Adelic Geometry via Topos Theory

by

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Abstract

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Our starting point has to do with a key tension running through number theory: although all completions of the rationals $\mathbb{Q}$ should be treated symmetrically, this is complicated by fundamental disanalogies between the p-adics vs. the reals. Whereas prior work has typically been guided by classical point-set reasoning, this thesis explores various ways of pulling this problem away from the underlying set theory, revealing various surprises that are obscured by the classical perspective. Framing these investigations is the following test problem: construct and describe the topos of completions of $\mathbb{Q}$ (up to equivalence).

Chapter 2 begins with the preliminaries: we set up the topos-theoretic framework of point-free topology, with a view towards highlighting the distinction between classical vs. geometric mathematics, before introducing the number-theoretic context. A key theme is that geometric mathematics possesses an intrinsic continuity, which forces us to think more carefully about the topological character of classical algebraic constructions.

Chapter 3 represents the first step towards constructing the topos of completions. Here, we provide a point-free account of real exponentiation and logarithms, which will allow us to define the equivalence of completions geometrically. Chapter 4 provides a geometric proof of Ostrowski's Theorem for both upper-valued abosolute values on $\mathbb{Z}$ as well as Dedekind-valued absolute values on $\mathbb{Q}$, along with some key insights about the relationship between the multiplicative seminorms and upper reals.

In a slightly more classical interlude, Chapter 5 extends these insights to obtain a surprising generalisation of a foundational result in Berkovich geometry. Namely, by replacing the use of classical rigid discs with formal balls, we obtain a classification of the points of Berkovich Spectra $\mathcal{M}(K\{R^{-1}T\})$ via the language of filters [more precisely, what we call: R-good filters] *even* when the base field K is trivially-valued.

Returning to geometricity, Chapter 6 builds upon Chapters 3 and 4 to investigate the space of places of $\mathbb{Q}$ via descent arguments. Here, we uncover an even deeper surprise. Although the non-Archimedean places correspond to singletons (as is classically expected), the Archimedean place corresponds to the subspace $\overleftarrow{[0,1]}$ of upper reals, a sort of blurred unit interval. The chapter then analyses the topological differences between the non-Archimedean vs. Archimedean places. In particular, we discover that while the topos corresponding to Archimedean place witnesses non-trivial forking in the connected components of its sheaves, the topos corresponding to the non-Archimedean place eliminates all kinds of forking phenomena. We then conclude with some insights and observations, framed by the question: "How should the connected and the disconnected interact?"

To my father, who always found ways to be proud of me

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Acknowledgments

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This thesis was written whilst I was a student of the School of Computer Science, and yet my natural curiosity led me much farther afield. After Ivan invited me to give a talk at SEEMOD in Dec '21, I was led to think more carefully about the logical perspectives in topos theory vs. model theory. Developing these thoughts led to the first two sections of Chapter 2, which reworked the basics of topos theory with the model theorist in mind. John Baldwin read a preliminary draft of the material and made various helpful remarks. Will Boney, Jinhe (Vincent) Ye, Konstantinos Kartas and Francesco Parente also listened patiently and were encouragingly receptive – thank you all. In addition, thanks also to François Loeser and Jinhe Ye for resolving some of my initial confusions about Hrushovski-Loeser theory.

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I'd be remiss if I didn't mention another major part of my life at Birmingham, of which I'm also immensely proud. During my first year at UoB, I was a member of the Lunar Society, an informal open discussion society, before we decided to rebrand as the Devil's Advocate Society. This was a last-bid attempt to save the society, which was on the brink of extinction due to not having enough members. I was then entrusted to be the founding President of this new society, and for the next two years, I fought for its survival. I, like many others amongst our small group of members, felt strongly that our society could play a unique and valuable role in university life. Over the past 10 years in academia, I'd grown weary of the name-calling and slogan-chanting embedded in its political discourse — despite all the lip service we pay to embracing diversity, we still seem unable to accept those who hold fundamentally different viewpoints from us. DA Society was my attempt to recenter campus dialogue on the deeper questions underlying our conflicts of opinion, to focus on the discussion of ideas rather than ideological loyalties. I am proud of all the work that we've done, from our collaboration with the UoB Jewish Society on the Israeli-Palestinian conflict, to our Speed-Debating event with the Philosophy Society where we pondered the paradox of virtuous selfishness, to our panel discussion on the geopolitical rise of China, and many many others. I am heartened by the incredible variety of people who have come to our events, and am grateful to those who have stayed. Trying to balance my commitments to this society and my PhD work was not easy, and I depended on the help of many. In particular, thanks to: Tiffany Idle, Megan Wright, Ben Sculfor, Chris Wright, Jake Stewart, Adam Belcher, Paul Dupont, Chloe Bruno and Dimitar Ferdinandov. Special thanks to Jonathan Korn, Alice Phillips, Lorna Lock and Leila Sallie, for all their support + hard work in keeping this society going. I've had many worries about what would become of this society once I stepped down as President; two years later, I am heartened to note that it still appears to be a healthy space of open & balanced dialogue at UoB, and I hope that will continue.

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Chapter 1

Introduction

1.1 Motivation from Number Theory

Much of the theory-building in Number Theory has been guided by a deep tension: while it is important to treat all the completions of the rationals[1] $\mathbb{Q}$ symmetrically (cf. the Hasse principle), it is also clear that there exist key disanalogies between the p-adics and the reals. The depth of these disanalogies can be measured by the fact that there are many powerful technologies that work well in one setting but not the other.[2] Indeed, as Mazur muses [Maz93]:

> "A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."

This leads to a natural question, which will guide the investigations of this thesis.

Question 1. What is the right perspective from which to understand this tension? That is, how can we treat the p-adics and the reals symmetrically whilst also accommodating their differences?

The number theorist is likely to have one of two reactions to Question 1 (and in fact, perhaps both). First, that our understanding of the reals and the p-adics should be guided by the function field analogy. Two, as already alluded to by Mazur, that we should strive to develop tools that work well for both settings. We discuss this in the context of Arakelov intersection theory [Ara74; PR21].

The Function Field Case. Consider a smooth affine curve C over an algebraically closed field k. Then, take the (unique) smooth compactification of C, which adds a finite number of points to yield a smooth projective curve $\overline{C}$. A *divisor* D on $\overline{C}$ is a finite formal linear combination of points on $\overline{C}$

$$D = \sum_{P \in \overline{C}} n_P \cdot P, \qquad n_P \in \mathbb{Z}. \tag{1.1}$$

[1] In fact, the same issue arises for a general number field, but we shall almost always restrict to $\mathbb{Q}$ for simplicity.

[2] One indication of this is that many choose to work with just the p-adics (e.g. via the finite adele ring $\mathbb{A}_{\mathbb{Q}}^{\text{fin}}$) and ignore the reals. See, for instance Balchin-Greenlees' work [BG20] on Adelic Models for tensor-triangulated categories, or Huber's work [Hub91] on the Beilinson-Parshin adeles, where she writes: "We want to stress that at this stage only a generalization of the finite adeles is found. It is not clear what one should take at infinity, or in fact even what the infinite 'places' should be."

In particular, for any non-zero *rational function* f on $\overline{C}$, one can define the divisor

$$(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) \cdot P, \tag{1.2}$$

where $\mathrm{ord}_P(f)$ denotes the multiplicity of f at P. One can then compute the *degree* of divisor (f) and deduce

$$\deg(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) = 0, \tag{1.3}$$

a key result that allows us to develop a good intersection theory of divisors.

The Number Field Case. Consider $\mathbb{Q}$ and the spectrum of the ring of integers $\mathrm{Spec}(\mathbb{Z})$. Notice the non-zero primes $p \in \mathrm{Spec}(\mathbb{Z})$ each corresponds to the p-adic numbers $\mathbb{Q}_p$. To account for $\mathbb{R}$, we formally add to $\mathrm{Spec}(\mathbb{Z})$ the set of complex embeddings $\sigma\colon \mathbb{Q} \hookrightarrow \mathbb{C}$; in which case, this gives a single embedding factoring through $\mathbb{R}$.[3] Denote this enlargement of $\mathrm{Spec}(\mathbb{Z})$ as $\Lambda_{\mathbb{Q}}$, which we shall call *the set of places of* $\mathbb{Q}$. Following standard conventions, we denote the "real prime" adjoined to $\mathrm{Spec}(\mathbb{Z})$ as ∞.

Next, define the *Arakelov divisor* D on $\Lambda_{\mathbb{Q}}$ as the following finite formal linear combination:

$$D = \sum_p n_p \cdot p + \alpha_\infty \cdot \infty, \qquad n_p \in \mathbb{Z},\ \alpha_\infty \in \mathbb{R}, \tag{1.4}$$

where the first sum runs over the set of non-zero primes in $\mathrm{Spec}(\mathbb{Z})$. As before, given any non-zero rational $f \in \mathbb{Q}$, one can define its Arakelov divisor (f), whose *Arakelov degree* can be computed to give

$$\widehat{\deg}(f) = \sum_{v \in \Lambda_{\mathbb{Q}}} \log |f|_v = 0. \tag{1.5}$$

Considered side-by-side, the analogy between the two setups becomes clear, but notice the formal nature of the number field case. In the function field case, we added points to the smooth affine curve C by performing a geometric construction on C ("smooth compactification"). By contrast, the number field case starts with a formal abstraction: take the underlying set of $\mathrm{Spec}(\mathbb{Z})$. It is this formal move that allows us to combine the set of primes with the set of complex embeddings (even though they are *a priori* different objects), giving a new set $\Lambda_{\mathbb{Q}}$ which we use to index the summands of the Arakelov divisor.

This style of point-set reasoning ("take the set of ...") is widely accepted in classical mathematics, but here it presents a challenge to our understanding. For one, extending the function field analogy, one would like to regard $\Lambda_{\mathbb{Q}}$ as the compactification of $\mathrm{Spec}(\mathbb{Z})$. But on what grounds? Strictly speaking, $\Lambda_{\mathbb{Q}}$ is just a set of elements with no topology — it is only by analogy that one might regard it as morally being a kind of compactified affine curve. Second, notice that the construction of $\Lambda_{\mathbb{Q}}$ is still guided by an obvious case-split between the p-adics vs. the reals. In fact, as pointed out in [Bak08], Arakelov intersection theory uses very different-looking tools to deal with these two components[4], raising sharp questions about the extent to which Arakelov theory successfully resolves the lack of symmetry between the p-adics and the reals.

[3] In the general case of a number field K, the construction involves adding $[K : \mathbb{Q}]$ many complex embeddings to $\mathrm{Spec}(\mathcal{O}_K)$.

[4] Baker's remark [Bak08] was made in the context of motivating the development of non-Archimedean potential theory, which aims to formulate an Arakelov theory that applies analytic methods from potential theory not only at the Archimedean places but also at the non-Archimedean places too. For details on how this works for curves, see, e.g. [BR10].

1.2 Connections to Topos Theory

Having provided some number-theoretic context, we now shift gears and discuss the connection to the logical aspects of topos theory. Our main point of leverage is the following structure theorem:

Theorem 2. Every (Grothendieck) topos $\mathcal{E}$ is a classifying topos of some geometric theory $\mathbb{T}_{\mathcal{E}}$. Conversely, every geometric theory $\mathbb{T}$ has a classifying topos $\mathcal{S}[\mathbb{T}]$.

The precise definitions of the relevant terms will be deferred to Chapter 2. For now, let us just say:

- A *theory* $\mathbb{T}$ is a set of logical axioms that describes structures of interest (e.g. groups, rings etc.);
- *Geometric logic* is a logic that is tailored to reflect topology, e.g. connectives $\wedge$ and $\bigvee$ to match intersection and union of opens. A *geometric theory* is a set of axioms expressed in geometric logic;
- A *model* $M_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is a structure satisfying the description expressed by $\mathbb{T}$;
- A *topos* $\mathcal{E}$ is some kind of category satisfying certain nice properties;[5]
- A *classifying topos of* $\mathbb{T}$, denoted $\mathcal{S}[\mathbb{T}]$, is a topos representing the universe of all models of $\mathbb{T}$. In particular, it contains a *generic model* $G_{\mathbb{T}}$, which is generic in the informal sense that it gives a blueprint from which all models $M_{\mathbb{T}}$ of $\mathbb{T}$ can be derived.[6]

This sets up the following question: does there exist a geometric theory $\mathbb{T}_{\text{comp}}$ whose models are the completions of $\mathbb{Q}$ (up to topological equivalence)? Notice if yes, then Theorem 2 gives a classifying topos of $\mathbb{T}_{\text{comp}}$ along with a generic model, which we shall call *the generic completion of* $\mathbb{Q}$.

Why might this be an interesting perspective? We give two natural reasons. First, the generic model $G_{\mathbb{T}}$ of any geometric theory $\mathbb{T}$ is conservative, i.e. given any property ϕ expressible in geometric logic, ϕ holds for $G_{\mathbb{T}}$ iff ϕ holds for all models of $\mathbb{T}$. There are a couple ways to read this in the present context. One interpretation: the generic completion of $\mathbb{Q}$ is a device that allows us to reason about properties that hold for all completions of $\mathbb{Q}$ in a symmetric manner — much like the adele ring $\mathbb{A}_{\mathbb{Q}}$ in classical number theory. Another interpretation: the generic completion of $\mathbb{Q}$ is a construction possessing no other properties besides being a completion of $\mathbb{Q}$. As such, if we wish to calibrate our understanding of the p-adics vs. the reals, it can be helpful to have a well-defined object that distills precisely what their shared similarities are.

The second, and more fundamental, reason is that the topos-theoretic perspective pulls Question 1 away from classical set theory, and opens it up to new tools from logic and category theory. This requires some explanation. To the uninitiated, the existence of serious interactions between number theory and logic may come as a surprise, but this itself is certainly not new. For instance, continuing with the function field analogy, a remarkable transfer theorem was proved by the model theorists back in the 1960s:

Theorem 3 (Ax-Kochen-Eršov Principle [AK65; Erš65])**.** As our setup,

- Let $\mathcal{U}$ be a non-principal (= contains all cofinite sets) ultrafilter on the set of primes;
- Let $\prod_p \mathbb{Q}_p/\mathcal{U}$ be the ultraproduct of p-adic fields $\mathbb{Q}_p$;
- Let $\prod_p \mathbb{F}_p((t))/\mathcal{U}$ be the ultraproduct of the fields of formal Laurent series over $\mathbb{F}_p$.

<u>Then</u>, $\prod_p \mathbb{Q}_p/\mathcal{U}$ and $\prod_p \mathbb{F}_p((t))/\mathcal{U}$ are elementarily equivalent.

[5] Convention: the unqualified term "topos" will always mean a Grothendieck 1-topos, unless stated otherwise. The expert reader may take the phrase "nice properties" to mean Giraud's axiomatic characterisation of a topos.

[6] More precisely: given any $\mathbb{T}$-model $M_{\mathbb{T}}$ living in any topos $\mathcal{E}$, there exists a functor $f^* \colon \mathcal{S}[\mathbb{T}] \to \mathcal{E}$, unique up to isomorphism, such that $f^*(G_{\mathbb{T}}) \cong M_{\mathbb{T}}$ whilst also preserving colimits and finite limits.

In plainer terms, the Ax-Kochen-Eršov Principle says: given any first-order logical statement ϕ about valued fields, there exists a finite set C of primes such that ϕ holds for $\mathbb{F}_p((t))$ iff ϕ holds for $\mathbb{Q}_p$ just in case $p \notin C$. As a beautiful application of this result, Ax and Kochen [AK65] proved that every homogeneous polynomial of degree d with more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$ for all but finitely many primes p. However, while this breakthrough result may be vindicating for the classical logician, its non-constructive aspects makes it problematic for the topos theorist. In particular, notice that the Ax-Kochen-Eršov Principle is formulated using non-principal ultrafilters, whose existence implies a weak form of choice and thus cannot be shown constructively.[7]

This discussion sets up an important organising principle of this thesis. Properly understood, Theorem 2 gives rise to a new understanding of a topos as a so-called "point-free space", which we define below:

Definition 4. A *(point-free) space* is a space X whose points are the models of a geometric theory. A map $f\colon X \to Y$ is defined by a geometric construction of points $f(x) \in Y$ out of points $x \in X$.

This unusual marriage of topology and logic, which we call "point-free topology", differs from the classical perspective in two important ways. One, it challenges the classical notion of a space as a set decorated with a chosen topology. Two, it generalises the classical notion of model as a set decorated with the logical data of relations and/or functions that have been singled out for study. Further details will be explained in due course, but notice that this perspective already gives some indication of how point-free topology systematically pulls our mathematics away from its underlying set theory.

Returning to our original context, what does the point-free perspective mean for Question 1? The methodological upshot: in order to work with models as if they were points of some kind of generalised space (embodied by the topos), we shall need to adhere to a strict regime of constructive mathematics known as *geometric mathematics* [Vic07b; Vic14].[8] In practice, "working geometrically" means abandoning many classical tools and principles in exchange for new ones. Unlike the model theorist, we do not have the axiom of choice, and so we shall prefer to work with the generic model of a theory rather than the ultraproducts of its models[9]; and unlike the classical number theorist, we cannot take the underlying set of $\mathrm{Spec}(\mathbb{Z})$ (at least, not without losing geometricity), and so we must find other ways of dealing with the places of $\mathbb{Q}$.

1.3 Overview of Thesis

Hereafter, the term "space" shall always mean a point-free space (cf. Definition 4) unless stated otherwise. As previously discussed, this thesis will focus on the following test problem:

Problem 5. Construct and describe the classifying topos of completions of $\mathbb{Q}$ (up to equivalence).

[7] This fact follows from a combination of two results: [HL67] shows that the strongest form of the ultrafilter lemma (= all filters can be extended to ultrafilters) does not imply the standard Axiom of Choice; [Bla77] shows that the weakest form of ultrafilter the lemma (= there exists a non-principal ultrafilter on some set) cannot be proved in ZF set theory.

[8] To work geometrically means to reason using constructions that are preserved by pullback along geometric morphisms between toposes. As will be explained in due course, this essentially means working with constructions/properties built out of finite limits and arbitrary colimits.

[9] Why the comparison between ultraproducts of models with the generic model? The short answer: because both constructions, properly understood, lead to representative models of their first-order theories. The case for the generic model is clear given the fact that it is conservative. The case for the ultraproduct construction is more involved — see [Mal19] for details, particularly the discussion on regular ultrapowers and Keisler's Order.

Step One: Point-free Real Exponentiation. The first step towards constructing this topos is understanding when a given completion K of $\mathbb{Q}$ is topologically equivalent to another completion K'. Classically, completions of $\mathbb{Q}$ are defined as point-set spaces comprising the Cauchy sequences of $\mathbb{Q}$ with respect to some kind of metric on $\mathbb{Q}$:

$$\begin{aligned} |\cdot|\colon \mathbb{Q} &\longrightarrow [0,\infty) \\ x &\longmapsto |x| \end{aligned} \tag{1.6}$$

known as an *absolute value*. Given two absolute values $|\cdot|_1, |\cdot|_2$, we can define an equivalence relation $\sim$ where $|\cdot|_1 \sim |\cdot|_2$ iff there exists some $\alpha \in (0,1]$ such that $|x|_1^\alpha = |x|_2$ or $|x|_2^\alpha = |x|_1$ for all $x \in \mathbb{Q}$ such that $x \neq 0$. Such an equivalence class of absolute values is called a *place*, and it turns out that two absolute values belong to the same place iff their completions are topologically equivalent. This reduces an *a priori* topological problem to an algebraic one, except that we shall first need a geometric account of real exponentiation. This is worked out in Chapter 3.

Theorem A. There exists an exponentiation map on the Dedekinds

$$\exp\colon (0,\infty)\times\mathbb{R} \to (0,\infty), \tag{1.7}$$

satisfying the usual exponent laws

$$\begin{aligned} x^{\zeta+\zeta'} &= x^\zeta x^{\zeta'}, \quad x^0 = 1 \\ x^{\zeta\cdot\zeta'} &= (x^\zeta)^{\zeta'}, \quad x^1 = x \\ (xy)^\zeta &= x^\zeta y^\zeta, \quad 1^\zeta = 1. \end{aligned} \tag{1.8}$$

The result itself is not surprising; the main challenge in the construction are the technical constraints imposed by geometricity. Just as we cannot take the underlying set of $\mathrm{Spec}(\mathbb{Z})$, we may not take the underlying set of the Dedekinds and treat exponentiation as a purely algebraic construction on its elements. Further, exponentiation x^ζ is monotonic in the exponent when $x \in (1,\infty)$ whereas it is antitonic in the exponent when $x \in (0,1)$. This indicates a natural case-splitting on the base, which requires careful justification since, working geometrically, we generally cannot assume the Law of Excluded Middle.

Something interesting that already emerges at this stage are the so-called one-sided reals, which are essentially semi-continuous versions of the usual Dedekinds. Note: while the points of the Dedekinds and one-sided reals more or less coincide classically[10], they are very different entities in geometric mathematics. In Chapter 3, the one-sided reals serve primarily as computational tools: our general approach involves developing exponentiation for the one-sideds, before lifting the result to the Dedekinds. A similar approach is adopted when developing a geometric account of logarithms:

Theorem B. Fix $b \in (1,\infty)$. We can then define one-sided logarithm maps[11]

$$\log_b\colon \overrightarrow{[0,\infty]} \to \overrightarrow{[-\infty,\infty]} \quad \text{and} \quad \log_b\colon \overleftarrow{[0,\infty]} \to \overleftarrow{[-\infty,\infty]} \tag{1.9}$$

inverse to the corresponding exponentiation maps $b^{(-)}$ on the one-sideds. These combine to yield an isomorphism on the Dedekinds

$$\log_b\colon (0,\infty) \xrightarrow{\sim} (-\infty,\infty). \tag{1.10}$$

[10] For the sake of argument, let us presently ignore the one-sided infinities.

[11] Convention: intervals of one-sided reals are indicated with an arrow on top, indicating the direction of the Scott topology with respect to the numerical order.

Step Two: Investigation of Absolute Values. The next step in tackling Problem 5 is to construct the topos of absolute values and provide a geometric proof of Ostrowski's Theorem. As before, we shall prove the result for the one-sided reals (in fact, just the upper reals) and the Dedekinds, but now the one-sided reals take on a conceptual significance.

Theorem C (Ostrowski's Theorem for $\mathbb{Z}$)**.** As our setup, denote:

- $\overleftarrow{[av]}$:= The space of absolute values on $\mathbb{Z}$, valued in upper reals.
- $\mathrm{ISpec}(\mathbb{Z})$:= The space of prime ideals of $\mathbb{Z}$.
- $\overleftarrow{[-\infty, 1]}$:= The space of upper reals bounded above by 1.

Define

$$\mathfrak{P}_\Lambda := \{(\mathfrak{p}, \lambda) \in \mathrm{ISpec}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]} \mid \lambda < 0 \leftrightarrow \exists a \in \mathbb{Z}_{\neq 0}.(a \in \mathfrak{p})\}. \tag{1.11}$$

Then, we get the following isomorphism of spaces:

$$\overleftarrow{[av]} \cong \mathfrak{P}_\Lambda. \tag{1.12}$$

Theorem D (Ostrowski's Theorem for $\mathbb{Q}$)**.** As our setup,

- Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$;
- Let $|\cdot|_\infty$ be the standard Euclidean absolute value, whose completion of $\mathbb{Q}$ yields the reals $\mathbb{R}$;
- Let $|\cdot|_p$ be the standard p-adic absolute value, whose completion of $\mathbb{Q}$ yields the p-adic field $\mathbb{Q}_p$.

Then, one of the following must hold:

(i) $|\cdot| = |\cdot|_\infty^\alpha$ for some $\alpha \in (0, 1]$; or
(ii) $|\cdot| = |\cdot|_p^\alpha$ for some $\alpha \in (0, \infty)$ and some prime $p \in \mathbb{N}_+$.

Setting aside the issues of geometricity, let us highlight the differences between Theorems C and D. Theorem D, which is the standard formulation of Ostrowski's Theorem, is a classification result on the absolute values on $\mathbb{Q}$. Since an absolute value $|\cdot|$ on $\mathbb{Q}$ is obviously determined by where it sends the non-zero integers $\mathbb{Z}$, this suggests a natural extension of Ostrowski's Theorem for absolute values defined on $\mathbb{Z}$, which gives Theorem C. Notice that Theorem C as formulated is not just a classification result but also a representation result: not only can we associate any absolute value $|\cdot|$ on $\mathbb{Z}$ to a pair $(p, \lambda) \in \mathfrak{P}_\Lambda$, but this association is also unique (up to unique isomorphism).

There is, however, a deeper point to be made. When defining the theory of absolute values on $\mathbb{Z}$

$$|\cdot| \colon \mathbb{Z} \to \overleftarrow{[0, \infty)}$$

we defined them as multiplicative seminorms valued in the upper reals; on the other hand, we chose to define absolute values on $\mathbb{Q}$ as being valued in the Dedekinds. Geometricity shows this to be canonical. In particular, if we wish to define absolute values valued in upper reals, then we lose the ability to axiomatise positive definiteness and so are forced to consider just the multiplicative seminorms on $\mathbb{Z}$; conversely, if we wish to define absolute values on $\mathbb{Q}$, then they must be valued in Dedekinds instead of the upper reals, which can be shown to give us positive definiteness for free.

The idea of a space whose points correspond to multiplicative seminorms, such as $\overleftarrow{[av]}$, is not new (see e.g. [Ber90]); what *is* new is the tight connection with the upper reals, revealing a subtle interplay between the topology and algebra that was previously hidden. In a slightly more classical interlude, we extend this insight to sharpen a foundational result in Berkovich geometry. In the language of Chapter 5,

Theorem E (Berkovich's Disc Theorem)**.** Fix K to be an algebraically closed field complete with respect to a non-Archimedean norm. Define $\mathcal{A}$ to be a ring of convergent power series, i.e.

$$\mathcal{A} := K\{R^{-1}T\} = \left\{ f = \sum_{i=0}^{\infty} c_i T^i \;\middle|\; c_i \in K, \lim_{i\to\infty} |c_i|R^i = 0 \right\}, \tag{1.13}$$

and define its Berkovich Spectrum $\mathcal{M}(\mathcal{A})$ to be the space of bounded multiplicative seminorms on $\mathcal{A}$.

Then, the space $\mathcal{M}(\mathcal{A})$ is classically equivalent to the space of R-good filters.

There is a hidden surprise here for the expert. Berkovich's original result holds that all points of $\mathcal{M}(\mathcal{A})$ can be described as nested descending sequences of rigid discs *so long as* the norm on K is non-trivial. It is well-known that his argument breaks down when we consider trivially-valued K. However, by using point-free techniques from [Vic05; Vic09], we found the correct modification of rigid discs and were thus able to eliminate the non-triviality hypothesis from Berkovich's result. Not only does this tighten the comparison between the classical Ostrowski's Theorem and Berkovich's Disc Theorem (for reasons we will explain in due course), it also shows that the previous *algebraic* hypothesis of K being non-trivially valued is in fact a *point-set* hypothesis, and is not essential to the underlying mathematics. The surprising aspects of this result hints at the clarifying potential of the point-free techniques (even when applied classically), and motivates a very interesting series of test problems on their interactions with non-Archimedean geometry, discussed in Section 5.3.

Step Three: Topos of Places of $\mathbb{Q}$. An important payoff for working geometrically is that, leveraging Theorem 2, we have at our disposal a deep collection of structure theorems for toposes, such as descent, that allows us to extract topological information from our logical setup.

This motivates Chapter 6, which investigates the topos of places of $\mathbb{Q}$. Here we explore the question: considered as a point-free space, what do the places of $\mathbb{Q}$ "look" like? A central theme of this chapter is that while it is clear that the exponentiation of absolute values gives an algebraic action, characterising the point-free spaces quotiented by this action is a subtler issue.

Applying the classification result of Theorem D, we first localise and define the topos of a single non-Archimedean place, denoted $\mathcal{D}$, associated to some prime p. By characterising $\mathcal{D}$ as an appropriate descent topos, we get the following result:

Theorem F. $\mathcal{D} \simeq \mathrm{Set} = \mathcal{S}\{*\}$.

In other words, a single non-Archimedean place corresponds to a singleton $\{*\}$, as one might expect classically. However, here comes the big surprise. When we apply a similar analysis to the topos of the Archimedean (i.e. the real) place, denoted $\mathcal{D}'$, we instead get:

Theorem G. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$.

This result overturns a longstanding classical assumption in number theory. Instead of corresponding to a singleton with no intrinsic features (as is assumed in, e.g. Arakelov geometry), Theorem G indicates that the Archimedean place corresponds to a sort of blurred unit interval comprising the upper reals.

