# The Topos Approach

## in the Qubit case

Isham/Butterfield/Döring
also
Heunen/Landsman/Spitters

Steve Vickers
School of Computer Science
University of Birmingham

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## The topos approach to quantum formalism

Describe quantum system by

$A$ — von Neumann algebra — Isham/Butterfield/Döring

$A$ — $C^*$-algebra — Heunen/Landsman/Spitters

Observable = self-adjoint element of $A$

in $A_{sa}$

## Classical case – A commutative

$A$ has a (Gelfand-Naimark) spectrum $\Sigma$,

$A \cong C(\Sigma, \mathbb{C}) \qquad A_{sa} \cong C(\Sigma, \mathbb{R})$

Element $\phi \in \Sigma$ is classically pure state – specifies value of every observable

Also – Dist $(\Sigma)$ = space of distributions on $\Sigma$

regular probability measures

$\phi \in$ Dist $(\Sigma)$ is mixed state – specifies probabilistic distribution of values of observables

## Quantum case – A non-commutative

e.g. $A$ = bdd linear operators on Hilbert space $\mathcal{H}$.

$\phi \in \mathcal{H}$ gives probabilistic distribution of values of observables

Born rule

BUT no spectrum of classically pure states

Kochen-Specker:
Cannot assign values to all observables consistent with their functional relationships

$a = f(b)$
$f: \mathbb{R} \to \mathbb{R}$

## Topos approach

- Define a topos $\mathcal{T}(A)$
- Define spectrum in internal mathematics of $\mathcal{T}(A)$
- Do internal mathematics, but extract external information.

What does this mean?

Let $\mathcal{C}(A)$ = posets of commutative subalgebras of $A$

— "classical viewpoints" of system

Each $C \in \mathcal{C}(A)$ has spectrum $\Sigma(C)$

Idea: classical treatment indexed by classical viewpoint

$\rightarrow$ quantum treatment

## Topos internal as bundles

Suppose topos $\mathcal{T} = Sh(X)$

Object of $\mathcal{T}$ = local homeomorphism over $X$

Internal space in $\mathcal{T}$ (use point-free topology)

= bundle over $X$

Topos-valid, point-free reasoning internally

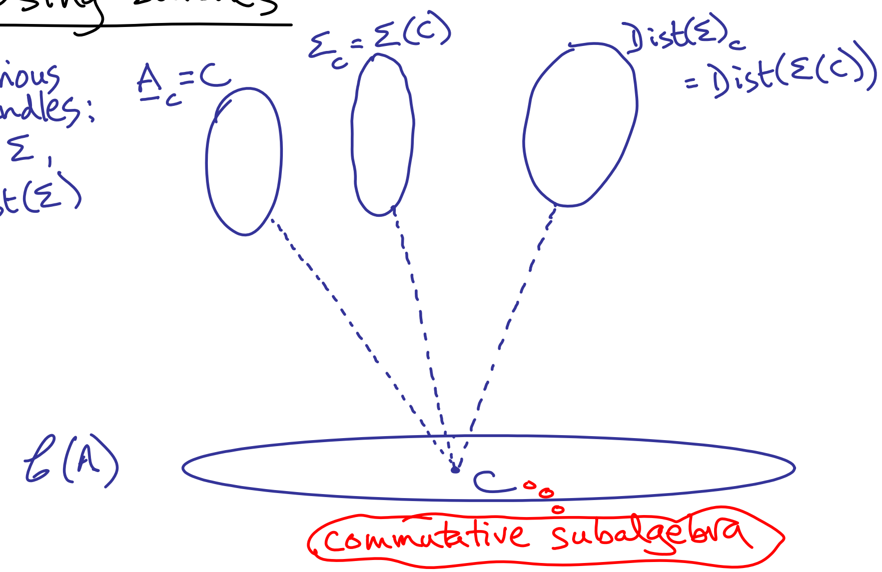
= bundles externally (geometricity)

Often — internal reasoning = fibrewise, indexed by base point

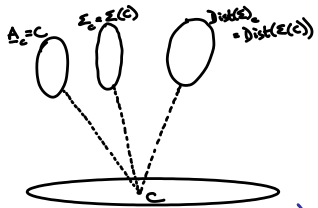
## Using bundles

Matches Landsman et al.

Various bundles: $\underline{A}$, $\Sigma$, $Dist(\Sigma)$



Kochen-Specker




$\Rightarrow$ $\Sigma$ has no cross-sections (continuous choice of point from every fibre = external point of bundle)

However — Dist($\Sigma$) does have cross-sections

Each quantum state produces one.

Internally: quantum states are distributions over classical states

Externally: this is impossible

Does topos support "neo realist" reasoning?

