# Fuzzy Sets & Geometric Logic

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HÖHLE: Fuzzy sets (valued in frames) ⊆ Sheaves

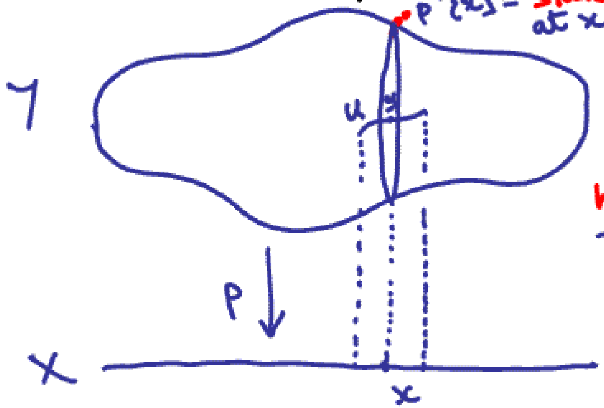
GEOMETRIC LOGIC:

Sheaves = Continuous, set-valued maps

# Sheaves

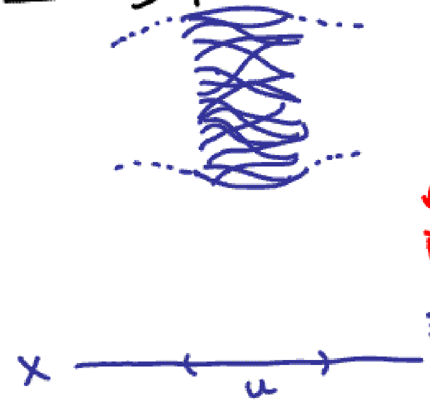
## Local homeomorphisms

$p^{-1}\{x\}$ = stalk at $x$



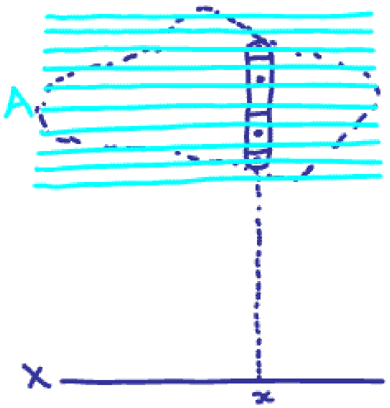

$\forall y \in Y$
$\exists$ open nbhd $U$
$p$ maps $U$
homeomorphically
to an open
in $X$

## Pasting presheaves




$U$ open in $X$
$\mathrm{Sect}_p(U)$
$= \{$local sections of $p$ on $U\}$
$U \subseteq V$:
restriction $\mathrm{Sect}_p(V) \to \mathrm{Sect}_p(U)$
pasting: $s_i \in \mathrm{Sect}_p(U_i)$
$s_i|U_i \cap U_j = s_j|U_i \cap U_j$
$\exists$ unique $s \in \mathrm{Sect}_p(\bigcup_i U_i)$
$s|U_i = s_i \quad \forall i$.