As such, our understanding of the mechanics underlying Question 1 has started to shift. At this critical juncture, we are still sorting through the implications of our results, but interesting fragments of the picture have emerged. Section 6.4 gives a topos-theoretic insight on the differences between the non-Archimedean vs. Archimedean place: in our language, whereas $\mathcal{D}$ eliminates all forms of non-trivial forking in its sheaves, upper-bound forking still persists in $\mathcal{D}'$. Section 6.5 identifies and discusses a key theme that has been hidden in plain sight in our investigations: namely, the interactions between the connected and the disconnected. This theme turns out to have a surprisingly far reach. On the topos-theoretic side, we discuss how its relation to Theorem F brings into focus an interesting limitation of classifying toposes, raising challenging questions about its intended role in modern applications. On the number-theoretic side, notice that Theorems F and G only give a characterisation of individual places and not of the entire space of places (much less the entire space of completions). In fact, as we discuss, the question of how the Archimedean and non-Archimedean components fit together is surprisingly subtle, and also appears bound up with questions about reconciling the connected with the disconnected. Nonetheless, some very interesting parallels have emerged between our work and Clausen-Scholze's framework of Condensed Mathematics, particularly in regards to the differences between solidity and p-liquidity. This gives us some useful clues on where to start looking for answers.

ROADMAP

This thesis investigates the test problem of constructing and describing the topos of completions of $\mathbb{Q}$ (up to equivalence). Chapter 2 begins with the preliminaries: we set up the framework of point-free topology, with a view towards highlighting the distinction between classical vs. geometric mathematics, before introducing the number-theoretic context. Chapter 3 provides a point-free account of real exponentiation and logarithms, which allows us to define the equivalence of completions geometrically. Chapters 4-5 describe various spaces of norms/seminorms, and classifies their points up to equivalence. Chapter 6 investigates the space of places of $\mathbb{Q}$ via descent arguments, before concluding with some insights and observations.

Chapter 2

Preliminaries

> *Model theory rarely deals directly with topology; the great exception is the theory of o-minimal structures, where the topology arises naturally from an ordered structure.*
>
> **— E. Hrushovski and F. Loeser [HL16]**

> *While geometric logic can be treated as just another logic, it is an unusual one. [...] To put it another way, the geometric mathematics has an intrinsic continuity.*
>
> **— S. Vickers [Vic14]**

There are (at least) two different levels on which to read the slogan: "A topos is a generalised space". The mainstream view is that a topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ is a category of sheaves on a Grothendieck site $(\mathcal{C}, J)$; it is a generalised space insofar as the Grothendieck site categorifies a topological space [more correctly, the lattice of opens on a topological space]. The second view, which will play a more central role in this thesis, is that a topos is a "point-free space". There are various understandings of this phrase in the literature (e.g. locales, formal topologies, etc.), but we believe they can be subsumed by the following definition.[12]

Definition 2.0.1. A *(point-free) space* is a space X whose points are the models of a geometric theory. A map $f\colon X \to Y$ is defined by a geometric construction of points $f(x) \in Y$ out of points $x \in X$.

As mentioned earlier, this definition differs from the classical perspective in two important ways. One, it challenges the standard notion of a space as a set decorated with a chosen topology. Two, it generalises the classical notion of model as a set decorated with the logical data of relations and/or functions singled out for study. Once made precise through the topos theory, this give rise to a new mathematical framework (let us say, "point-free topology"), where algebra and logic now become *intrinsically* linked with topology.

Section 2.1 develops this perspective, paying special attention to the gap between classical mathematics vs. what we shall call "geometric mathematics". This sets up a key tension between algebraic intuitions and strict topological constraints, which underscores much of our work in the later chapters. Section 2.2 focuses on a particularly well-behaved class of spaces known as localic spaces, before establishing some

[12] To elaborate: locales [actually, frames] can be understood as corresponding to the Lindenbaum Algebras of propositional geometric theories (whose points are the completely prime filters) whereas formal topologies can be understood as presenting the geometric theory directly, with the base as signature and covers as axioms. For more details on these connections, see [Vic07c].

new tools of analysis, such as the Lifting Lemma 2.2.55. The key points of discussion from the two sections are then summarised in an interlude (Section 2.3), with a view towards the questions raised in Chapter 1. Shifting gears in Section 2.4, we then discuss the number-theoretic context for our work by introducing the Local-Global Principle, which essentially asks: how do we pass from knowledge about local structures (e.g. the completions of $\mathbb{Q}$) to knowledge about global structures (e.g. $\mathbb{Q}$ itself)? The final Section 2.5 ties the various threads together, and describes our research programme.

2.1 Point-free Topology

The main difference between point-set topology vs. point-free topology is one of priority: what are the basic units for defining a space? In the case of a point-set space, one starts with a set of elements, before defining the topology on it in the usual way. In the case of point-free spaces, we start by defining the topological structure — this structure implicitly carries the notion of the "generic point", which may later be instantiated as other points of the space.

This section reviews various key concepts and structure theorems from topos theory that guided the development of this perspective. Most of the material is not new, although the view that, e.g. one can leverage the theory of classifying toposes to define point-free spaces (in the sense of Definition 2.0.1) is perhaps a more modern understanding [Vic99; Vic07b; Vic14; Vic22]. For standard references, see e.g. [AJ21; Joh77b; Joh02a; Joh02b; LM94].

2.1.1 Toposes as a Category of Sheaves. The first shift in perspective can be traced back to the fundamental notion of a sheaf on a topological space X, which already highlights the role of the topological structure of X over its underlying set of points.

Definition 2.1.1. Let X be an ordinary topological space, and define $\mathcal{O}(X)$ to be its category of open sets, characterised by:

Objects: Opens U of the space X
Morphisms: $U \to V$ if $U \subseteq V$

(i) A *presheaf on* X is a contravariant functor

$$F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}.$$

(ii) A *sheaf on* X is a presheaf satisfying two extra conditions:

- *(Locality)* If $\{V_i\}$ is an open covering of U, and if $s, t \in \mathcal{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$.
- *(Gluing)* Given any open covering $\{V_i\}$ of U, and any family of sections $\{s_i \in \mathcal{F}(V_i)\}_{i \in I}$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all $i, j \in I$, then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$.

(iii) A *morphism* of sheaves $f\colon F \to G$ is a natural transformation of functors. We write $\mathbf{Sh}(X)$ for the category of sheaves on X.

In particular, notice that the basic unit used in defining a sheaf over X is the open $U \subseteq X$ and *not* the point $x \in X$. This sets up the first level of abstraction through the definition of frames/locales.

Definition 2.1.2.

(i) A *frame* is a complete lattice A possessing all small joins $\bigvee$ and all finite meets $\wedge$, such that the following distributivity law holds

$$a \wedge \bigvee S = \bigvee \{a \wedge b \,|\, b \in S\}$$

where $a \in A, S \subseteq A$.

(ii) A *frame homomorphism* is a function between frames that preserves arbitrary joins and finite meets.

Frames and frame homomorphisms form the category Frm. We define the category $\mathrm{Loc} := \mathrm{Frm}^{\mathrm{op}}$. We shall refer to the objects in Loc as *localic spaces*, or often more simply as *spaces*.

The category of sheaves over a localic space can be defined in the obvious way. Of particular importance to us, especially in Chapter 6, is that sheaves over spaces can also be described as local homeomorphisms, or what we call *étale bundles*. Call a localic map $f\colon E \to L$ an *étale map* if E can be covered by open sublocales U such that the composite $U \rightarrowtail E \to L$ is isomorphic to the inclusion of an open sublocale of L. This sets up the following structure theorem.

Theorem 2.1.3. *As our setup,*

- *Denote L to be a localic space and **Sh**(L) to be the category of sheaves on L;*
- *Denote Et/L to be the category of étale bundles over L, which is characterised by:*

  ***Objects:** Étale maps $f\colon E \to L$ with codomain L;*
  ***Morphisms:** Localic maps $\theta\colon E' \to E$ such that the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{\quad\theta\quad} & E \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f} & \\ & L & \end{array}$$

  where f', f are both étale maps.

<u>*Then,*</u> ***Sh**$(L) \simeq \mathrm{Et}/L$.*

Proof. See [Joh02b, Theorem C1.3.11]. □

Remark 2.1.4. It is instructive to compare the (point-free) definition of an étale mapping of locales with the classical (point-wise) definition of a local homeomorphism, i.e. a continuous map $f\colon Y \to X$ of topological spaces such that every point $y \in Y$, there exist open neighbourhoods $U \subseteq Y$ and $f(U) \subseteq X$ such that $y \in U$ and f induces a homeomorphism $U \cong f(U)$. In fact, as the reader may have anticipated, an analogue of Theorem 2.1.3 can also be proved for topological spaces [Bor94, §2.4].

Abstracting one step further, one may ask: what was the category-theoretic data used in defining a sheaf? Examining Definition 2.1.1 reveals two main ingredients: (a) a small category $\mathcal{O}(X)$; and (b) a topology $J_{\mathcal{O}(X)}$ on $\mathcal{O}(X)$ that tells us how the opens are covered by the other opens. Categorifying this insight, one is naturally led to the notion of a Grothendieck topos.

Definition 2.1.5.

(i) A (small) *Grothendieck site* $(\mathcal{C}, J)$ is a small category $\mathcal{C}$ equipped with a *Grothendieck topology* J, which (informally speaking) is a structure on the morphisms of $\mathcal{C}$ satisfying certain axioms that are also satisfied by the usual lattice of opens of a topological space.

(ii) A *(Grothendieck) topos* is a category $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ equivalent to the category of sheaves constructed on a small Grothendieck site $(\mathcal{C}, J)$.

The full details of the definition will not be needed, and have been suppressed (but see, e.g. [Joh77b, §0.3]); for this thesis, it suffices to simply view a topos $\mathcal{E}$ as a generalised category of sheaves. As an indication of the naturalness of Definition 2.1.5, we include Giraud's Theorem which (remarkably) characterises a topos entirely in terms of its intrinsic categorical properties.[13]

Theorem 2.1.6 (Giraud's Theorem, [Gir63])**.** *Let $\mathcal{E}$ be a category. Then, the following conditions are equivalent:*

(i) $\mathcal{E}$ is a Grothendieck topos.

(ii) $\mathcal{E}$ satisfies the following properties:

- *(a) $\mathcal{E}$ has finite limits.*
- *(b) $\mathcal{E}$ has all set-indexed coproducts, and they are disjoint and universal.*
- *(c) Every equivalence relation in $\mathcal{E}$ is effective & every epimorphism in $\mathcal{E}$ is a coequaliser.* } *$\mathcal{E}$ is an ∞-pretopos*
- *(d) Equivalence relations in $\mathcal{E}$ have universal coequalisers.*
- *(e) $\mathcal{E}$ has small hom-sets.*
- *(f) $\mathcal{E}$ has a set of generators.*

Convention 2.1.7 ("Topos")**.** For the expert reader:

- Unless stated otherwise, the unqualified term "topos" will always mean a Grothendieck 1-topos. This is in contrast to more general definitions, e.g. an elementary topos or an ∞-topos.
- Another generality: one sometimes defines a topos as a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ over a fixed elementary topos $\mathcal{S}$ with natural number object (nno) (e.g., as in [Joh02a]). Phrased in this language, the standing assumption of this thesis is that a Grothendieck topos is a bounded $\mathcal{S}$-topos where $\mathcal{S} = \mathrm{Set}$. Nonetheless, most of our work (with the possible exception of Chapter 5, which is slightly more classical) remains valid relative to any elementary topos $\mathcal{S}$ with nno.

2.1.2 Toposes as a Point-free Space. Although point-free topology pulls the classical notion of "space" away from its underlying set theory, we emphasise that working "point-free" does *not* mean working point*less*ly, i.e. without mentioning points at all. This is not obvious, and some work is required to show: (a) what is the correct notion of points in the present context; and (b) what are the acceptable vs. unacceptable ways of reasoning with these generalised points. An early indication of this insight can be found in Moerdijk's paper *The Classifying Topos of a Continuous Groupoid, I.* [Moe88], where (reworded slightly) he writes:

> "In presenting many arguments concerning generalized, 'point-free' spaces, I have tried to convey the idea that by using change-of-base techniques *and* exploiting the internal logic of a Grothendieck topos, point-set arguments are perfectly suitable for dealing with point-free spaces (at least as long as one stays within the 'stable' part of the theory)." [14]

[13] We remark that Giraud's characterisation of toposes seems particularly well-suited to proving various 2-categorical properties about the 2-category of Grothendieck toposes $\mathfrak{Top}$, e.g. regarding the existence of colimits [Moe88].

[14] The motivation is in the telling, and so we have taken the liberty to tweak some of the terminology to reduce confusion. In the original text, Moerdijk [Moe88, p. 629] uses the phrase "pointless spaces" instead of "point-free".

Moerdijk's approach was to illustrate by example; it was left implicit as to what the "stable" part of the theory actually was. For us, "staying within the 'stable' part of the theory" essentially means "working geometrically", a framework developed using the tight connection between toposes and geometric logic. This goal of this subsection is to develop and justify this remark.

Throughout this thesis, we shall frequently refer to certain constructions or properties as being geometric or non-geometric. In a strict sense, one may call something geometric if it can be characterised up to isomorphism using the structure and axioms of geometric theories.

Definition 2.1.8 (Geometric Logic)**.** Let Σ be a first-order signature of sorts, relations and functions. Then over Σ, we define the following:

- Each *term* t carries a specification of which sort it belongs to; this will be denoted as $t \in A$.
- An *atomic formula* is an expression ϕ built recursively from the following two clauses:
  (a) *Equality.* $t_1 = t_2$, if t_1 and t_2 are terms belonging to same sort.
  (b) *Relations.* $R(t_1, \ldots, t_n)$, if $R \rightarrowtail A_1 \cdots A_n$ is a relation symbol and $t_1 \in A_1, \ldots, t_n \in A_n$ are terms.
- Let $\vec{x} = (x_1, \ldots, x_n)$ be a (finite) vector of distinct variables, each with a given sort.

  A *geometric formula over* Σ *in context* $\vec{x}$ is a formula built from atomic formulae using truth $\top$, **finite** conjunctions $\wedge$, **arbitrary** (possibly infinite) disjunctions $\bigvee$, and $\exists$.
- A *geometric sequent* is an expression of the form

$$\forall x_1, \ldots x_n.(\phi \rightarrow \psi),$$

  where ϕ and ψ are geometric formulae in the same (finite) context $(x_1, \ldots, x_n)$.
- A *geometric theory* over Σ is a set $\mathbb{T}$ of geometric sequents. We call these sequents the *axioms* of $\mathbb{T}$.

Remark 2.1.9 (Geometric Syntax)**.** The general syntax of Definition 2.1.8 should be familiar to the model theorist, but let us also note the main differences with classical first-order logic. They are:

- The two-level distinction between formulas vs. sequents, which prevent nested implications or universal quantification in the axioms of a theory.
- The absence of negation $\neg$ in the connectives;[15]
- The allowance for possibly infinite disjunctions $\bigvee$.

Moving on to the semantics, how should we define a model of a geometric theory $\mathbb{T}$? An explicit definition using categorical semantics (e.g. following [Joh02b, D1.2]) is possible, but for this thesis, the following working definition will suffice.

Definition 2.1.10. By a *model of* $\mathbb{T}$, or a $\mathbb{T}$-*model*, we shall mean some mathematical structure M that satisfies the axioms of $\mathbb{T}$. The model M may live in the universe Set (i.e. M is a set satisfying the axioms

[15]This means that negation is absent from geometric formulae; however, negation can still be present in the sequents if expressed as an implication ("$\phi \rightarrow \bot$", where false can be represented as $\bot = \bigvee \emptyset$). This gives another way of reading the difference between geometric formulae vs. geometric sequents.

of $\mathbb{T}$, as is typically assumed by model theorists), or it may live in some other universe, let us say a topos $\mathcal{E}$.[16] We denote $\mathbb{T}\text{-mod}(\mathcal{E})$ to be the category of $\mathbb{T}$-models in the topos $\mathcal{E}$.

Following Definition 2.0.1, this gives an explicit characterisation of the points of a *point-free space*: namely, as mathematical structures for Σ that satisfy a fixed set of axioms. Of course, this account is still wanting in explanation — e.g. why should models even be regarded as points of a generalised space? — which will be provided in due course. For now, let us fix some important conventions.

Convention 2.1.11 ("Space")**.**

- All theories shall be assumed to be geometric, unless stated otherwise. If $\mathbb{T}$ denotes a geometric theory, we write $[\mathbb{T}]$ for the space of models of $\mathbb{T}$.
- In this thesis, the unqualified term "space" should be taken to mean a *point-free space* (i.e. a space of models for some $\mathbb{T}$) — if we wish to specify a point-set space, we shall be explicit about this.

We illustrate with a couple of examples. Example 2.1.12 takes Definition 2.1.10 at face value, and gives an illustration of how geometric logic works in practice. Example 2.1.13 is subtler, and gives an explicit account of the difference between the model theorist's vs. the topos theorist's notion of a "model".

Example 2.1.12 (Commutative Rings)**.** The usual algebraic laws of commutative rings (with 1) can be formulated as geometric axioms, yielding a geometric theory $\mathbb{T}_{\text{com}}$. For instance, declaring R as our sort, and including $0, 1, +, \cdot$ as the obvious function symbols in Σ, we may express the distributivity law as

$$\forall xyz \in R. (\top \to x \cdot (y + z) = (x \cdot y) + (x \cdot z)).$$

The space of $\mathbb{T}_{\text{com}}$-models includes *all* discrete commutative rings living in *all* toposes.[17] Standard examples of $\mathbb{T}_{\text{com}}$-models include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_p$ etc.[18]

Example 2.1.13 (Models of Classical Propositional Logic)**.** As our setup:

- Σ is a propositional signature (= no sorts);
- Sen_Σ denotes the set of sentences constructed over Σ using first-order classical logic;
- $\mathbb{T}_{\text{prop}}$ is a theory over Σ, i.e. $\mathbb{T}_{\text{prop}}$ is a subset of Sen_Σ.

By way of motivation, we recall the following discussion by Chang and Keisler. Rewritten in our notation:

[16] We sketch the full definition here but there are no real surprises. Start by defining a Σ-*structure* M in a category $\mathcal{C}$: this assigns to each sort "A" in Σ an object $[[A]]_M \in \mathcal{C}$, to each relation "$R \subseteq A_1 \times \cdots \times A_n$" a subobject $[[R]]_M \rightarrowtail [[A_1]]_M \times \cdots \times [[A_m]]_M$ in $\mathcal{C}$, and finally to each function "$f\colon A_1 \times \cdots \times A_n \to B$" a morphism $[[f]]_M \colon [[A_1]]_M \times \cdots \times [[A_n]]_M \to [[B]]_M$ in $\mathcal{C}$. Notice this categorifies the classical notion of a model of Σ as a pair $(M, \mathcal{I})$ where M is a set and $\mathcal{I}$ is an interpretation of Σ in M [CK90, §1.3]. Next, a formula "$\phi(\vec{x})$" of $\mathbb{T}$ in a context $\vec{x} = (x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$ is interpreted as a subobject $[[\phi]]_M \rightarrowtail [[A_1]]_M \times \cdots \times [[A_n]]_M$. Finally, the requirement that M satisfies the axiom "$\forall \vec{x} \in \vec{A}.(\phi \to \psi)$" is interpreted as an inclusion of subobjects $[[\phi]]_M \rightarrowtail [[\psi]]_M$. To summarise: when we say "a $\mathbb{T}$-model M is a structure living in a topos $\mathcal{E}$", we mean that M is a Σ-structure of $\mathcal{E}$ satisfying the axioms of $\mathbb{T}$ in the manner just described. For additional details, see [Joh02b, D1].

[17] The model theorist may wish to view our notion of "space" as a topos-theoretic generalisation of "elementary class".

[18] Warning: the Dedekind reals $(\mathbb{R}, +, \times)$ equipped with the usual addition and multiplication is *not* a $\mathbb{T}_{\text{com}}$-model. Classically, one is used to reasoning with the reals as if they were elements of a set (ignoring the topology); geometrically, however, one must always recognise the Dedekinds form a topological space, and so $(\mathbb{R}, +, \times)$ cannot be regarded as a discrete commutative ring.

"The first thing which comes to mind [on what a model might be] is a function F which associates with each simple statement S [from Sen_Σ] one of the truth values 'true' or 'false'. Stripping away the inessentials, we shall instead take a model to be a subset A of Sen_Σ; the idea is that $S \in A$ indicates that the simple statement S is true, and $S \notin A$ indicates that the simple statement S is false. [...] We shall say that A is a model of $\mathbb{T}_{\mathrm{prop}}$, $A \models \mathbb{T}_{\mathrm{prop}}$, iff every sentence $\phi \in \mathbb{T}_{\mathrm{prop}}$ is true in A." — [CK90, §1.2]

This quote, which may strike the modern model theorist as an anachronism, is interesting for a couple of reasons. Firstly, [CK90] notes that a model may be regarded as a special kind of set (a characterisation familiar to the model theorist[19]) *or*, equivalently, as a function into an object of truth values (a characterisation more familiar to the topos theorist). Secondly, both candidate definitions of a model by [CK90] are implicitly classical. In particular, they assume there exists only two truth values ("true" or "false"), which sets up a subsequent appeal to the Law of Excluded Middle ("ϕ is either true or false in A"). While this may be valid in classical logic, the situation is subtler in topos theory. In every topos $\mathcal{E}$, there exists a so-called *subobject classifier* Ω, which functions as our object of truth values.[20] Importantly, although $\Omega \cong \{0, 1\}$ holds in certain toposes (e.g. Set), the isomorphism does *not* hold in a general topos (see Footnote 23).

Where does this leave us? Although the equivalence between "$\{\in, \notin\}$" and the object of truth values does not extend to our setting, we can still define a model as a function that evaluates the truth of the relevant propositions.

We follow [Vic07b]. Let B be a Boolean algebra. We define a *Σ-structure in B* to be a function

$$M \colon \Sigma \to B.$$

It is clear this extends uniquely to a function

$$\overline{M} \colon \mathrm{Sen}_\Sigma \to B$$

where the connectives are evaluated via the corresponding operations on B, e.g. $\phi \wedge \psi \mapsto M(\phi) \wedge M(\psi)$. We then define a *$\mathbb{T}_{\mathrm{prop}}$-model in B* to be a Σ-structure such that $\overline{M}_{\mathbb{T}_{\mathrm{prop}}}(\phi) = 1$ for all $\phi \in \mathbb{T}_{\mathrm{prop}}$.

In particular, notice:

- The standard Set-based models are recovered when we take $B = \{0, 1\}$.
- Unlike [CK90], we distinguish between a Σ-structure vs. a model of a theory $\mathbb{T}_{\mathrm{prop}}$ over Σ, instead of calling both structures "models".[21]

[19] Although, perhaps, not quite in the same way as presented here: model theorists typically regard a model $(M, \mathcal{I})$ as a set M equipped with an interpretation $\mathcal{I}$ of the signature Σ. For details, see e.g. [CK90, §1.3] or [Mar02, Ch. 1]).

[20] Why is this? Recall that the subobject classifier Ω of a topos $\mathcal{E}$ can be characterised as the power object of the terminal object $\mathbf{1}$ in $\mathcal{E}$, and so the subobjects of A in $\mathcal{E}$ are equivalent to their characteristic morphisms $A \to \Omega$ [Joh02a, A2]. Rephrased in the internal logic of $\mathcal{E}$ (cf. Footnote 16), this means that the question of, e.g. whether a formula ϕ of theory $\mathbb{T}$ is satisfied by some $\mathbb{T}$-model M now becomes a question of whether its corresponding characteristic morphism $\chi_\phi \colon [[A_1]]_M \times \cdots \times [[A_n]]_M \to \Omega$ maps to $\mathbf{1} \in \Omega$. Compare this with this example's definition of $\overline{M}_{\mathbb{T}_{\mathrm{prop}}}$.

[21] This signals a difference in priority analogous to the difference in priority between point-set vs. point-free topology. Whereas a topos theorist always defines a model to be a structure satisfying the axioms of a chosen theory $\mathbb{T}$, Chang and Keisler [CK90, §1.2 - 1.3] simply define a model M to be a structure that interprets some signature Σ — in which case, the fact that such an M may be considered a model of a theory becomes an *a posteriori* judgement. Interestingly, traces of this attitude are still visible in modern model theory, even though many are now more careful to distinguish terminologically between "structures" vs. "models" (see e.g. [Mar02; Kir19]). Consider, for instance, the well-known construction $\mathrm{Th}(M)$, which starts with a Σ-structure M before defining its theory ("$\mathrm{Th}(M)$") as the set of Σ-sentences satisfied by M. By contrast, $\mathrm{Th}(M)$ is not a standard construction in topos theory, likely due to complications arising from the incompleteness of geometric logic (see Footnote 31).

- In analogy with how models might live in different toposes (cf. Definition 2.1.10), our definition of a model here similarly allows for non-standard models of $\mathbb{T}_{\text{prop}}$ in Boolean Algebras other than $B = \{0, 1\}$.[22] Of course, by completeness of classical first-order logic it suffices to consider just the standard models, but the availability of non-standard models will become important once we move into incomplete logics (cf. Discussion 2.1.25).

Discussion 2.1.14 (Decidability and Topology)**.** Let ϕ be a formula definable in $\mathbb{T}$, and $M_{\mathbb{T}}$ be a model of $\mathbb{T}$. Classically, the Law of Excluded Middle automatically gives $M_{\mathbb{T}} \models \phi \vee \neg\phi$. However, in geometric logic, we are unable to even express this principle as stated (much less affirm its validity) since our syntax lacks negation. We must therefore look for alternative formulations.

In our setting, the validity of non-constructive principles (such as LEM) gets reframed as a question of topology. To illustrate, suppose ϕ is a propositional formula and denote $[\mathbb{T}]$ to be the space of models of $\mathbb{T}$. In point-free topology, we regard "the collection of models in $[\mathbb{T}]$ satisfying ϕ" as an open subspace of $[\mathbb{T}]$. Hence, the question of whether the axiom

$$\top \to \phi \vee \neg\phi$$

holds in $\mathbb{T}$ should properly be understood as asking if ϕ has an *open* Boolean complement in $[\mathbb{T}]$. We call ϕ *decidable* just in case the Boolean complement of ϕ is also open (and so ϕ is clopen).[23] As we shall later see, in situations where a desired property is not decidable, this often creates subtleties when trying to prove analogues of classical results in our setting. For more details on decidability, see [Joh02a, A1.4].

So far, we've treated $[\mathbb{T}]$ as a black box: it denotes the space of *all* models of $\mathbb{T}$ in *all* relevant universes, but we've not defined what $[\mathbb{T}]$ is mathematically. For us, $[\mathbb{T}]$ actually corresponds to a (Grothendieck) topos [more correctly, the points of the classifying topos of $\mathbb{T}$]. This requires some explanation. The standard Definition 2.1.5 says: a topos is a category of sheaves constructed over a Grothendieck site $(\mathcal{C}, J)$. To see its connection with logic, we shall need the fundamental notion of a *geometric morphism*.

Definition 2.1.15. Let $\mathcal{E}, \mathcal{F}$ be toposes. A *geometric morphism* $f\colon \mathcal{E} \to \mathcal{F}$ is a pair of adjoint functors

$$\mathcal{E} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{F}$$

such that f^* preserves **finite** limits and **arbitrary** colimits. We call f^* the *inverse image functor* of f, and denote $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$ to be the category of geometric morphisms $f\colon \mathcal{E} \to \mathcal{F}$.

[22] For the model theorist with a good background in set theory: the models defined in this example should be compared with the "Boolean-valued models" presented in, e.g. [Jec06, Ch. 14]. To understand how Boolean-valued models may play a serious role in the classification of first-order theories, see for instance Ulrich's work on Keisler's Order [Ulr18].

[23] What does this mean in the internal logic of the topos $\mathcal{E}$? Recall (e.g. from Footnote 20): the object of truth values in $\mathcal{E}$ is its subobject classifier Ω, which is the power object of $\mathbf{1}$ in $\mathcal{E}$. Recall also: in the internal logic of $\mathcal{E}$, a formula ϕ gets interpreted as a characteristic morphism into Ω. Since $\mathbf{1}$ may be regarded as a singleton, this means that the truth value of ϕ in $\mathcal{E}$ is a subobject of a singleton. This prompts the natural question: what exactly *are* the subobjects of $\mathbf{1}$?

The answer, of course, depends on the topos. In the topos Set, it is clear that the only subsets of the singleton $\mathbf{1}$ are just $\{\emptyset, \mathbf{1}\}$, and so we deduce $\Omega \cong \{0, 1\}$, i.e. we only have "true" or "false" as truth values (cf. Example 2.1.13). In a general sheaf topos, however, there may exist other subsheaves of $\mathbf{1}$ and so the logic becomes more subtle. Nonetheless, it is well-known that the subobjects of $\mathbf{1}$ for any $\mathcal{E}$ form a frame (cf. Definition 2.1.2). In the context of point-free topology, this suggests that ϕ should be regarded as an *open subspace* of $[\mathbb{T}]$ whose models satisfy ϕ (as claimed). It also signals to us that the complement of ϕ should be regarded as a closed subspace $[\mathbb{T}]$, which is generally not open (and thus does not show up as a truth value in Ω). Nonetheless, just in case the truth value of ϕ in Ω is either $\emptyset$ or $\mathbf{1}$ itself (which are trivially clopen), then ϕ is *decidable* in the sense just described.

