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FORMAL TOPOLOGY and GEOMETRIC LOGIC

Steve Vickers

School of Computer Science
University of Birmingham

- I Space = geometric theory
- II Map = geometric transformation of points to points
- III Bundle = geometric transformation of points to spaces

Watch out for —

- Obvious link is with propositional geometric logic
- Big insights come from predicate logic
  - toposes
  - geometric type theory
- Bundle ideas have consequences even for classical topology

I Spaces

A space is a geometric theory

Point = model

Open = proposition

Specialization = model homomorphism

Geometric logic

Propositional

positive, infinitary

Signature Σ: Propositional symbols

Formulae ϕ: use $\top, \wedge, \bot, \bigvee$

disjunctions can be infinite

Sequent $\phi \vdash \psi$ ϕ, ψ both formulae

axioms

Theory T over Σ: set of sequents

Follow account in Elephant)

Sequent based: because no $\rightarrow$.
No other surprises

Inference rules

$$\frac{}{\phi \vdash \phi} \qquad \frac{\phi \vdash \psi \quad \psi \vdash \chi}{\phi \vdash \chi} \qquad \frac{}{\phi \vdash \top}$$

$$\frac{}{\phi \wedge \psi \vdash \phi} \qquad \frac{}{\phi \wedge \psi \vdash \psi} \qquad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi}$$

$$\frac{}{\phi \vdash \bigvee S} \ (\phi \in S) \qquad \frac{\phi \vdash \psi \ (\text{all } \phi \in S)}{\bigvee S \vdash \psi}$$

$$\frac{}{\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}}$$

Need this in the absence of $\rightarrow$

Example : Reals

Signature : propositions P_{qr} $(q, r \in \mathbb{Q})$

Topology:
P_{qr} - open interval (q, r)

Axioms:

$$P_{qr} \wedge P_{q'r'} \dashv\vdash \bigvee \{ P_{st} \mid \max(q, q') < s < t < \min(r, r') \}$$

$$\top \vdash \bigvee \{ P_{q-\varepsilon, q+\varepsilon} \mid q \in \mathbb{Q} \}$$

$$(0 < \varepsilon \in \mathbb{Q})$$

read
$\bigvee$ as $\bigcup$
$\wedge$ " $\cap$
$\vdash$ " $\subseteq$

Example: Discrete spaces
Example: Sierpinski $

Example: GRD system

Triangle of sets & functions

disjuncts

generators

relations

finite power set – $\mathcal{F}G$

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ \mathcal{F}G & & R \arrow[ll, "\lambda"] \end{tikzcd}$$

Signature: $\Sigma = G$

Axioms: for each $r \in R$, $\quad \bigwedge(\lambda(r)) \vdash \bigvee_{\pi(d)=r} \bigwedge(\rho(d))$

Any geometric theory can be presented in this form.

Example: Formal topology

Example: DL-site

bounded

L - a distributive lattice, $\lhd_0 \subseteq L \times \mathcal{P}L$ such that for every $x \lhd_0 U$ in $\lhd_0$, $y \in L$

- U is directed
- $x \wedge y \lhd_0 \{u \wedge y \mid u \in U\}$ $\lhd_0$ $\wedge$-stable
- $x \vee y \lhd_0 \{u \vee y \mid u \in U\}$ $\lhd_0$ $\vee$-stable

Propositional symbols: $P_x \quad (x \in L)$

Axioms: $P_\top \dashv\vdash \top, \quad P_{x \wedge y} \dashv\vdash P_x \wedge P_y$

$P_\bot \dashv\vdash \bot, \quad P_{x \vee y} \dashv\vdash P_x \vee P_y$

$P_x \vdash \bigvee_{u \in U} P_u \quad \text{if } x \lhd_0 U$

Geometric logic

predicate

proposition = nullary predicate

First order, many sorted, positive, infinitary

Signature Σ: Sorts, functions, predicates

Formulae ϕ: use $\top, \wedge, \bot, \bigvee, =, \exists$

disjunctions can be infinite

Formulae in context $(\vec{x}.\,\phi)$

finite list of sorted variables

All free variables are in $\vec{x}$

Sequent $\phi \vdash_{\vec{x}} \psi$ $\quad$ $(\vec{x}.\phi)$, $(\vec{x}.\psi)$ both formulae in context

