LOCALES via BUNDLES

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- Old ideas: Grothendieck, Fourman, Scott, Joyal, Tierney...
- Bundle $p: Y \to X$ is parametrized space Y_x $(x \in X)$
- = space in mathematics of parametrized sets
- ... but must replace spaces by locales.
- Key notion: geometricity

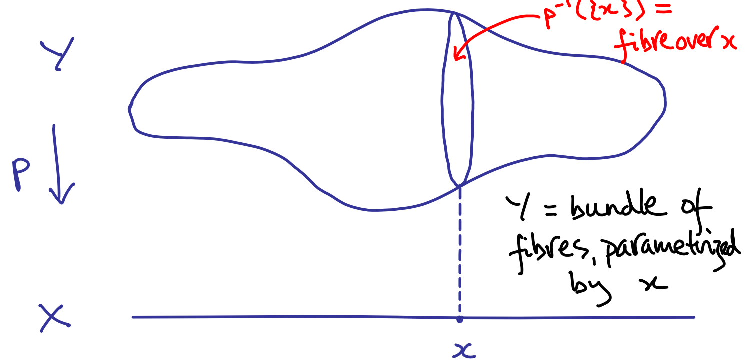
Bundles ("Fibrewise topology")

maps assumed continuous

Bundle = arbitrary map $p: Y \to X$

Think: space (fibre $Y_x = p^{-1}(\{x\})$) parametrized by $x \in X$

e.g. tangent bundle – fibre = tangent space at x



Idea

- Replace sets by "set bundles" over X
- Get modified set theory
- "Do topology" in it
- ? Get bundles over X ?
- ? Does it all work fibrewise ?

Hurdles

- Replace sets by "set bundles" over X

  Sheaves / local homeomorphisms
- Get modified set theory

  Internal in topos of sheaves

  Non-classical – topos-valid
- "Do topology" in it

  Replace spaces by locales
- ? Get bundles over X ?

  YES! Joyal/Tierney 1984 & earlier
- ? Does it all work fibrewise ?

  Restrict to "geometric" reasoning

FIBRES — PULLBACKS — GEOMETRIC

Fibred product / pullback

$$\begin{tikzcd} & Y_1 \times_X Y_2 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$= \{(y_1, y_2) \mid p_1(y_1) = p_2(y_2)\}$ with subspace topology from $Y_1 \times Y_2$

Fibre of $Y_1 \times_X Y_2$ = product of fibres of Y_1 and Y_2

Fibres are pullbacks

pullback

fibre

$$\begin{tikzcd} 1 \times_X Y = p^{-1}(\{x\}) \arrow[r] \arrow[d, "!"'] & Y \arrow[d, "p"] \\ 1 \arrow[r, "x"'] & X \end{tikzcd}$$

"Pullbacks are fibres"

pullback

fibre

$$\begin{tikzcd} 1 \times_X Y = p^{-1}(\{x\}) \arrow[r] \arrow[d, "!"'] & Y \arrow[d, "p"] \\ 1 \arrow[r, "x"'] & X \end{tikzcd}$$

General pullback is "generalized fibre"

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "x"'] & X \end{tikzcd}$$

generalized point

Geometricity

- Reason geometrically: in way that's preserved by pullback
- Constructions/properties of bundles work fibrewise

More subtly —

- Work uniformly for "arbitrary fibre"
- Includes generalized fibres over generic point $\mathrm{Id}: X \to X$ – i.e. whole bundles

$$\boxed{\text{Geometric} = \text{fibrewise}}$$

"SET BUNDLES" = LOCAL HOMEOMORPHISMS = SHEAVES

When is $p: Y \to X$ "fibrewise discrete"?

- Must have each fibre $p^{-1}(\{x\})$ discrete in subspace topology
- Best to require a little more.
  - local homeomorphism
- Concept is geometric
  - local homeo property preserved by pullback

S discrete $\Leftrightarrow$ $\Delta(S)$ open in $S \times S$

$\Delta: S \to S \times S$, $\Delta(x) = (x,x)$

$\Leftarrow$: Suppose $\Delta(S)$ open

If $x \in S$ then $(x,x) \in \Delta(S)$

$\therefore \exists\, U, V$ open in S, $(x,x) \in U \times V \subseteq \Delta(S)$

$U \times \{x\} \subseteq \Delta(S)$, $\therefore$ $U = \{x\}$

$\therefore \{x\}$ open for all x

$\therefore S$ discrete

For $Y \xrightarrow{p} X$ "fibrewise discrete"

- Say $\Delta : Y \to Y \times_X Y$ is open map

  If $y \in Y$: $\exists\, U, V$ open in Y,

  $(y,y) \in (U \times V) \cap (Y \times_X Y) \subseteq \Delta(Y)$

  If $y_1, y_2 \in U \cap V$, $p(y_1) = p(y_2)$ then $y_1 = y_2$

  $\therefore$ p is continuous & 1-1 on $U \cap V$.