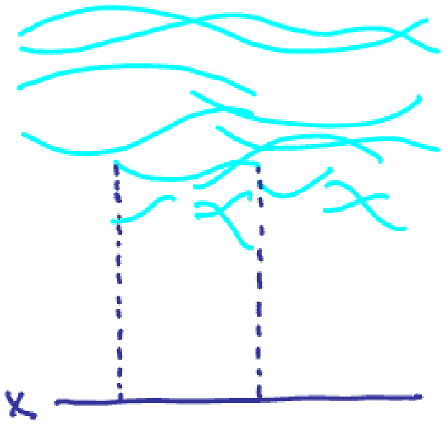
## $\Omega X$-valued sets




For each $x$:
partial equivalence relation $\sim_x$ on $A$

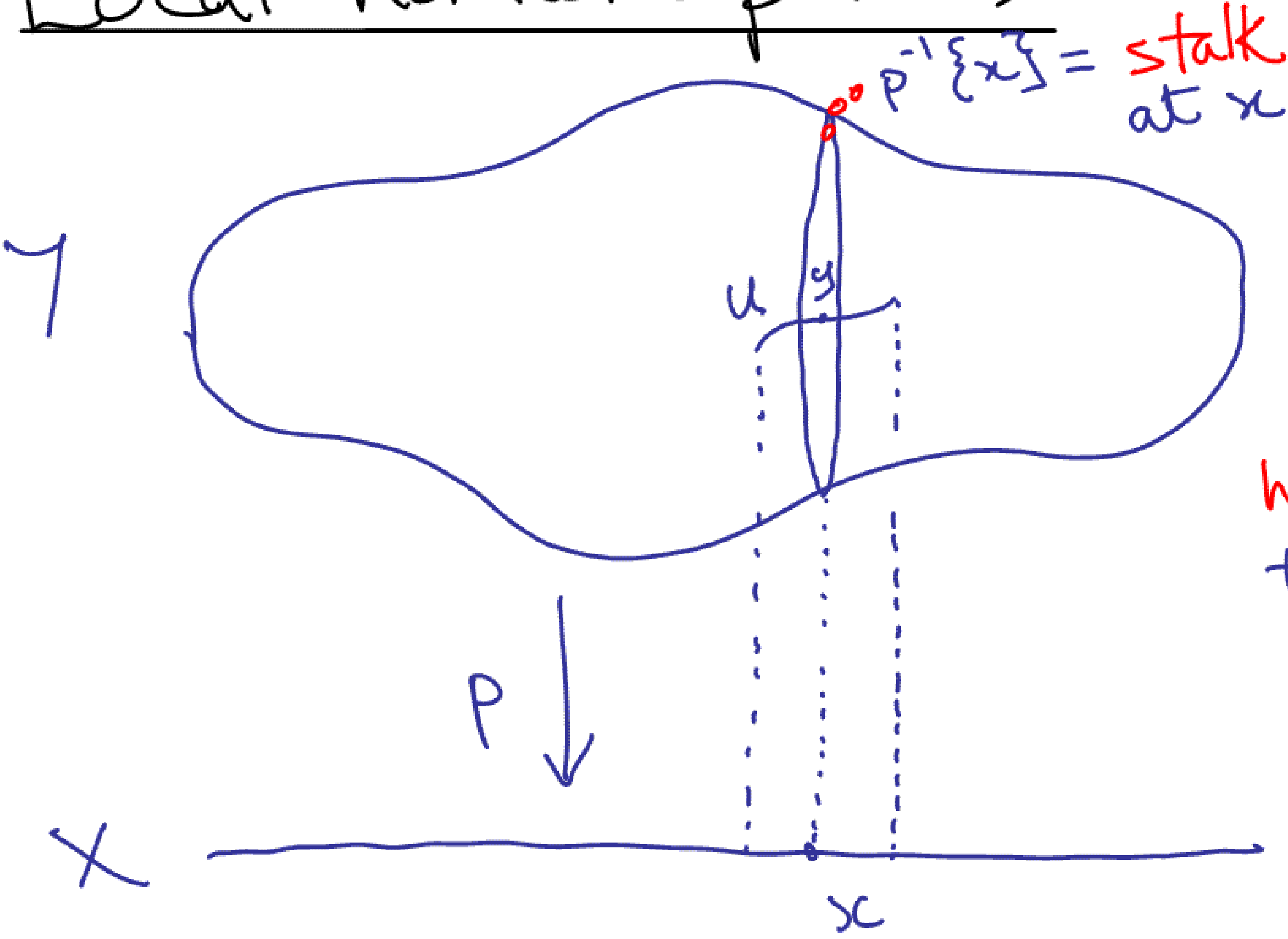
Stalk $= A/\sim_x$
$a \sim_x a$ : $a$ exists at $x$
$a \sim_x b$ : $a = b$ at $x$
$E(a,b) = \{x \mid a \sim_x b\}$
"continuity of stalks" – $E(a,b)$ open
$\sim_x$ a p.e.r. —
$E(a,b) = E(b,a)$
$E(a,b) \wedge E(b,c) \subseteq E(a,c)$

## Complete $\Omega X$-valued sets




$A$ is complete if in bijection with set of all local sections.
$E(s,t) = \bigvee\{U \mid s|U = t|U\}$

# Local homeomorphisms




$\forall y \in Y$
$\exists$ open nhbd $U$
$p$ maps $U$
homeomorphically
to an open
in $X$