Discussion 2.1.16 (Geometric Morphism = Generalised Continuous Map)**.** Two key points:

(i) The finite limits and arbitrary colimits in Definition 2.1.15 should be understood as corresponding to the finite conjunctions and arbitrary disjunctions we saw in Definition 2.1.8;

(ii) The fact that the inverse image functor f^* of a geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ preserves this geometric structure is analogous to the pre-image f^{-1} of an ordinary continuous map $f\colon X \to Y$ preserves the opens of Y. In particular, recalling our distinction between point-set vs. point-free topology, note that the pre-image f^{-1} is generally not well-defined on the set of *elements* of Y but it is well-defined on its lattice of *opens*.

Item (ii) justifies the perspective that a geometric morphism is a generalised continuous map. Combined with (i), this justifies the view that geometric logic possesses an intrinsic continuity.

Convention 2.1.17 ("Geometricity")**.** Discussion 2.1.16 suggests a less syntactic notion of what it means to work geometrically. In this thesis, we shall refer to a construction as "geometric" if it is preserved by inverse image functors (or equivalently, if it is preserved by pullback along geometric morphisms) — essentially, if it is constructed from finite limits and arbitrary colimits.[24]

Discussion 2.1.16 is significant because it highlights how geometric morphisms possess both a logical *and* a topological character. Leveraging this insight, we can now formulate other key notions (e.g. points, classifying toposes) and structure theorems to establish the topos-theoretic basis for the point-free perspective.

Definition 2.1.18 (Points = Generalised Maps)**.** Let $\mathcal{E}, \mathcal{F}$ be toposes.

(i) We define a *generalised point* of $\mathcal{F}$ to be a geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ whose codomain is $\mathcal{F}$.

(ii) In the special case where $f\colon \mathrm{Set} \to \mathcal{F}$, we call f a *global point* of $\mathcal{F}$.

For more details justifying this choice of terminology, see [Vic07b].

Definition 2.1.19. A *classifying topos* of a geometric theory $\mathbb{T}$ is a topos $\mathcal{S}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in the following sense — for any topos $\mathcal{E}$, we have the equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E}),$$

natural in $\mathcal{E}$. That is, for any geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$, we have a commutative square up to natural isomorphism

$$\begin{array}{ccc} \mathbf{Geom}(\mathcal{F}, \mathcal{S}[\mathbb{T}]) & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{F}) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle f^*(-)} \\ \mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{E}) \end{array}$$

In particular, notice: the global points of $\mathcal{S}[\mathbb{T}]$ correspond to the standard models of $\mathbb{T}$ in Set.

Theorem 2.1.20. *Every Grothendieck topos $\mathcal{E}$ is a classifying topos of some geometric theory $\mathbb{T}_{\mathcal{E}}$. Conversely, every geometric theory $\mathbb{T}$ is classified by a topos $\mathcal{S}[\mathbb{T}]$.*

[24] A side-note: there are certain constructions, e.g. frames, which are technically not geometric but can be given a presentation which *are* geometric. Such constructions will continue to play a role in geometric mathematics – see. e.g. our use of Moerdijk's Stability Theorem 6.1.13 in Chapter 6.

Sketch of Proof. Let $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ be a topos. By Diaconescu's Theorem and the fact that any topos is a subtopos of a presheaf topos, deduce that $\mathcal{E}$ classifies the geometric theory of J-continuous flat functors from $\mathcal{C}$ (see [LM94, §VII.7] or [Car17, Theorem 2.1.11]). Conversely, given any geometric theory $\mathbb{T}$, one can construct its syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. It can then be verified that $\mathcal{S}[\mathbb{T}] := \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, the category of sheaves on this site, is a classifying topos of $\mathbb{T}$ (see [Joh02b, D3.1.12]). □

We now explain the topos theory underlying Definition 2.0.1.

Discussion 2.1.21 (Topos = Generalised Space)**.** Theorem 2.1.20 says the following: since any geometric theory $\mathbb{T}$ has a classifying topos $\mathcal{S}[\mathbb{T}]$, its logical data (in terms of $\mathbb{T}$-models) is exactly encoded by topological data (in terms of maps into $\mathcal{S}[\mathbb{T}]$).[25] Combined with Definitions 2.1.18 and 2.1.19, this means the models of $\mathbb{T}$ correspond to the points of $\mathcal{S}[\mathbb{T}]$, justifying the view that the universe of all the models of $\mathbb{T}$ is a generalised space.

Convention 2.1.22 (Notation: $[\mathbb{T}]$ vs. $\mathcal{S}[\mathbb{T}]$)**.** We were slightly loose in our language when we said "$[\mathbb{T}]$ corresponds to a topos", but were still careful to distinguish between $\mathcal{S}[\mathbb{T}]$ and $[\mathbb{T}]$. Why? As should be clear from examining definitions, $\mathcal{S}[\mathbb{T}]$ denotes an actual topos (indeed, the classifying topos of $\mathbb{T}$) *whereas* $[\mathbb{T}]$ denotes the space of points belonging to $\mathcal{S}[\mathbb{T}]$. Still, in light of Discussion 2.1.21, one can view $\mathcal{S}[\mathbb{T}]$ and $[\mathbb{T}]$ as representing two equivalent perspectives on what a topos "is", each supporting different intuitions. Again, we will not need the technical definition of $[\mathbb{T}]$ — it suffices for this thesis to view it as some meta-universe containing all the $\mathbb{T}$-models — but for the curious reader, see [Joh02a, B4.2].[26] For a deeper discussion on this dual perspective of toposes, we recommend [AJ21].

A crucial fact in geometric mathematics is the existence of a so-called generic model of our theories:

Fact 2.1.23 (The Generic Model)**.** For every theory $\mathbb{T}$, there exists a model known as the *generic model* of $\mathbb{T}$, which we denote as $G_{\mathbb{T}}$, that lives in the classifying topos $\mathrm{Set}[\mathbb{T}]$. It is "generic" in the sense that it has no other geometric properties[27] other than being a model of $\mathbb{T}$ [more precisely, the geometric sequents valid in $G_{\mathbb{T}}$ are precisely those provable in $\mathbb{T}$].

The generic model is essentially a syntactic construction, and plays a key role in the logical study of toposes. Although well-known to topos theorists, the construction will be unfamiliar to many classical logicians working in Set, e.g. the model theorists. For those interested in an explicit description of the generic model, see Example 2.1.24. Otherwise, the reader may wish to move on to Discussion 2.1.25, which gives a more conceptual perspective.

Example 2.1.24. We continue with the setup of Example 2.1.13. Let us recall:

- $\mathbb{T}_{\text{prop}}$ denotes a classical propositional theory over its signature Σ;
- A *model of* $\mathbb{T}_{\text{prop}}$ *in a Boolean algebra* B is defined as a function $M\colon \Sigma \to B$ such that $\overline{M}(\phi) = 1$ for all $\phi \in \mathbb{T}_{\text{prop}}$.

[25] For the algebraic topologist: this is analogous to how vector bundles over finite CW complexes (geometric data) are classified by maps into the classifying space (homotopical data) [Hat, Theorem 1.16].

[26] More explicitly, one can view $[\mathbb{T}]$ as a pseudo-functor $[\mathbb{T}]\colon \mathfrak{Top}^{\text{op}} \to \mathfrak{CAT}$ that assigns to each topos $\mathcal{E}$ a category $\mathbb{T}(\mathcal{E})$, whose objects are the $\mathbb{T}$-models in $\mathcal{E}$, with the topos $\mathcal{S}[\mathbb{T}]$ as the representing object for the pseudo-functor. Put otherwise, $[\mathbb{T}]$ is a $\mathfrak{Top}$-indexed category cataloguing all the points of $\mathcal{S}[\mathbb{T}]$.

[27] Note: this holds true only for *geometric* properties. In particular, $G_{\mathbb{T}}$ may satisfy non-geometric properties not shared by other models. See [Joh77a, §3] for an interesting discussion on the sense in which the generic non-trivial ring can be regarded as a field.

Notice that our definition of a model works for any Boolean algebra B. In particular, consider the Lindenbaum Algebra $\mathcal{LA}(\Sigma, \mathbb{T}_{\text{prop}})$ for $\mathbb{T}_{\text{prop}}$, i.e. the set of sentences Sen_Σ quotiented by the equivalence relation $\mathbb{T}_{\text{prop}} \vdash \phi \leftrightarrow \psi$. We then define the *generic model* $G_{\mathbb{T}_{\text{prop}}}$ of $\mathbb{T}_{\text{prop}}$ to be a $\mathbb{T}_{\text{prop}}$-model in the Lindenbaum algebra that interprets each propositional symbol $P \in \Sigma$ as the equivalence class of P as a sentence.[28] In particular, it possesses a universal property: any model $M_{\mathbb{T}_{\text{prop}}}$ can be obtained by applying some Boolean algebra homorphism $f_{M_{\mathbb{T}_{\text{prop}}}}$ to the generic model:

$$\begin{array}{ccc} \Sigma & \xrightarrow{G_{\mathbb{T}_{\text{prop}}}} & \mathcal{LA}(\Sigma, \mathbb{T}_{\text{prop}}) \\ & \searrow^{M_{\mathbb{T}_{\text{prop}}}} & \downarrow f_{M_{\mathbb{T}_{\text{prop}}}} \\ & & B \end{array}$$

For additional context, see [Vic07b, §2.1]. For details on how to generalise this construction to the setting of geometric theories and toposes, see [Joh02b, D1.4].[29]

Discussion 2.1.25. Let us highlight a few key aspects of the generic model.

(i) *$G_{\mathbb{T}}$ as the universal model of* $\mathbb{T}$. The question of universality asks: given a theory $\mathbb{T}$, does there exist a “nice” model from which we can obtain all other $\mathbb{T}$-models? In model theory, a universal model is typically some suitably large model into which all models (of bounded size) embed as elementary substructures — e.g. the monster model, the homogeneous universal model of the Fraïssé Limit, etc. In topos theory, we work inversely by starting with the “smallest” possible model instead.

Let us elaborate. By Theorem 2.1.20, any $\mathbb{T}$-model living in any topos $\mathcal{E}$ corresponds to a geometric morphism $f\colon \mathcal{E} \to \mathcal{S}[\mathbb{T}]$. For the generic model $G_{\mathbb{T}}$, this corresponds to the identity morphism

$$\text{id}_{\mathcal{S}[\mathbb{T}]}\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}].$$

For an arbitrary $\mathbb{T}$-model $M_{\mathbb{T}}$ in topos $\mathcal{E}$, with corresponding geometric morphism $f\colon \mathcal{E} \to \mathcal{S}[\mathbb{T}]$, Definition 2.1.19 of the classifying topos gives the following diagram:

$$\begin{array}{ccc} \text{id}_{\mathcal{S}[\mathbb{T}]} \in \mathbf{Geom}(\mathcal{S}[\mathbb{T}], \mathcal{S}[\mathbb{T}]) & \overset{\sim}{\longrightarrow} & \mathbb{T}\text{-mod}(\mathcal{S}[\mathbb{T}]) \ni G_{\mathbb{T}} \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle f^*(-)} \\ f \in \mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) & \overset{\sim}{\longrightarrow} & \mathbb{T}\text{-mod}(\mathcal{E}) \ni f^*(G_{\mathbb{T}}) \end{array}$$

[28] Why does this define a model? Note: by definition of the Lindenbaum Algebra, $\mathbb{T}_{\text{prop}} \vdash \phi \leftrightarrow \top$ for any $\phi \in \mathbb{T}_{\text{prop}}$, and so $\overline{G}_{\mathbb{T}_{\text{prop}}}(\phi) = 1$.

[29] We sketch the construction here. Given a geometric theory $\mathbb{T}$, start by constructing the generic model $G_{\mathbb{T}}$ in its syntactic category $C_{\mathbb{T}}$ (see [Joh02b, pp. 844-845]). Next, construct the topos $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$, where $(C_{\mathbb{T}}, J_{\mathbb{T}})$ is the syntactic site. As it turns out, $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$ is also the classifying topos of $\mathbb{T}$ (up to equivalence), and the generic model living in $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$ can thus be obtained as the image of $G_{\mathbb{T}}$ in the syntactic category $C_{\mathbb{T}}$ under the Yoneda embedding $C_{\mathbb{T}} \hookrightarrow \mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$.

Two natural questions. First, if the generic model already exists in $C_{\mathbb{T}}$, why do we embed it into a topos $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$? Answer: because constructing the sheaves over $(C_{\mathbb{T}}, J_{\mathbb{T}})$ introduces the colimits that were absent in original syntactic category $C_{\mathbb{T}}$. This richer structure allows us to e.g. geometrically define equivalence relations (cf. Giraud's Theorem 2.1.6). The model theorist should understand this embedding as being analogous to the elimination of imaginaries (for details, see [Har11]). Second, why do we work with $C_{\mathbb{T}}$ instead of the Lindenbaum Algebra of $\mathbb{T}$? Answer: the Lindenbaum Algebra defined for a general (= not necessarily propositional) theory $\mathbb{T}$ is the poset of provable-equivalence classes of formulae *in the empty context*. In general, when constructing the generic model of $\mathbb{T}$, we will also need to account for the formulae occurring in non-empty contexts, hence our choice to work in the syntactic category. Nonetheless, in the case where $\mathbb{T}$ is propositional (and so the context is empty by default), then this is no longer a problem — indeed, the Lindenbaum Algebra of $\mathbb{T}$ and $C_{\mathbb{T}}$ turn out to be equivalent [Joh02b, Remark D1.4.14].

In English, this diagram says: any $\mathbb{T}$-model $M_{\mathbb{T}}$ can be represented as $M_{\mathbb{T}} \cong f^*(G_{\mathbb{T}})$, where f^* is a functor preserving colimits and finite limits (and thus all geometric constructions, cf. Convention 2.1.17).

(ii) *$G_{\mathbb{T}}$ generally does not live in* Set. We emphasise that $G_{\mathbb{T}}$ lives in the classifying topos $\mathcal{S}[\mathbb{T}]$, which in general is **not** Set. This gives another way of reading the difference between $G_{\mathbb{T}}$ and, say the homogeneous universal model of the Fraïssé construction — despite their family resemblance, the latter lives in Set whereas the former generally does not.[30]

(iii) *Incompleteness of Infinitary Logics.* Like many other infinitary logics, geometric logic is incomplete, i.e. not all consistent sets of geometric sequents are satisfied by a standard Set-based model.[31] Nonetheless, we claim that this is not a deficiency of geometric logic, but rather a deficiency of Set to provide us with enough models. This follows from recalling: (a) the geometric sequents valid in $G_{\mathbb{T}}$ are precisely those provable in $\mathbb{T}$, essentially by construction; and (b) $G_{\mathbb{T}}$ generally lives in a topos that is *not* Set. Put together, one sees that geometric logic is complete *once* we consider $\mathbb{T}$-models in *all* toposes, justifying our generalisation of models in Definition 2.1.10.

Reformulating item (i) of Discussion 2.1.25 in the language of point-free topology, we obtain the following key principle.

Discussion 2.1.26 (Maps as Point-Transformers)**.** Consider a *map* $f\colon [\mathbb{T}] \to [\mathbb{T}']$, as described in Definition 2.0.1. To define it, we declare "let x be a point of $[\mathbb{T}]$", and then work geometrically to construct a point $f(x)$ of $[\mathbb{T}']$. In the particular case where x is the generic model $G_{\mathbb{T}}$, we get a geometric construction $f(G_{\mathbb{T}})$ in $\mathcal{S}[\mathbb{T}]$. Since geometric constructions are precisely those which are preserved by inverse image functors of geometric morphism, and since all $\mathbb{T}$-models $M_{\mathbb{T}}$ can be represented as $M_{\mathbb{T}} \cong g^*(G_{\mathbb{T}})$ for some appropriate geometric morphism $g\colon \mathcal{E} \to \mathcal{S}[\mathbb{T}]$, it follows that the generic construction suffices to describe all the instances for more specific points of $[\mathbb{T}]$. We thus see that the generic point $G_{\mathbb{T}}$ plays the role of *formal parameter* x in the definition of $f(x)$, and actual parameters are substituted by transporting constructions along the functors.

To summarise: a map $f\colon [\mathbb{T}] \to [\mathbb{T}']$ defines a point $f(G_{\mathbb{T}})$ of $[\mathbb{T}']$, constructed geometrically in $\mathcal{S}[\mathbb{T}]$. But that is in turn equivalent to a functor $f^*\colon \mathcal{S}[\mathbb{T}'] \to \mathcal{S}[\mathbb{T}]$ – note the reversal of direction – that preserves colimits and finite limits, and takes $G_{\mathbb{T}'}$ to $f(G_{\mathbb{T}})$. From preservation of colimits we can get a right adjoint $f_*\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}']$, and we have arrived at the usual definition of geometric morphism. This shows that our point-free maps do in fact correspond to geometric morphisms, as the reader may have already anticipated by Discussion 2.1.16.

Convention 2.1.27. We shall often suppress the logical notation and write:

- "$x \in X$" to mean that x is a point of a point-free space X (i.e. a model of some theory)
- "$f\colon X \to Y$" to mean that f is a map that transforms points $x \in X$ to points $f(x) \in Y$, in the sense described in Discussion 2.1.26.

This suggestive convention is meant to highlight the topological character of our framework. Nonetheless, a couple of warnings. The reader should not take "$x \in X$" to mean that x denotes an element of the underlying set of X. Also, $f\colon X \to Y$ expresses how a geometric morphism acts as a map on the *points*

[30] One can, however, use the machinery of classifying toposes to reinterpret the Fraïssé Construction [Car14]. Let us also remark that there is a deep connection between the generic model and the set-theoretic notion of forcing [Šče84].

[31] A classic example: the theory of all surjections from the natural numbers $\mathbb{N}$ to the reals $\mathbb{R}$, which obviously has no Set-based models yet has non-trivial models in other toposes. For details, see e.g. [Ble20, §1.1].

of a generalised space; this action should not be confused with that of either of the adjoint functors f^*, f_*, which act on the *objects* of a topos instead (cf. Remark 2.1.22).

This completes our discussion of Definition 2.0.1. We end this section with three important structure theorems. These can be read as examples of the theme: "Topology, category theory and logic interact nicely with each other in the context of point-free topology".

The first result elaborates on our claim in Remark 2.1.22 that $\mathcal{S}[\mathbb{T}]$ and $[\mathbb{T}]$ represent dual perspectives on how to think about a topos. We summarise the result with the following slogan: two toposes are equivalent as categories iff they are equivalent as point-free spaces.[32]

Proposition 2.1.28. *Let $\mathcal{E}, \mathcal{F}$ be toposes. Then $\mathcal{E} \simeq \mathcal{F}$ iff for any topos $\mathcal{W}$, there exists an equivalence of categories $\textbf{\textit{Geom}}(\mathcal{W}, \mathcal{E}) \simeq \textbf{\textit{Geom}}(\mathcal{W}, \mathcal{F})$, natural in $\mathcal{W}$.*

Proof. Suppose $\mathcal{E} \simeq \mathcal{F}$ are equivalent as categories, induced by functors

$$\mathcal{E} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{F}$$

so that $GF \cong 1_{\mathcal{E}}$ and $FG \cong 1_{\mathcal{F}}$. Without loss of generality, assume F, G are an adjoint equivalence and fix some topos $\mathcal{W}$. Since adjunctions compose, any geometric morphism $x \colon \mathcal{W} \to \mathcal{E}$ whereby $x = (x^* \dashv x_*)$ can be extended to a new pair of adjoint functors

$$\mathcal{W} \underset{x^*G}{\overset{Fx_*}{\rightleftarrows}} \mathcal{F} \ .$$

Since F, G preserve all limits and colimits, $(x^*G \dashv Fx_*)$ defines a geometric morphism, which we denote as $\widetilde{x} \colon \mathcal{W} \to \mathcal{F}$. Similarly, given a geometric morphism $y \colon \mathcal{W} \to \mathcal{F}$ whereby $y = (y^* \dashv y_*)$, one can extend it to a geometric morphism $\overline{y} \colon \mathcal{W} \to \mathcal{E}$ whereby $\overline{y} = (y^*F \dashv Gy_*)$. It is clear by inspection that these two constructions are inverse to each other, showing that the functors F, G induce an equivalence $\textbf{Geom}(\mathcal{W}, \mathcal{E}) \simeq \textbf{Geom}(\mathcal{W}, \mathcal{F})$. In fact, the equivalence is natural in $\mathcal{W}$ since it is preserved by any geometric morphism between two toposes $f \colon \mathcal{W}' \to \mathcal{W}$.

Conversely, given any topos $\mathcal{W}$, suppose we have an equivalence $\textbf{Geom}(\mathcal{W}, \mathcal{E}) \simeq \textbf{Geom}(\mathcal{W}, \mathcal{F})$, natural in $\mathcal{W}$. We claim that this equivalence is induced by composition of geometric morphisms

$$\mathcal{E} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{F}$$

[Why? By hypothesis, we know that $\textbf{Geom}(\mathcal{E}, \mathcal{E}) \simeq \textbf{Geom}(\mathcal{E}, \mathcal{F})$. For explicitness, suppose the equivalence is induced by some functor

$$F \colon \textbf{Geom}(\mathcal{E}, \mathcal{E}) \longrightarrow \textbf{Geom}(\mathcal{E}, \mathcal{F}).$$

[32] An aside: this result was already observed in [Joh02b, p.850], but in a different language: "We define *Morita equivalence* for [geometric] theories to mean equivalence of categories $\mathcal{G}_{\mathbb{T}}$ rather than of the syntactic categories themselves. Of course, this is a weaker notion than equivalence of the syntactic categories but it suffices to ensure that the theories have equivalent model categories in any (cocomplete) topos, which is our main concern. (We shall see in 3.1.12 and 3.3.8 below that the converse is true)." Nonetheless, for clarity, we opt to give a direct proof of the result here rather than bringing in other structure theorems.

Denote

$$f := F(\mathrm{id}_{\mathcal{E}})\colon \mathcal{E} \to \mathcal{F}$$

to be the geometric morphism associated to $\mathrm{id}_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E}$ under F. Then, given any $x\colon \mathcal{W} \to \mathcal{E}$, naturality yields

$$\begin{array}{ccc} \mathrm{id}_{\mathcal{E}} \in \mathbf{Geom}(\mathcal{E},\mathcal{E}) & \xrightarrow{\quad F \quad} & \mathbf{Geom}(\mathcal{E},\mathcal{F}) \ni f \\ \Big\downarrow{\scriptstyle -\circ x} & & \Big\downarrow{\scriptstyle -\circ x} \\ x \in \mathbf{Geom}(\mathcal{W},\mathcal{E}) & \xrightarrow{\quad \overline{F} \quad} & \mathbf{Geom}(\mathcal{W},\mathcal{F}) \ni \overline{F}(x) \cong f \circ x \end{array}.$$

Put otherwise, naturality implies that the induced functor $\overline{F}\colon \mathbf{Geom}(\mathcal{W},\mathcal{E}) \xrightarrow{\sim} \mathbf{Geom}(\mathcal{W},\mathcal{F})$ acts by composition with f. Since $\overline{F}$ induces an equivalence, it has an inverse $\overline{G}\colon \mathbf{Geom}(\mathcal{W},\mathcal{F}) \to \mathbf{Geom}(\mathcal{W},\mathcal{E})$ which (by the same argument) also acts by composing with some $g\colon \mathcal{F} \to \mathcal{E}$.]

A straightforward check then verifies that f, g induce an equivalence $\mathcal{E} \simeq \mathcal{F}$. [Why? Since $\overline{F}, \overline{G}$ are inverse to each other, this means

$$\overline{G}(\overline{F}x) \cong x, \qquad \text{for all points } x\colon \mathcal{W} \to \mathcal{E}.$$

Since $\overline{F}, \overline{G}$ act by composition, this means

$$gfx \cong x, \qquad \text{for all points } x\colon \mathcal{W} \to \mathcal{E}.$$

In particular, this yields

$$gf \cong \mathrm{id}_{\mathcal{E}}.$$

The same argument yields $fg \cong \mathrm{id}_{\mathcal{F}}$. Put together, conclude that $\mathcal{E} \simeq \mathcal{F}$, as claimed.]

□

The second result makes explicit the logical meaning of "Let X be a subspace of $[\mathbb{T}]$".

Definition 2.1.29. Let $\mathbb{T}$ be a geometric theory over a signature Σ and $\mathcal{S}[\mathbb{T}]$ be its classifying topos.

(i) A *quotient* of $\mathbb{T}$ is a syntactic equivalence class of the geometric theories $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$. Informally: a quotient of $\mathbb{T}$ is the theory $\mathbb{T}$ but with additional axioms (modulo provable equivalence).

(ii) A *subtopos* of $\mathcal{S}[\mathbb{T}]$ is an equivalence class of geometric inclusions $\mathcal{E} \hookrightarrow \mathcal{S}[\mathbb{T}]$. A *subspace* of $[\mathbb{T}]$ corresponds to the space of points of a subtopos of $\mathcal{S}[\mathbb{T}]$.

Theorem 2.1.20 leads us to expect that quotients and subspaces ought to correspond to each other. The following proposition justifies this hunch:

Proposition 2.1.30 (Quotients of $\mathbb{T}$ = Subspaces of $[\mathbb{T}]$)**.** *Let $\mathbb{T}$ be a geometric theory. Then the quotients of $\mathbb{T}$ correspond bijectively to the subspaces of $[\mathbb{T}]$ (up to equivalence).*

Proof. First, we claim that subtoposes of $\mathcal{S}[\mathbb{T}]$ are in bijection with the quotients of $\mathbb{T}$. The case when $\mathbb{T}$ is propositional can be found in [Vic89, Prop. 6.2.2]; this was later[33] generalised by [Car17, Theorem 3.2.5] to any geometric theory $\mathbb{T}$. The result then follows from Proposition 2.1.28, which holds that two subtoposes of $\mathcal{S}[\mathbb{T}]$ are equivalent iff they are equivalent as spaces. □

[33] Another aside: [Joh02a, Example B4.2.8(i)] also proves that subtoposes of a classifying topos $\mathcal{S}[\mathbb{T}]$ correspond to the quotients of a geometric theory $\mathbb{T}$, except that Johnstone uses a rather non-syntactic notion of a geometric theory whereas in [Car17, Theorem 3.2.5], a quotient of $\mathbb{T}$ is required to have the *same* signature as $\mathbb{T}$, and so gives a clearer semantic-syntax correspondence.

The third result works out the point-free perspective on what it means to work over (or "in") a topos, accomplished via the language of bundles.

Definition 2.1.31. A *bundle*

$$\begin{array}{c} \Sigma_{x\in[\mathbb{T}]}[\mathbb{U}(x)] \\ \big\downarrow{\scriptstyle p} \\ [\mathbb{T}] \end{array}$$

is defined by a *geometric* construction of geometric theories $\mathbb{U}(x)$ (i.e. the *fibre* over x) out of arbitrary points x of $[\mathbb{T}]$. Here the *bundle space* corresponds to a theory that extends $\mathbb{T}$ with the ingredients of $\mathbb{U}(x)$: a point is a pair (x, y) where x is a point of $[\mathbb{T}]$ and y is a point of $[\mathbb{U}(x)]$. As a map, p acts by model reduction — it forgets y.

Remark 2.1.32. The language of "bundles" (Definition 2.1.31) and "maps" (Definition 2.0.1) give two topological perspectives on how geometric constructions work. Two salient points of comparison:

(i) To define a map $f\colon [\mathbb{T}] \to [\mathbb{T}']$ of spaces, we start with a point x of $[\mathbb{T}]$ before geometrically constructing a point $f(x)$ of $[\mathbb{T}']$; notice, in particular, that the formal parameter x lives in the domain space $[\mathbb{T}]$. For a bundle $p\colon (\Sigma_{x\in[\mathbb{T}]}[\mathbb{U}(x)]) \to [\mathbb{T}]$, the converse is true: we start with $x \in [\mathbb{T}]$ in the codomain as our parameter, and define a family of spaces $\Sigma_{x\in[\mathbb{T}]}[\mathbb{U}(x)]$ constructed from x.

(ii) Notice: in both cases, working point-free dispenses with the need for continuity proofs. To motivate, recall that in point-set topology, one typically defines ...

- ... a *map* $f\colon X \to Y$ by first defining f as a function on the underlying set of points, before verifying that f satisfies the required continuity properties.
- ... a *bundle space* $p\colon Y \to X$ by first forming the point-set coproduct $\Sigma_{x\in X}\left(p^{-1}(x)\right)$, defining an appropriate topology on it (which in general is not the coproduct topology), before proving that p is continuous with respect to it.

