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The fibrewise Vietoris hyperspace – and why it needs constructive point-free topology

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- Bundles = topology in a topos
- Topos-valid constructivism
- Geometricity $\Rightarrow$ fibrewise

The Vietoris hyperspace VX

Hausdorff (metric spaces)
Vietoris

X a space

Point of hyperspace = subspace of X

Topology: subbasic opens $\Box U, \Diamond U$ $(U \in \Omega X)$

$K \in \Box U \Leftrightarrow K \subseteq U$

$K \in \Diamond U \Leftrightarrow K \cap U \neq \emptyset$ $\quad K \between U$

- $\Box(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} \Box U_i$ $\quad$ • $\Diamond(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} \Diamond U_i$
- $\Box U \wedge \Diamond V \leq \Diamond(U \wedge V)$ $\quad$ • $\Box(U \vee V) \leq \Box U \vee \Diamond V$
- Not T_0: $\Diamond$ can't distinguish K from its closure
  $\Box$ " " K " " saturation

Vietoris powerlocale

Johnstone

$\Omega VX = \mathrm{Fr}\langle \Box U, \Diamond U \ (U \in \Omega X) \mid$

$\Box$ preserves finite meets

$\Box$ - - - - - directed joins

$\Diamond$ - - - - - - all joins

$\Box U \wedge \Diamond V \leq \Diamond(U \wedge V)$

$\Box(U \vee V) \leq \Box U \vee \Diamond V \rangle$

Otherwise doesn't work well

Points

$\Omega VX = \mathrm{Fr}\langle \Box U, \Diamond U \ (U \in \Omega X) \mid$
$\Box$ preserves finite meets
$\Box$ - - - - - directed joins
$\Diamond$ - - - - - all joins
$\Box U \wedge \Diamond V \leq \Diamond(U \wedge V)$
$\Box(U \vee V) \leq \Box U \vee \Diamond V \rangle$

$\Box$ - gives Scott open filter of ΩX

$\approx$ compact fitted sublocale of X

Johnstone-Hofmann-Mislove

meet of opens (in filter) – analogue of saturated

$\Diamond$ - gives $\bigvee$-inaccessible upset of ΩX

classically

$\approx$ closed sublocale of X

complement of join of opens not in upset

Altogether: compact semifitted sublocale

meet of closed & fitted

Bundles

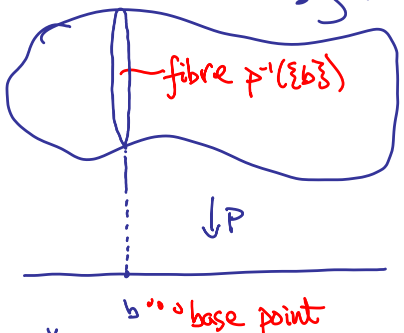
Bundle = map ⇝ that's all

BUT think: space (fibre) parametrized by base point

bundle space X

$\downarrow p$

base space B



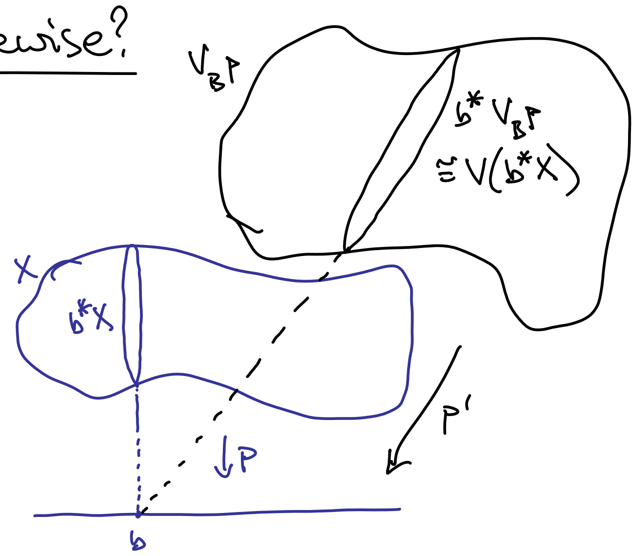
Locally fibre = pullback

fibre at b ⇝

$$\begin{tikzcd} b^*X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

Vietoris fibrewise?

$$\begin{tikzcd} X \arrow[dr, "p"'] & & V_B p \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$



Naively:

- Calculate V for each fibre
- Take disjoint union of them all

BUT –

- Really need topology to glue fibres together
  cf. tangent bundles
- Point-free: what if B not spatial?