- Say also p is open map

  Then p a homeomorphism on $U \cap V$

$\boxed{p \text{ a local homeomorphism}}$ i.e. p, Δ both open

Equivalent notions

- Local homeomorphism with codomain X

  Think: - bundle of sets over X

  - set parametrized by point of X
  - continuous map $X \to$ Sets

- Sheaf over X

  $Sh(X)$ = topos of sheaves over X

"Internal in $Sh(X)$" = "maths of set bundles"

Some logical principles not topos-valid

Excluded middle $\quad P \vee \neg P$

Proof by contradiction $\quad \neg\neg P \to P$

Axiom of choice

$$R \subseteq X \times Y$$

$$\forall x \in X\ \exists y \in Y \quad (x,y) \in R$$

$$\Downarrow$$

$$\exists f : X \to Y \quad \forall x \in X\ (x, f(x)) \in R$$

Geometricity for set constructions (on local homeomorphisms)

Geometric constructions can be done fibrewise

e.g. $\times$, $+$, pullback, quotients, $\mathbb{N}$, $\mathbb{Q}$, $\mathcal{F}$

Some topos-valid construction not geometric – can't be done fibrewise

e.g. $\mathcal{P}$, function sets.

$$(Z^Y)_x = \{\text{germs of maps } U \times_X Y \to Z \mid U \text{ open nbhd of } x\}$$

LOCALES

Point-set topology (usual approach)

Space = set of points + extra structure

For bundle $p : Y \longrightarrow X$:

sheaf of points

"set of points" is a local homeomorphism

$$pt(p) \longrightarrow X$$

$\therefore$ approximates general map by local homeo

— loses information

Technically – can calculate that

fibre $pt(p)_x$ = set of

non-geometric

germs of maps

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ U \arrow[ur] \arrow[r, hook] & X \end{tikzcd}$$

local sections of p on U

for U open nbhd of x

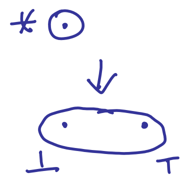
Example

$\$$ = Sierpiński space $\{\bot, \top\}$

opens $\emptyset$, $\{\top\}$, $\{\bot, \top\}$

$$p : 1 = \{*\} \longrightarrow \$$$

$$* \longmapsto \bot$$



No local sections $U \to 1$ if $U \neq \emptyset$

pt p has both fibres empty

$\therefore$ as bundle over $\$$, p is not "$pt(p)$ + topology"

Use opens instead of points

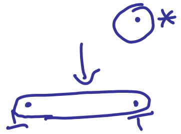
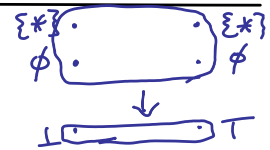
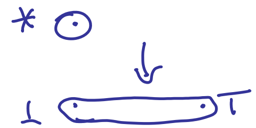
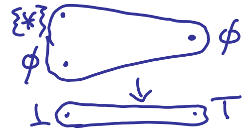
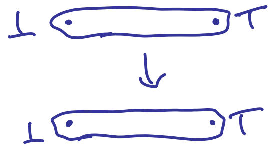
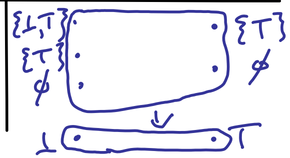
There's a sheaf $\Omega(p)$ with fibres

$\Omega(p)_x$ = set of germs of opens $V \subseteq p^{-1}(U)$, U open nbhd of x

(if $V_i \subseteq p^{-1}(U_i)$, $i = 1, 2$: $V_1 \sim V_2$ iff $V_1 \cap p^{-1}(U_2) = V_2 \cap p^{-1}(U_1)$)

NB If $X = 1$ then unique fibre is set of opens of Y.