By contrast, recall that point-free topology does *not* separate the points from the space as an underlying set; points for us are the models of a theory. The beauty of Definition 2.1.19 of a classifying topos is that any geometric construction on the generic model of a space $[\mathbb{T}]$ automatically extends to the rest of the points of $[\mathbb{T}]$, i.e. geometric mathematics possesses an intrinsic continuity. It is for this reason that, e.g. a geometric construction of $\mathbb{U}(x)$ where x is the generic point of $[\mathbb{T}]$ automatically gives a bundle that is a continuously-indexed family of spaces.

One important fact is that geometric constructions on bundles work fibrewise.

Proposition 2.1.33 ([Vic22])**.** *Let $p'\colon (\Sigma_{x'\in[\mathbb{T}']}[\mathbb{U}(x')]) \to [\mathbb{T}']$ be a bundle, and $f\colon [\mathbb{T}] \to [\mathbb{T}']$ a map. Then the following diagram of spaces is a pullback. The top map takes (x, y) to $(f(x), y)$.*

$$\begin{array}{ccc} \Sigma_{x\in[\mathbb{T}]}[\mathbb{U}(f(x))] & \longrightarrow & \Sigma_{x'\in[\mathbb{T}']}[\mathbb{U}(x')] \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} \\ [\mathbb{T}] & \xrightarrow{\quad f \quad} & [\mathbb{T}'] \end{array}$$

Proof. This proposition essentially reformulates a well-known result that the 2-category of toposes $\mathfrak{Top}$ possesses pullbacks. For a proof of the original result, see, e.g. [Joh77b, Corollary 4.48].[34] For details on how to reformulate this in the language of point-free topology, see [Vic22, §8- 9]. □

Notice: if $\mathbb{T}'$ is the empty theory (no sorts or symbols, no axioms) then $[\mathbb{T}']$ is 1, the 1-point space, and $\mathbb{U}$ is a plain theory (no x' to depend on) and the pullback is $[\mathbb{T}] \times [\mathbb{U}]$. This has an interesting methodological consequence:

Convention 2.1.34 ("Fixing x"). Suppose we wish to construct a map with multiple arguments, such as

$$f\colon [\mathbb{T}] \times [\mathbb{U}] \to [\mathbb{U}'].$$

To do this, we shall often say "fix $x \in [\mathbb{T}]$" and then, by the usual process, construct a map

$$f_x\colon [\mathbb{U}] \to [\mathbb{U}'].$$

The declaration "fix $x \in [\mathbb{T}]$" means that we are working over $[\mathbb{T}]$ (technically, in the topos of sheaves $\mathcal{S}[\mathbb{T}]$), so that $[\mathbb{U}]$ and $[\mathbb{U}']$ are transported to their products with $[\mathbb{T}]$, so we are actually defining a commutative triangle as follows — but that is equivalent to the f we wanted.

$$\begin{array}{ccc} [\mathbb{T}] \times [\mathbb{U}] & \xrightarrow{\langle p, f\rangle} & [\mathbb{T}] \times [\mathbb{U}'] \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & [\mathbb{T}] & \end{array}$$

The reader may notice that we are doing is reminiscent of dependent type theory, except cast in a topos-theoretic language.

2.2 Localic Spaces and Essentially Propositional Theories

Of particular interest to us is a class of geometric theories known as (essentially) propositional theories. As we explain in this section, propositional theories occupy a certain sweet spot in which the links between lattice theory, topology and logic are made especially clear. We then shift focus to two important examples of such theories: (a) the localic reals; and (b) the localic primes. (The model theorist may be interested to learn that the Dedekind reals, a prototypical example of types in model theory, show up as honest models of a geometric theory in our context.) Finally, we establish a "toolkit" of new results and lemmas, and indicate their usage in the later chapters.

2.2.1 Background. Recall from Definition 2.1.2 that *localic spaces* are defined to be complete distributive lattices. Given our particular understanding of "spaces" from Section 2.1, this calls for some justification. As it turns out, localic spaces actually correspond to *propositional theories*. For clarity, we give the full definition:

Definition 2.2.1. A (geometric) theory $\mathbb{T}$ is called a *propositional theory* if its signature Σ has no sorts [so there can be no variables or terms, nor existential quantification]. In particular, its axioms are constructed only from constant symbols in Σ, $\top$ (true), finite $\wedge$ and arbitrary $\bigvee$.

[34] In fact, [Joh77b, Corollary 4.48] proves the more general result that $\mathfrak{BTop}/\mathcal{S}$ possesses all finite limits.

The connection between geometric logic and topology now becomes apparent: the propositional formulae of $\mathbb{T}$ correspond to the opens, the finite $\wedge$ to the finite intersections of opens and the arbitrary $\bigvee$ to their arbitrary unions. As for the points of the localic space, they are best characterised as *completely prime filters*. A *filter*, one may recall, is a collection of subsets satisfying certain formal properties also satisfied by the collection of open neighbourhoods of any point $x \in X$ in a point-set topological space. The hypothesis of *completely prime* enforces a kind of coherence within the filter with respect to $\bigvee$. More precisely:

Definition 2.2.2. Let S be an infinite set.

(i) A *filter on* S is a collection of subsets $\mathcal{F} \subseteq \mathcal{P}(S)$ such that:

  (a) $A \subseteq B \subseteq S$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$;
  (b) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$; and
  (c) $S \in \mathcal{F}$

(ii) A filter $\mathcal{F}$ is *prime* if for every finite set index set I:

$\bigcup_i A_i \in \mathcal{F}$ implies there exists some $j \in I$ such that $A_j \in \mathcal{F}$.

A filter $\mathcal{F}$ is *completely prime* if the same holds true for *any* index set I (including when I is infinite).

(iii) A filter $\mathcal{F}$ is called an *ultrafilter* (or a *maximal filter*) if it has an opinion on all subsets of S:

If $A \subset S$, then either A or its complement $S \setminus A$ belongs to $\mathcal{F}$ (but not both).

Given the connections between conjunctions/intersections and disjunctions/unions, one easily translates Definition 2.2.2 of completely prime filters to the setting of locales. (As for ultrafilters, translating $S \setminus A$ becomes problematic if our lattice isn't Boolean, but see Discussion 2.2.19). This gives a transparent way of understanding the interactions between topology and logic. In our present setting, one easily checks that the Lindenbaum algebra of a propositional theory $\mathbb{T}$ (i.e. the set of geometric formulae modulo equivalence provable from $\mathbb{T}$) yields a frame $\Omega_{\mathbb{T}}$, which may be regarded as the frame of opens for the space $[\mathbb{T}]$, whose points are the completely prime filters. We record these connections (along with several others) in the following summary theorem.

Summary Theorem 2.2.3 ([Joh02b, C1.3–4], [Vic07b, §2.2])**.**

(i) Let $\mathcal{E}_L$ be the category of sheaves constructed over a localic space L. Then $\mathcal{E}_L$ is a topos that classifies the propositional theory $\mathbb{T}$ of completely prime filters of L.

*(ii) Furthermore, given any pair of locales L, L', $\mathrm{Loc}(L, L') \simeq$ **Geom**$(\mathcal{E}_L, \mathcal{E}_{L'})$, i.e. any geometric morphism $\mathcal{E}_L \to \mathcal{E}_{L'}$ corresponds to a locale morphism $L \to L'$ and vice versa.*

(iii) Conversely, let $\mathbb{T}$ be a propositional theory. Then, the Lindenbaum algebra of $\mathbb{T}$, denoted $\Omega_{[\mathbb{T}]}$, is a frame. In particular, $\mathcal{S}[\mathbb{T}] \simeq \mathcal{E}_{\Omega_{[\mathbb{T}]}}$, and so $\Omega_{[\mathbb{T}]}$ may be viewed as the frame of opens for $[\mathbb{T}]$.

There is also a natural weakening of the notion of propositional theories. Call $\mathbb{T}$ an *essentially propositional* theory if there exists a propositional theory $\mathbb{T}'$ such that $\mathcal{S}[\mathbb{T}] \simeq \mathcal{S}[\mathbb{T}']$. By Proposition 2.1.28, note this is equivalent to saying that $\mathbb{T}$ and $\mathbb{T}'$ have equivalent models, even if $\mathbb{T}$ possesses sorts in its signature. Indeed, there exist various kinds of sorts that can be added to a theory's signature without essentially changing its models (although, a complete description of such sorts appears to be still unknown). One important class of such sorts are the free algebra constructions:

Fact 2.2.4 ([Vic07b; Vic17], but see also [Joh02a, pp. 108])**.**

(i) The free algebra constructions are geometric constructions (cf. Convention 2.1.17), and are uniquely determined up to isomorphism. These include the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$, along with their usual arithmetic structure (e.g. addition, multiplication, strict order etc.).

(ii) Let $\mathbb{T}$ be a geometric theory. If we perform free algebra constructions on the sorts already present in its signature to construct new "derived sorts", we obtain a new theory whose models are equivalent to those of $\mathbb{T}$. In particular, if $\mathbb{T}$ has only free algebra constructions as is its sorts (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, etc.), then it is essentially propositional.

Discussion 2.2.5. The fact that we can determine free algebra constructions up to isomorphism is a striking feature of geometric mathematics — we really *do* mean $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. Our ability to express this comes from geometric logic possessing arbitrary disjunctions — see, e.g. [Vic07b, §3.4] for such an explanation regarding $\mathbb{N}$. This should be contrasted with classical model theory, where finitary logic *cannot* determine any infinite structure up to isomorphism due to the Löwenheim-Skolem theorem.

Convention 2.2.6. If $\mathbb{T}$ is an essentially propositional theory, we will also call $[\mathbb{T}]$ a localic space, even if $\mathbb{T}$ has sorts in its signature.

In practice, the presence of sorts in a theory's signature often allows for a much nicer axiomatisation of the models, whereas the absence of sorts indicates that the theory is less logically complex and so potentially easier to work with (cf. Summary Theorem 2.2.3). By working with essentially propositional theories (as opposed to just propositional theories), we enjoy the best of both worlds; a case in point would be the localic reals, which we turn to in the next subsection.

2.2.2 The Localic Reals. This thesis uses two different types of reals: Dedekind reals and the so-called 'one-sided reals'. Both reals are built up from the rationals but in different ways; this results in different topologies and therefore different subtleties in their analysis.

2.2.2.1 Basic Definitions. Denote by $\mathbb{Q}$ the set of rationals, by Q_+ the positive rationals, and by Q the non-negative rationals. We denote by $\mathbb{R}$ the space of points of the theory of Dedekind reals, which we explicitly define in the following:

Definition 2.2.7. The theory of Dedekind Reals, with space $\mathbb{R}$, comprises two relations $L, R \subset \mathbb{Q}$ which satisfy the following axioms:

1. $\exists r \in \mathbb{Q}.R(r)$ (Right Inhabitedness)
2. $\forall r \in \mathbb{Q}.\big(R(r) \leftrightarrow \exists r' \in \mathbb{Q}.(r' < r \wedge R(r'))\big)$ ($\leftarrow$ Upward closure; $\rightarrow$ Roundedness)
3. $\exists q \in \mathbb{Q}.L(q)$ (Left Inhabitedness)
4. $\forall q \in \mathbb{Q}.\big(L(q) \leftrightarrow \exists q' \in \mathbb{Q}.(q' > q \wedge L(q'))\big)$ ($\leftarrow$ Downward closure; $\rightarrow$ Roundedness)
5. $\forall q, r \in \mathbb{Q}.(L(q) \wedge R(r) \rightarrow q < r)$ (Separatedness)
6. $\forall q, r \in \mathbb{Q}.(q < r \rightarrow L(q) \vee R(r))$ (Locatedness)

The two relations L, R correspond to the left and right Dedekind sections of a real number.

Convention 2.2.8. We shall often denote a point of $\mathbb{R}$ as x, instead of explicitly writing out the pair of relations representing it: (L_x, R_x). We will also use $q < x$ to mean $L_x(q)$ and use $x < r$ to mean $R_x(r)$.

Remark 2.2.9. It is known (e.g. see [MV12]) that Axiom (6) is equivalent to the following axiom:

$$\forall \epsilon \in \mathbb{Q}.\big(\epsilon > 0 \rightarrow \exists q, r \in \mathbb{Q}.(L(q) \wedge R(r) \wedge r - q < \epsilon)\big).$$

Notice: we used the rationals $\mathbb{Q}$ as our sort for the theory of $\mathbb{R}$, and so by Fact 2.2.4 the theory of Dedekind reals is an essentially propositional theory (and thus $\mathbb{R}$ is a localic space). Further, since the strict order $<$ on $\mathbb{Q}$ is geometrically defined, this can be exploited to give various (constructive) ways of comparing two points with each other, be they in $\mathbb{Q}$ or $\mathbb{R}$:

Fact 2.2.10 (Archimedean Property)**.** Given two non-negative rationals $x, y \in Q$, where $x > 0$ there exists at least one natural number $N \in \mathbb{N}$ such that $Nx > y$, or equivalently $x > \frac{y}{N}$.

Definition 2.2.11. Given two Dedekind reals x and y, we denote:

- $x < y$ if there exists some rational number q such that $x < q < y$. In particular, this defines a strict order on $\mathbb{R}$, and the space of all pairs of Dedekinds satisfying $x < y$ defines an open subspace of $\mathbb{R} \times \mathbb{R}$.
- $x \geq y$ if (x, y) belongs to the closed complement of $<$ in $\mathbb{R} \times \mathbb{R}$.

Remark 2.2.12 (Equality of Dedekinds)**.** Syntactically, two models of a propositional theory are isomorphic if they satisfy exactly the same propositions. Consequently, in the case of $\mathbb{R}$, this means that $x = y$ if the following condition holds: for any $q \in \mathbb{Q}$, we have that $q < x$ iff $q < y$ and $x < q$ iff $y < q$.

Discussion 2.2.13 (Decidability of order)**.** Definition 2.2.11 says that there is a sense in which the strict order $<$ relation on $\mathbb{Q}$ lifts to yield another relation on $\mathbb{R}$. However, there is an issue of decidability here, i.e. whether an open has an *open* Boolean complement (cf. Discussion 2.1.14). In particular, $<$ is decidable on $\mathbb{Q}$ (where, because $\mathbb{Q}$ is discrete, open subspaces of $\mathbb{Q}$ are just subsets), but not on the reals.

There are also two main classes of spaces/geometric theories closely related to $\mathbb{R}$ that will be of interest to us in this thesis. The first important class are its subspaces:

Definition 2.2.14 (Subspaces of $\mathbb{R}$)**.** Recall from Section 2.1 that a subspace of $\mathbb{R}$ are the Dedekinds which satisfy additional axioms.

(i) Denote $(0, \infty)$ to be the open subspace of positive Dedekinds: this is the subspace of $\mathbb{R}$ satisfying the axiom '$\top \to L(0)$'. Likewise for q rational, denote (q, ∞) and $(-\infty, q)$ for the subspaces satisfying $\top \to L(q)$ and $\top \to R(q)$ respectively.
(ii) Denote $[0, \infty)$ to be the closed complement of $(-\infty, 0)$, satisfying $R(0) \to \bot$. Using the axioms of $\mathbb{R}$ we see that this is equivalent to $\forall q \in \mathbb{Q}.\ (q < 0 \to L(q))$. Similarly we write $[q, \infty)$ and $(-\infty, q]$.
(iii) Finally, we extend the notation in the obvious way. For example, $(0, 1] = (-\infty, 1] \wedge (0, \infty)$ has the axioms of both $(-\infty, 1]$ and $(0, \infty)$.

The second important class of spaces related to $\mathbb{R}$ are the one-sided reals:

Definition 2.2.15. Recall the axioms defining the Dedekind reals in Definition 2.2.7. Then:

(i) The *upper reals* is a space whose points just satisfy Axiom (2).
(ii) The *lower reals* is a space whose points just satisfy Axiom (4).

Note that this allows the upper (resp. lower) reals to be empty, which correspond to ∞ (resp. $-\infty$). We can exclude these cases by using Axiom (1) (resp. Axiom (3)). Informally, an inhabited upper real (resp. lower real) approximates a number from above (resp. below), whereas a Dedekind real approximates the number from both directions.

Convention 2.2.16. Extending Convention 2.2.8, given an upper real x (resp. a lower real x), we often write $x < q$ (resp. $q < x$) to mean that q belongs to the subset of rationals constituting x. We shall also often refer to the one-sided reals as just the 'one-sideds'.

Convention 2.2.17 (Subspaces of One-Sided Reals)**.**

(i) The one-sided reals are spaces with the corresponding Scott topologies: for lower reals, $x \sqsubseteq y$ iff $x \leq y$ whereas for upper reals $x \sqsubseteq y$ iff $x \geq y$. Observe that the specialisation order $\sqsubseteq$ for the lower reals agrees with the numerical order, whereas for the upper reals it is the opposite. To reflect this, we shall use arrows on top of the spaces to show the direction of the refinement under their respective specialisation orders. For instance, consider the space $(0, \infty)$ — we then denote the corresponding space of lower reals as $\overrightarrow{(0, \infty]}$ and the corresponding space of upper reals as $\overleftarrow{[0, \infty)}$.

(ii) Notice that the previous one-sided intervals were closed at the arrowhead — e.g. ∞ was included in $\overrightarrow{(0, \infty]}$ and 0 in $\overleftarrow{[0, \infty)}$. In fact, this is canonical — *all* one-sided intervals must be closed at the arrowhead. Why? Answer: the one-sided reals possess the Scott topology, and so all subspaces of the one-sideds must be closed under arbitrary directed joins with respect to $\sqsubseteq$.

Fact 2.2.18. There exist natural maps

$$L\colon \mathbb{R} \longrightarrow \overrightarrow{(-\infty, \infty]}$$
$$R\colon \mathbb{R} \longrightarrow \overleftarrow{[-\infty, \infty)}$$

where given a Dedekind real $x = (L_x, R_x)$, L sends $x \mapsto L_x$ and R sends $x \mapsto R_x$.

2.2.2.2 Interlude on the Classical vs. Geometric Perspective. The claim that Dedekind reals can be characterised as models of a first-order theory will be provocative to the model theorist. In classical model theory, Dedekind reals typically arise not as models but as *types* over the model $M = (\mathbb{Q}, <)$, i.e. the rationals considered as a dense linear order. We contextualise this via the language of filters:

Discussion 2.2.19 (Types vs. Models as Filters)**.** Informally, a (complete) type p over a model M corresponds to an ultrafilter of the Boolean Algebra of definable subsets of M, which we denote as $\mathcal{B}_M$. Analogously, Summary Theorem 2.2.3 tells us that the models of a propositional theory $\mathbb{T}$ correspond to the completely prime filters of $\Omega_{[\mathbb{T}]}$. The appearance of filters in both contexts is suggestive, but there is a subtlety. Since the geometric syntax does not have negation (cf. Remark 2.1.9), the Lindenbaum Algebra $\Omega_{[\mathbb{T}]}$ is generally not Boolean (and so its completely prime filters are generally not ultrafilters either). However, when $\Omega_{[\mathbb{T}]}$ is in fact Boolean, then the prime filters of $\Omega_{\mathbb{T}}$ turn out to be precisely its ultrafilters and so the two notions coincide[35].

We can also phrase this in the language of points (in the sense of Definition 2.1.18). Whereas a complete type over M may be characterised as a Boolean homomorphism $\mathcal{B}_M \to \{0, 1\}$ to the two-element Boolean algebra, a global point of a localic space $[\mathbb{T}]$ corresponds to a frame homomorphism $\Omega_{[\mathbb{T}]} \to \Omega$, where Ω is the frame of truth values (cf. Footnote 23 and Summary Theorem 2.2.3 once more).

Discussion 2.2.20. Viewed logically, the example of the Dedekind reals brings into focus a challenging connection between the model theorist's type spaces and the topos theorist's point-free spaces. Parsing their similarities and differences reflects the contrasting legacies of Shelah vs. Grothendieck on the development

[35] Some care, however, needs to be taken regarding the distinction between prime vs. completely prime filters.

of modern logic. On the one hand, the model theorist and the topos theorist have developed very different understandings of what constitutes logical complexity[36]; on the other hand, there appears to be some convergence in attitudes regarding the desired structure of these logical spaces (the interested reader may wish to compare the Galois-theoretic ideas in Joyal-Tierney's monograph [JT84] with Hrushovski's recent preprints [Hru19; Hru21]). Further development of these structural connections appear important, and will be the subject of future work.

Warning 2.2.21. One should not jump to conclusions about the space of Dedekinds being equivalent to the Stone space of 1-types over $(\mathbb{Q},<)$, which we denote as $\mathrm{S}_1(\mathbb{Q},<)$. Unlike the latter, the space of Dedekinds does **not** contain infinities or infinitesimals — its models are just the real numbers belonging to the interval $(-\infty,\infty)$. We can also understand the difference topologically: $\mathrm{S}_1(\mathbb{Q},<)$ is a totally disconnected compact space whereas the space of Dedekind reals $\mathbb{R}$ is connected and non-compact.

Aside from Warning 2.2.21, there is a more fundamental reason to be cautious about overextending the informal picture of "Models of a geometric theory $\approx$ Types arising from model theory". As already mentioned, completely prime filters are generally not ultrafilters. In other words, there exist geometric theories, such as the upper/lower reals, whose models do not correspond to anything classical.

Discussion 2.2.22. Recall that an upper real is a real number that only records the rationals strictly larger than itself and nothing else. This results in some striking differences when compared with the Dedekinds. Certainly the standard upper real does not correspond to a complete type over $(\mathbb{Q},<)$ (unless, of course, the upper real happens to be ∞ or $-\infty$). Further, since an upper real is blind to the rationals less than itself, we also cannot say when one upper real is *strictly* smaller/greater than another. Moreover, recall that an upper reals is defined as a well-behaved subset of $\mathbb{Q}$, which are collectively ordered by subset inclusion. As such, the upper reals combine to form a non-Hausdorff space whose points 'contain' each other.

Discussion 2.2.22 may lead the reader to view the one-sided reals as pathological, but in fact they occupy a computational sweet spot in our point-free calculations. On the one hand, they correspond more closely to our intuitive notion of a 'real number' compared to the rationals $\mathbb{Q}$. On the other hand, unlike the Dedekinds, the one-sideds can also be viewed as honest *subsets* of $\mathbb{Q}$. The upshot is that there is often a direct sense in which properties can be lifted from the rationals to the one-sideds, so long as they respect the order relation. This is a powerful insight once we realise that a Dedekind real is entirely determined by its left or right sections.

We end this subsection with an easy justification of our claim that Dedekinds are determined by their one-sided representations. (Our claim about lifting results from the rationals to the one-sideds requires more work, and will be deferred to Section 2.2.4.)

Lemma 2.2.23. *The following are equivalent for Dedekinds x, y:*

(i) $x \leq y$
(ii) $L_x \sqsubseteq L_y$

[36] Whereas model theorists typically tie the logical complexity of the theory to combinatorial questions (e.g. number of types, number of non-isomorphic models etc.), the topos theorist typically ties the logical complexity of a theory to its expressiveness (e.g. coherent, geometric, propositional vs. predicate etc. — see [Joh02b, Remark D1.4.14]). As such, in the case of Dedekind reals, the model theorist regards them as evidence that the theory $\mathrm{Th}(\mathbb{Q},<)$ is complex [more precisely, $\mathrm{Th}(\mathbb{Q},<)$ is unstable], since $|\mathbb{Q}| = \aleph_0$ but $|S_1(\mathbb{Q},<)| = 2^{\aleph_0} > \aleph_0$. By contrast, as we've already discussed, the topos theorist regards the theory of Dedekind reals as being particularly well-behaved [more precisely, it is an essentially propositional theory].

(iii) $R_x \sqsupseteq R_y$

Proof. (i) ⇒ (ii): Suppose $q < L_x$. By roundedness, we can find $q' \in \mathbb{Q}$ such that $q < q' < L_x$. By locatedness, either $q < L_y$ or $R_y < q'$. If $q < L_y$, then done; else if $R_y < q'$, then $y < x$, contradicting (1).

(ii) ⇒ (i): Suppose for contradiction that there exists some q such that $y < q < x$. This means there exists $q \in L_x$ and $q \notin L_y$, contradicting (2). Since (2) implies that $y < x$ does not hold, and $x \leq y$ is the closed complement of $y < x$, this implies (1).

(i) ⇔ (iii): Analogous to above. □

By symmetry, we obtain the following corollary, which refines Remark 2.2.12:

Corollary 2.2.24. *The following are equivalent for Dedekinds x, y:*

(i) $x = y$
(ii) $L_x = L_y$
(iii) $R_x = R_y$.

2.2.2.3 Basic Operations on the Reals. We now define some basic arithmetic operations on the reals (Dedekind and one-sided), before collecting some familiar facts on how these operations interact. More details (including the proofs, which we have omitted) can be found in [Ray14]. We start with addition and subtraction:

Definition 2.2.25. Reals x and y can be added by the rules

$$q < x + y \leftrightarrow \exists s < x \wedge \exists r < y.(q \leq s + r)$$

$$q > x + y \leftrightarrow \exists s > x \wedge \exists r > y.(q \geq s + r)$$

where $q, s, r \in \mathbb{Q}$.

Definition 2.2.26. Reals x can be negated by the rules

$$-x < q \leftrightarrow -q < x$$

$$-x > q \leftrightarrow -q > x$$

where $q \in \mathbb{Q}$. Note that negation reverses orientation: if x is a lower real, then its negation yields an upper real, and vice versa. Nonetheless, if x is a Dedekind real (which comprises both the left and right Dedekind sections), then its negation yields another Dedekind real. As such, given two Dedekind reals x, y, we define their subtraction $x - y$ as $x + (-y)$.

We next define multiplication and inverses for non-negative reals. For convenience, we shall make use of the following representation of non-negatives:

Convention 2.2.27.

(i) A non-negative lower real x is determined by the positive rationals, as follows:

$$x := \{q \in Q_+ \mid q < x\}$$

In particular, notice that the lower real 0 corresponds to the empty set whereas the lower real ∞ corresponds to the whole set of positive rationals.

(ii) A non-negative upper real x is similarly determined by the positive rationals:

$$x := \{q \in Q_+ \mid q > x\}$$

In particular, notice that the upper real 0 corresponds to the whole set of positive rationals whereas the upper real ∞ corresponds to the empty set.

We shall have more to say about this particular style of representing one-sideds in Section 2.2.4.

Definition 2.2.28. Non-negative reals x, y can be multiplied by the rules:

$$q < x \cdot y \leftrightarrow \exists s < x \wedge \exists t < y.(q < s \cdot t)$$

$$q > x \cdot y \leftrightarrow \exists s > x \wedge \exists t > y.(q > s \cdot t).$$

Multiplication of arbitrary reals (not necessarily non-negative) is more involved, and will not be used in this thesis.

Definition 2.2.29. The inverse of a non-negative one-sided real x is defined as:

$$q < x^{-1} \leftrightarrow x < q^{-1}$$

$$x^{-1} < r \leftrightarrow r^{-1} < x.$$

Just as in the case of subtraction, inverting reverses orientation, sending lowers to uppers and vice versa. The definition gives that ∞ and 0 are inverses. One easily checks that $(—)^{-1}$ is an isomorphism, with $((—)^{-1})^{-1}$ the identity. They combine to give inverses of positive Dedekinds (where 0 must be excluded since ∞ is not a Dedekind), with x^{-1} the unique positive Dedekind real such that $x \cdot x^{-1} = 1$.

Remark 2.2.30. It is an easy exercise to verify that the additive and multiplicative operations defined above turn $[0, \infty)$, $\overrightarrow{[0, \infty]}$ and $\overleftarrow{[0, \infty]}$ into semirings, $\mathbb{R}$ into a field, and $(0, \infty)$ a group. As such, they satisfy all the expected arithmetic identities and inequalities — e.g. $x \cdot (y + z) = x \cdot y + x \cdot z$. Further, it is also straightforward to verify that multiplication preserves strict order on positive Dedekind reals (i.e. $x < y \implies x \cdot z < y \cdot z$) and non-strict order on the one-sided reals (i.e. $x \sqsubseteq y \implies x \cdot z \sqsubseteq y \cdot z$).

Finally, we define the standard $\min$ and $\max$ operations.

Definition 2.2.31. Given two Dedekinds $x := (L_x, R_x)$ and $y := (L_y, R_y)$, we define

$$\max(x, y) := (L_x \cup L_y, R_x \cap R_y)$$

$$\min(x, y) := (L_x \cap L_y, R_x \cup R_y).$$

The extension to the one-sided reals is obvious, but there is a subtlety: if $\max$ corresponds to taking an intersection of two upper reals, then $\sup$ corresponds to taking arbitrary intersections of upper reals, which is no longer geometric. On the other hand, since $\min$ corresponds to taking unions of upper reals, this turns out to be a well-defined operation:

Observation 2.2.32. The space of upper reals has arbitrary (set-indexed) $\inf$s. Analogously, the space of lower reals has arbitrary (set-indexed) $\sup$s.