Generalized points / fibres

fibre ⇝

$$\begin{tikzcd} b^*X \arrow[r, "\text{pullback}"] \arrow[d] & X \arrow[d, "p"] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

global point
"ordinary"

generalized fibre

$$\begin{tikzcd} b^*X \arrow[r, "\text{pullback}"] \arrow[d, "b^*p"'] & X \arrow[d, "p"] \\ W \arrow[r, "b"'] & B \end{tikzcd}$$

generalized point
(W arbitrary)

"Generalized fibrewise" = preserved by pullbacks

$$\begin{tikzcd}
& V_W(b^*p) \arrow[d, phantom, "\cong"] \arrow[ddd, bend right=10] & & \\
& b^*(V_B p) \arrow[rr] \arrow[ddl, "b^*p'"] & & V_B p \arrow[ddl, "p'"] \\
b^*X \arrow[rr] \arrow[d, "b^*p"'] & & X \arrow[d, "p"] & \\
W \arrow[rr, "b"'] & & B &
\end{tikzcd}$$

coherence? Which isomorphisms?

Problem restated

Can we find general V construction on bundles that

① is preserved by pullback ⟹ works fibrewise

② agrees with ordinary V when base $B = 1$

Solution

① Work constructively ⟹ topos-valid + geometric

② and point-free.

Everything point-free from now on

Fundamental result

Fourman, Scott, Joyal, Tierney

Bundle over B

$\approx$ internal locale in $Sh(B)$ — topos of sheaves over B

$\rightarrow$: $\begin{array}{c} X \\ p \downarrow \\ B \end{array}$ $\approx$ geometric morphism $\begin{array}{c} Sh(X) \\ p^* \uparrow \dashv \downarrow p_* \\ Sh(B) \end{array}$

$p_*(\Omega_{Sh(X)})$ is a frame in $Sh(B)$

$\leftarrow$: Map $B \xrightarrow{!} 1$ gives

geometric morphism $Sh(B) \underset{!_*}{\overset{!^*}{\rightleftarrows}} Set = Sh(1)$

Frame homomorphism $\Omega_{Sh(B)} \to A$ in $Sh(B)$

gives $\Omega B \cong !_*\Omega_{Sh(B)} \to !_*A = \Omega X$, hence $B \leftarrow X$

Discrete locales in $Sh(B)$

Set $X \longmapsto$ discrete locale, $\Omega X = \mathcal{P}X$

In topos: object of topos ... powerobject

In $Sh(B)$: sheaf

$\approx$ local homeomorphism with codomain B

Local homeomorphisms are the bundle form of internal discrete locales

still works point-free

$p: X \to B$ local homeo.

$\Leftrightarrow$ p open and $\Delta: X \to X \times_B X$ open

p open $\Leftrightarrow$ Ωp has left adjoint $\exists_p$

$\exists_p(a \wedge \Omega p(b)) = \exists_p a \wedge b$

(Joyal-Tierney)

Internal frames in toposes A

- Finite meets $\top : 1 \to A$, $\wedge : A \times A \to A$
- Certain diagrams must commute for semilattice
- Arbitrary joins $\bigvee : \mathcal{P}A \to A$

Need $\mathcal{P}$ – impredicative non-geometric

- More properties so $\bigvee$ gives joins w.r.t semilattice order
- + frame distributivity
- Presentations (generators & relations) still work
  - "set of generators" now object in topos (etc)