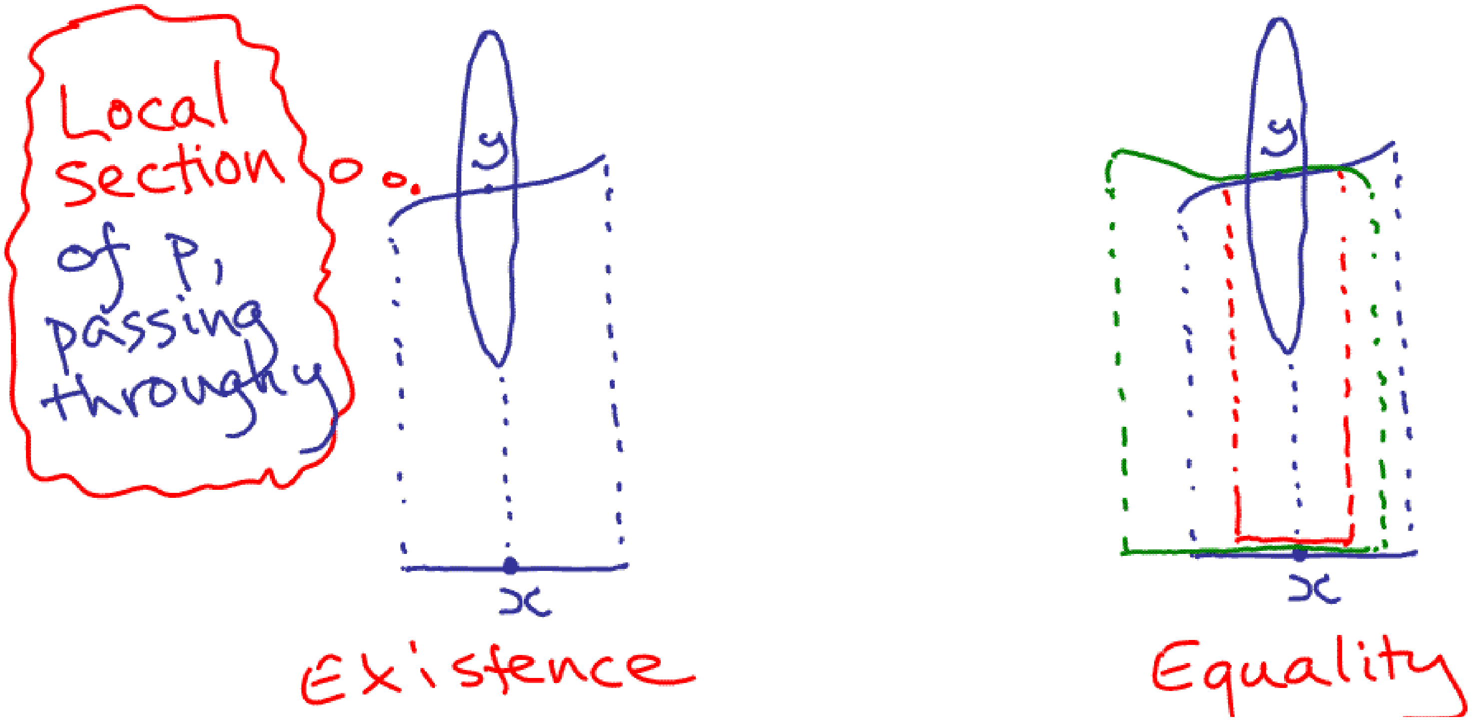
NB each stalk is discrete

Stalks give —

"<u>Continuous set-valued map</u>" on X.

Intuition only — {sets} not a topological space.

Existence & equality of stalk elements extend from points to neighbourhoods.



# Another characterization

$p : Y \longrightarrow X$ a local homeomorphism iff

- $p$ an open map (direct image preserves openness)