Proof. For the upper reals, this follows from the fact that: (a) upper reals have numerically codirected $\inf$s, because those are the directed joins with respect to $\sqsubseteq$; and (b) they have finite $\inf$s due to $\min$. The lower real case is entirely analogous. □

This sets up the following definition.

Definition 2.2.33. Given $\{x_i\}_{i\in I}$ an indexed set of reals, we define

$$\inf_{i\in I} x_i := \bigcup_{i\in I}\{q \in \mathbb{Q} \,|\, x_i < q\}$$

$$\sup_{i\in I} x_i := \bigcup_{i\in I}\{q \in \mathbb{Q} \,|\, q < x_i\}$$

Notice: a set-indexed $\inf$ of Dedekinds/upper reals thus defines an upper real; analogously, a set-indexed $\sup$ of Dedekinds/lower reals defines a lower real.

2.2.3 The Localic Primes. The basic algebraic notion of localisation is well-understood and versatile: given a commutative ring R, take a *multiplicative subset*[37] $S \subseteq R \setminus \{0\}$ and define "*the localisation of R away from S*" as the set of equivalence classes

$$S^{-1}R := \{r/s \,|\, r \in R, s \in S\}/\sim \quad ,$$

where $r/s \sim r'/s'$ iff $rs' = r's$, and $S^{-1}R$ is equipped with the obvious ring structure. The prototypical example of this construction involves primes of the ring R: given a prime ideal $\mathfrak{p} \subset R$, one easily verifies $R \setminus \mathfrak{p}$ is a multiplicative subset, and so we define "*R localised at $\mathfrak{p}$*" as:

$$R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R.$$

The deep organising influence of this construction in areas such as Number Theory and Homotopy Theory, particularly when viewed through the lens of Local-Global Principles (see Section 2.4), makes it especially interesting to us. But beyond the obvious translation of these ideas to geometric mathematics, what else might the point-free perspective bring to the algebraic study of primes?

Our answer: topology. This section explains how ideas from point-free topology provide different levels on which to understand the interaction between topology and algebra in the setting of primes. On a more stringent level, working geometrically means that the availability of certain classical principles in the algebra gets recast as topological statements (cf. Discussion 2.1.14). This is reflected in the different kinds of spectra definable for the ring R — namely, the Zariski, the coZariski, and the constructible. Alternatively, even if one is content to work classically, the language of frames still provides a clean abstraction of spectral spaces, giving a sharper view of the underlying mathematics.

[37] Recall: a multiplicative subset $S \subseteq R \setminus \{0\}$ is a subset which $1 \in S$ and $a, b \in S$ implies $a \cdot b \in S$. Here, we will always assume that a commutative ring has $1 \neq 0$. Of course, if we want to work geometrically, we shall need to rephrase this using categorical semantics, e.g. defining R to be a (commutative) ring object and S to be a multiplicative subobject, but the core idea remains the same. For details, see [Tie76], which justifies the point-set notation used in this section.

2.2.3.1 Three Examples of Spectra. Classically, the spectrum of a commutative ring R, which we denote $\mathrm{spec}(R)$ (with small s), is defined as the set of all prime ideals of R, sometimes decorated with the Zariski topology. Once equipped with a suitable structure sheaf, this construction gives a solution to the representation problem:

Problem. Given a commutative ring R, find a space X and a sheaf of local rings on X such that R is the ring of global sections of this sheaf.

In fact, the Spec construction can be extended to a functor sending a ringed topos to a canonical locally ringed topos

$$\mathrm{Spec}\colon \textbf{RingedTopos} \longrightarrow \textbf{LRingedTopos},$$

which was later observed by Hakim [Hak72] to be right adjoint to the natural inclusion:

$$i\colon \textbf{LRingedTopos} \hookrightarrow \textbf{RingedTopos}.$$

This insight was subsequently generalised by Cole in [Col16]:

Theorem 2.2.34 (Cole's Spectra)**.** *As our setup,*

- *Let $\mathbb{S}$, $\mathbb{T}$ be a pair of algebraic theories*[38] *where $\mathbb{S}$ is a quotient theory of $\mathbb{T}$;*
- *Let $\mathbb{A}$ be an admissible class of morphisms of $\mathbb{S}$-models.*

Then, the inclusion functor $i\colon \mathbb{A}\text{-}\mathfrak{Top} \hookrightarrow \mathbb{T}\text{-}\mathfrak{Top}$ has a right adjoint $\mathrm{Spec} : \mathbb{T}\text{-}\mathfrak{Top} \to \mathbb{A}\text{-}\mathfrak{Top}$.

To avoid burdening this thesis with too much theory, we leave the details as a dark grey box, if not a black one (for the interested reader, [Joh77b, §6.5] gives a clear presentation). It suffices for us to regard Cole's Spectra as some kind of general machinery for constructing a space[39] out of an algebraic theory $\mathbb{T}$ and a quotient theory $\mathbb{S}$. This was used in [Joh77a] to give a clear insight into how the algebraic properties of the quotient theory $\mathbb{S}$ may be reflected in the topology of the Spectrum space. Of particular relevance to us are the following three examples:

Example 2.2.35 (The Zariski Spectrum)**.** Let $\mathbb{T}_{\mathrm{com}}$ be the theory of commutative rings (Example 2.1.12), and $\mathbb{S}_{\mathrm{loc}}$ be the theory of local rings, i.e. the theory $\mathbb{T}_{\mathrm{com}}$ plus the axiom

$$(\forall a, a' \in R).\, a + a' \in U_R \to a \in U_R \vee a' \in U_R$$

where U_R denotes the group of units of the ring R.[40] A ring morphism $f : A \to B$ is said to be *local* if it reflects the property of being a unit, i.e.

$$f(a) \in U_B \to a \in U_A, \qquad \text{where } U_A, U_B \text{ denote the group of units of } A, B \text{ respectively.}$$

If we take the admissible class of morphisms to be the local morphisms between the local rings, then Cole's construction yields the space $\mathrm{LSpec}(R)$ for commutative ring R, whose points are the *prime filters* of R, i.e. they are the subobjects[41] $S \rightarrowtail R$ satisfying the axioms:

[38] In fact, this construction works more generally for finitely-presented geometric theories, i.e. theories with a finite set of sorts, relations, functions and axioms.

[39] In fact, a category of sheaves on this space plus a suitable structure sheaf.

[40] Recall: $u \in R$ is called a unit iff $\exists u^{-1} \in R$ such that $u \cdot u^{-1} = 1$.

[41] The reader may wish to substitute mentions of "subobject" with "subsets" without too much harm — see Footnote 37.

- $\top \to 1 \in S$, and $0 \in S \to \bot$;
- $(\forall a, a' \in R).\ aa' \in S \leftrightarrow a \in S \wedge a' \in S$;
- $(\forall a, a' \in R).\ a + a' \in S \leftrightarrow a \in S \vee a' \in S$.

Notice: in Set, these axioms say that S is the complement of a prime ideal of R. Hence, restricting ourselves to the classical setting, the underlying set of points of $\mathrm{LSpec}(R)$ is equivalent to $\mathrm{spec}(R)$, i.e. the set of prime ideals of R. As a point-set topological space, $\mathrm{LSpec}(R)$ is $\mathrm{spec}(R)$ decorated with the *Zariski topology*, which is generated by the basic Zariski open sets $D(a) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid a \notin \mathfrak{p}\}$.

Example 2.2.36 (The coZariski Spectrum)**.** We now consider a different quotient theory of $\mathbb{T}_{\mathrm{com}}$, the theory $\mathbb{S}_{\mathrm{int}}$ of integral domains, i.e. $\mathbb{T}_{\mathrm{com}}$ plus the axiom

$$(\forall a, a' \in R).\ a \cdot a' = 0 \to a = 0 \vee a' = 0.$$

A ring morphism $f\colon A \to B$ is said to *integral* if it reflects the property of being equal to zero, i.e.

$$f(a) = 0 \to a = 0.$$

In other words, the integral morphisms are the monics. If we take the admissible class of morphisms to be the monics between integral domains, then Cole's construction yields the space $\mathrm{ISpec}(R)$, whose points are the *prime ideals* of R. Regarded as a point-set space, $\mathrm{ISpec}(R)$ is $\mathrm{spec}(R)$ decorated with the *coZariski topology*, which is generated by the sub-basic open set $V(a) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid a \in \mathfrak{p}\}$.[42]

Example 2.2.37 (The Constructible Spectrum)**.** Finally, we consider the quotient theory of geometric fields $\mathbb{S}_{\mathrm{fld}}$, i.e. $\mathbb{T}_{\mathrm{com}}$ plus the axiom

$$\top \to a = 0 \vee a \in U.$$

In other words, for any $a \in R$, it is decidable if $a = 0$ or a is a unit. One easily verifies that "ring homomorphism", "monic" and "local morphism" all coincide for morphisms between geometric fields — and so any one of these will work for the required admissible class of morphisms. Cole's construction thus yields the space $\mathrm{FSpec}(R)$, whose points are the *complemented prime ideals* of R. More explicitly, the points of $\mathrm{FSpec}(R)$ are pairs (P, S) where P is a prime ideal, S is a prime filter and P and S are complements of each other (as subobjects of R). Regarded as a point-set space, $\mathrm{FSpec}(R)$ is $\mathrm{spec}(R)$ decorated with the *constructible topology*, which is the join of the Zariski and coZariski topologies.

Remark 2.2.38 (On "admissible" morphisms)**.** Informally, a class $\mathbb{A}$ of $\mathbb{S}$-model morphisms is said to be *admissible* if it witnesses all relevant factorisations of $\mathbb{T}$-model morphisms $R \xrightarrow{f} L$ where L is an $\mathbb{S}$-model. We give a quick sketch in the case of local rings. First, verify that any ring homomorphism $R \xrightarrow{f} L$ to a local ring L factors through a localisation of R:

$$f\colon R \longrightarrow S^{-1}R \longrightarrow L, \tag{2.1}$$

where the multiplicative subobject S is formed by pullback

$$\begin{array}{ccc} S & \longrightarrow & U_L \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & L \end{array} \tag{2.2}$$

[42]Why do we call $V(a)$ a sub-basic open rather than basic? Answer: because $V(a)$ as defined is not closed under finite intersections.

with U_L being the group of units of local ring L. Next, check that $S^{-1}R$ is local, and that the factorisation $S^{-1}R \to L$ is in fact a local morphism. Conclude that the class of local morphism between local rings $\mathbb{A}$ witnesses all the factorisations of the form Equation (2.1), and check that this suffices to satisfy the technical requirement of admissibility (for details, see [Joh77a, Lemma 4.1]). Notice the role of the localisation construction in this factorisation argument (which was hidden in our previous discussion of LSpec).

Examples 2.2.35 - 2.2.37 combine to give the following picture. Viewed as *point-free spaces*, the points of the three spectral spaces are clearly different. However, when viewed classically as *point-set spaces*, then the topology and points begin to separate. On the one hand, their underlying set of points become classically equivalent: since every subset in Set is complementable, there is no longer a meaningful difference between, e.g. a complementable prime ideal vs. a prime ideal in Set. On the other hand, their respective topologies (Zariski, coZariski, constructible) are still different, although the algebraic reasons for this difference are now obscured.

Convention 2.2.39 ("The space of primes")**.** In the Introduction, we presented our reasons for wanting to work with the primes of $\mathbb{Z}$ geometrically, as opposed to classically. In which case, our prior discussion indicates that the correct space to use is the coZariski spectrum $\mathrm{ISpec}(\mathbb{Z})$. In particular, we remark:

- This is different from the standard choice of the Zariski spectrum in classical algebraic geometry.
- For general commutative ring R, the space $\mathrm{ISpec}(R)$ will not be localic. However, since $\mathbb{Z}$ is a free algebra construction, it is a geometric sort and so $\mathrm{ISpec}(\mathbb{Z})$ *will* be a localic space (cf. Fact 2.2.4).

2.2.3.2 Prime Ideals of $\mathbb{Z}$. In Chapter 4, we shall be interested in analysing the absolute values on $\mathbb{Q}$, which are determined by their values on the integers $\mathbb{Z}$, which we now single out for study. Denote $(\mathbb{Z}, +, \cdot, 1)$ to be the set of integers equipped with the obvious addition and multiplication operations.

Definition 2.2.40 (Ideals of $\mathbb{Z}$)**.**

(i) An *ideal* I is a subset $\mathbb{Z}$ such that the following hold:

- $0 \in I$
- *(Closure under Multiplication)* $\forall n \in \mathbb{Z}.\, i \in I \to i \cdot n \in I.$
- *(Closure under Addition)* $\forall i, j, \in I.\, i + j \in I.$

(ii) An ideal I is *non-trivial* if there exists a non-zero integer $a \in \mathbb{Z}_{\neq 0}$ such that $a \in I$.

(iii) An ideal I is *principal* if $\exists a \in \mathbb{Z}$ such that $I = (a) := \{a \cdot n | n \in \mathbb{Z}\}$.

(iv) An ideal I is *prime* if:

- $\forall i, j \in I.\, i \cdot j \in I \to i \in I$ or $j \in I.$
- $1 \notin I.$

The powerset of $\mathbb{Z}$, denoted $\mathcal{P}(\mathbb{Z})$, can be regarded as a localic space whose global points correspond to the subsets of $\mathbb{Z}$ (see, e.g. [JT84, §1.3]). Since: (a) all the properties listed in Definition 2.2.40 can be formulated as geometric axioms[43]; and (b) the quotients of a theory correspond to subspaces (Proposition 2.1.30), we can thus define:

Definition 2.2.41. Denote $\mathrm{ISpec}(\mathbb{Z})$ to be the space of prime ideals of $\mathbb{Z}$ — it is the subspace of $\mathcal{P}(\mathbb{Z})$ whose points satisfy items (i) and (iv) of Definition 2.2.40.

[43] The property "$1 \notin I$" can be read as "$1 \in I \to \bot$".

Particularly useful is the following representation lemma of $\mathrm{ISpec}(\mathbb{Z})$:

Lemma 2.2.42. *Let $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{Z})$ be a non-trivial prime ideal. Then $\mathfrak{p} = (p)$ for some prime number $p \in \mathbb{N}_+$.*

Proof. Suppose $a \in \mathfrak{p}$ where $a \neq 0$. Define the following algorithm:

- Step 1: By unique prime factorisation, represent a as
$$a = (-1)^{\mathrm{sgn}(a)} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$
where $\mathrm{sgn}(a) = 0$ if a is positive and $\mathrm{sgn}(a) = 1$ if negative.
- Step 2: Since $\mathfrak{p}$ is closed under multiplication, we know that $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \in \mathfrak{p}$. Pick the first prime factor p_1. By primeness, we know either $p_1 \in \mathfrak{p}$ or $p_1^{\alpha_1 - 1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \in \mathfrak{p}$. If $p_1 \in \mathfrak{p}$, then stop. Otherwise, set $a = p_1^{\alpha_1 - 1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \in \mathfrak{p}$ and repeat Steps 1 - 2.

Note that this algorithm eventually terminates (since we have at most $\sum_i^n \alpha_i$ many checks to do) and yields a single prime p as its output. Since $p \in \mathfrak{p}$, this shows that $(p) \subseteq \mathfrak{p}$.

To show that $\mathfrak{p} \subseteq (p)$, suppose we have two elements $a, b \in \mathfrak{p}$ and the algorithm associates them to primes p and p'. If $p = p'$, then $a, b \in (p)$. Otherwise, suppose (without loss of generality) that $p' < p$. Recall that Bézout's Identity can be (constructively) obtained from inverting the Euclidean Algorithm and performing the relevant substitutions. One can thus verify that there exist $m, n \in \mathbb{Z}$ such that
$$mp' + np = \gcd(p, p'). \tag{2.3}$$
By primeness, we know $\gcd(p, p') = 1$, and so m, n must both be non-zero integers. Further, since $mp, np' \in \mathfrak{p}$, this implies that $1 \in \mathfrak{p}$, contradicting the requirement that $1 \notin \mathfrak{p}$ in Definition 2.2.40(iv). Hence, it must be the case that $a, b \in (p)$ for all $a, b \in \mathfrak{p}$, i.e. $\mathfrak{p} \subseteq (p)$. ☐

Remark 2.2.43. Classically, one typically proves that $\mathfrak{p} = (p)$ for some prime $p \in \mathbb{Z}$ by obtaining it as an easy corollary of the more general result that all ideals of $\mathbb{Z}$ are principal. However, proving the latter typically invokes the assumption that we can pick the least element of any non-trivial ideal $I \subset \mathbb{Z}$ (see, e.g. [Wae91a, §3.7]), which is a non-geometric assumption since membership of I is not decidable.

Remark 2.2.44. An analogue of Lemma 2.2.42 also exists for prime ideals of the positive integers $\mathbb{N}_+$, except that now:

- Instead of *non-trivial* prime ideals of $\mathbb{Z}$ we consider *inhabited* prime ideals of $\mathbb{N}_+$;
- We shall need to explicitly require that these prime ideals $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{N}_+)$ are also closed under formal subtraction, that is:
$$\forall i, j \in \mathfrak{p}, \forall n \in \mathbb{N}_+ \,.\, i + n = j \to n \in \mathfrak{p}$$

Then, the same argument works, so long as we are careful about handling the negative coefficients. To elaborate: given elements $a, b \in \mathfrak{p}$ that have been associated primes p, p' respectively via our algorithm, the Bézout representation of Equation (2.3) generally does not yield positive integer coefficients m, n, which is a problem if we want to work entirely within $\mathbb{N}$. To get around this, one shall need to do some rearranging of terms to obtain e.g.
$$\gcd(p, p') + mp' = np$$
so that m, n are both indeed positive integers. One then uses formal subtraction to deduce that $\gcd(p, p') = 1 \in \mathfrak{p}$, obtaining the desired contradiction to show that there cannot exist two distinct primes in $\mathfrak{p}$. Hence, whenever we speak of a prime number $p \in \mathbb{N}_+$, the reader can take this to mean some $p \in \mathbb{N}_+$ whereby (p) is a prime ideal in $\mathbb{N}_+$ in the above sense.

2.2.3.3 Hochster Duality and Zariski Frames. Let us pick out one final postcard regarding the point-free perspective on primes.

Definition 2.2.45 (Spectral Spaces)**.** For a commutative ring R, denote $\mathrm{spec}_Z(R)$ to be the classical Zariski spectrum (= the set of prime ideals of R equipped with the Zariski topology).

(i) A *spectral space* is a (point-set) topological space X homeomorphic to $\mathrm{spec}_Z(R)$ for some commutative ring R.[44]

(ii) The *Hochster dual* of a spectral space X, denoted $X^\vee$, is the space with the same underlying set of points as X but whose basic opens are the closed sets with quasi-compact complements.

Example 2.2.46. For any commutative ring R, $\mathrm{spec}_Z(R)$ is obviously a spectral space. Its Hochster dual is the set of prime ideals of R equipped with the coZariski topology.

The language of frames gives a particularly nice perspective on spectral spaces.

Fact 2.2.47. Let X be a spectral space.

(i) Denote the frame of open sets in X as Ω_X. Then, the underlying set of X corresponds to the classical points of its frame:
$$x\colon \Omega_X \to \{0,1\}.$$

(ii) The Hochster dual $X^\vee$ is a spectral space, and its Hochster dual is X itself, i.e. $(X^\vee)^\vee = X$.

Proof Sketch/Discussion. Item (i) is a straightforward exercise, but is significant because the underlying set of an arbitrary space X does not always correspond to its frame-theoretic points. This shows that spectral spaces belong to a nice class of spaces (i.e. sober spaces) whose analysis is particularly amenable to frame-theoretic analysis. Item (ii) was proved in [Hoc69, Proposition 8] but the argument simplifies considerably if we work frame-theoretically. In which case, the result essentially follows from the fact that Ω_X corresponds to a (bounded) distributive lattice F (equivalent to the lattice of quasi-compact opens in X), and that the Hochster dual of a distributive lattice is simply the opposite lattice; see [Koc07; KP17]. □

Notice: the analysis just presented is classical and *not* geometric. This can be seen from Definition 2.2.45's point-set formulation of $\mathrm{spec}_Z(R)$, or Fact 2.2.47's definition of Ω_X-points as $x\colon \Omega_X \to \{0,1\}$ (as opposed to $x\colon \Omega_X \to \Omega$, cf. Footnote 23). Nonetheless, as indicated below in Example 2.2.48, the point-free perspective can still provide a powerful clarity by tuning out certain kinds of (set-theoretic) noise in our analysis. In fact, this idea that point-free topology may play a serious role even when unmoored from constructive mathematics will find resonance in our work in Chapter 5.

Example 2.2.48 (Applications to Balmer Spectra)**.** Informally, a *tensor-triangulated category* [hereafter: tt-category] is a triangulated category T equipped with a compatible symmetric monoidal structure $(\otimes, \mathbb{1})$ — this notion can be understood as abstracting key structural features of many important categories encountered in homotopy theory, algebraic geometry and beyond. When the tt-category T is essentially small, its structural properties are controlled by its *Balmer spectrum* $\mathrm{Spc}(\mathsf{T})$, defined as the set of *prime thick tensor ideals* of T (i.e. triangulated subcategories $\mathsf{J} \subsetneq \mathsf{T}$ closed under tensoring $\otimes$ and summands $\oplus$, satisfying the condition: if $a \otimes b \in \mathsf{J}$ then $a \in \mathsf{J}$ or $b \in \mathsf{J}$).

[44] Experts may recognise that this is not the original definition of a spectral space (i.e. a T_0-space whose quasi-compact open subsets form a sub-lattice that is a basis for the topology). To improve readability, we have chosen to give an equivalent characterisation instead, first proved by Hochster in his PhD thesis [Hoc67, Theorem 0.0].

Precise details can be found in [Bal05; Bal10], but already the analogy between the Balmer Spectrum $\mathrm{Spc}(\mathsf{T})$ and the usual Zariski Spectrum $\mathrm{spec}_Z(R)$ should be clear — in fact, $\mathrm{Spc}(\mathsf{T})$ is a spectral space. Nonetheless, there is one crucial difference: the topology on $\mathrm{Spc}(\mathsf{T})$ is generated by open basics of the form:

$$U(a) := \{\mathsf{J} \in \mathrm{Spc}(\mathsf{T}) \,\big|\, a \in \mathsf{J}\}.$$

Reviewing Example 2.2.36, one immediately recognises this as corresponding to the coZariski topology (instead of the Zariski topology).

The observation that Hochster's theory of spectral spaces may be relevant to the study of tt-categories led to [KP17], where the authors applied point-free techniques to obtain new conceptual proofs of classical theorems of Hopkins-Neeman and Thomason, as well as clarifying various standard results regarding Balmer spectra. Further work along similar lines have also been carried out in [BKS20; BS21].

2.2.4 Toolkit: Gluing Principles and Lifting Lemmas. This section establishes some new tools of analysis that will guide our work in subsequent chapters. There are two general kinds of results here. The first are the Lifting Lemmas, which establishes conditions for when we can lift constructions/properties from (subsets of) the rationals to (subspaces) of the one-sided reals — this plays a key role in our point-free analysis of exponentiation (Chapter 3) and places of $\mathbb{Q}$ (Chapter 6), but also finds resonance in, e.g. lifting results from rational closed discs to irrational closed discs in non-Archimedean geometry (Chapter 5).

The second kind of results are various gluing principles, which deal with the topological subtleties regarding case-splitting analysis. We already saw in the Introduction (Chapter 1) on how certain case-splittings (e.g. between $\mathbb{Q}_p$ vs $\mathbb{R}$) can present a serious challenge to a deeper understanding of the mathematics; we also saw in Discussion 2.1.14 how geometricity is sensitive to case-splittings insofar as not all geometric properties ϕ are decidable (= the classical case split of $\phi \vee \neg\phi$ is generally unavailable). Some non-trivial work, therefore, has to be done to determine how specific case-splittings – even if they may be "algebraically obvious" – are in fact topologically justified. This comes in the form of Proposition 2.2.60 and Lemma 2.2.63. Once established, these results justify the application of classical algebraic intuitions to the geometric setting, and streamline the main proofs of Chapters 3 and 4.

2.2.4.1 The Lifting Lemma. Although perhaps unusual to the classical mathematician, the one-sided reals arise as natural examples of a well-known construction in domain theory known as *rounded ideal completions* [Smy77; Vic93].

Definition 2.2.49 (Rounded Ideal Completions)**.** Consider $(Y, \prec)$ where Y is a set equipped with a dense[45] transitive order $\prec$. We emphasise that we do not require $\prec$ to be linear or strict here.

(i) An *ideal* in Y is a subset $I \subseteq Y$ that is downward-closed and contains an upper bound for each of its finite subsets (with respect to $\prec$). In particular, if the set $I_q := \{q' \in Y | q' \prec q\}$ is an ideal for all $q \in Y$, then we call $(Y, \prec)$ an R-*structure*.
(ii) A subset $S \subseteq Y$ is called *rounded* if for any $q \in S$, there exists $q' \in S$ such that $q \prec q'$. It is clear that all ideals of Y are rounded. Hereafter, we shall typically refer to the ideals of Y as *rounded ideals*.
(iii) The *rounded ideal completion* of an R-structure $(Y, \prec)$ is the space $\mathsf{RIdl}(Y, \prec)$ of all ideals of Y. The specialisation order $\sqsubseteq$ is then the partial order by inclusion, inducing an order topology known as the *Scott topology*.

[45] Recall: $(Y, \prec)$ is said to have a *dense* order if for any $q, q' \in Y$ such that $q \prec q'$, there exists $q'' \in Y$ such that $q \prec q'' \prec q'$. This property goes by a variety of names — e.g. in [Vic93, Definition 2.1], the same property is referred to as being 'interpolative'.

A subset $D \subset Y$ is called *directed* if it is inhabited, and any two elements of D have an upper bound in D. The *directed join* (of ideals) over a directed subset D is defined as $\bigsqcup^{\uparrow}_{q\in D} I_q := \bigcup_{q\in D} I_q$. For more details on why we single out directed joins for study as opposed to arbitrary joins, see [Vic89]. For this thesis, it suffices to observe that directed joins interact with the topology of $\mathsf{RIdl}(Y, \prec)$ in a natural way:

Fact 2.2.50. Let $(Y, \prec)$ be an R-structure. Given a rounded ideal $I \in \mathsf{RIdl}(Y, \prec)$, a space Z equipped with a specialisation order, then the following is true:

(i) $I = \bigsqcup^{\uparrow}_{q\in I} I_q$.

(ii) The space $\mathsf{RIdl}(Y, \prec)$ is closed under directed joins, i.e. if $I = \bigsqcup^{\uparrow}_{q\in D} I_q$ for any directed subset $D \subset Y$, then $I \in \mathsf{RIdl}(Y, \prec)$.

(iii) Any continuous map

$$f\colon \mathsf{RIdl}(Y, \prec) \longrightarrow Z$$

preserves directed joins, i.e. $f(\bigsqcup^{\uparrow}_{q\in D} I_q) = \bigsqcup^{\uparrow}_{q\in D} f(I_q)$.

(iv) Suppose we have two continuous maps

$$f\colon \mathsf{RIdl}(Y, \prec) \to Z$$

$$g\colon \mathsf{RIdl}(Y, \prec) \to Z$$

such that $f(I_q) \sqsubseteq g(I_q)$ for all $q \in I$. Then $f(I) \sqsubseteq g(I)$.

Proof. (i) and (ii) are obvious. (iii) is [Vic89, Theorem 7.3.1]. For (iv), note that (i) and (iii) give

$$f(I) = \bigsqcup_{q\in I}{}^{\uparrow} f(I_q) \quad \text{and} \quad g(I) = \bigsqcup_{q\in I}{}^{\uparrow} g(I_q).$$

Now suppose $t \in f(I)$. Then there exists $q' \in I$ such that $t \in f(I_{q'})$. But since $f(I_{q'}) \sqsubseteq g(I_{q'})$ by hypothesis, this implies $t \in g(I)$, and so $f(I) \sqsubseteq g(I)$. ☐

The fact that Definition 2.2.49 does not require $\prec$ to be a strict order gives us considerable flexibility. In fact, a subspace of one-sided reals is often representable as a rounded ideal completion $\mathsf{RIdl}(Y, \prec)$, where Y is some subset of $\mathbb{Q}$ and $\prec$ is the standard order $<$ on $\mathbb{Q}$ except possibly reversed or modified to permit edge cases. For this thesis, the most relevant examples are:

Example 2.2.51 (One-Sideds as Rounded Ideals)**.**

- $\overrightarrow{(0, \infty]} \cong \mathsf{RIdl}(Q_+, <)$ and $\overleftarrow{[0, \infty)} \cong \mathsf{RIdl}(Q_+, >)$ where $<$ is the standard strict order.
- $\overleftarrow{[0, 1]} \cong \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$, where $\mathbb{Q}_{(0,1]} := \{q \in \mathbb{Q} \,|\, 0 < q \leq 1\}$ and $x \prec y$ iff $x > y$ or $x = y = 1$.
- $\overrightarrow{[0, \infty]} \cong \mathsf{RIdl}(Q, <)$, where Q denotes the non-negative rationals, and and so we modify $<$ to allow $0 < 0$. Same for $\overleftarrow{[0, \infty]} \cong \mathsf{RIdl}(Q, >)$.
- $\overrightarrow{[-\infty, \infty]} \cong \mathsf{RIdl}(\mathbb{Q} \cup \{-\infty\}, <)$, where we add an additional $-\infty$ symbol to $\mathbb{Q}$ and modify $<$ to allow $-\infty < -\infty$.
- Similarly, we have $\overleftarrow{[-\infty, \infty]} \cong \mathsf{RIdl}(\mathbb{Q} \cup \{\infty\}, >)$ and $\overleftarrow{[-\infty, \infty)} \cong \mathsf{RIdl}(\mathbb{Q}, >)$, with the obvious modifications to the order.