Examples for $X = \$$

p	Bundle	$\Omega(p)$
$\top : 1 \to \$$		
$\perp : 1 \to \$$		
$\mathrm{Id} : \$ \to \$$		

Point-free topology

Can't rely on "set" of points.

∴ use abstract algebraic structure capturing set of opens

frame – operations $\wedge \bigvee$ to match $\cap \bigcup$ of open sets

- complete lattice
- $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$ (frame distributivity)

homomorphisms of frames preserve $\wedge \bigvee$

Example

Ω = frame of truth values (classically, $\{ff, tt\}$)

= topology for 1 (1-point space)

Locale = frame pretending to be space

$$\underline{\text{Locale}} \;=\; \text{frame}$$
$$X \qquad \Omega X$$

Map $f : X \longrightarrow Y$

is frame homomorphism

$\Omega f : \underbrace{\Omega X \longleftarrow}_{} \Omega Y$

reverse direction!

like inverse image for opens

Point of X - map $1 \to X$

- homomorphism $\Omega X \to \Omega$

Is all topology got this way?

(1) Space $X \longmapsto$ frame (topology) = locale

every $x \in X \longmapsto$ locale point

BUT X can omit locale points

- or have two points with same locale point

X is sober if this doesn't happen

Policy: Topology lives in sober spaces

Non-sober space given by map

set of point labels $\longrightarrow$ sober space

Is all topology got this way?

(2) Locale might be non-spatial

- not enough points

cf. $1 : 1 \to \$$ (but examples even in ordinary sets)

Locale not always topology on set of points

Point-free & point-set topology not equivalent

- but point-free works well for bundles

Sh(X) replaces sets with local homeomorphisms to X

Working "internally in Sh(X)" —

$\boxed{\text{Theory of frames still works}}$

Joyal & Tierney 1984

Internal locales

Can still define:
- $Sh(X)$
- local homeos to X

Suppose X a locale

Joyal/Tierney: duality between
- internal frames in $Sh(X)$
- locale bundles $p: Y \to X$

Contravariance:

Frame hom

$A_1 \leftarrow A_2$

Bundle map

$$\begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

GEOMETRIC THEORIES

Locale = describe points using geometric theory

Logical theory
- propositional symbols – interpreted as true/false
- axioms – constrain allowable interpretations

Geometric axioms of form $\phi \to \psi$ (implies)

ϕ, ψ built using
- propositional symbols
- $\wedge$ (and), $\vee$ (or), $\top$ (true), $\perp$ (false)

ϕ, ψ are geometric formulae

Why geometric axioms?

Property of satisfying those axioms is preserved by pullback.

BUT... "geometric" first applied to axioms.
I have generalized it to preservation by pullback.

Example

Symbols R A G

Axioms $\top \to R \vee A \vee G$

$R \wedge G \to \bot$

$A \wedge G \to \bot$

Model = interpretation that respects axioms

Example

Symbols R A G

Axioms $\top \to R \vee A \vee G$

$R \wedge G \to \bot$

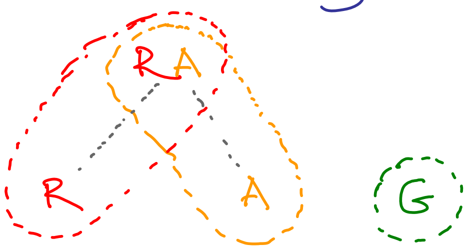
$A \wedge G \to \bot$

Interpretations

Models		R	A	G
	RA	t	t	f
	R	t	f	f
	A	f	t	f
	G	f	f	t

Topological space of models

Propositional symbols - subbase of opens



General opens:

- got using $\cap, \cup$
- up-closed subsets (w.r.t.)
- 10 in total

Example: real numbers

Propositional symbols: $L_q, R_q \quad (q \in \mathbb{Q})$

Axioms: $L_q \to L_{q'}$ $R_{q'} \to R_q \quad (q' \leq q)$

$L_{q'} \to \bigvee_{q' < q} L_q$ $R_q \to \bigvee_{q' < q} R_{q'}$

$\top \to \bigvee_q L_q$ $\top \to \bigvee_q R_q$

$L_q \wedge R_q \to \bot$

$\top \to L_{q'} \vee R_q \quad (q' < q)$

Models?