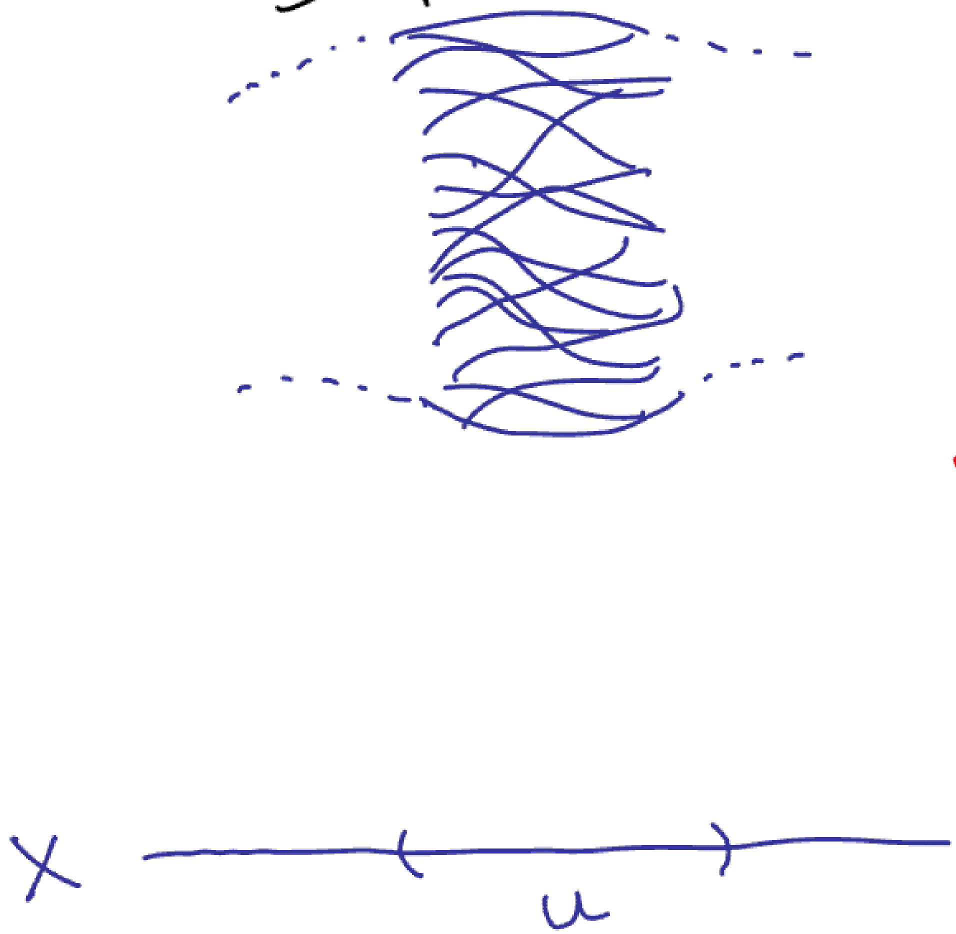
- $\Delta : Y \longrightarrow Y \times_X Y$ open
  $y \longmapsto (y, y)$

$= \{(y_1, y_2) / p(y_1) = p(y_2)\}$
fibred product, pullback

$$(U \times U) \cap (Y \times_X Y) \subseteq \Delta(U)$$

$$\Longleftrightarrow p \text{ 1-1 on } U$$

# Pasting presheaves

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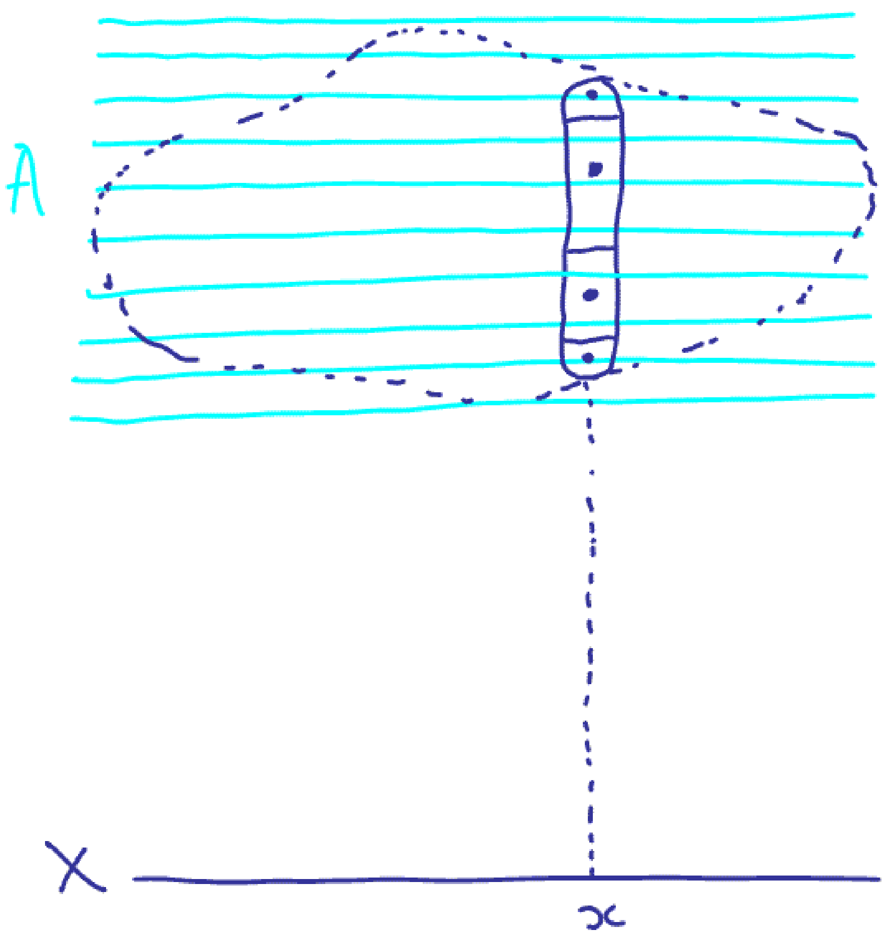
# $\Omega X$ - valued sets