Proof of these isomorphisms are straightforward, if involved. A more thorough discussion of the relevant ideas can be found in [Smy77; Vic93].

Convention 2.2.52. Example 2.2.51 justifies the view that, e.g. a point $\gamma \in \overleftarrow{[0,1]}$ is simultaneously an upper real in the usual sense, as well as a rounded ideal $I_\gamma \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$ in the sense of Definition 2.2.49. In this thesis (especially Chapter 6) we shall use both representations interchangeably, depending on convenience.

The language of rounded ideal completions allows us to reduce many questions about the one-sided reals to questions about the rationals, which are comparatively easier to work with. The following series of results develop this remark.

Lemma 2.2.53. *Let $f\colon X \to Y$ be a map of generalised spaces — in particular, X and Y need not be localic. Then f preserves filtered colimits of points.*

Proof. Standard, but we elaborate. Denote $\mathrm{colim}_{i\in J}\, x_i$ to be a set-indexed filtered colimit of W-points of X, i.e. each point x_i can be represented as:

$$W \xrightarrow{x_i} X \xrightarrow{f} Y$$

Since the filtered colimit is computed pointwise, it is clear that $f(\mathrm{colim}_{i\in J}\, x_i) \cong \mathrm{colim}_{i\in J}\, f(x_i)$.[46] □

Lemma 2.2.54. *Let $\mathsf{RIdl}(Y, \prec)$ be the rounded ideal completion of R-structure $(Y, \prec)$. Then, there exists a canonical map*

$$\begin{aligned} \psi\colon Y &\longrightarrow \mathsf{RIdl}(Y, \prec) \\ q &\longmapsto I_q := \{q' \in Y \mid q' \prec q\}, \end{aligned} \tag{2.4}$$

which is an epimorphism of spaces.

Proof. It is clear that the canonical map ψ is well-defined. [Why? Note that Y is an R-structure, and so I_q is a rounded ideal of Y by definition.] To show that ψ is an epimorphism of spaces, suppose we have two maps $g_1, g_2\colon \mathsf{RIdl}(Y, \prec) \to Z$ such that $g_1 \circ \psi \cong g_2 \circ \psi$. Note: every ideal $I \in \mathsf{RIdl}(Y, \prec)$ can be represented as a directed join $I = \bigsqcup^{\uparrow}_{q\in I} I_q$, which is a filtered colimit. Hence, apply Lemma 2.2.53 to compute

$$g_1(I) = g_1(\bigsqcup_{q\in I}{}^{\uparrow} I_q) \cong \operatorname*{colim}_{q\in I} g_1(I_q) \cong \operatorname*{colim}_{q\in I} g_1 \circ \psi(q)$$

and

$$g_2(I) = \operatorname*{colim}_{q\in I} g_2 \circ \psi(q).$$

Since $g_1 \circ \psi \cong g_2 \circ \psi$ by hypothesis, it follows that $g_1 \cong g_2$, i.e. ψ is indeed an epimorphism. □

Lemma 2.2.55 (Lifting Lemma)**.** *As our setup,*

- *Let $\mathcal{E}$ be a generalised space [in particular, $\mathcal{E}$ need not be localic];*
- *Let $(Y, \prec)$ be an R-structure.*

Then, the epimorphpism from Lemma 2.2.54 induces an equivalence between:

(i) A map $f\colon Y \longrightarrow \mathcal{E}$ satisfying the following continuity conditions:

[46] Note that this generalises Fact 2.2.50 (iii), except we now use isomorphisms rather than writing $f(\mathrm{colim}_{i\in J}\, x_i) = \mathrm{colim}_{i\in J}\, f(x_i)$, since spaces here may not be localic.

- (Cocycle condition) *For all* $q, q' \in Y$, *we have* f *maps* $q' \prec q$ *to a map* $\theta_{q'q}\colon f(q') \to f(q)$ *such that if* $q'' \prec q' \prec q$ *then* $\theta_{q''q} = \theta_{q'q} \circ \theta_{q''q'}$*;*
- (Colimit condition) *The map* $\theta_q\colon \operatorname*{colim}_{q' \prec q} f(q') \to f(q)$ *is an isomorphism.*

(ii) A map $\overline{f}\colon \mathsf{RIdl}(Y, \prec) \longrightarrow \mathcal{E}$.

Proof. The proof splits into two main stages.

Step 1: Transforming the given map. Suppose $f\colon Y \longrightarrow \mathcal{E}$ is a map satisfying the continuity conditions of the lemma. Then for any $I \in \mathsf{RIdl}(Y, \prec)$, one easily checks that $\operatorname*{colim}_{q \in I} f(q)$ is a filtered colimit due to the cocycle condition. As such, since toposes possess all set-indexed filtered colimits of their points [Joh77b, Corollary 7.14], the following map is well-defined:

$$\begin{aligned} \overline{f}\colon \mathsf{RIdl}(Y, \prec) &\longrightarrow \mathcal{E} \\ I &\longmapsto \operatorname*{colim}_{q \in I} f(q). \end{aligned} \tag{2.5}$$

Conversely, suppose we have a map $\overline{f}\colon \mathsf{RIdl}(Y, \prec) \to \mathcal{E}$. We can then define a map

$$\begin{aligned} f\colon Y &\longrightarrow \mathcal{E} \\ q &\longmapsto \overline{f}(I_q) \end{aligned} \tag{2.6}$$

where I_q as in Equation (2.4). That f as defined in Equation (2.6) satisfies the cocycle condition is immediate from functoriality of $\overline{f}$. That f also satisifes the colimit condition follows from applying Lemma 2.2.53, which gives

$$\operatorname*{colim}_{q' \prec q} f(q') = \operatorname*{colim}_{q' \prec q} \overline{f}(I_{q'}) \cong \overline{f}(I_q) = f(q).$$

Step 2: Proving Equivalence. Suppose we are given $f\colon Y \to \mathcal{E}$. Following Step 1, define

$$\begin{aligned} g\colon Y &\longrightarrow \mathcal{E} \\ q &\longmapsto \operatorname*{colim}_{q' \in I_q} f(I_q). \end{aligned}$$

We claim that $g \cong f$. Why? By Lemma 2.2.54, there exists a canonical epimorphism $\psi\colon Y \to \mathsf{RIdl}(Y, \prec)$ such that $\psi(q) = I_q$. Note that $g(q) = \overline{f} \circ \psi(q)$ where $\overline{f}$ is defined as in Equation (2.5). As such,

$$g(q) = \overline{f} \circ \psi(q) = \overline{f}(I_q) = \operatorname*{colim}_{q' \prec q} f(q') \cong f(q),$$

where the final isomorphism follows from the colimit condition.

Conversely, suppose we are given $\overline{f}\colon \mathsf{RIdl}(Y, \prec) \to \mathcal{E}$. By Step 1, define

$$\begin{aligned} \overline{g}\colon \mathsf{RIdl}(Y, \prec) &\longrightarrow \mathcal{E} \\ I &\longmapsto \operatorname*{colim}_{q \in I} \overline{f}(I_q) \end{aligned}$$

By Lemma 2.2.53, it is clear that $\overline{g} \circ \psi \cong \overline{f} \circ \psi$. Since ψ is epi by Lemma 2.2.54, this gives $\overline{f} \cong \overline{g}$, finishing the proof. □

For later quotation in Chapter 3, we specialise Lemma 2.2.55 to the following case:

Lemma 2.2.56. *As our setup,*

- *Let* $(X, \sqsubseteq)$ *be any localic space in* Loc *with specialisation order* $\sqsubseteq$ *on its points.*
- *Let* $(Y, <)$ *be an* R*-structure.*

Then, there exists a surjection $\psi\colon Y \to \mathsf{RIdl}(Y, <)$ *in* Loc *inducing an equivalence between:*

(i) Maps $\overline{f}\colon \mathsf{RIdl}(Y, <) \to X$
(ii) Maps $f\colon Y \to X$ *satisfying the following two lifting conditions for all* $q, q' \in Y$*:*

  - **Monotonicity:** $q' < q \implies f(q') \sqsubseteq f(q)$.
  - **Continuity**: $f(q) = \bigsqcup^{\uparrow}_{q'<q} f(q')$.

Remark 2.2.57. A *surjection* of localic spaces is an epimorphism in Loc; these maps are most simply characterised by the inverse image function $\Omega\phi$ being one to one. Notice, however, that in contrast to point-set topology, this does *not* imply that every point in the codomain has a preimage in the domain. This should be clear from Lemma 2.2.54, which constructs surjections such as

$$\begin{aligned} \psi\colon \mathbb{Q} &\longrightarrow \overrightarrow{(-\infty, \infty]} \\ q &\longmapsto I_q := \{q' \in \mathbb{Q} | q' < q\}, \end{aligned}$$

sending each rational q to its one-sided representative.

As an important application of the general Lifting Lemma 2.2.55, we get the following characterisation of sheaves on $\overleftarrow{[0,1]}$, which will be important in Chapter 6.

Observation 2.2.58. As our setup,

- Let F be an object in the category of sheaves $\mathcal{S}\overleftarrow{[0,1]}$ (cf. Remark 2.1.22)
- Denote $\mathbb{O}$ to be the theory of objects, i.e. it has one sort, and no functions, predicates or axioms.
- Denote $[\mathbb{O}]$ to be the *object classifier*, i.e. the space of models of $\mathbb{O}$.[47]

Then, F can be equivalently characterised as:

(i) F is a sheaf over $\overleftarrow{[0,1]}$;
(ii) $F\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$;
(iii) $F\colon \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$ is a map satisfying the continuity conditions of Lemma 2.2.55.

Proof. (i) $\iff$ (ii): Any model of $\mathbb{O}$ in any topos $\mathcal{E}$ corresponds to an object of $\mathcal{E}$ (i.e. a sheaf over the point-free space), essentially by construction.[48]

(ii) $\iff$ (iii): Immediate from the fact that $\overleftarrow{[0,1]} \cong \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$ (Example 2.2.51) and the Lifting Lemma 2.2.55. □

[47] Warning: not to be confused with the *subobject classifier*, which is an *object* living in each topos $\mathcal{E}$. By contrast, the object classifier $[\mathbb{O}]$ [more correctly, $\mathcal{S}[\mathbb{O}]$] corresponds to an actual topos.

[48] Following our discussion in Convention 2.1.7, let us remark that any 2-category $\mathfrak{BTop}/\mathcal{S}$ has an object classifier so long as $\mathcal{S}$ is an elementary topos with natural number object [Joh02a, Theorem B4.2.11].

2.2.4.2 Gluing. In Chapter 3 on exponentiation (specifically Theorem A), we would like to perform a construction for generic $x \in (0,\infty)$, except that it must be done along a case-splitting for $x \leq 1$ and $x \geq 1$, with agreement at $x = 1$. In point-set topology, this immediately gives a *function*, after which a continuity proof is needed. Working point-free, however, means we must work more carefully to maintain geometricity because, reasoning in $\mathcal{S}(0,\infty)$, the conditions $x \leq 1$ and $1 \leq x$ are not geometric formulae.[49]

Categorically, what we need to prove is that the left-hand diagram below is a pushout square:

$$\begin{array}{ccc} \{1\} & \hookrightarrow & (0,1] \\ \downarrow & & \downarrow \\ [1,\infty) & \hookrightarrow & (0,\infty) \end{array} \qquad \{1\} \underset{1_L}{\overset{1_R}{\rightrightarrows}} (0,1] \amalg [1,\infty) \xrightarrow{P} (0,\infty) \tag{2.7}$$

Or, equivalently, that the right-hand diagram is a coequaliser, where 1_L and 1_R are the global points 1 in the left and right components, and P is the copairing of the two natural inclusions.

We justify this by applying Vermeulen's work [Ver94] on proper maps. There are various equivalent characterizations of "proper", showing the connection with the point-set notion, and the most relevant for our purposes is that a map $f\colon Y \to X$ is proper iff it is fibrewise compact[50]. In particular, the class of proper surjections (cf. Remark 2.2.57) possesses many nice categorical properties, such as:

Proposition 2.2.59 ([Ver94, Propositions 4.2 and 5.4])**.**

(i) Proper surjections are coequalisers (of their kernel pair).
(ii) Proper surjections are stable under pullback, i.e. in a pullback square of spaces

$$\begin{array}{ccc} W & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

if f is a proper surjection, then so is h.

We now state and prove our gluing principle for $(0,1]$ and $[1,\infty)$ (but the same principle holds for any interval of $\mathbb{R}$ divided at a point).

Proposition 2.2.60 (Gluing Principle)**.** *The right-hand diagram of* (2.7) *is a coequaliser, stable under pullbacks.*

Proof. Our main step is to show that P is a proper surjection, and hence the stable coequaliser of its kernel pair. After that it remains to show that it is the stable coequaliser of the pair in the statement.

In fact we prove a stronger property of P, that it is an *entire* surjection. "Entire" means fibrewise Stone: in other words, each fibre $P^{-1}(x)$ is the spectrum (i.e. the space of prime filters) of a Boolean algebra B_x (of clopens). We define B_x to be the Boolean algebra presented by one generator α subject to the following relations:

$$B_x = BA\left\langle \alpha \;\middle|\; \begin{array}{ll} \alpha = 1 & (\text{if } x < 1) \\ \alpha = 0 & (\text{if } x > 1) \end{array} \right\rangle.$$

[49]Why not? Notice: $x \leq 1$ and $1 \leq x$ do not give open subspaces of $(0,\infty)$.

[50]One proves geometrically that $f^{-1}(x)$ is compact for a generic x, and this shows that the corresponding internal space in the topos of sheaves $\mathcal{S}X$ is compact.

We show that the fibre of P over each x is isomorphic to $\mathsf{Spec} B_x$.

For a geometric description of points of the coproduct, we write them as $(p?\, y \colon z)$, an abbreviation of the notation **if** p **then** y **else** z used in [Vic99, §2.2.6]. Here, p is a decidable proposition, y is a point of $(0, 1]$ defined if p holds, and z a point of $[1, \infty)$ defined if $\neg p$. Then, as described in [Vic99], a copairing map $[f, g]$ maps $(p?\, y \colon z)$ to the directed join $\bigsqcup^{\uparrow}(\{f(y) \mid p\} \cup \{g(z) \mid \neg p\})$.

For P, it follows that a point of the fibre $P^{-1}(x)$ must be of the form $(p?\, x \colon x)$. Thus if $x < 1$ then p must be true, since x is not defined as point of $[1, \infty)$; and, similarly, if $1 < x$ then p must be false:

$$P^{-1}(x) = \begin{cases} 2, & \text{if } x = 1 \\ 1, & \text{if } x < 1 \text{ or } 1 < x \end{cases}$$

To map $P^{-1}(x)$ to $\mathsf{Spec} B_x$, we map $(p?\, x \colon x) \mapsto \{1\} \cup \{\alpha \mid p\} \cup \{\neg\alpha \mid \neg p\}$. To show that this subset F of B_x is a prime filter, the main non-trivial check is to show it does not contain 0. To see this, consider if $\alpha = 0$. Then $x > 1$, hence $\neg p$, and $\alpha \neq F$. The case $\neg\alpha = 0$ is similar.

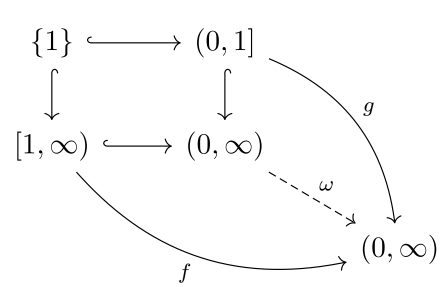
For the reverse direction we map $F \mapsto (\alpha \in F?\, x \colon x)$. Note that $\alpha \in F$ is decidable; its complement is $\neg\alpha \in F$. If $\alpha \in F$ then $\alpha \neq 0$, so $x \leq 1$ and x is defined as point of $(0, 1]$. Similarly, if $\neg\alpha \in F$ then x is defined as point of $[1, \infty)$. It follows that $(\alpha \in F?\, x \colon x)$ is a point of $P^{-1}(x)$.

The two maps are mutually inverse, which proves our claim that P is entire. It is surjective because every B_x is non-degenerate (i.e. it has $1 \neq 0$), essentially because we cannot have both $x < 1$ and $x > 1$.

Now we know that P is the coequaliser of its kernel pair, it remains to show that the kernel pair and the pair $(1_R, 1_L)$ have the same coequalisers. The kernel pair, the pullback of P against itself, can be calculated as the coproduct of four pairwise pullbacks of the components of $(0, 1] \amalg [1, \infty)$. Since both $(0, 1]$ and $[1, \infty)$ are embedded in $(0, \infty)$, their kernel pairs are just the reflexive parts and are irrelevant to the coequaliser. The pullback $(0, 1] \times_{(0,\infty)} [1, \infty)$, the space of pairs (x, x) such that $1 \geq x \geq 1$, is just $\{1\}$, and the remaining component is just the reverse of that, and implied by symmetry. Hence the kernel pair has the same coequaliser as the pair $(1_L, 1_R)$. □

The following corollary gives an explicit translation of the Gluing Principle:

Corollary 2.2.61. *The left-hand diagram in* (2.7) *is a pushout square. In particular, given two maps* $f \colon [1, \infty) \to (0, \infty)$ *and* $g \colon (0, 1] \to (0, \infty)$ *such that* $f(1) = g(1)$, *we can glue them together to obtain a map* $\omega \colon (0, \infty) \to (0, \infty)$ *via the pushout property:*



Discussion 2.2.62 (Stability under pullback)**.** Why ask for the coequaliser to be stable in the Gluing Principle? The short answer: geometricity. To elaborate, the Gluing Principle is meant to provide a geometric justification for the case-splitting along $x \leq 1$ and $x \geq 1$: the pushout property gives a framework for gluing the two cases together, whereas stability under pullback tells us the gluing is geometric (cf. Convention 2.1.17).

2.2.4.3 The Case-Splitting Lemma. One may also wonder if we can glue two pieces of a construction defined on two different spaces if the spaces do not overlap at a single point, e.g. if spaces U and U^{c} turn out to be complements of each other in a space $X = U \vee U^{\mathsf{c}}$. The following Case-Splitting Lemma establishes the general conditions when this is permitted, and will be important in our geometric proof of Ostrowski's Theorem (where we have to case-split between the Archimedean vs. non-Archimedean case).

Lemma 2.2.63 (Case-Splitting Lemma)**.** *Consider the following cospan in* Loc

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{2.8}$$

where i is an inclusion. Further, suppose that:

- *$X = U \vee U^{\mathsf{c}}$, where U is an open subspace of X and U^{c} is its closed complement.*[51]
- *There exist the following maps*
  - (Inclusions) $i_1 \colon U \hookrightarrow X$, $i_2 \colon U^{\mathsf{c}} \hookrightarrow X$*; and*
  - (Transformations) $f_1 \colon U \hookrightarrow Y$, $f_2 \colon U^{\mathsf{c}} \hookrightarrow Y$,

  such that $f \circ i_1 = i \circ f_1$ and $f \circ i_2 = i \circ f_2$.

Then*, the pullback P of the cospan in Equation* (2.8) *is isomorphic to X.*

Proof. Since Loc possesses all pullbacks, we know that the pullback P of Diagram (2.9) exists:

$$\begin{array}{ccc} P & \xrightarrow{\widehat{f}} & Y \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{2.9}$$

Further, recall from [Joh82, §II.2.1] that inclusion maps of (localic) spaces are precisely the regular monics in Loc. Since regular monics are preserved by pullback, this implies the map $p \colon P \to X$ of Diagram (2.9) is a regular monic as well. In English, this means: the pullback P is a subspace of X.

Exploiting the universal pullback property, we obtain the following diagrams:

$$\begin{array}{cccc} U & \overset{f_1}{\curvearrowright} & & \\ & \overset{\theta_1}{\searrow} & & \\ & P & \xrightarrow{\widehat{f}} & Y \\ {\scriptstyle i_1} & {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle i} \\ & X & \xrightarrow[f]{} & Z \end{array} \qquad \begin{array}{cccc} U^{\mathsf{c}} & \overset{f_2}{\curvearrowright} & & \\ & \overset{\theta_2}{\searrow} & & \\ & P & \xrightarrow{\widehat{f}} & Y \\ {\scriptstyle i_2} & {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle i} \\ & X & \xrightarrow[f]{} & Z \end{array} \tag{2.10}$$

[51] Here we exploit the following fact: given a localic subspace X, its open and closed subspaces generate a Boolean algebra. For more details, particularly on the constructive/predicative aspects of this result, see [Vic07c]. Note that this builds on Discussion 2.1.14 and Footnote 23, where we first commented on the interaction between constructive questions and topology.

Since $i_1 = p \circ \theta_1$ and $i_2 = p \circ \theta_2$ are regular monics, this implies θ_1 and θ_2 are regular monics as well, i.e. U and U^{c} are subspaces of P. Since spaces are closed under finite joins of their subspaces[52], this implies $U \vee U^{\mathsf{c}}$ is a subspace of P. But since $U \vee U^{\mathsf{c}} = X$ (by hypothesis), and since P is also a subspace of X, conclude that $P \cong X$. □

2.3 Interlude: A Walk Between Two Worlds

Let's step back for a moment. What were the key ideas developed in the previous sections, and how do they relate to the issues raised in Chapter 1? Recalling Question 1, we want a framework that treats the **reals** $\mathbb{R}$ and the **p-adics** $\mathbb{Q}_p$ symmetrically whilst also being sensitive to their differences. Chapter 1 gave the example of Arakelov Geometry as one such possible framework, but also pointed out some issues with its set-theoretic approach. We subsequently remarked that if there exists a classifying topos of completions of $\mathbb{Q}$, then this pulls Question 1 away from classical set theory and opens it up to new tools of analysis.

Both Sections 2.1 and 2.2 substantiate this remark. We saw how the topos can be regarded as a generalised space whose points correspond to models of a geometric theory (Definition 2.0.1). We also saw how the generic model is a powerful device for reasoning about all models of a theory simultaneously, which can be leveraged to investigate how these generalised spaces interrelate. Along the way, we were careful to highlight various parallels and differences between geometric vs. classical mathematics. One way to view these differences is that certain tools, e.g. the Axiom of Choice, are unavailable to us if we wish to work geometrically. However, another way of seeing things is that geometric mathematics is sensitive to certain nuances that are elided by classical assumptions. This was seen (for instance) in Discussion 2.1.14 (regarding the topological significance of decidability), but also emerged in our analysis of localic reals and primes (see e.g. the discussion contrasting the one-sided vs. Dedekind reals).

These insights combine to support the following picture. Suppose there exists a topos $\mathcal{S}[\mathbb{T}]$ that classifies the completions of $\mathbb{Q}$ up to equivalence. By Fact 2.1.23, we know that $\mathcal{S}[\mathbb{T}]$ possesses a generic model $U_{\mathbb{T}}$ whereby any geometric property ϕ is satisfied by $U_{\mathbb{T}}$ iff ϕ is satisfied by all completions of $\mathbb{Q}$.[53] Put otherwise, the so-called generic completion $U_{\mathbb{T}}$ allows us to reason about all completions of $\mathbb{Q}$ in a symmetric manner, so long as we do so geometrically. In addition, since:

(a) There exists deep interactions between logic and number theory (see e.g. the Ax-Kochen Theorem);
(b) Geometric mathematics detects various subtleties that classical mathematics does not,

the potential for seeing new things in this topos-theoretic framework is also significant. Of course, we still have yet to properly motivate the importance of developing a unifying framework that treats all the completions of $\mathbb{Q}$ symmetrically. Let us therefore turn to the next section, which provides the number-theoretic context for this thesis.

[52] In fact, the subspaces of a localic space X form a co-frame [PP12, Theorem 3.2.1].

[53] In anticipation of later results, we are being deliberately loose with our language here. One may ask: since $[\mathbb{T}]$ classifies the completions of $\mathbb{Q}$ up to equivalence, should we think of the $\mathbb{T}$-models as completions of $\mathbb{Q}$ or the equivalence classes of completions of $\mathbb{Q}$ here? This question turns out to be surprisingly subtle in the geometric setting – see Footnote 67.

2.4 The Local-Global Principle

> *An important point is that the p-adic field, or respectively the real or complex field, corresponding to a prime ideal, plays exactly the role in arithmetic that the field of power series* in the neighbourhood of a point *plays in the theory of functions: that is why one calls it a* local field.
>
> **— André Weil, letter to his sister [Wei05]**

In mathematics, one often tackles a problem by breaking it up into smaller pieces, solving each of the smaller pieces, before reassembling the answers to obtain a solution to the original problem. This raises two natural questions:

(a) How do we account for all the different pieces of the problem?
(b) How/when can we glue the pieces of the solution together? In particular, what are the obstructions to reassembling a global solution from its local pieces?

These questions present a significant challenge to our understanding, revealing a deep nerve connecting many important conjectures in mathematics. In this section, we focus on the so-called *Local-Global Principle*, and its organising influence on reassembly problems in Arithmetic Geometry and Homotopy Theory.

2.4.1 Arithmetic Geometry and Rational Points.

2.4.1.1 Local vs. Global Solubility. Consider a polynomial, say

$$X^n + Y^n + Z^n = 0, \qquad n > 2.$$

It is natural to ask: does this polynomial have non-trivial $\mathbb{Q}$-solutions? That is, are there $x, y, z \in \mathbb{Q}$ such that $x^n + y^n + z^n = 0$ and x, y, z are not all 0?

This is a difficult question in general. Nonetheless, recall that one may define a metric on $\mathbb{Q}$, which allows us to construct new fields containing the points of $\mathbb{Q}$ plus some additional 'new points', providing a richer geometry. Such fields are called *completions of* $\mathbb{Q}$. By Ostrowski's Theorem, the only non-trivial completions of $\mathbb{Q}$ (up to equivalence) are the **reals** $\mathbb{R}$ and the p**-adic numbers** $\mathbb{Q}_p$ for all primes p.

Observation 2.4.1.

(i) $\mathbb{Q}$ embeds into its completions $\mathbb{R}$ and $\mathbb{Q}_p$; hence, the existence of a $\mathbb{Q}$-solution implies the existence of a solution in *all* completions of $\mathbb{Q}$.
(ii) It is easier to determine if a given polynomial has $\mathbb{R}$-solutions or $\mathbb{Q}_p$-solutions as opposed to determining if it has a $\mathbb{Q}$-solution — this is due to the completions' richer structure.

In light of this, we may reformulate our original question: if the existence of $\mathbb{Q}$-solutions implies the existence of solutions in *all* completions of $\mathbb{Q}$, when does the converse hold? More explicitly, define the following Local-Global Principle:

Definition 2.4.2 (Hasse Principle)**.** A polynomial over $\mathbb{Q}$ is said to follow the *Hasse Principle* just in case it has $\mathbb{Q}$-solutions iff it has solutions for all (non-trivial) completions of $\mathbb{Q}$.

When then does the Hasse Principle hold? The answer is frustratingly cryptic: sometimes. We illustrate with two contrasting examples.

Theorem 2.4.3 (Hasse-Minkowski Theorem [Ser73, Theorem IV.8])**.** *Let f be a quadratic form over $\mathbb{Q}$, i.e. f is of the form*

$$f(x_1, \dots, x_n) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j, \qquad \text{for } a_{ij} \in \mathbb{Q}.$$

Then, f has a $\mathbb{Q}$-solution iff it has solutions over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p.

Example 2.4.4 (Lind [Lin40], Reichardt [Rei42])**.** The polynomial

$$2Y^2 = X^4 - 17Z^4 \tag{2.11}$$

is a counter-example to the Hasse principle.

Proof. We need to show that the polynomial has local solutions everywhere yet has no rational solutions. We follow the argument of [BTL18].

No rational solutions. Without loss of generality, assume x, y, z are integer solutions [by clearing denominators] with $\gcd(x, z) = 1$ and $y > 0$. Which primes divide y? If p is an odd prime and $p|y$, then

$$x^4 \equiv 17z^4 \bmod p,$$

and so 17 is a non-zero square $\bmod\ p$.