Propositional symbols: $L_q, R_q \quad (q \in \mathbb{Q})$

Axioms:
$$L_q \to L_{q'} \qquad R_{q'} \to R_q \qquad (q' \leq q)$$
$$L_{q'} \to \bigvee_{q' < q} L_q \qquad R_q \to \bigvee_{q' < q} R_{q'}$$
$$\top \to \bigvee_q L_q \qquad \top \to \bigvee_q R_q$$
$$L_q \wedge R_q \to \bot$$
$$\top \to L_{q'} \vee R_q \quad (q' < q)$$

Determined by:

$L = \{q \in \mathbb{Q} \mid L_q \text{ interpreted as tt}\}$

$R = \{q \in \mathbb{Q} \mid R_q \quad \text{..} \quad \text{..} \quad \text{..}\}$

Axioms assert:

L down-closed $\qquad R$ up-closed

L, R rounded, inhabited

$L \cap R = \emptyset$

If $q' < q$ then $q' \in L$ or $q \in R$

$\Rightarrow$ L, R come arbitrarily close

Model = (variant of) Dedekind section of rationals

= real number x

$L = \{q \in \mathbb{Q} \mid q < x\}$

$R = \{q \in \mathbb{Q} \mid x < q\}$

Topology: subbase $L_q = (q, \infty)$

$R_q = (-\infty, q)$

$\therefore$ usual topology on reals

Summary

Propositional geometric theory $\mapsto$ Topological space

Point = model

Subbasic open = propositional symbol

General open made using $\cap$ $\cup$

Lindenbaum algebras

Lindenbaum algebra $\Omega[\mathbb{T}]$ of theory $\mathbb{T}$

= formulae modulo equivalence under axioms

- canonical represention of theory.

Theories equivalent iff Lindenbaum algebras isomorphic.

Universal characterization of $\Omega[\mathbb{T}]$

- Generalized model of $\mathbb{T}$ in frame A
  - interprets symbols as elements of A
  - respecting axioms
- Generic model of $\mathbb{T}$ in $\Omega[\mathbb{T}]$
  $P \longmapsto$ equiv. class of P in $\Omega[\mathbb{T}]$
- For any model M in A:
  unique homomorphism $\Omega[\mathbb{T}] \to A$
  transforming generic model to M

$$\boxed{\text{Models of } \mathbb{T} \text{ in } A \;\cong\; \text{homs } \Omega[\mathbb{T}] \to A}$$

Frame homomorphism as model transformer

$$\begin{tikzcd} \Omega[\mathbb{T}_2] \arrow[r, "\alpha"] \arrow[rr, bend right, dashed] & \Omega[\mathbb{T}_1] \arrow[r, dotted] & A \end{tikzcd}$$

α transforms models of $\mathbb{T}_1$ (in any A)
to models of $\mathbb{T}_2$

$\therefore$ locale map $[\mathbb{T}_1] \to [\mathbb{T}_2]$ goes in right
direction for model transformer

Internal geometric theories

$\mathbb{T}_{00} \longmapsto$ internal frame $\Omega[\mathbb{T}]$

(local homeomorphisms for sets of prop's, axioms etc.)

Identify corresponding internal locale with
bundle

$$\begin{array}{c} [\mathbb{T}] \\ \downarrow \\ X \end{array}$$

NB local homeomorphism $\Omega[\mathbb{T}] \to X$ is different!

Geometricity

$\mathbb{T} \longmapsto \Omega[\mathbb{T}]$ not geometric - uses $\mathcal{P}$

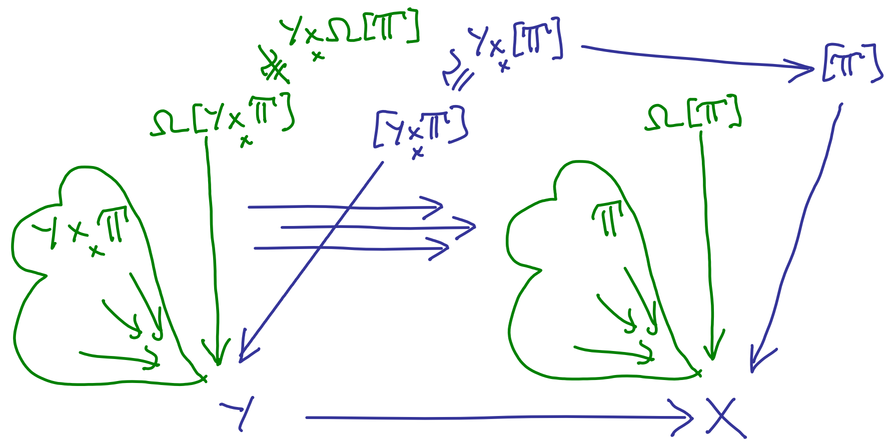
$\mathbb{T} \longmapsto [\mathbb{T}]$ is geometric

