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FORMAL TOPOLOGY and GEOMETRIC LOGIC

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I Space = geometric theory

• II Map = geometric transformation of points to points

III Bundle = geometric transformation of points to spaces

What is a map?

$f : X \to Y$

Transforms points to points – BUT

Central difficulty: geometric logic incomplete

$\Rightarrow$ insufficient points

Inverse image question:

Given x – point of X

P – propositional symbol for Y

What must x satisfy to ensure $f(x) \models P$?

What is inverse image $f^{-1}(P)$? or $f^{*}(P)$

Continuity: $f^{-1}(P)$ a geometric (open) formula for X

Frames

Complete lattice with frame distributivity.

Frame homomorphism preserves $\wedge, \bigvee$

Frame $\longmapsto$ propositional geometric theory

Theory X $\longmapsto$ frame ΩX = Lindenbaum algebra
$= (\Sigma, T)$ = formulae modulo $\dashv\vdash$

impredicative

Universal property of ΩX

A another frame.

Think: A = frame of non-standard truth values

Model of (Σ_X, T_X) in A

- interpretation $\Sigma_X \longrightarrow A$
- satisfying axioms in T_X

Theorem Bijection between

- models of (Σ_X, T_X) in A
- frame homs $\Omega X \longrightarrow A$

Algebraically: ΩX presented as $\mathrm{Fr}\langle \Sigma_X, T_X \rangle$

Hence:

Point of X = model of (Σ_X, T_X) = frame hom $\Omega X \longrightarrow \Omega$

Answering inverse image question

Using frames: $\Omega X \xleftarrow{f^*} \Omega Y$ frame hom

$$\begin{array}{ccc} \Omega X & \xleftarrow{f^*} & \Omega Y \\ & \nwarrow & \uparrow \\ & & \Sigma_Y \end{array}$$

Model of (Σ_Y, T_Y) in ΩX

Map = frame hom in opposite direction

More generally: $\Sigma_Y \longrightarrow$ formulae in X respecting axioms

Maps f, g equal if $f(P) \longmapsto g(P)$ for all P.

Canonical form?

Hypothesizing

X — theory (Σ, T)

Let x be a point of X.

What logic lives in this box?
Geometrically?

- Truth values "$x \models P$" $(P \in \Sigma)$
- More got by $\wedge, \vee$
- ΩX is Ω for box
- x is generic point:

P interpreted as corresponding $P \in \Omega X$

Map

X, Y - theories (Σ_X, T_X), (Σ_Y, T_Y)

> Let x be a point of X.
>
> ⋮
>
> (geometrically) defines point $f(x)$ of Y
>
> ($f(x)$ = model of (Σ_Y, T_Y) in ΩX)

Have defined frame hom $\Omega Y \to \Omega X$

i.e. map $f: X \to Y$

Generic construction works on **all** points.

Map = geometric transformation of points.

Generalized points – of Y

Ordinary (global) point = model in Set
= map $1 \to Y$

Generalized point "at stage X" = model in $\mathcal{S}X$
= map $X \to Y$

Generic point = identity map $Y \to Y$

And predicate theories?

Let x be a point of X

Maths in box (geometrically)

- a carrier set for each sort in Σ_x
- predicates, functions
- more, by geometric constructions
- working in classifying topos $\mathcal{S}X$ for theory

Maps $f : X \longrightarrow Y$

Geometric tranformation of x to $f(x)$

$=$ model of (Σ_Y, T_Y) in $\mathcal{S}X$

$\sim$ functor $\mathcal{S}Y \xrightarrow{f^*} \mathcal{S}X$ preserving geometric constructions

$\sim$ geometric morphism $\mathcal{S}X \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{S}Y$

$\sim$ frame hom $\Omega X \xleftarrow{f^*} \Omega Y$ ⇝ (for propositional theories)

Geometric morphism = topos generalization of map

∴ again say map = geometric transformation of points

Predicativity

$\mathbb{S}X$ certainly not small! (cf. topos-valid maths: frame is a set)

Constructing $f(x)$ out of x: geometric

- don't use impredicative features of topos

Impredicativity ⇒ construction isn't generic

Crucial lesson from topos theory is not "be impredicative" but -

Admit predicate theories as spaces

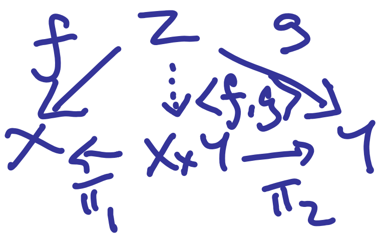
Example: product of spaces $X \times Y$

Theory for $X \times Y$ is $(\Sigma_X + \Sigma_Y, T_X + T_Y)$

Point = pair (x, y), where x, y points of X, Y

Projection $\pi_1 : X \times Y \to X$ $\quad \pi_1(x,y) = x$

Pairing



$\langle f, g \rangle (z) = (f(z), g(z))$

Example: equalizer $E \hookrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$

Theory for E is that for X, with extra axioms

$f^*(P) \dashv\vdash g^*(P)$ $\quad (P \in \Sigma_Y)$

Point = point x of X such that $f(x) = g(x)$

Sheaf = set-valued map

set = theory with one sort, nothing else

[set] = corresponding space

point = set (or object in non-standard universe)

sheaf on X = map $X \longrightarrow [\text{set}]$ ∴ = object in 𝒮X

e.g.:

constant sheaves $X \to 1 \to [\text{set}]$

opens $X \longrightarrow \$ \longrightarrow [\text{set}]$ ∴ subsheaves of constant 1

$\bot, \top \mapsto \phi, \{*\}$

Local homeomorphisms (in ordinary topology)

$p : Y \to X$ s.t. every $y \in Y$ has open nbhd U with p restricting to a homeomorphism of U onto an open nbhd of $p(y)$

Fact: each fibre $p^{-1}(\{x\})$ is discrete

Intuition: defⁿ expresses "continuity" of set-valued map $x \mapsto p^{-1}(\{x\})$

Thm: p local homeomorphism iff p & $\Delta : Y \to Y \times_X Y$ both open

For point-free spaces

propl geometric theories

- Can define openness of maps
- Then define $p : Y \to X$ local homeomorphism if p & $\Delta : Y \longrightarrow Y \times_X Y$ both open
- Thm: Equivalence between sheaves over X & local homeomorphisms with codomain X

$$\begin{array}{ccc} & & [\mathrm{set}, \mathrm{elt}] \\ & & \downarrow \\ X & \xrightarrow[s]{\quad\quad} & [\mathrm{set}] \end{array}$$