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"continuity of stalks" – $E(a,b)$ open

$\sim_x$ a p.e.r. —

$$E(a,b) = E(b,a)$$

$$E(a,b) \cap E(b,c) \subseteq E(a,c)$$

# Morphisms $\quad A \longrightarrow B$

Want: stalk functions

$$f_x : A/{\sim_x} \longrightarrow B/{\sim_x}$$

$$\underline{f(a,b)} = \{x \mid a \sim_x a \text{ and } f_x[a] = [b]\}$$

$$f(a,b) \le E(a,a) \wedge E(b,b)$$

$$E(a',a) \wedge f(a,b) \wedge E(b,b') \le f(a',b') \quad \text{extensional}$$

$$f(a,b) \wedge f(a,b') \le E(b,b') \quad \text{single-valued}$$

$$E(a,a) \le \bigvee_b f(a,b) \quad \text{total}$$

# Singletons

$a \in A$ gives local section $\tilde{a}$ over $E(a,a)$: $\tilde{a}(x) = [a]_{\sim_x}$

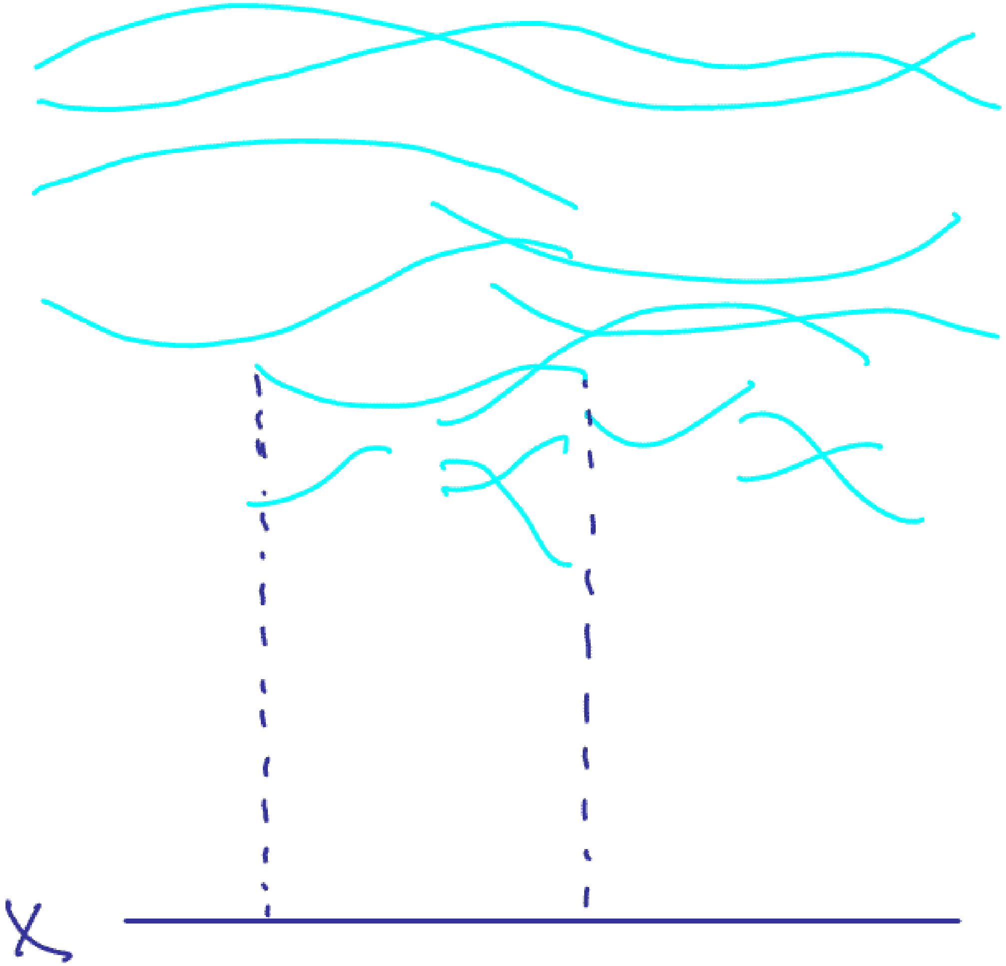
General local section = singleton $s : A \to \Omega X$

$$s(a) \subseteq E(a,a)$$
$$s(a) \cap E(a,b) = s(b)$$
$$s(a) \cap s(b) \subseteq E(a,b)$$