Next, recall the Legendre symbol, a function which defines for an integer a and odd prime p:

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \equiv n^2 \bmod p \text{ for some integer } n \text{ and } a \not\equiv 0 \bmod p \\ 0 & \text{if } a \equiv 0 \bmod p \\ -1 & \text{if otherwise.} \end{cases}$$

In particular, the law of quadratic reciprocity says: for all distinct odd primes p, p', we have

$$\left(\frac{p}{p'}\right)\left(\frac{p'}{p}\right) = (-1)^{\frac{p'-1}{2} \cdot \frac{p-1}{2}}.$$

Hence, since $\left(\frac{17}{p}\right) = 1$, deduce that p is a square $\bmod\ 17$. Similarly, notice that $2, 1, -1$ are also squares $\bmod\ 17$. As such, since the Legendre symbol is multiplicative in the numerator, deduce that $\left(\frac{y}{17}\right) = 1$. Hence, write $y \equiv y_0^2 \bmod 17$. Plugging this into Equation (2.11), we get

$$2y_0^4 \equiv x^4 \bmod 17,$$

and so 2 is a fourth power $\bmod\ 17$. But this is not true, and so rational solutions cannot exist.

All local solutions exist. The existence of real solutions is obvious — take, e.g. $(x, y, z) = (3, \sqrt{32}, 1)$. The existence of p-adic solutions require more work. Recall: for any smooth projective curve C of genus g over a finite field $\mathbb{F}_p$, the number of $\mathbb{F}_p$-points of C satisfies the Hasse-Weil bound

$$|\#C(\mathbb{F}_p) - (p + 1)| \leq 2g\sqrt{p}. \tag{2.12}$$

Next, by giving variable Y weight 2, one checks that the polynomial defines a smooth, projective curve of genus 1 (in a weighted projective space). Note this implies the existence of $\mathbb{Q}_p$-solutions for all $p \geq 3$ *so long as* Equation (2.11) has smooth reduction $\bmod p$. Why? Applying the Hasse-Weil Bound, we know the (smooth) reduction must have at least $p + 1 - 2\sqrt{p} > 0$ points over $\mathbb{F}_p$, and that these all lift to a point over $\mathbb{Q}_p$ via Hensel's Lemma.

It remains to check the (finitely) many primes of bad reduction, namely $p = 2, 17$. In which case, one can apply Hensel's Lemma once more to find an explicit solution[54] — take, e.g. $(x, y, z) = (\sqrt[4]{17}, 0, 1)$ for a $\mathbb{Q}_2$-solution and $(x, y, z) = (1, \frac{1}{\sqrt{2}}, 0)$ for a $\mathbb{Q}_{17}$-solution. □

Discussion 2.4.5 (Quadratic Reciprocity)**.** Example 2.4.4 illustrates that the existence of local solutions does not guarantee the existence of global solutions. Notice, however, the argument itself is not entirely local since we use quadratic reciprocity, which links the behaviour at one prime with behaviour at another. This indicates that the local pieces do not behave independently, a crucial fact exploited in the analysis of many other examples, e.g. [BTL18, Example 2.3.5].

Discussion 2.4.6 (Finitely many local checks)**.** Analyses of local solutions for varieties over $\mathbb{Q}$ are typically guided by the following general facts[55]:

(a) A smooth variety over $\mathbb{Q}$ has smooth reduction at almost all primes.
(b) A smooth variety over a finite field $\mathbb{F}_q$ has points over $\mathbb{F}_q$ when q is sufficiently large.
(c) Smooth points over the residue field lift to points over the completion via Hensel's Lemma.

Collectively, Facts (a) - (c) imply a remarkable result: any smooth variety X over $\mathbb{Q}$ automatically has local solutions almost everywhere. Hence, as we saw in Example 2.4.4, once the appropriate hypotheses have been verified, the task of checking that X has local solutions everywhere reduces to an explicit check at finitely many places. Of course, much work is first needed to establish these facts in the appropriate generality, particularly Fact (b) – see Discussion 2.4.7.

Discussion 2.4.7 (Point-Counting and the Weil Conjectures)**.** Notice we were able to deduce that Example 2.4.4 has local solutions at almost all primes p because the smooth projective model of the polynomial was a curve, allowing us to apply the Hasse-Weil bound. For more general varieties, one can use the Weil conjectures (more specifically, the Grothendieck-Lefschetz Trace formula) to bound the number of points of a variety over a finite field.

We develop this remark with a brief summary. Let X_0 be a smooth projective[56], geometrically irreducible[57] variety of dimension d over finite field $\mathbb{F}_q$ of q elements. Let ℓ be a prime $\ell \nmid q$, and let $\overline{X}_0$ denote the base change of X_0 to an algebraic closure of $\mathbb{F}_q$. One can then define the ℓ-adic cohomology groups

[54] In particular, suppose we have a p-adic integer $a \in \mathbb{Z}_p$ where $a \not\equiv 0 \bmod p$. Then, applying Hensel's Lemma, we get that $x^2 = a$ has a solution in $\mathbb{Z}_p$ iff a is a quadratic residue $\bmod\, p$ (if p is odd) *or* $a \equiv 1 \bmod 8$ (if $p = 2$).

[55] In fact, these general facts hold for any number field. For details, see e.g. [BTL18, §2.2].

[56] For affine schemes, we shall need to replace étale cohomology with *compactly supported* étale cohomology.

[57] That is, X is irreducible over the algebraic closure of $\mathbb{F}_q$.

using étale cohomology. By work of Grothendieck and others [Mil80, §VI.13], it can be shown that the number of points in $X_0(\mathbb{F}_q)$ is given by the Grothendieck-Lefschetz Trace Formula

$$\#X_0(\mathbb{F}_q) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(\mathrm{Frob}^*|_{H^i(\overline{X}_0,\mathbb{Q}_\ell)}), \tag{2.13}$$

where $\mathrm{Frob}^*\colon H^i(\overline{X}_0,\mathbb{Q}_\ell) \to H^i(\overline{X}_0,\mathbb{Q}_\ell)$ is the endomorphism on $H^i(\overline{X}_0,\mathbb{Q}_\ell)$ induced by the q-power Frobenius morphism[58] on $\overline{X}_0$. We next record the following key facts:

(a) The eigenvalues of Frob^* acting on $H^i(\overline{X}_0,\mathbb{Q}_\ell)$ are algebraic integers α such that $|\alpha| = q^{\frac{i}{2}}$ [Del74, Lemma 1.7].
(b) $H^i(\overline{X}_0,\mathbb{Q}_\ell)$ are finite-dimensional vector spaces over $\mathbb{Q}_l$.
(c) The trace of an endomorphism is the sum of its eigenvalues.
(d) $H^i(\overline{X}_0,\mathbb{Q}_\ell) = 0$ for $i > 2d$, while $H^{2d}(\overline{X}_0,\mathbb{Q}_\ell)$ and $H^0(\overline{X}_0,\mathbb{Q}_\ell)$ have dimension 1 since X_0 is geometrically irreducible.

Facts (a) - (c) tell us that we may bound $\mathrm{Tr}(\mathrm{Frob}^*|_{H^i(\overline{X}_0,\mathbb{Q}_\ell)})$ via the dimensions of $H^i(\overline{X}_0,\mathbb{Q}_\ell)$. Combined with Fact (d), this gives the inequality:

$$|\#X_0(\mathbb{F}_q) - (q^d+1)| \leq \sum_{i=1}^{2d-1} q^{\frac{i}{2}} \dim H^i(\overline{H}_0,\mathbb{Q}_\ell). \tag{2.14}$$

The reader should compare Equation (2.14) with Equation (2.12). For details, we recommend [FK88; Mil80].

2.4.1.2 Classical Definitions. Our discussion reveals a tight connection between the different places of $\mathbb{Q}$, and how this may be leveraged to obtain number-theoretic insights. This is obviously true when the Hasse Principle holds, but we also saw how e.g. quadratic reciprocity was important in understanding how it fails (Discussion 2.4.5). Nonetheless, certain issues raised in Chapter 1 remain unresolved. In particular:

(a) Given the differences between the p-adics vs. the reals, what language should we use to describe all the local pieces of the problem?
(b) We also saw how point-set and analogical reasoning sometimes combine to give formal constructions that obscure certain parts of the mathematics – e.g. the Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$, which essentially treats the "real prime" as a black box. Given the prevalence of point-set reasoning in classical mathematics, do similar issues arise here?

Both concerns track a deeper issue: although the problem of finding $\mathbb{Q}$-solutions to a polynomial appears algebraic in formulation, it also has a topological character, especially once we start asking about local solutions over the completions of $\mathbb{Q}$. This becomes clear once one properly examines the definitions; let us therefore pause to review the basics.

Definition 2.4.8.

(i) An *absolute value of* $\mathbb{Q}$ is a map $|\cdot|\colon \mathbb{Q} \to [0,\infty)$ satisfying the following axioms:

[58] Recall: for any variety X defined over $\mathbb{F}_q$, one can define a Frobenius automorphism on every affine chart given by $\mathrm{Frob}_q(x_1,\ldots,x_k) = (x_1^q,\ldots,x_k^q)$.

- *Positive Definite.* $|x| = 0$ iff $x = 0$
- *Multiplicative.* $|xy| = |x||y|$
- *Triangle Inequality.* $|x + y| \leq |x| + |y|$

(ii) If the absolute value also satisfies the *ultrametric inequality*, that is

$$|x - y| \leq \max\{|x|, |y|\},$$

then we call $|\cdot|$ *non-Archimedean*. Otherwise, we call $|\cdot|$ *Archimedean*. Notice the ultrametric inequality implies triangle inequality.

(iii) A *completion of* $\mathbb{Q}$ (with respect to an absolute value $|\cdot|$) is a metric space X, defined as[59]:

Underlying set of X = the set of $|\cdot|$-Cauchy sequences quotiented by the set of $|\cdot|$-Cauchy sequences converging to 0;
Topology on X = metric topology induced by $|\cdot|$.

Example 2.4.9. We record the standard examples; x here will always denote a rational.

(i) The *trivial absolute value* on $\mathbb{Q}$, denoted $|\cdot|_0$, is defined as

$$|x|_0 = 1, \qquad \text{for all } x \neq 0.$$

Any other absolute value is called *non-trivial*. The completion of $\mathbb{Q}$ with respect to $|\cdot|_0$ is $\mathbb{Q}$ itself.

(ii) The *Euclidean absolute value* on $\mathbb{Q}$, denoted $|\cdot|_\infty$, is defined as the usual norm

$$|x|_\infty = x, \qquad \text{for all } x \in \mathbb{Q}.$$

In particular, $|\cdot|_\infty$ is Archimedean, and the completion with respect to $|\cdot|_\infty$ is the reals $\mathbb{R}$.

(iii) Fix a prime p of the integers $\mathbb{Z}$. If $n \in \mathbb{Z}$ and $n \neq 0$, we define its *p-adic ordinal* as

$$\mathrm{ord}_p(n) := \max\{r \in \mathbb{N} \,\big|\, p^r \text{ divides } n\}.$$

If $\frac{a}{b} \in \mathbb{Q}$ is a non-zero rational with $a, b \in \mathbb{Z}$, we define the natural extension

$$\mathrm{ord}_p\left(\frac{a}{b}\right) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b).$$

This allows us to define the *p-adic absolute value* on $\mathbb{Q}$, denoted $|\cdot|_p$, as

$$|x|_p = p^{-\mathrm{ord}_p(x)}, \qquad \text{for all } x \neq 0.$$

In particular, $|\cdot|_p$ is non-Archimedean, and the completion with respect to $|\cdot|_p$ is the p-adics $\mathbb{Q}_p$.

Next, notice the Hasse principle asks for solutions in non-trivial completions of $\mathbb{Q}$ up to equivalence. The following fact shows that the (topological) equivalence of completions has an algebraic characterisation via the key definition of a *place*:

Fact 2.4.10. A *place of* $\mathbb{Q}$ is defined as an equivalence class of absolute values, where $|\cdot|_1 \sim |\cdot|_2$ iff there exists $\alpha \in (0, 1]$ such that $|\cdot|_1^\alpha = |\cdot|_2$ or $|\cdot|_2^\alpha = |\cdot|_1$. In particular:

[59]There is an alternative definition, which says: $(X, \widetilde{|\cdot|})$ is a completion of $(\mathbb{Q}, |\cdot|)$ if X is complete as a metric space, $\widetilde{|\cdot|}$ extends $|\cdot|$ and $\mathbb{Q}$ is dense in X. However, since completions with respect to $|\cdot|$ are unique (up to isomorphism), we chose to give an explicit construction instead.

(i) Let $|\cdot|_1, |\cdot|_2$ be two absolute values of $\mathbb{Q}$ belonging to the same place. Then, $|\cdot|_1, |\cdot|_2$ define homeomorphic completions of $\mathbb{Q}$.
(ii) The non-trivial completions of $\mathbb{Q}$ are precisely $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p (up to equivalence).

Proof. (i) follows from checking that $|\cdot|_1, |\cdot|_2$ define the same conditions for convergence: $|x_n - x|_1 \to 0$ iff $|x_n - x|_2 \to 0$. (ii) follows from Ostrowski's Theorem [which holds: all non-trivial absolute values are equivalent to $|\cdot|_\infty$ or $|\cdot|_p$ for some prime p] and applying (i). □

Since the Hasse Principle asks about solutions over all non-trivial completions of $\mathbb{Q}$, it is helpful to have a device that allows us to reason about properties that hold simultaneously for *all* such completions of $\mathbb{Q}$ (up to equivalence). This sets up the following definition:

Definition 2.4.11 (Adele Ring)**.** As our setup,

- Denote $\widehat{\mathbb{Z}}_p := \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$ as the ring of p-adic integers.
- Denote $\Lambda_\mathbb{Q} \setminus \{0\}$ as the set of all non-trivial places of $\mathbb{Q}$; note this excludes the trivial place but includes the Archimedean real place, denoted as ∞.
- For $v \in \Lambda_\mathbb{Q} \setminus \{0\}$, we define:

$$\mathbb{Q}_v := \begin{cases} \mathbb{Q}_p, & \text{if } v = p \text{ for some prime } p \text{ in } \mathbb{Z} \\ \mathbb{R}, & \text{if } v = \infty \end{cases} .$$

Then, the *adele ring of* $\mathbb{Q}$ — denoted $\mathbb{A}_\mathbb{Q}$ — can be equivalently characterised as ...

(i) ... the *restricted product* of all (non-trivial) completions of $\mathbb{Q}$:

$$\mathbb{A}_\mathbb{Q} := \prod_{v \in \Lambda_\mathbb{Q} \setminus \{0\}}' \mathbb{Q}_v := \left\{ (x_v) \in \prod_{v \in \Lambda_\mathbb{Q} \setminus \{0\}} \mathbb{Q}_v \;\middle|\; x_v \in \widehat{\mathbb{Z}}_p \text{ at all but finitely many places } v \right\}$$

(ii) ... the tensor product:

$$\mathbb{A}_\mathbb{Q} := \left(\mathbb{Q} \otimes_\mathbb{Z} \prod_p \widehat{\mathbb{Z}}_p \right) \times \mathbb{R}$$

The adelic construction is technical, but parsing its details reveals interesting insights.

Discussion 2.4.12 (Discrete vs. Topological Algebra)**.** Why the use of the restricted product in Definition 2.4.11? If the intention was to provide a construction that accounts for all completions of $\mathbb{Q}$ simultaneously, why not e.g. define the adele ring as the obvious direct product

$$\mathbb{A}_\mathbb{Q} := \prod_{v \in \Lambda_\mathbb{Q} \setminus \{0\}} \mathbb{Q}_v \qquad ? \tag{2.15}$$

The textbook answer, interestingly, involves an appeal to topology. Going back to Tate's thesis [Tat50], it is well-known that techniques from harmonic analysis can be applied to the adeles to study e.g. the functional equations for ζ-functions over number fields. However, note:

(a) Harmonic analysis is conventionally defined on *locally compact* topological groups.[60]
(b) The restricted product of (non-trivial) completions of $\mathbb{Q}$ is locally compact, whereas their direct product fails to be.[61]
(c) When X is a projective variety, then $X(\mathbb{A}_{\mathbb{Q}}) = \prod_v X(\mathbb{Q}_v)$, i.e. X is everywhere locally soluble iff it has an adelic solution (for details, see [BTL18, §2.2]).

Put together, this makes a strong case for favouring the restricted product over the direct product when defining the adeles. But more fundamentally, it reinforces a recurrent theme of this chapter: namely, that it is worth paying attention to the topological character of algebraic constructions, as opposed to just viewing them as discrete structures.

Discussion 2.4.13 (p-adics vs. Reals)**.** There is an asymmetry in both definitions of the adele ring: there does not exist an analogue of the p-adics integers $\widehat{\mathbb{Z}}_p$ for $\mathbb{R}$. This is true regardless of which characterisation of $\widehat{\mathbb{Z}}_p$ one chooses to use: one easily checks that $\{x \in \mathbb{R} \,\big|\, |x|_\infty \leq 1\}$ does not define a ring in $\mathbb{R}$, and it is not even clear how one might implement the more algebraic characterisation

$$\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

when $p = \infty$. It appears our decision to regard the real place as a formal prime has caught up with us. This raises sharp questions about the extent to which the language of primes is suitable for describing the places of $\mathbb{Q}$. In particular, if the usual finite primes measure divisibility of an integer, what exactly is the infinite prime meant to measure?

Finally, let us briefly mention a more mysterious construction, known as the *Tate-Shafarevich group*[62], which gives a deeper insight into the failure of the Hasse principle.

Definition 2.4.14. Let A be an abelian variety over $\mathbb{Q}$.[63] The *Tate-Shafarevich Group* of A is

$$Ш(A) := \ker\big(H^1(\mathbb{Q}, A) \xrightarrow{\gamma_A} \prod_{v \in \Lambda_{\mathbb{Q}} \setminus \{0\}} H^1(\mathbb{Q}_v, A)\big),$$

where γ_A is the induced global-to-local map in Galois cohomology.

Classically, $Ш(A)$ can be viewed as defining the set of A-torsors (modulo equivalence) that have local solutions over every completion of $\mathbb{Q}$; in particular, non-zero elements of $Ш(A)$ correspond to varieties that do *not* have a rational point, and so give counter-examples to the Hasse principle.[64] Many natural questions can be asked about this group, but the most urgent seems to be the following:

[60] Why? Informally: harmonic analysis extends the usual Fourier analysis to study functions defined on domains other than the real line. To do this, one requires sufficient structure on the domain such that one can define a suitable analogue of the Fourier Transform. This is supplied by the so-called Haar measure, which exists (uniquely) on any locally compact topological group. For additional background, see e.g. [DE14], which the reader may wish to cross-reference with [Tat50, §1.3].

[61] This can be deduced from the following fact [Mun99, §29, Exercise 2]: if $\prod_{s\in S} X_s$ be a product of non-empty spaces, then $\prod_{s\in S} X_s$ is locally compact iff all X_s are locally compact *and* each X_s is compact for all but finitely many values of s.

[62] For those wondering (like the author of this thesis did) why many call it the "Tate-Shafarevich Group" rather than "Shafarevich-Tate Group", this is because the first letter Ш of Shafarevich apparently comes after T in the Cyrillic alphabet — I learnt this from Bjorn Poonen's monograph on rational points [Poo17]. Still, both terms appear frequently in the literature.

[63] That is, smooth projective varieties over $\mathbb{Q}$ which are also algebraic groups, with the group law $A \times A \to A$ defined over $\mathbb{Q}$. Again, the definition of the Tate-Shafarevich group can be extended to general number fields, but we restrict to $\mathbb{Q}$ for readability.

[64] For more details, see [Maz93] for a historical survey, which gives a progressively technical account of the Tate-Shafarevich group and related ideas; for a more concise presentation, see [Sil86, Chapter X].

Conjecture 2.4.15 (Tate-Shafarevich Conjecture)**.** For every abelian variety A over $\mathbb{Q}$, $Ш(A)$ is finite.

Although widely believed to be true, the finiteness of $Ш(A)$ has only been shown in special cases [Kol88; Rub87]; the original conjecture is still wide open, even for elliptic curves over $\mathbb{Q}$. The urgency of this conjecture lies in its strucutral implications. Just to quote a few well-known examples:

(a) If true, then the conjecture gives a sharper picture of why the Hasse Principle fails: suppose C is a smooth proper curve C over $\mathbb{Q}$ with Jacobian $\mathrm{Jac}(C)$ such that C has no K-rational divisor of degree 1. Then, if $Ш(\mathrm{Jac}(C))$ is finite then the so-called Brauer-Manin obstruction is the *only* obstruction to the Hasse principle for C [Sko01, Cor. 6.2.5].

(b) Denote $L(A, s)$ to be the L-function of an abelian variety A. Then, the Birch and Swinnerton-Dyer conjecture (another deep open problem) predicts that the leading coefficient of the Taylor series of $L(A, s)$ at $s = 1$ equals a product of various arithmetic invariants of A. Importantly, one of these invariants is the cardinality of $Ш(A)$, which is assumed to be finite. See [Poo17, §5.7.7] for a brief discussion, or [Tat66] for a more in-depth account.

All this has motivated a great deal of research on $Ш(A)$. Still, while our understanding of its structural aspects has improved over the years, progress towards actually proving the Tate-Shafarevich conjecture remains slow. Read in the present context, this reopens a basic question about definitions: what *is* the right way to characterise the class of locally-trivial A-torsors, and why? We shall return to this later in Section 2.5.

2.4.2 Fracture Theorems in Homotopy Theory.

> *Il y a là la possibilité d'une étude locale (au sens arithmétique!) des groupes d'homotopie …*
>
> **— J.P. Serre [Ser53]**

The approach of "working one prime at a time" also finds resonance in homotopy theory. Interestingly, homotopy theorists typically ignore the reals and work with just the *finite adeles*, defined as

$$\mathbb{A}^{\mathrm{fin}}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p.$$

The key insight is that $\mathbb{A}^{\mathrm{fin}}_{\mathbb{Q}}$ naturally breaks into two pieces: the rationals $\mathbb{Q}$ and the product of p-adic integers $\prod_p \widehat{\mathbb{Z}}_p$. The data can then be assembled into a pullback square, also known as the *Arithmetic Square*[65]:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \widehat{\mathbb{Z}}_p \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p \end{array} \tag{2.16}$$

This features two basic algebraic constructions:

[65] Interestingly, several homotopy theorists have recently taken to calling this pullback square the "Hasse Square" after the Hasse principle from the previous section — see e.g. [BB19; Gre19]. We shall prefer Sullivan's original name "Arithmetic Square" since the Hasse principle requires us to consider $\mathbb{R}$, which is missing here. Still, the suggestive name gives impetus to Problem 7.

(a) p-*Localisation.* Notice that $\mathbb{Q} \cong \mathbb{Z}_{(0)}$, where $\mathbb{Z}_{(0)}$ is the localisation of $\mathbb{Z}$ at (0) [and so inverts all primes $p \in \mathbb{Z}$]. Further, $\mathbb{Z}$ is the limit of the following diagram of p-localisations [Sul05, Prop. 1.12]:

$$\begin{array}{ccccc} \mathbb{Z}_{(2)} & & \mathbb{Z}_{(3)} & & \mathbb{Z}_{(5)} \quad \dots \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{Z}_{(0)} \cong \mathbb{Q} & & \end{array}$$

(b) p-*Completion.* Given a decreasing sequence of ideals in a ring R (with unit)

$$R := I_0 \supset I_1 \supset I_2 \supset \dots$$

such that

$$\bigcap_{j=1}^{\infty} I_j = \{0\},$$

one can canonically define the following metric on R

$$d(x, y) = e^{-k}, \quad e > 1$$

where $x - y \in I_k$ but $x - y \notin I_{k+1}$. This allows us to obtain a completion of R in the usual way. Notice: in the case where $I_j = (p^j) \subseteq \mathbb{Z}$, the completion is [isomorphic to] the usual ring of p-adic integers $\widehat{\mathbb{Z}}_p$. In particular, $\prod_p \widehat{\mathbb{Z}}_p$ denotes the product of all p-completions of $\mathbb{Z}$.

Put together, this gives a new variation on the Local-Global Principle. Since we know that the p-adic integers and the rationals map into a common domain (i.e. the finite adeles)

$$\begin{array}{ccc} & & \prod_p \widehat{\mathbb{Z}}_p \\ & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{A}_{\mathbb{Q}}^{\text{fin}} \end{array} \tag{2.17}$$

and since the integers $\mathbb{Z}$ can be recovered as a pullback of this cospan, one may ask: can questions about the integers first be answered over $\mathbb{Q}$ and $\prod_p \widehat{\mathbb{Z}}_p$, before being reassembled to yield an answer over $\mathbb{Z}$? Or, phrased more generally:

Question 6. Given a nice object X, to what extent can information about X be recovered from its localisations and completions?

There are many interesting extensions of Question 6 when X is a recognisably algberaic object (e.g. a ring, an R-module etc.); what was exciting about Sullivan's groundbreaking work in the 1970s was that he showed that Question 6 still made sense when X is a topological space. The crux move involves looking at spaces on the level of their homotopy/homology groups. For instance, given a simply connected space X, there exists a *rational space* $X_{\mathbb{Q}}$, unique up to homotopy equivalence, with a map $X \to X_{\mathbb{Q}}$ inducing an isomorphism on homotopy groups *once* tensored with $\mathbb{Q}$, i.e.

$$\pi_i(X_{\mathbb{Q}}) \cong \pi_i(X) \otimes \mathbb{Q}, \qquad \text{for all } i.$$

This construction $X_{\mathbb{Q}}$ is called the *rationalisation of* X. Analogously, one can construct for (simply connected) X its *profinite completion* $\widehat{X}$ as well as its *adele space* X_A, which assemble into the following diagram:

$$\begin{array}{ccc} & & \widehat{X} \\ & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & X_A \end{array} \tag{2.18}$$

For details on the constructions, see [Sul05, Ch. 3]. The punchline is Sullivan's remarkable result:

Theorem 2.4.16 ([Sul05, Prop. 3.20])**.** *When X is simply connected, the following diagram*

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & X_A \end{array} \tag{2.19}$$

is a homotopy pullback.

This theorem establishes the following Local-Global Principle: any sufficiently nice space X can be understood as being built from infinitely many p-adic pieces and one rational piece. This basic idea has been substantially developed to prove so-called "Fracture Theorems" in various homotopical contexts — from chromatic homotopy theory (with a view towards computing the stable homotopy groups of spheres) [BB19] to the setting of tensor-triangulated categories [BG20; BG].

Nonetheless, there are still some loose ends from Sullivan's original work. Sullivan was well aware that $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p$ was not the complete adele ring $\mathbb{A}_{\mathbb{Q}}$, and remarked [Sul05, pp. 87-88] that we ought to (somehow) incorporate $\mathbb{R}$ and work with the "complete adele type"

$$X_A \times X_{\mathbb{R}}.$$

However, progress on this problem appears to have stalled. New subtleties, of course, arise when working with real homotopy types, but there's also a more basic issue: it is unclear what p-localisation/completion means when $p = \infty$.[66] This leaves us with the following test problem:

Problem 7. How do we augment the Arithmetic Square such that it includes $\mathbb{R}$?

2.5 Adelic Geometry via Topos Theory

In our discussion of the Hasse Principle in Section 2.4, topos theory already played a background role. It is well-known that the topos was originally conceived by Grothendieck as a framework for developing étale cohomology, which was in turn motivated by the Weil conjectures. Example 2.4.4 gave a prototype argument for why some polynomials fail the Hasse principle; this was later expanded upon in Discussion 2.4.7, which gave the link between the existence of p-adic solutions and the Weil conjectures.

In this thesis, we set up and explore a different interaction between topos theory and local-global questions, this time with a view towards understanding both the p-adics *and* the reals. We already saw in Section 2.4 how the Hasse Principle gives clear motivation for developing constructions that allow us to

[66] In fact, this issue was already alluded to in Footnote 2 of the Introduction.

reason about behaviour at *all* places of $\mathbb{Q}$ simultaneously — examples of this included the adele ring (Definition 2.4.11) and the Tate-Shafarevich group (Definition 2.4.14). On the other hand, we continued to notice some issues with the default policy of describing places as primes — in particular, denoting the real place as the prime at infinity. While this may suffice for indexing the summands of the Arakelov divisor (as we saw in Chapter 1), the formal description of $p = \infty$ is less useful/applicable for more involved algebraic constructions – e.g. defining an Archimedean analogue of the p-adic integers (Discussion 2.4.13) or the p-localisation/completion of $\mathbb{Z}$ (cf. Problem 7).