Vietoris is topos-valid

$\Omega VX = \mathrm{Fr}\langle \Box u, \Diamond u \ (u \in \Omega X) \mid$
$\Box$ preserves finite meets
$\Box$ ----- directed joins
$\Diamond$ ----- all joins
$\Box u \wedge \Diamond v \leq \Diamond(u \wedge v)$
$\Box(u \vee v) \leq \Box u \vee \Diamond v \rangle$

- Presentation still works in toposes
- Important properties (e.g. a monad) still OK
- Points of VX constructively are

compact, overt, weakly semifitted sublocales of X

classically: all locales are overt

meet of weakly closed & fitted sublocales

Vietoris construction on internal locales gives construction on bundles

$$\begin{array}{ccc} X & & V_B(p) \\ p \downarrow & \longmapsto & \downarrow \\ B & & B \end{array}$$

Weakly semifitted sublocales of X

compact, overt, weakly semifitted sublocales of X
meet of weakly closed & fitted sublocales

– as extra relations on ΩX

Open sublocale for $a \in \Omega X$ presented by

- $\top \leq a$

Set of such relations = meet of open sublocales
= fitted sublocale

- $a \leq {!}^* p = \bigvee\{\top \mid p\}$ $(a \in \Omega X, p \in \Omega)$

Set of such relations = weakly closed sublocale (for $p = \bot$ – closed)

Together – weakly semifitted

Localic Hofmann-Mislove Theorem

Johnstone
Preframe
proof: Vickers

X locale

Thm Scott open filters of $\Omega X \Leftrightarrow$ compact fitted sublocales of X

$F \mapsto \bigwedge_{a \in F} a$ $\qquad \{a \mid Y \leq a\} \leftarrow Y$

Difficult part: given F, $Y = \bigwedge_{a \in F} a$, then $a \in F \Leftrightarrow Y \leq a$

$\Omega Y = \mathrm{Fr}\langle \Omega X \text{ (qua Fr)} \mid \top \leq (a) \ (a \in F) \rangle$

$\wedge \vee \bigvee^{\uparrow}$ $= \mathrm{Fr}\langle \Omega X \text{ (qua } \vee\text{-semi)} \mid \top \leq (\top) \quad (\bigvee^{\uparrow}_i a_i) \leq \bigvee^{\uparrow}_i (a_i)$
$(a) \wedge (b) \leq (a \wedge b) \quad \top \leq (a) \ (a \in F) \rangle$

$\wedge \bigvee^{\uparrow}$ $\cong \mathrm{PreFr}\langle \Omega X \text{ (qua poset)} \mid \text{same relations} \rangle$

preframe coverage theorem

Preframe hom $\alpha : \Omega Y \to \Omega$
$\alpha(a) = $ "$a \in F$"

Then ${!}^* \dashv \alpha$ (check: ${!}^* \circ \alpha(a) \leq (a)$, $p \Rightarrow \alpha \circ {!}^*(p)$)

It follows that $a \in F \Leftrightarrow (a) = \top$ in ΩY.

Geometricity

Is $\begin{array}{c} X \\ {\scriptstyle p}\downarrow \\ B \end{array} \longmapsto \begin{array}{c} V_B P \\ \downarrow \\ B \end{array}$ preserved by pullbacks?

($\Rightarrow$ gives ordinary V on each fibre)

$$\begin{tikzcd}
 & V_W(b^*p) \arrow[d, phantom, "\cong" rotate=90] & & \\
 & b^*(V_B P) \arrow[rr] \arrow[dl, "b^*p'"] & & V_B P \arrow[dl, "p'"] \\
b^*X \arrow[rr] \arrow[d, "b^*p"'] & & X \arrow[d, "p"] & \\
W \arrow[rr, "b"] & & B &
\end{tikzcd}$$

YES

Proof: show how construction works on presentations of frames

"Frames are not geometric – but presentations are"

Three ways to describe internal locale:

Described using objects of $Sh(B)$ hence local homeomorphisms

frame

frame presentation — bundle

Pulling back local homeomorphisms

along $b : W \to B$

agrees with b^* in corresponding geometric morphism

$$Sh(W) \underset{b_*}{\overset{b^*}{\underset{\perp}{\leftrightarrows}}} Sh(B)$$

b^* preserves colimits, finite limits, free algebra constructions.