$(\forall \vec{x})(\phi \to \psi)$

axioms

Theory $\mathbb{T}$ over Σ: set of sequents

Example Commutative rings

Algebra

Signature: sort – R

functions – $0, 1 : 1 \longrightarrow R$

$- : R \longrightarrow R$

$+, \cdot : R^2 \longrightarrow R$

Axioms:

$T \vdash_{xyz} x + (y+z) = (x+y) + z$ $\quad T \vdash_{xyz} x \cdot (y \cdot z) = (x \cdot y) \cdot z$

etc.

$T \vdash_{xy} x + y = y + x$ $\quad T \vdash_{xy} x \cdot y = y \cdot x$

$T \vdash_{x} x + 0 = x$ $\quad T \vdash_{x} x \cdot 1 = x$

$T \vdash_{x} x + (-x) = 0$

$T \vdash_{xyz} x \cdot (y+z) = x \cdot y + x \cdot z$

Example: Commutative local rings

Neither topology nor pure algebra

Signature: Same as commutative rings

Axioms: Same as commutative rings

$$+ \quad (\exists z)\ (x+y)\cdot z = 1 \vdash_{xy} (\exists z)\, x\cdot z = 1 \vee (\exists z)\, y\cdot z = 1$$

$$0 = 1 \vdash \bot$$

Invertibles form complement of a proper ideal

Inference rules

Always assuming formulae in correct contexts

same as propositional

$$\frac{}{\phi \vdash_{\vec{x}} \phi} \qquad \frac{\phi \vdash_{\vec{x}} \psi \quad \psi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \chi} \qquad \frac{}{\phi \vdash_{\vec{x}} \top}$$

$$\frac{}{\phi \wedge \psi \vdash_{\vec{x}} \phi} \qquad \frac{}{\phi \wedge \psi \vdash_{\vec{x}} \psi} \qquad \frac{\chi \vdash_{\vec{x}} \phi \quad \chi \vdash_{\vec{x}} \psi}{\chi \vdash_{\vec{x}} \phi \wedge \psi}$$

$$\frac{}{\phi \vdash_{\vec{x}} \bigvee S} \ (\phi \in S) \qquad \frac{\phi \vdash_{\vec{x}} \psi \ (\text{all } \phi \in S)}{\bigvee S \vdash_{\vec{x}} \psi}$$

$$\frac{}{\phi \wedge \bigvee S \vdash_{\vec{x}} \bigvee \{\phi \wedge \psi \mid \psi \in S\}}$$

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$$

($\vec{s}$ a vector of terms in context $\vec{y}$ with sorts matching $\vec{x}$)

$$\frac{}{\top \vdash_{x} x = x} \qquad \frac{}{\vec{x} = \vec{y} \wedge \phi \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]}$$

$$\begin{array}{c} \phi \vdash_{\vec{x}y} \psi \\ \hline\hline (\exists y)\phi \vdash_{\vec{x}} \psi \end{array} \qquad \frac{}{\phi \wedge (\exists y)\psi \vdash_{\vec{x}} (\exists y)(\phi \wedge \psi)}$$

Again no surprises except ...

Suppose have $T \vdash_x \phi$ as axiom $(\forall x)\phi$

Deduce:

$$\dfrac{T \vdash_x \phi \qquad \dfrac{\overline{(\exists x)\phi \vdash (\exists x)\phi}}{\phi \vdash_x (\exists x)\phi}}{T \vdash_x (\exists x)\phi}$$