Local section over $E(s) = \bigcup_{a \in A} s(a)$

If $x \in E(s)$, $x \in s(a)$, say, $s_x = [a]_{\sim_x} \in$ stalk at $x$

# Complete $\Omega X$-valued sets




A is complete if in bijection with set of all local sections.

$$E(s,t) = \bigvee\{U \mid s|U = t|U\}$$

# Direct & inverse image of sheaves

$$f : X \longrightarrow Y \qquad \mathcal{S}X \underset{f_* \circ\circ \text{ direct}}{\overset{f^* \circ\circ \text{ inverse}}{\rightleftarrows}} \mathcal{S}Y$$

- $f^*$ left adjoint to $f_*$
  - so $f^*$ preserves colimits
    (and $f_*$ preserves limits)
- also $f^*$ preserves finite limits

# $f^*$ - inverse image

| Local homeomorphisms | $\Omega X$-valued sets |
| --- | --- |
| pullback | $f^*A$ : same underlying set $A$ |
| $$\begin{tikzcd} \{(x,z) \mid fx = pz\} \arrow[r] \arrow[d, "f^*p"'] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$ | $E_{f^*A}(a,b) = \Omega f(E_A(a,b))$ |
| $(f^*p)^{-1}(x) \cong p^{-1}(f(x))$ | $a \sim_x b \iff a \sim_{f(x)} b$ |

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\text{stalks for } f^*S"'] & Y \arrow[r, "\text{stalks for } S"] & \text{Sets} \end{tikzcd}$$

# $f_*$ - direct image

| Pasting presheaves $F$ | Complete $\Omega X$-valued sets |
| --- | --- |
| $f_* F(U) = F(\Omega f(U))$ | $f_* A = \{(a,U) \in A \times \Omega Y \mid E(a,a) = \Omega f(U)\}$ |
| | $E_{f_*A}((a_1,U_1),(a_2,U_2)) = \bigvee\{V \subseteq U_1 \cap U_2 \mid \Omega f(V) \subseteq E(a_1,a_2)\}$ |

## Frames

Frame =

complete lattice, with

$\wedge$ distributes over $\bigvee$

Frame homomorphism

preserves $\wedge$, $\bigvee$

finite meets

arbitrary joins

e.g. lattice of opens

Topology $\Omega X$

$X$ a topological space

$\wedge = \cap$ $\quad \bigvee = \bigcup$

Inverse image function

$\Omega f : \Omega Y \longrightarrow \Omega X$

when $f : X \to Y$ continuous

$\Omega f(U) = f^{-1}(U)$

$= \{x \mid f(x) \in U\}$

<u>Locale</u> = Frame pretending to be space

Locale $X$ = frame $\Omega X$

separate spatial language from lattice language

Locale map $f: X \to Y$

= frame homomorphism $\Omega f: \Omega Y \to \Omega X$

Point of $X$ = map $1 \to X$ ... global point

$\Omega 1 = \Omega = \{\text{truth values}\}$

But also – map $f: X \to Y$ is generalized point of $Y$ (at stage $X$)

## e.g. pullbacks

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Pullback property defines $Y \times_X Z$ (up to isomorphism) in terms of its generalized points: pairs $(y, z)$ such that $f \circ y = g \circ z$.

