and suplattices

Examples of tractable localic surjections

Split surjections (retractions)

f has one-sided inverse g

$$
\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}
$$

$$g ; f = \mathrm{Id}_Y$$

$\theta = \Omega g$ is a frame homomorphism

Examples of tractable localic surjections

Open surjections

f open if Ωf has left adjoint θ satisfying Frobenius condition

$$\theta(a \wedge \Omega f(b)) = \theta(a) \wedge b$$

θ preserves all joins (suplattice hom)
- hence defines map g: Y -> P_L(X)   lower powerlocale
- if Ωf;θ = Id then g factors through positive lower powerlocale

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dll, "g"] \\ P_L^+ X & & \end{tikzcd}$$

g provides non-deterministic splitting
- don't choose a particular pre-image
- but show there is a non-empty space of them

Examples of tractable localic surjections

Proper surjections

e.g. $c : 2^\omega \to [-1, 1]$

f proper if Ωf has Scott continuous left adjoint θ satisfying Frobenius condition

$$\theta(a \vee \Omega f(b)) = \theta(a) \vee b$$

θ preserves finite meets and directed joins (preframe hom)
(in fact it preserves all meets)

Get non-deterministic splittings using upper powerlocale

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[dl, "g"] \\ P_U^+ X & \end{tikzcd}$$

Examples of tractable localic surjections

Triquotients Plewe

triquotients are surjections

there is also a non-surjective version

f triquotient if there is

- Scott continuous $\theta : \Omega X \to \Omega Y$
- satisfying both Frobenius conditions

$$\theta(a \wedge \Omega f(b)) = \theta(a) \wedge b$$

$$\theta(a \vee \Omega f(b)) = \theta(a) \vee b$$

Non-deterministic splitting uses double powerlocale.

e.g. can define point-free space Cauchy(Q) of Cauchy sequences
(with specified rate of convergence)

and map lim: Cauchy(Q) -> R = Dedekind reals

Then lim is triquotient (Vickers)

$c : 2^\omega \longrightarrow [-1, 1]$ is proper surjection

- we know it is proper, as both spaces are compact regular
- need description of θ = right adjoint of Ωc
- manipulate presentation of $\Omega 2^\wedge\omega$ into form with join-semilattice of generators, join-stable set of relations
- intricate!
- define action of θ on generators
- show it really is right adjoint of Ωc
- show it splits Ωc

Section 6 of my paper.

Can it be simplified?

Section 7 also proves c is a surjection

c as coequalizer

General result:
any proper surjection is coequalizer of its kernel pair

discrete space of finite sequences

More usefully:
get c as coequalizer of two maps from 2*

$$c(-+^{\omega}) = c(+-^{\omega}) = 0$$

$$\therefore \quad c(s-+^{\omega}) = c(s+-^{\omega}) \quad \text{for all finite } s$$

$$c(s-+^{\omega}) \overset{\text{def}}{=} u_-(s), \qquad c(s+-^{\omega}) \overset{\text{def}}{=} u_+(s)$$

$$2^* \underset{u_-}{\overset{u_+}{\rightrightarrows}} 2^{\omega} \xrightarrow{c} \mathbb{I} \quad \text{coequalizer}$$

I is interval object

Suppose A an iterative midpoint algebra, with two given points a_-, a_+

Want unique midpoint hom N: I -> A with $\pm 1 \mapsto a_\pm$

* *If* N exists, then c;N = M   easy

$$
\begin{tikzcd}
2^* \arrow[r, shift left, "u_+"] \arrow[r, shift right, "u_-"'] & 2^\omega \arrow[rr, "c = M_{-1,+1}"] \arrow[dr, "M = M_{a_- a_+}"'] & & I \arrow[dl, "N"] \\
& & A &
\end{tikzcd}
$$

* M composes equally with u^-, u^+   easy
* Hence can define N

I is interval object

Must show N preserves midpoints

$$
\begin{tikzcd}
2^{\omega} \times 2^{\omega} \arrow[r, "c \times c"] \arrow[rr, bend left, "M \times M"] & \mathbb{I} \times \mathbb{I} \arrow[r, "N \times N"] \arrow[d, "m"'] & A \times A \arrow[d, "m"] \\
 & \mathbb{I} \arrow[r, "N"'] & A
\end{tikzcd}
$$

proper surjection

suffices to show square commutes when composed with cxc

Midpoint map on 2^ω

$$m_s : 2^\omega \times 2^\omega \longrightarrow 3^\omega, \quad 3 = \{-, 0, +\}$$

$$m_s(\pm s, \pm s') = \pm\, m(s, s')$$

$$m_s(\pm s, \mp s') = 0\, m(s, s')$$

Also $M_0 : 3^\omega \longrightarrow A$ with

$$M_0(0s) = m(m(a_-, a_+), M_0(s))$$

$$M_0(\pm s) = m(a_\pm, M_0(s))$$

top triangle commutes by definition

outermost square commutes

so does this square (special case, I for A)

$$
\begin{tikzcd}
2^\omega \times 2^\omega \arrow[r, "c \times c"] \arrow[d, "m_s"'] \arrow[rr, bend left, "M \times M"] & \mathbb{I} \times \mathbb{I} \arrow[r, "N \times N"] \arrow[d, "m"'] & A \times A \arrow[d, "m"] \\
3^\omega \arrow[r, "c_0"'] \arrow[rr, bend right, "M_0"'] & \mathbb{I} \arrow[r, "N"'] & A
\end{tikzcd}
$$

bottom triangle commutes

=> right-hand square commutes when composed with cxc

=> it commutes anyway (cxc surjective)

Conclusions

* Interesting characterization of compact real interval
  - should lead more generally to treatment of convexity

* works point-free
  - locales
  - should also work for inductively generated formal topologies

* examples of point-free surjections
  - *not* constructively surjective on points
  - "non-deterministic splittings" using hyperspaces

References

Escardo and Simpson
A universal characterization of the closed Euclidean interval
Proceedings of LICS 2001, pp.115-125

Jung, Moshier and Vickers
Presenting dcpos and dcpo algebras
Proceedings of MFPS XXIV, ENTCS 218 (2008) pp. 209-229

Vickers
The localic compact interval is an Escardo-Simpson interval object
To appear in Mathematical Logic Quaterly