CANNOT conclude $T \vdash (\exists x)\phi$ even though no free variables

Rules work correctly for empty carriers

Geometric types

- characterized uniquely up to iso by geometric structure & axioms

- predicative fragment of topos-valid maths

Geometric types

characterized uniquely upto iso by geometric structure & axioms

e.g. Sorts A, L, $\quad$ $L = \mathrm{List}(A)$

Functions $\mathrm{nil}: 1 \to L$, $\mathrm{cons}: A \times L \to L$

Axioms $\quad \mathrm{cons}(a,l) = \mathrm{nil} \vdash_{a,l} \perp$

$\mathrm{cons}(a,l) = \mathrm{cons}(a',l') \vdash_{aa'll'} a = a' \wedge l = l'$

$\top \vdash_{l} \bigvee_{n \in \mathbb{N}} (\exists a_0, \ldots, a_{n-1} : A)$ $\quad$ abbreviate $[a_0, \ldots, a_{n-1}]$

$$l = \underbrace{\mathrm{cons}(a_0, \mathrm{cons}(a_1, \ldots\ldots \mathrm{cons}(a_{n-1}, \mathrm{nil}) \ldots))}_{\text{recursively defined family of formulae, indexed by } n}$$

Interpretation of L has to be parametrized list object of that of A.

Impossible with finitary logic

Proof sketch Given functions $f : B \longrightarrow Y$

$g : A \times Y \longrightarrow Y$

want unique $r = \mathrm{rec}(f,g) : L \times B \to Y$

$$\begin{tikzcd}
B \arrow[r, "\langle \mathrm{nil}!{,}\, \mathrm{id}\rangle"] \arrow[dr, "f"'] & L \times B \arrow[d, "r"] & A \times L \times B \arrow[l, "\mathrm{cons} \times B"'] \arrow[d, "A \times r"] \\
& Y & A \times Y \arrow[l, "g"]
\end{tikzcd}$$

Logically define graph of r, $\gamma \subseteq L \times B \times Y$

- $\gamma(l, b, y) \equiv \bigvee_n (\exists a_0, \ldots, a_{n-1}) \big(l = [a_0, \ldots, a_{n-1}]$
  $\wedge\, y = g(a_0, g(a_1, \ldots, g(a_{n-1}, f(b)) \ldots)) \big)$
- Prove γ total & single valued
- Appeal to unique choice to get r
- Prove uniqueness of r.

Geometric constructions - e.g.

finite limits, } cf. Giraud's theorem characterizing Grothendieck toposes
set-indexed colimits

free algebras - eg. $\mathbb{N}$, list objects
finite power set = free semilattice

also get finitely bounded $\forall$

$\mathbb{Z}, \mathbb{Q}$

Non-geometric constructions
exponentials (function types)
Ω, powersets $\mathcal{P}X$
$\mathbb{R}, \mathbb{C}$ (as sets)

"set" = object of topos

various kinds

Geometric types : two views

Syntactic sugar

Nice but not strictly necessary.

Can do it all with infinite disjunctions

Improve foundations

Avoid dependence on external infinities (at least for countable V)

Arithmetic universes

Useful either way

Example: the reals

Sorts: none

none declared – but $\mathbb{Q}$ constructed out of nothing

Predicates: $L, R \subseteq \mathbb{Q}$

Axioms: $\top \vdash (\exists q:\mathbb{Q})\, L(q)$ $\qquad \top \vdash (\exists r:\mathbb{Q})\, R(r)$

$$L(q) \dashv\vdash_{q:\mathbb{Q}} (\exists q':\mathbb{Q})(q<q' \wedge L(q')) \qquad R(r) \dashv\vdash_{r:\mathbb{Q}} (\exists r':\mathbb{Q})(r>r' \wedge R(r'))$$

$$L(q) \wedge R(q) \vdash_{q:\mathbb{Q}} \bot \qquad q<r \vdash_{q,r:\mathbb{Q}} L(q) \vee R(r)$$

- Directly describes Dedekind sections
- Equivalent to propositional version

Arithmetic Universes °° Joyal

AUs - categorical side of "arithmetic" type theory
- finitary geometric logic °° coherent
- type theory for finite limits, finite colimits,
  list objects
- $\exists$ but not P
- No function types or Π-types

Maths severely constrained, but
Maietti - Vickers 2012 "An induction principle for consequence in AUs" - evidence that good maths can be done.