# Is a locale really a space?

In general - NO.
Not enough (global) points

BUT - more like a space
if include generalized points

# Maps transform points

$f : X \longrightarrow Y$

Global points

$$1 \xrightarrow{x} X \quad \mapsto \quad \begin{tikzcd} 1 \arrow[r, "x"] \arrow[rr, bend right, "f \circ x"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Generalized points

$$W \xrightarrow{x} X \quad \mapsto \quad \begin{tikzcd} W \arrow[r, "x"] \arrow[rr, bend right, "f \circ x"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

# Does point transformation define map?

On global points: NO (maybe too few)

On generalized points: YES

Generic point

$$X \xrightarrow{\mathrm{id}} X \quad \mapsto \quad X \xrightarrow{f} Y$$

Map = point transformer that <u>commutes with change of stage</u> $\alpha^*$

$\alpha : W_1 \to W_2 \qquad x : W_2 \to X$

$$\begin{tikzcd} W_1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha^* x"'] & W_2 \arrow[r, "x"] & X \end{tikzcd}$$

$f \circ (\alpha^* x) = f \circ x \circ \alpha = \alpha^*(f \circ x)$

Suppose $\forall W \quad F_W : Loc(W, X) \longrightarrow Loc(W, Y)$

$F_{W_1}(\alpha^* x) = \alpha^* F_{W_2}(x)$

Define $f = F_X(Id_X) : X \to Y$

Then $F_W(x) = F_W(x^* Id_X) = x^* F_X(Id) = x^* f = f \circ x$

Previous slide —

Map = point transformer that <u>commutes with change of stage</u> $\alpha^*$

$\alpha: W_1 \to W_2 \quad x: W_2 \to X \quad W_1 \xrightarrow{\alpha} W_2 \xrightarrow{x} X$

$f \circ (\alpha^* x) = f \circ x \circ \alpha = \alpha^*(f \circ x)$

Suppose $\forall W \quad F_W : Loc(W, X) \to Loc(W, Y)$

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Then $F_W(x) = F_W(x^* Id_X) = x^* F_X(Id) = x^* f = f \circ x$

Looks complicated – but trivial really

General category theory (associativity)

Not so trivial:

geometric logic ⇒

logical conditions

to ensure commutes with change of stage.

# Sheaves over locales

Pasting presheaves, $\Omega$-sets: same definition

Local homeomorphisms: $p, \Delta$ open maps

$f : X \to Y$ open if $\Omega X \xleftarrow[\Omega f]{} \Omega Y$

has left adjoint $\exists_f : \Omega X \to \Omega Y$

with $\exists_f (a \wedge \Omega f(b)) = \exists_f a \wedge b$

Stalks: by pullback

$$
\begin{tikzcd}
p^{-1}(x) \arrow[r] \arrow[d, "x^* p"'] & Y \arrow[d, "p"] \\
1 \arrow[r, "x"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
W \times_X Y \arrow[r] \arrow[d, "x^* p"'] & Y \arrow[d, "p"] \\
W \arrow[r, "x"'] & X
\end{tikzcd}
$$

discrete

loc. homeo.

FUZZY SETS & GEOMETRIC LOGIC

# INTERLUDE

# Geometric logic

- Many-sorted, first order.
  Signature $\Sigma$ specifies sorts, predicates, functions
- Terms built up in usual way
- Formulae built up using $=, \wedge, \bigvee, \exists$
  infinitary disjunction
- Sequents of form
  $\forall x_1 \dots x_n (\phi \longrightarrow \psi)$
  formulae, free variables amongst $x_i$s
- Geometric theory $(\Sigma, T)$
  $T$ a set of sequents

# Propositional geometric theory $(\Sigma, T)$

NO SORTS – so no functions

$\Sigma$ a set of propositional symbols

Formula – disjunction of finite conjunctions

sequent – $\phi \to \psi$

$(\Sigma, T)$ presents a frame $Fr\langle \Sigma \mid T \rangle$

generators relations

Write $[\Sigma, T]$ for locale.

$$\Omega[\Sigma, T] = Fr\langle \Sigma \mid T \rangle$$

# Models of $(\Sigma, T)$

- Interpret prop$^l$ symbols as truth values
  $\Sigma \longrightarrow \Omega$
- satisfy sequents in $T$
  i.e. frame homomorphism $Fr\langle \Sigma | T \rangle \to \Omega$
  i.e. global point of $[\Sigma, T]$

More generally
generalized point of $[\Sigma, T]$ at stage $W$
= "model of $(\Sigma, T)$ in $\Omega W$"

Models of $(\Sigma, T)$ = points of $[\Sigma, T]$

# Maps $[\Sigma_1, T_1] \xrightarrow{f} [\Sigma_2, T_2]$

= model of $(\Sigma_2, T_2)$ in $\Omega[\Sigma_1, T_1]$

Symbols of $\Sigma_2$ interpreted as geometric combinations $(\wedge, \bigvee)$ of symbols of $\Sigma_1$

|  |
| --- |
| Let $x$ be a point of $[\Sigma_1, T_1]$<br>Then $f(x)$ defined [geometrically] as<br>. . . . . . . |

Technically, inside box have non-standard logic of $\Omega[\Sigma_1, T_1]$ and $x$ is generic point.