$\mathbb{T}$ made of "bundles of sets"

$[\mathbb{T}]$ a bundle

Way in which $\mathbb{T}$ presents $[\mathbb{T}]$ is
preserved under pullback

NB No longer saying locale is frame



local homeomorphisms are green

Restoring points to locales

Fact: internal model of $\mathbb{T}$ equivalent to section of $\begin{array}{c}[\mathbb{T}]\\ \downarrow\\ X\end{array}$.

Suppose have geometric transformation
models of $\mathbb{T}_1 \longrightarrow$ models of $\mathbb{T}_2$

Pulling back to $[\mathbb{T}_1]$, gives:

section of
$$\begin{tikzcd} {[\mathbb{T}_1]\times_X[\mathbb{T}_2]} \arrow[d, shift left] \\ {[\mathbb{T}_1]} \arrow[u, dotted, shift left] \end{tikzcd}$$
i.e. bundle map
$$\begin{tikzcd} {[\mathbb{T}_1]} \arrow[rr, dashed] \arrow[dr] & & {[\mathbb{T}_2]} \arrow[dl] \\ & X & \end{tikzcd}$$

$\boxed{\text{Locale map = geometric point transformer.}}$

For internal mathematics:
use locales, not spaces

BUT — to do it geometrically (fibrewise)

use theories, not frames.

BONUS Important theorems for spaces (Tychonoff, Heine-Borel, ---) not topos-valid
Localic versions are topos-valid

Case study:

POWERLOCALE = FIBREWISE HYPERSPACE

COMPACTNESS

Hyperspaces

Y a space

A hyperspace for Y has

point = subspace of Y

Powerlocale = localic hyperspace

point = sublocale

Upper powerlocale

form of hyperspace

If Z a locale:

theory $\mathbb{T}$: propositions $\Box U$ $(U \in \Omega Z)$

axioms $\Box U \to \Box V$ $(U \leq V)$

$\top \to \Box \top$

$\Box U \wedge \Box V \to \Box(U \wedge V)$

$\Box(\bigvee_i^{\uparrow} U_i) \to \bigvee_i^{\uparrow} \Box U_i$ (U_i a directed family)

Define $P_U Z = [\mathbb{T}]$

Points = compact fitted sublocales of Z

meet of opens

localic subspace

Compactness for locale

Finite subcover property for frame:

if $\top = \bigvee_{i \in I} a_i$

then $\top = \bigvee_{i \in I_0} a_i$

for some finite $I_0 \subseteq I$

But not geometric! (Uses frame)

Z as compact sublocale of itself

Theorem: Z compact (A) $\Longleftrightarrow$

$\exists$ point $1 \xrightarrow{\perp} P_U Z$ (B)

left adjoint to unique map $P_U Z \to 1$

(A) uses frame - non-geometric

$\therefore$ (A) $\Leftrightarrow$ (B) non-geometric

In (B): adjointness property of $\perp$ geometric

BUT Is $Z \mapsto P_U Z$ geometric?

Is $Z \mapsto P_u Z$ geometric? Yes!

Remember: $\Omega Z \mapsto P_u Z$ was not.

Theorem If $Z = [\mathbb{T}_Z]$ then $P_u Z \cong [\mathbb{T}]$
where $\mathbb{T}_Z \mapsto \mathbb{T}$ geometric

$$\begin{tikzcd}
& P_u(Y \times_X Z) \cong Y \times_X P_u Z \cong [Y \times_X \mathbb{T}] \arrow[rr] \arrow[ddl] & & P_u Z = [\mathbb{T}] \arrow[ddl] \\
Y \times_X Z \cong [Y \times_X \mathbb{T}_Z] \arrow[rr] \arrow[d] & & Z = [\mathbb{T}_Z] \arrow[d] & \\
Y \arrow[rr, "x"] & & X &
\end{tikzcd}$$

Geometric characterization of compactness

Z compact $\Leftrightarrow$ unique map $P_u Z \to 1$ has right adjoint

For a bundle:

internally compact $\Rightarrow$ fibrewise compact

For a bundle

$$\begin{tikzcd}
Z \arrow[dr, "p"'] & & P_u\left(\begin{smallmatrix} Z \\ \downarrow \\ X \end{smallmatrix}\right) \arrow[dl] \\
& X &
\end{tikzcd}$$

Required point $T : X \longrightarrow P_u\left(\begin{smallmatrix} Z \\ \downarrow \\ X \end{smallmatrix}\right)$ calculates:

for each pt x of X,
$T(x)$ = fibre over x
as compact sublocale of itself.