Isham

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Isham/Döring : $\mathcal{J}(A) = \mathbf{Set}^{\mathcal{C}(A)^{op}} = \mathrm{Sh}(\mathrm{Filt}(\mathcal{C}(A)))$

filter completion

Heunen/Landsman/Spitters :

ideal completion

$\mathcal{J}(A) = \mathbf{Set}^{\mathcal{C}(A)} = \mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$

Both: – include classical viewpoints as base points (principal filters/ideals)

– respect order on $\mathcal{C}(A)$ but not topology

Idea: use topology on $\mathcal{C}(A)$

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Case $A = M_2(\mathbb{C})$

Let $\mathcal{C}_2(A) = \{C \in \mathcal{C}(A) \mid \dim C = 2\}$ greatest dimension possible

$C \in \mathcal{C}_2(A) \Rightarrow C \cong \mathbb{C}^2$, spectrum = 2

C determined by two projectors $E$, $I-E$ self-adjoint idempotents

corr. to $(1,0)$, $(0,1)$

Spectrum $= \{E, I-E\}$

Projector has – eigenvalues 0 or 1

$\therefore$ trace 0, 1 or 2

trace 0, 2 are $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ & $\begin{pmatrix}1&0\\0&1\end{pmatrix}$

We seek projectors $E$ with trace 1.

$E$, $I-E$ both describe same $C$.

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Projectors via unitaries

$E$ idempotent $\Leftrightarrow$ $U = 2E - I$ has $U^2 = I$

$E$ a projector $\Leftrightarrow$ $U$ s.a. unitary

$\mathrm{tr}\, E = 1$ $\Leftrightarrow$ $\mathrm{tr}\, U = 0$

s.a. unitaries, trace 0 :

$U = \begin{pmatrix} c & a+ib \\ a-ib & -c \end{pmatrix}$, $a^2+b^2+c^2 = 1$

$E, I-E$ corr. to $U, -U$

$\mathcal{C}_2(A) \cong$ sphere $S^2$ with antipodes identified

= real projective plane $\mathbb{RP}^2$

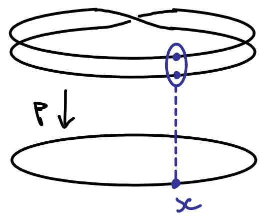
## Take $\mathcal{T}(A) = Sh(\mathcal{C}_2(A)) = Sh(\mathbb{R}P^2)$

Spectral bundle is obvious cover $\begin{array}{c} S^2 \\ \downarrow \\ \mathbb{R}P^2 \end{array}$

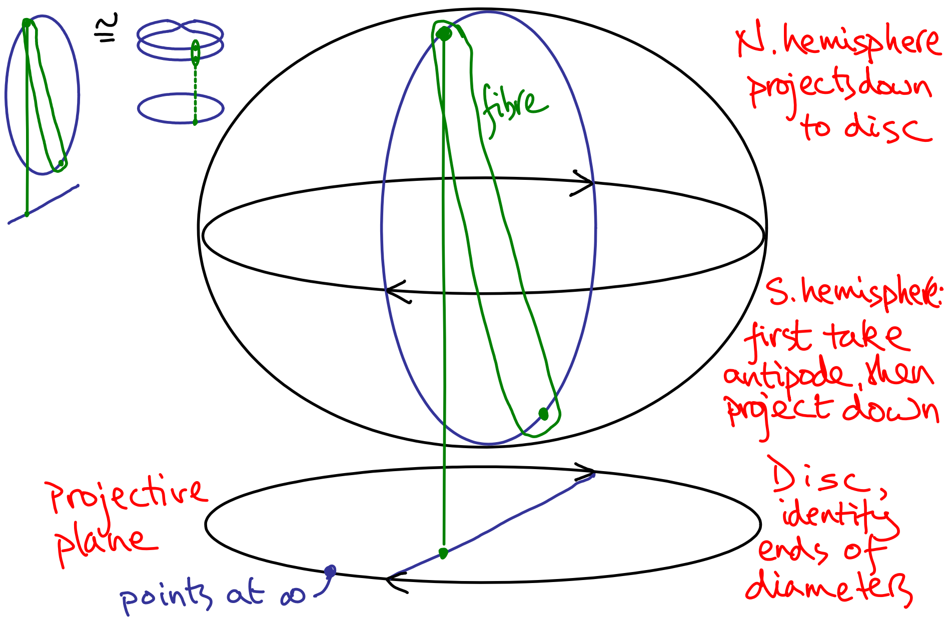
- A local homeomorphism (object of $\mathcal{T}(A)$)
- Each fibre has two points
- No continuous global sections
- cf. Kochen-Specker (doesn't apply to $M_2(A)$, but doesn't require continuity)

## No global sections

$$\begin{tikzcd} S^2 \arrow[d] & \supseteq & S^1 \arrow[d] \\ \mathbb{R}P^2 & \supseteq & S^1 \end{tikzcd}$$




Global section would give continuous choice of edge of Möbius strip.