## Summary

- A locale is the "space of models" of a propositional geometric theory
- A map is a geometric transformation of models.

No continuity proof needed!
Valid even if locale lacks global points!

CONTINUITY = GEOMETRICITY

# Change of stage

Map = point transformer that <u>commutes with change of stage</u> $\alpha^*$

$\alpha : W_1 \to W_2 \qquad x : W_2 \to X \qquad W_1 \xrightarrow{\alpha} W_2 \xrightarrow{x} X$, $\alpha^* x : W_1 \to X$

$f \circ (\alpha^* x) = f \circ x \circ \alpha = \alpha^*(f \circ x)$

Suppose $\forall W$ $F_W : Loc(W, X) \longrightarrow Loc(W, Y)$

$F_{W_1}(\alpha^* x) = \alpha^* F_{W_2}(x)$

Define $f = F_X(1d_X) : X \to Y$

Then $F_W(x) = F_W(x^* 1d_X) = x^* F_X(1d) = x^* f = f \circ x$

Geometricity (use $\wedge, \vee$)

$\Rightarrow$ preserved by $\Omega\alpha$

$\Rightarrow$ commutes with change of stage

# Predicate geometric logic

| Logic | Interpretation: Usual | Generalized |
|---|---|---|
| Propositional | $\Omega$ (truth values) | $\Omega W$ (opens) |
| Predicate | Sets | $\mathcal{S}W$ (sheaves) |

NB predicate extends propositional:
open $\sim$ subsheaf of $1$

# Geometricity and inverse image functors

To interpret geometric logic in $\mathcal{S}W$:
depends on categorical properties
$\mathcal{S}W$ is a Grothendieck topos
Structure involved: colimits, finite limits

Consequence: Suppose $(\Sigma, T)$ a geometric theory
$\alpha : W_1 \to W_2$
If $M$ a model of $(\Sigma, T)$ in $\mathcal{S}W_2$ then
$\alpha^* M$ - - - - - - - - - - - - $\mathcal{S}W_1$.

# Constructions: topos-valid v. geometric

Set-theoretic constructions meaningful in Grothendieck toposes.

e.g.

- products, pullbacks ✓ (finitary only)
- coproducts, quotients ✓
- set of truth values (= subobject classifier $\Omega$) ✗
- power sets ✗
- function sets $X^Y$ ✗
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ ✓
- $\mathbb{R}$ ✗
- free algebras ✓
- finite powerset ✓

Some are geometric - preserved by inverse image functors

# Sheaves as geometric constructions of sets

X a locale

Let $x$ be a point of X
Then set $S_x$ defined [geometrically] as ....

(1) Apply construction to generic pt of X in $\mathcal{S}X$. Get sheaf S.
[Topos-valid good enough here?]

(2) Geometricity ⇒ for any point $W \xrightarrow{x} X$, stalk constructed same way in $\mathcal{S}W$ (apply $x^*$)

# Sheaf = continuous set-valued map

- To define a sheaf:
  define its stalks geometrically
- To define a sheaf morphism:
  define stalk functions geometrically
- To show two sheaves isomorphic:
  show isomorphic stalkwise, geometrically.

# Examples

- A with crisp equality gives constant sheaf. $X \times A \to X$

  $a \sim_x b \iff x \text{ in } V\{\top \mid a=b\} \iff a=b$

  $\therefore A/\sim_x \;\cong\; A$

- Function $u : A \longrightarrow \Omega X$ gives subsheaf of constant sheaf — fuzzy set

$$E(a,b) = u(a) \wedge u(b) \wedge E_c(a,b)$$

$$a \sim_x b \iff a=b \text{ and } x \in u(a)$$

$$A/\sim_x \;\cong\; \{a \in A \mid x \in u(a)\}$$

# (Examples)

- $\Omega X$-valued equality on A gives quotient of subsheaf of A
- Pullbacks, coequalizers, coproducts, finite powersets, list objects are ... (anything geometric)

Do obvious construction & then check it works correctly on stalks

# Conclusions

- Sheaf = continuous (= geometric) set-valued function point $\mapsto$ stalk (include generalized points)
- $\Omega X$-valued set = way to describe sheaf as subquotient of constant sheaf
- For local homeomorphism or $\Omega X$-valued set, can describe geometric constructions stalkwise.