Topos for algebraic quantum theory $\left\{ \begin{smallmatrix} \text{as} \\ \text{bundles} \end{smallmatrix} \right.$

Heunen / Landsman / Spitters $\leftarrow$ Banaschewski / Mulvey
cf. Döring / Isham

A — C^*-algebra (non-commutative)
$\mathcal{C}(A)$ — poset of commutative subalgebras
$\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$
$\mathrm{Idl}(\mathcal{C}(A))$ = ideal completion $<$ space - sober / locale
– comm. subalgebras C ~ principal ideals
$\mathcal{T}(A) \simeq \mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$
Internal in $\mathcal{T}(A)$ = bundles over $\mathrm{Idl}(\mathcal{C}(A))$

Spectral bundle

$\underline{A}(C) = C$

H/L/S proved it's internal C^*-algebra (commutative)

B/M – get spectrum $\underline{\Sigma}(\underline{A})$ – a locale

$$\begin{tikzcd} \underline{A} \arrow[dr, "\text{loc. homeo.}"'] & & \underline{\Sigma}(\underline{A}) \arrow[dl] \\ & \mathrm{Idl}(\mathcal{B}(A)) & \end{tikzcd}$$

Fibrewise?

$$\begin{tikzcd} C \cong C^*(\underline{A}) \arrow[dr] & & \Sigma(C) \cong C^*(\underline{\Sigma}(\underline{A}))? \arrow[dl] & \underline{A} \arrow[dr, "\text{loc. homeo.}"'] & & \underline{\Sigma}(\underline{A}) \arrow[dl] \\ & 1 \arrow[rrr, "C"'] & & & \mathrm{Idl}(\mathcal{B}(A)) & \end{tikzcd}$$

Geometricity

B/M: $\underline{A} \longmapsto \underline{\Sigma}(\underline{A})$ geometric (geometric theory)

But – completeness of C^*-algebra not geometric

localic account of C^* algebras?

For general $x: Z \to \mathrm{Idl}(\mathcal{B}(A))$
$x^*(\underline{A})$ might not be C^*-algebra

However – doesn't matter.
Apply construction $\underline{A} \mapsto \underline{\Sigma}(\underline{A})$ direct to $x^*(\underline{A})$
– same answer as if complete first
(Coquand/Spitters)

Valuations "measures defined on opens"

$\mu: \Omega X \longrightarrow [0,1]$

- preserves directed joins
- $\mu(X) = 1$
- $\mu(\emptyset) = 0$
- $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$

$\mathrm{Val}(X)$ – space of valuations on X

$X \longmapsto \mathrm{Val}(X)$ geometric (& localic)

Jones/Plotkin; Heckmann; Vickers; Coquand/Spitters
Achim Jung

Sections

$$\begin{array}{ccc} \underline{\Sigma}(\underline{A}) & \xrightarrow{\text{Dirac}} & \mathrm{Val}(\underline{\Sigma}(\underline{A})) \\ & \searrow \quad \swarrow & \\ & \mathrm{Idl}(\mathcal{C}(A)) & \end{array}$$

$\underline{\Sigma}(\underline{A})$: no global sections

Kochen/Specker

"no classically pure states"

$\mathrm{Val}(\underline{\Sigma}(\underline{A}))$: lots of global sections

"classically mixed states"

e.g. derived from states of A

Example - & modify topos

$A = M_2(\mathbb{C})$

Commutative subalgebras - $\underbrace{\text{dim } 1}_{\mathbb{C}I}$ or 2

Dim 2: defined by two matrices E, $I - E$

where E self adjoint, idempotent, trace 1

Matrices E ~ points of real sphere S^2

Commutative subalgebras (ignore $\mathbb{C}I$) :

real projective plane $\mathbb{R}P^2$

Spectrum - 2 points in each fibre

$$\begin{array}{c} S^2 \\ \downarrow \\ \mathbb{R}P^2 \end{array}$$

no continuous global sections