## Distributions

point-free i.e. locale

Internally: space $X \mapsto$ space $Dist(X)$ of distributions (regular probability measures)

Heunen/Landsman/Spitters

Externally: bundle $\mapsto$ distribution bundle

Logical argument

geometricity of Distr – big result!

$\Rightarrow$ construction works fibrewise

## Global distributions

General feature of topos approach

Internally: have spectrum $\Sigma$
element = classically pure state
BUT no global elements – can't specify results of all observables simultaneously

Also have $\mathrm{Dist}(\Sigma)$
element = classically mixed state
DOES have global elements – e.g. each quantum pure state provides one.

(External) quantum pure = classically mixed – but only internally.

## e.g. for $A = M_2(\mathbb{C})$

On fibres:

$$\mathrm{Dist}(2) = \{(p_0, p_1) \in [0,1] \mid p_0 + p_1 = 1\} \cong [0,1]$$

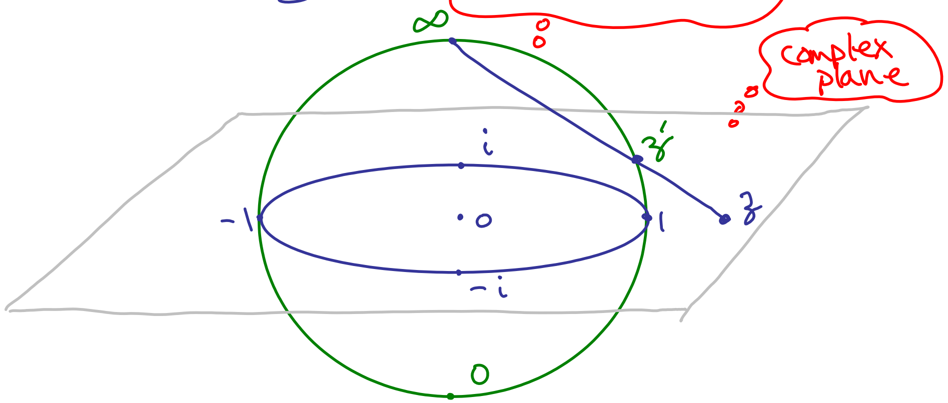
probabilities of two points of spectrum

Show how quantum pure state $\in \mathbb{C}^2$ gives global section of $\mathrm{Distr}\left(\begin{smallmatrix} S^2 \\ \downarrow \\ \mathbb{RP}^2 \end{smallmatrix}\right)$

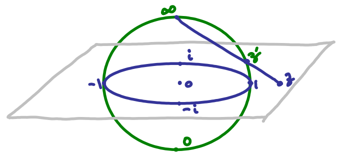
## States $\phi \in \mathbb{C}^2$

Scale invariant, $\therefore$ in complex projective line $\mathbb{CP}^1$

$\cong S^2$ Riemann sphere




If $z' = (a,b,c)$ on sphere
$a^2 + b^2 + c^2 = 1$
$c \neq 1$




then $z = \dfrac{a}{1-c} + i\,\dfrac{b}{1-c}$ on plane

$$(a,b,c) = c(0,0,1) + (1-c)\left(\tfrac{a}{1-c}, \tfrac{b}{1-c}, 0\right)$$

– corresponds to 1-dim subspace $\mathbb{C}\begin{pmatrix} a+ib \\ 1-c \end{pmatrix}$ in $\mathbb{C}^2$

= eigenspace for $E(a,b,c)$ with eigenvalue 1

$$\frac{1}{2}\begin{pmatrix} 1+c & a+ib \\ a-ib & 1-c \end{pmatrix}\begin{pmatrix} a+ib \\ 1-c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (a+ib)(1+c+1-c) \\ a^2+b^2+1-2c+c^2 \end{pmatrix} = \begin{pmatrix} a+ib \\ 1-c \end{pmatrix}$$

**Consider:** projector $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ – North Pole $E\ (0,0,1)$

state $\phi = \begin{pmatrix}a+ib\\1-c\end{pmatrix}$

Born rule: probability of going to $\begin{pmatrix}1\\0\end{pmatrix}$

= expected value of $E$

$$= \frac{(a-ib,\ 1-c)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a+ib\\1-c\end{pmatrix}}{(a-ib,\ 1-c)\begin{pmatrix}a+ib\\1-c\end{pmatrix}} = \frac{(a-ib,\ 1-c)\begin{pmatrix}a+ib\\0\end{pmatrix}}{a^2+b^2+1-2c+c^2}$$

$$= \frac{a^2+b^2}{2(1-c)} = \frac{1+c}{2}$$

**In general**

$E$, $\phi$ two points on $S^2$

projector    state

Let $P$ = equatorial plane taking $E$ as North pole

Expected value $p_E$ of $E$ at $\phi = \frac{1+c}{2}$ where $c$ = distance of $\phi$ from $P$, taking $E$ to be on positive side.