Those constructions on local homeomorphisms are geometric (preserved under pullback)
– so they work fibrewise.

Locales under pullback

Connection broken. b^* frame not even a frame

b^* frame

b^* presentation — b^* bundle

presents new frame in $Sh(W)$

connection preserved

Problem presenting a frame adds "arbitrary" joins
– but meaning of "arbitrary" depends on topos
For geometricity use presentations, not frames

Presentations

e.g. free dist. lattice is geometric construction

Geometrically, can manipulate any presentation into form

$$\Omega X = \mathrm{Fr}\langle L \text{ (qua dist. lattice)} \mid \text{relations } a \le {\textstyle\bigvee^{\uparrow}} U \rangle$$

Relations need $\wedge$ and $\vee$ stability:

if $a \le \bigvee^{\uparrow} U$ a relation then so are

$$a \wedge b \le {\textstyle\bigvee^{\uparrow}} \{u \wedge b \mid u \in U\}$$
$$a \vee b \le {\textstyle\bigvee^{\uparrow}} \{u \vee b \mid u \in U\}$$

Then $\Omega X \cong \mathrm{PreFr}\langle L \text{ (qua } \wedge\text{-semilattice)} \mid \text{same relations} \rangle$

$\cong \mathrm{Suplat}\langle L(\cdots \vee \cdots\cdots) \mid \cdots\cdots \rangle$

non-trivial "coverage theorems"

Vietoris

presentation constructed geometrically from that for X

$$\underbrace{\Omega VX}_{=\,A\text{ (say)}} \cong \mathrm{Fr}\langle \Box a, \Diamond a \ (a \in L) \mid \begin{array}{l} \Box \text{ preserves } \top, \wedge \\ \Diamond \text{ preserves } \bot, \vee \\ \Box a \wedge \Diamond b \le \Diamond(a \wedge b),\ \Box(a \vee b) \le \Box a \vee \Diamond b \\ \Box a \le \bigvee^{\uparrow}_{u \in U} \Box u,\ \Diamond a \le \bigvee^{\uparrow}_{u \in U} \Diamond u \\ \text{(for each relation } a \le \bigvee^{\uparrow} U) \end{array} \rangle$$

Proof sketch: Every element of ΩX of form $\bigvee^{\uparrow}_i a_i$, $a_i \in L$

Coverage theorems give preframe & suplattice homs $\Omega X \to A$

$\bigvee^{\uparrow}_i a_i \mapsto \bigvee^{\uparrow}_i \Box a_i$ and $\bigvee_i a_i \mapsto \bigvee_i \Diamond a_i$

Can deduce frame hom $\Omega VX \to A$

Inverse $A \to \Omega VX$ is easier.

On presentations

$$\text{Presentation of } X \xmapsto{\text{geometric}} \text{presentation of } V(X)$$

$$\therefore \quad \text{bundle for } X \longmapsto \text{bundle for } V(X)$$

also geometric

$$\begin{tikzcd}
V_W(b^*p) \arrow[ddd] \arrow[dr, phantom, "\cong" sloped] & & & \\
& b^*(V_B p) \arrow[rr] \arrow[ddl, "b^*p'"] & & V_B p \arrow[ddl, "p'"] \\
b^*X \arrow[rr] \arrow[d, "b^*p"'] & & X \arrow[d, "p"] & \\
W \arrow[rr, "b"'] & & B &
\end{tikzcd}$$

Conclusions

- There is a Vietoris powerlocale construction on localic bundles
- It works fibrewise
- The proof is that V is
  - (i) topos valid
  - (ii) geometric on frame presentations

Constructive reasoning with classical payoff.

References

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