Continuous, & coordinate independent

**Fix state $\phi$**

For each $C \in \mathbb{R}P^2$ –

- let $E, E'$ be antipodal points in its fibre in $S^2$
- $p_E + p_{E'} = 1$ (both calculated for $\phi$)
- Get point in fibre over $C$ of distribution bundle
- Gives continuous global section of distribution bundle.

**Product distributions**

Conjecture: still works internally

$$\mathrm{Dist}(X) \times \mathrm{Dist}(Y) \longrightarrow \mathrm{Dist}(X\times Y)$$

e.g. $X, Y$ finite discrete

$$(p_x)_{x\in X} \quad (q_y)_{y\in Y} \longmapsto (p_x q_y)_{\substack{x\in X\\ y\in Y}}$$

$$\sum_x p_x = 1,\ \sum_y q_y = 1 \implies \sum_{xy} p_x q_y = 1$$

## Not all distributions on product are product distributions

Distributions on —

$$\left.\begin{array}{lll} X & - & X\times 1 \text{ column vector } p \\ Y & - & 1\times Y \text{ row vector } q \\ X\times Y & - & X\times Y \text{ matrix } M \end{array}\right\} \text{ with elements in } [0,1] \text{ and summing to } 1$$

Product distribution for $p, q$ has matrix $pq$

— rank $= 1$

$\therefore \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix}, \begin{pmatrix} 0 & \frac12 \\ \frac12 & 0 \end{pmatrix}$ not product distributions – exhibit "entanglement" between $X, Y$ components.

## Product spaces

For $\mathcal{H}\otimes\mathcal{H}'$ with $A\otimes A'$:

- dont know all commutative subalgebras
- if $C, C' \in \mathcal{C}(A), \mathcal{C}(A')$ then $C\otimes C' \in \mathcal{C}(A\otimes A')$
- what can we get from $\begin{array}{c}\Sigma\times\Sigma' \\ \downarrow \\ \mathcal{C}(A)\times\mathcal{C}(A')\end{array}$ ?

## $\mathcal{H} = \mathcal{H}' = \mathbb{C}^2$

Product states $\phi\phi'$ $(= \phi\otimes\phi')$, $\phi, \phi'$ unit vectors

If $C$ given by projectors $E, I-E$

" $C'$ " " " $E', I-E'$

$\phi\phi'$ gives distribution matrix

$$\begin{pmatrix} \langle\phi\phi', (E\otimes E')\phi\phi'\rangle & \langle\phi\phi', (E\otimes(I-E'))\phi\phi'\rangle \\ \langle\phi\phi', ((I-E)\otimes E')\phi\phi'\rangle & \langle\phi\phi', ((I-E)\otimes(I-E'))\phi\phi'\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle\phi, E\phi\rangle \\ \langle\phi, (I-E)\phi\rangle \end{pmatrix} \left( \langle\phi', E'\phi'\rangle, \langle\phi', (I-E')\phi'\rangle \right)$$

- product distribution

## Entangled state?

e.g. $\phi = (\uparrow\uparrow + \downarrow\downarrow)/\sqrt{2}$

$\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$, $\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

If $E = \frac12 \begin{pmatrix} 1+c & a+ib \\ a-ib & 1-c \end{pmatrix}$

need

$$\langle\uparrow, E\uparrow\rangle = \frac{1+c}{2}$$

$$\langle\downarrow, E\downarrow\rangle = \frac{1-c}{2}$$

calculated before

$$\langle\uparrow, E\downarrow\rangle = \frac{a-ib}{2}$$

$$\langle\downarrow, E\uparrow\rangle = \frac{a+ib}{2}$$

$$\langle \psi, (E \otimes E')\psi \rangle = \frac{1 + cc' + aa' - bb'}{4}$$

coordinate dependent?

Four projector pairs – take antipodes

Distribution $\frac{1}{4}\begin{pmatrix} 1+cc'+aa'-bb' & 1-cc'-aa'+bb' \\ 1-cc'-aa'+bb' & 1+cc'+aa'-bb' \end{pmatrix}$

e.g. for $c, c' = \pm 1$

Distribution $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

not a product – rank = 2

## Conclusions

- Illustrated external bundle view of topos internal reasoning
- Example using topology of $\mathcal{G}(A)$
- "Kochen-Specker" even in dimension 2, if require continuity
- Global distributions
- Some manifestation of entanglement.