This discussion highlights the same tension articulated by Question 1 at the start of this thesis: how do we create a framework that treats all the places of $\mathbb{Q}$ symmetrically whilst also accommodating their differences? In response to this challenge, various generalisations of commutative rings have been proposed (e.g. [Dur07; Har07]), each suggesting a different characterisation of the so-called "real prime". Our approach, by contrast, is topological in its orientation. In the language of Section 2.1, we ask: what does the (point-free) space of places of $\mathbb{Q}$ look like? More broadly, we are guided by the following test problem stated in the Introduction:

Problem 5. Construct and describe the classifying topos of completions of $\mathbb{Q}$ (up to equivalence).

The following informal discussion gives the motivation for our approach.

What can the point-free perspective tell us about the p-adics and the reals? We complete the discussion started in Section 2.3. Suppose there does exist such a topos $\mathcal{S}[\mathbb{T}]$ that classifies the completions of $\mathbb{Q}$ (up to equivalence). Then:

(a) $\mathcal{S}[\mathbb{T}]$ *possesses a generic model $U_{\mathbb{T}}$ (i.e. "the generic completion").* In particular, $U_{\mathbb{T}}$ is generic in the sense that any geometric sequent ϕ holds for $U_{\mathbb{T}}$ iff ϕ holds for all completions of $\mathbb{Q}$. The reader may have noticed a certain resonance between the generic completion and the adele ring $\mathbb{A}_{\mathbb{Q}}$ insofar as both constructions allow us to reason about all completions of $\mathbb{Q}$ simultaneously (cf. Discussion 2.4.12). However, there is a key difference: once we work geometrically, the emphasis shifts from reasoning about properties that hold over *all* completions (i.e. universal quantification) to properties that hold for the generic completion (i.e. generic reasoning). Furthermore, working generically pulls us away from treating the places of $\mathbb{Q}$ as an indexing set, an issue already flagged in Chapter 1 and still present in Definition 2.4.11 of the adele ring.

(b) $\mathcal{S}[\mathbb{T}]$ *corresponds to a (point-free) space* $[\mathbb{T}]$. In particular, the points of $[\mathbb{T}]$ correspond to the completions of $\mathbb{Q}$. This perspective gives a new language for investigating how the p-adics and the reals may interrelate. For instance, how do the various completions of $\mathbb{Q}$ fit together as points of $[\mathbb{T}]$? How might the differences between $\mathbb{R}$ and $\mathbb{Q}_p$ be reflected in this space? More broadly, how might analysing the geometry of $[\mathbb{T}]$ yield interesting insights into the completions of $\mathbb{Q}$ themselves?[67]

[67] The expert reader may recognise a resonance between this particular line of questions and the way we use moduli spaces to analyse a suitable family of geometric objects (equipped with a notion of equivalence) — in particular, it is well-known that analysing the geometric structure of a moduli space often yields valuable insights into the parametrised objects themselves. There is, however, a surprising difference in our setting. Whereas points of a moduli space are in bijection with the equivalence classes of parametrised objects, we do not expect the global points of $[\mathbb{T}]$ here to correspond bijectively to the equivalence classes of completions of $\mathbb{Q}$. In fact, as we shall later prove in Theorems F and G, quotienting a family of (equivalent) structures by their equivalence relation may or may not correspond to a single point, depending on the algebraic nature of the equivalence relation.

What can the generic place/completion of $\mathbb{Q}$ tell us about Local-Global questions? Much of this author's original excitement about this project laid in the potential applications once we construct the desired topos of completions. Two particularly interesting possibilities stood out:

(a) *Hasse Principle as a Transfer Principle.* Recall the Ax-Kochen-Eršov Principle (Theorem 3), which says: given any logical statement ϕ about valued fields, there exists a finite set of primes C such that ϕ holds for $\mathbb{Q}_p$ iff ϕ holds for $\mathbb{F}_p((t))$ just in case $p \notin C$. As discussed in Chapter 1, this transfer principle gives a powerful example of how logical methods can be applied to solve problems in number theory.

    It is therefore interesting to ask: can we use geometricity to establish analogous transfer principles to tackle Local-Global questions? Are there interesting/useful geometric properties ϕ exist such that ϕ holds for $\mathbb{Q}$ iff ϕ holds for the generic completion of $\mathbb{Q}$? How do they relate e.g. to the existence of polynomial solutions? As a warm-up problem, can we use genericity to give a different proof of the Hasse-Minkowski Theorem (Theorem 2.4.3)?

(b) *Tate-Shafarevich Group.* Let us make explicit the role of torsors in point-free topology. By Diaconsescu's Theorem, we know that the presheaf topos $[\mathcal{C}, \mathrm{Set}]$ classifies the flat functors from the small category $\mathcal{C}$. In particular, when $\mathcal{C}$ is a discrete group G, one easily checks that a flat functor from G gives the usual G-torsor arising in first cohomology groups [Joh77b, §8.3]. More generally, any pro-discrete groupoid G can be associated to a topos $\mathcal{E}$ whose points are the G-torsors; see [AGV72] or more generally [Bun90].

    Now recall that the Tate-Shaferevich group $Ш(A)$ is the subgroup of $H^1(\mathbb{Q}, A)$ whose elements are the A-torsors that are everywhere locally trivial. In particular, notice that the condition of "everywhere local" is enforced by the definition's use of a (set-indexed) direct product $\prod$. Read in the present context, one may wonder: can we eliminate $\prod$ by reasoning generically? More explicitly, can we reformulate the Tate-Shaferevich group as follows:

$$Ш(A) := \ker\left(H^1(\mathbb{Q}, A) \to H^1(\mathbb{Q}_v, A)\right),$$

    where $\mathbb{Q}_v$ now denotes the generic completion? Now that we are freed from having to keep track of infinitely many local pieces, does this reformulation open up a productive new line of attack on the Tate-Shafarevich conjecture?

Unfortunately (but also quite interestingly), both items (a) and (b) turn out to be too naive as stated. By convention, when the Hasse Principle asks if a result holds over $\mathbb{Q}$ iff it holds over all completions of $\mathbb{Q}$, it is of course assumed that we are asking about *non-trivial* completions of $\mathbb{Q}$. While excluding the trivial place/completion from consideration is no issue for the classical number theorist, our situation turns out to be much more subtle. As we shall see in Chapter 6, Theorem G tells us that the Archimedean place is equivalent to $\overleftarrow{[0,1]}$, indicating that the trivial and Archimedean place cannot be definably separated.

Where does this leave us? It appears we have traded one difficulty for another. Whereas the formal description of $p = \infty$ made it unclear how to properly incorporate the reals into an algebraic framework, the point-free perspective introduces new difficulties in separating the global (i.e. the trivial completion $\mathbb{Q}$) from the local (i.e. the non-trivial completions). Nonetheless, while immediate applications to Local-Global problems remain presently out of reach[68], the result that the Archimedean place is equivalent to

[68] Although, let us remark that the inseparability between the trivial place and the Archimedean place appears to be less of an immediate issue for the Fracture Theorems in homotopy theory. See also Section 6.5.2.2.

some blurred unit interval $\overleftarrow{[0,1]}$ is itself striking and opens up new urgent questions. Much more work will be needed to sort out the implications, some of which has already been started in Chapter 6. Perhaps one reason why previous generalisations of commutative algebra have not found much success in solving new number theory problems (despite offering an explicit candidate definition of the "real prime") is that there are important topological issues regarding the algebra of places that need to be understood first.

Chapter 3

Point-free Real Exponentiation

Our primary objective here is self-explanatory: we wish to develop an account of exponential and logarithmic functions

$$\begin{aligned} \exp\colon (0,\infty)\times\mathbb{R} &\longrightarrow (0,\infty) \\ (\,x\,,\,\zeta\,) &\longmapsto x^{\zeta} \end{aligned}$$

$$\begin{aligned} \log\colon (1,\infty)\times(0,\infty) &\longrightarrow \mathbb{R} \\ (\,b\,,\,y\,) &\longmapsto \log_b(y) \end{aligned}$$

that is *geometric* – that is to say, it is valid over any topos, and moreover is preserved by pullback along geometric morphisms. As explained before, an important motivation behind this is so that we can (geometrically) define the exponentiation of absolute values $|\cdot|^{\alpha}$ in order to analyse the places of $\mathbb{Q}$.

As established in Chapter 2, geometricity cannot be achieved using point-set topology, and instead a point-free approach must be taken — though in our geometric methodology the points will still play a major role. Indeed, this philosophy will guide our construction of the exponentiation and logarithm maps on the Dedekind reals, as well as our development of some of their obvious algebraic properties. As far as we are aware, this is the first time these maps have been defined in a point-free setting.

The heuristic behind our construction is simple, and involves building up to real exponentiation in increasing levels of (topological) complexity to get to the general case:

Step 1: Define natural number exponentiation for non-negative rationals: x^a for $x \in Q$ and $a \in \mathbb{N}$.
Step 2: Define natural number exponentiation for non-negative reals: x^a for $x \in [0,\infty)$ and $a \in \mathbb{N}$.
Step 3: Define rational exponentiation for non-negative reals: x^q for $x \in [0,\infty)$ and $q \in \mathbb{Q}$.
Step 4: Define real exponentiation for positive reals: x^{ζ} for $x \in (0,\infty)$ and $\zeta \in \mathbb{R}$.

However, Step 4 presents several geometric issues. For one, working with real exponents creates continuity issues at $x = 0$, forcing us to work with positive Dedekind base. Additionally, exponentiation can either be monotone or antitone, depending on whether $x > 1$ or $x < 1$. These different cases require individual treatment, which gives rise to a piecewise account of exponentiation, raising further continuity issues. In light of this, we rely on new lifting and gluing techniques for localic spaces (developed in Section 2.2.4) which allow us to glue these different cases of exponentiation together to obtain a continuous map. The results of this section have been recently been published in [NV22].

3.1 Rational Exponents

In this section, we develop the consequences of two pairs of basic *exponent laws:*

$$x^{\zeta+\zeta'} = x^{\zeta}x^{\zeta'}, \quad x^0 = 1 \tag{3.1}$$

$$x^{\zeta\cdot\zeta'} = (x^{\zeta})^{\zeta'}, \quad x^1 = x \tag{3.2}$$

In describing a map $(x, \zeta) \mapsto x^{\zeta}$ as an *exponentiation*, we shall mean that it satisfies the above exponent laws. They are enough to prescribe what x^{ζ} has to be for ζ rational. With a a natural number, x^a must be by repeated multiplication; for b a positive natural number, $x^{\frac{1}{b}}$ must be a radical, and $x^{\frac{a}{b}}$ combines those; and $x^{-\zeta}$ is $(x^{\zeta})^{-1}$. For completeness, we shall also prove the following *base product* law:

$$(xy)^{\zeta} = x^{\zeta}y^{\zeta}, \quad 1^{\zeta} = 1 \tag{3.3}$$

These identities recover the familiar (and standard) algebraic properties of exponentiation. Hereafter, we shall refer to Equations (3.1) - (3.3) collectively as *the Basic Equations*.

Before proceeding, however, first some obligatory remarks about the appropriate range for the base x. Clearly, without complex numbers we cannot hope to deal with radicals of negative reals, so we shall have to assume $x \geq 0$. Further, in later sections, we shall also find two additional problems with the case $x = 0$. The first (Section 3.1.5) is for negative exponents and Dedekind base x since x will need to be invertible. The second (Section 3.2) is that while our definition $x^0 = 1$ is OK for rational exponents (cf. Discussion 2.2.13), this causes continuity issues for real exponents; indeed, this reflects the classical fact that 0^0 is not well-defined. Nonetheless, so long as we work with just non-negative rational exponents, the assumption that $x \geq 0$ is OK.

3.1.1 Natural Number Exponentiation of Discrete Monoids. Let M be a set equipped with a multiplicative monoid structure. Let $x \in M$ be an element of such a set-based multiplicative monoid. By the universal property of $\mathbb{N}$ being the free monoid generated by 1, we obtain a unique monoid homomorphism corresponding to the set-based function sending 1 to x in M. This yields the following map:

$$\begin{aligned} M \times \mathbb{N} &\to M \\ (x, a) &\mapsto x^a \end{aligned}$$

Proposition 3.1.1. *If M is commutative, then exponentiation (as defined above) satisfies the Basic Equations.*

Proof. Let $x \in M$, and $a, a' \in \mathbb{N}$. Equation (3.1), and the second part of Equation (3.2), come straight from the definition. The others are by induction on a or a'. □

Obviously it will be our aim to show these equations for real exponentiation. As our first step towards this goal, recall that Q denotes the set of non-negative rationals. It is known geometrically that Q is a monoid with respect to multiplication, hence we obtain the following exponentiation map as a special case of the previous construction:

$$\begin{aligned} Q \times \mathbb{N} &\to Q \\ (x, a) &\mapsto x^a \end{aligned}$$

We finish this subsection by establishing some important (and useful!) algebraic properties of this exponentiation map:

Lemma 3.1.2 (Monotonicity)**.** *Denote $\mathbb{N}_+$ to be the set of positive natural numbers. If $a \in \mathbb{N}_+$, then the map $(—)^a$ on non-negative rational base preserves and reflects the strict order, and is also unbounded.*

Proof. We prove that $(—)^a$ preserves the strict order (i.e. is strictly monotonic) on Q by induction on a. The base case is trivial since $x < y \implies x^1 < y^1$ for any $x, y \in Q$ by the basic exponent laws. For the inductive step, suppose that $x < y$ implies that $x^a < y^a$. Then this yields: $x^{a+1} = x^a \cdot x < x^a \cdot y < y^a \cdot y = y^{a+1}$. To prove that $(—)^a$ reflects the strict order, we have to show that $x^a < y^a \implies x < y$. Decidability of $<$ on Q means that either $x < y$ or $x \geq y$. Since monotonicity means $x \geq y \implies x^a \geq y^a$, contradicting our hypothesis that $x^a < y^a$, this means that the remaining case $x < y$ must be true. Finally, we show that $(—)^a$ is unbounded. Given any $x \in Q$ such that $1 < x$, note that monotonicity implies that $1 < x^a$. A simple inductive argument easily shows that $x < x^a$. □

Lemma 3.1.3. *For any pair of non-negative rationals $q, r \in Q$ such that $q < r$, and any positive natural number $a \in \mathbb{N}_+$, there exists a positive rational $s \in Q_+$ so that $q < s^a < r$.*

Proof. Given $q, r \in Q$ such that $q < r$, denote $\epsilon := \frac{r-q}{2}$. By Lemma 3.1.2, we know that $(—)^a$ is unbounded on Q — thus there exists some $v \in Q$ such that $v^a > r$. Consider the function $(—)^a$ on the set of rationals from 0 to v. For any two rationals x, y such that $0 \leq x < y \leq v$, we have:

$$\begin{aligned} y^a - x^a &= (y-x) \cdot (y^{a-1} + y^{a-2} \cdot x + \cdots + y \cdot x^{a-2} + x^{a-1}) \\ &\leq (y-x) \cdot (v^{a-1} + v^{a-2} \cdot v + \cdots + v \cdot v^{a-2} + v^{a-1}) \\ &= (y-x) \cdot a \cdot v^{a-1}. \end{aligned}$$

By the Archimedean property (Fact 2.2.10), there exists some $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\epsilon}{a \cdot v^a}$. Denoting $s_i := \frac{i \cdot v}{M}$, it is clear that:

$$\bigcup_{0 \leq i \leq M} [s_i, s_{i+1}] = [0, v]$$

and that $s_{i+1} - s_i = \frac{v \cdot (i+1-i)}{M} = \frac{v}{M} < \frac{\epsilon}{a \cdot v^{a-1}}$, and so $s^a_{i+1} - s^a_i < \epsilon$.

Next, one easily verifies that since:

- $s^a_0 = 0 \leq q$;
- $s^a_M = v^a > r > q$; and
- $\{s^a_i\}$ is strictly monotone in $i \in \{0, 1., ..., M\}$;

there thus exists a unique j such that $s^a_j \leq q < s^a_{j+1}$. Recalling that $\epsilon := \frac{r-q}{2}$, this consequently yields:

$$q < s^a_{j+1} \leq s^a_j + \epsilon \leq q + \epsilon < r$$

proving the lemma, i.e. $\exists s \in Q_+$ such that $q < s^a < r$. □

3.1.2 Natural Number Exponentiation of Non-negative Reals.

Unfortunately, the previous exponentiation map is geometric only for monoid structures on sets, as opposed to topological monoids. Hence, in order to generalise x^a to the case where x is real, adjustments will be needed. The good news is there is a way in which we can lift the exponentiation map from the rational case to the real case, which we make precise in the following proposition:

Proposition 3.1.4. *The map $(—)^a$ that sends $(x, a) \mapsto x^a$ on rationals extends to two maps on one-sided reals:*

$$\overrightarrow{[0,\infty]} \times \mathbb{N} \longrightarrow \overrightarrow{[0,\infty]}$$
$$\overleftarrow{[0,\infty]} \times \mathbb{N} \longrightarrow \overleftarrow{[0,\infty]}.$$

Each map is unique subject to being monoid homomorphism for n and with $x^1 = x$. In fact, all the Basic Equations (3.1) – (3.3) *hold.*

Proof. Let us fix some $a \in \mathbb{N}$. Then, consider the diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\;f_a\;} & Q \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \overrightarrow{[0,\infty]} & \overset{\overline{f_a}}{\dashrightarrow} & \overrightarrow{[0,\infty]} \end{array}$$

where $f_a\colon Q \to Q$ is the map that sends $x \mapsto x^a$, and ψ is the canonical surjection associated to $\overrightarrow{[0,\infty]} \cong \mathsf{RIdl}(Q, <)$, where $<$ is the usual strict order on Q but modified to allow $0 < 0$ (cf. Example 2.2.51). We now check that $\psi \circ f_a$ satisfies the two conditions of the Lifting Lemma 2.2.56.

For monotonicity, suppose $q < q'$ for $q, q' \in Q$. By Lemma 3.1.2, this yields the inequality $q^a \leq (q')^a$, which is preserved by ψ, and so $q < q' \implies \psi \circ f_a(q) \sqsubseteq \psi \circ f_a(q')$. To verify continuity, this amounts to showing that if $r < (q')^a$, then $\exists q < q'$ such that $r < q^a$ (where $r, q, q' \in Q$). For $a = 0$, $r < (q')^0 = 1$, then we can let $q = \frac{q'}{2}$ since $r < 1 = q^0$. For $a \geq 1$, we know by Lemma 3.1.3 that there exists some $t \in Q$ such that $r < t^a < (q')^a$. Since $(—)^a$ still reflects the (modified) order $<$ on Q, we have that $t < q'$ and $r < t^a$, as desired. With the appropriate hypotheses verified, we apply the Lifting Lemma to obtain the (unique) map $\overline{f}\colon \overrightarrow{[0,\infty]} \to \overrightarrow{[0,\infty]}$. Viewing this map externally (cf. Convention 2.1.34), we get an exponentiation map $\overrightarrow{[0,\infty]} \times \mathbb{N} \to \overrightarrow{[0,\infty]}$ sending $(x, a) \mapsto x^a$. The Basic Equations, and also the uniqueness, follow from surjectivity of ψ, since we already know that they hold for rational x and y.

A similar argument (using $Q_+ \cup \{\infty\}$) works for the upper reals, thus defining x^a for some upper real $x \in \overleftarrow{[0,\infty]}$. □

More explicitly, Proposition 3.1.4 extends natural number exponentiation $(—)^a$ from the rationals to the one-sided reals, yielding:

$$q < x^a \leftrightarrow q < 0 \vee \exists q' \in Q.(q' < x \wedge q < (q')^a)$$
$$x^a < r \leftrightarrow \exists r' \in Q_+.(x < r' \wedge (r')^a < r).$$

on $\overrightarrow{[0,\infty]}$ and $\overleftarrow{[0,\infty]}$ respectively. Putting everything together, we now define natural number exponentiation on the Dedekind reals as follows:

Proposition 3.1.5. *Natural number exponentiation on the one-sided reals (as per Proposition 3.1.4) combine to yield the following map on the Dedekind reals:*

$$[0,\infty) \times \mathbb{N} \to [0,\infty)$$
$$(x, a) \mapsto x^a = (L_x^a, R_x^a).$$

The Basic Equations (3.1) - (3.3) *hold for Dedekind x and y.*

Proof. It remains to verify the inhabitedness, separatedness and locatedness axioms. Right inhabitedness essentially follows from the unboundedness of $(—)^a$ on $\mathbb{Q}$ (Lemma 3.1.2) whereas we get left inhabitedness (and non-negativity) for free since for any negative rational $q < 0$, we get $q < x^a$ by construction.

For separatedness, suppose $q < x^a < r$. Notice that $r \in Q_+$ by construction, hence if $q \leq 0$, then $q < r$ automatically. Suppose instead that $q > 0$ and there exist non-negative rationals s, t such that $s < x < t$, whereby $q < s^a$ and $t^a < r$. Since x is a *separated* Dedekind, $s < x < t \implies s < t$. Hence, since $s^a = 1 = t^a$ if $a = 0$, and $s^a < t^a$ if $a > 0$ by Lemma 3.1.2, this combines to yield the inequality $q < s^a \leq t^a < r$, proving separatedness.

For locatedness, suppose we have $q, r \in \mathbb{Q}$ such that $q < r$. If $q < 0$, then $q < x^a$ automatically, so assume $q \geq 0$ for the remainder of this proof.

In the case where $a > 0$, by Lemma 3.1.3 we can find $q', r' \in Q_+$ such that $q < (q')^a < (r')^a < r$. Since $(—)^a$ reflects strict order (Lemma 3.1.2), we get $(q')^a < (r')^a \implies q' < r'$. Further, since x is a *located* Dedekind real, this implies that either $q' < x$ (and thus $q < x^a$) or that $r' > x$ (and thus $r > x^a$). Alternatively, suppose that $a = 0$. Then locatedness is obvious, since $x^0 = 1$ is located. Hence, in either case ($a = 0$ or $a > 0$), locatedness holds.

The Basic Equations follow immediately, because we know they hold for the lower and upper parts. □

Convention 3.1.6 ("Non-negative Reals")**.** Whenever we state that a result holds for the "non-negative reals", we shall mean that it holds for both the non-negative one-sideds (i.e. $x \in \overrightarrow{[0, \infty]}$ or $x \in \overleftarrow{[0, \infty]}$) and the non-negative Dedekinds (i.e. $x \in [0, \infty)$) — which are the cases for which we have defined exponentiation in Propositions 3.1.4 and 3.1.5. Whenever we wish to prove a sharper result that holds just in the case of non-negative Dedekinds, we shall signpost this explicitly.

We end this section by generalising the monotonicity principle of Lemma 3.1.2:

Lemma 3.1.7 (Monotonicity)**.** *Let $a \in \mathbb{N}_+$ be a positive natural number. Then the map $(—)^a$ preserves and reflects non-strict order on the non-negative reals. Further, it is also an unbounded map that preserves and reflects strict order on the non-negative Dedekind reals.*

Proof. Consider $(—)^a$ on the lower reals $\overrightarrow{[0, \infty]}$. Preservation of non-strict order follows immediately from the fact that any continuous map preserves specialisation order. To show that $(—)^a$ reflects non-strict order, suppose that $x^a \sqsubseteq y^a$. This yields the computation:

$$\begin{aligned}
q < x &\implies q^a < x^a && \text{[by unwinding the definition of } (—)^a\text{]} \\
&\implies q^a < y^a && \text{[since } x^a \sqsubseteq y^a\text{]} \\
&\implies \exists q' \in Q.(q' < y \wedge q^a < (q')^a) && \text{[by construction/definition of } (—)^a\text{]} \\
&\implies q < q' && \text{[since } (—)^a \text{ reflects strict order on } Q\text{]} \\
&\implies q < y && \text{[by downward closure of lower reals]}
\end{aligned}$$

Next, to show that $(—)^a$ also preserves strict order on the non-negative Dedekinds, suppose $x < y$, and so there exists rationals q, q' such that $x < q < q' < y$. Then since $(—)^a$ preserves non-strict order, and using Lemma 3.1.2, we get $x^a \leq q^a < (q')^a \leq y^a$ and hence $x^a < y^a$.

On the other hand, suppose $x^a < y^a$. Since $0 \leq x^a$, we may apply Lemma 3.1.3, to obtain positive rationals $s, s' \in Q_+$ such that $x^a < s^a < (s')^a < y^a$. Since $s < s'$ iff $s^a < (s')^a$, and since $(—)^a$ reflects non-strict order, we get $x \leq s < s' \leq y$, which shows that $(—)^a$ also reflects strict order on the non-negative Dedekinds.

Unboundedness follows directly from the rational case, Lemma 3.1.2.

□

3.1.3 Radicals of Non-Negative Reals. Next, given some $b \in \mathbb{N}_+$, we would like to define the bth-root of a non-negative real. Unlike the previous subsection, we shall define this directly (as opposed to first working with the rationals before lifting to the reals):

Proposition 3.1.8. *Define maps* $(x, b) \mapsto x^{\frac{1}{b}}$ *on the non-negative reals using:*

$$q < x^{\frac{1}{b}} \leftrightarrow q < 0 \vee \left(q^b < x \wedge q \in Q\right)$$

$$r > x^{\frac{1}{b}} \leftrightarrow r^b > x.$$

Then $x^{\frac{1}{b}}$ *is a real of the same kind as* x *(lower, upper, or Dedekind).*

Proof. Let x be a non-negative lower real. Non-negativity of $x^{\frac{1}{b}}$ is immediate from definition. When $q < 0$, downward closure and roundedness are obvious, so we shall assume that $q^b < x \wedge q \in Q$. In which case, downward closure says that if $q' < q$ for $q' \in \mathbb{Q}$, then $q < x^{\frac{1}{b}} \implies q' < x^{\frac{1}{b}}$. If $q' < 0$, this is obvious. If $q' \geq 0$, this follows immediately from the monotonicity of exponentiation by $b \in \mathbb{N}_+$ (Lemma 3.1.2). As for roundedness, we must show if $q^b < x$ then there exists rational $q' > q$ such that $q^b < (q')^b < x$. We know there exists $r \in Q_+$ such that $q^b < r < x$. By Lemma 3.1.3, there exists $q' \in Q_+$ such that $q^b < (q')^b < r < x$, and we are done.

The case for non-negative upper reals is analogous.

As before, to define $(—)^{\frac{1}{b}}$ on non-negative Dedekinds, we shall need to verify the rest of the axioms from Definition 2.2.7. Left inhabitedness comes for free since $q < x^{\frac{1}{b}}$ for all negative rationals q. Right inhabitedness follows from the unboundedness of $(—)^b$ on the rationals (Lemma 3.1.2).

For separatedness, suppose $q < x^{\frac{1}{b}} < r$. Since x is non-negative, and since $(—)^b$ reflects strict order, this immediately implies that $r > 0$. As such, if $q < 0$, then $q < r$ automatically. Hence, suppose instead that $q \geq 0$, and that $q^b < x < r^b$. Since x is a *separated* Dedekind, this implies that $q^b < r^b$, and so we get $q < r$ (again by Lemma 3.1.2).

Finally, we check locatedness. Suppose $q < r$. Again, if $q < 0$, then we get that $q < x^{\frac{1}{b}}$ for free, so suppose $q \geq 0$. By Lemma 3.1.2, we know that $0 \leq q^b < r^b$. Since x is a *located* Dedekind, this implies that either $q^b < x \vee x < r^b$, which in turn implies (by construction) that $q < x^{\frac{1}{b}} \vee x^{\frac{1}{b}} < r$, proving the axiom. ☐

The key property of bth roots, of course is that taking the root is inverse to raising to the power.

Proposition 3.1.9. *Given a non-negative real* x*, and* $0 \neq b \in \mathbb{N}$*, we have that*

$$x = (x^{\frac{1}{b}})^b = (x^b)^{\frac{1}{b}}.$$

Proof. The proof of these identities for the one-sided reals is analogous, so we shall only prove it for the upper reals, which will also automatically extend the result to the Dedekind reals (cf. Corollary 2.2.24).

To prove $x = (x^{\frac{1}{b}})^b$ for the upper reals, suppose $q > (x^{\frac{1}{b}})^b$. This means that $\exists q' \in Q_+$ such that $(q')^b > x$ and $q > (q')^b$, which implies that $q > x$. Conversely, suppose $q > x$. By roundedness, we know that there exists some $q'' \in Q_+$ such that $q > q'' > x \geq 0$. By Lemma 3.1.3, we know that there exists some $q' \in Q_+$ such that $q > (q')^b > q'' > x$, proving that $q > (x^{\frac{1}{b}})^b$.

To prove $x = (x^b)^{\frac{1}{b}}$, note that $x = (x^{\frac{1}{b}})^b$ implies that $\left((x^b)^{\frac{1}{b}}\right)^b = x^b$. The result then follows since $(—)^b$ reflects the (non-strict) order on the upper reals by Lemma 3.1.7. ☐

Corollary 3.1.10. *The following equations hold for x, y non-negative reals, and $a, b, d \in \mathbb{N}$ where $b, d \neq 0$:*

$$(x^a)^{\frac{1}{b}} = (x^{\frac{1}{b}})^a$$
$$x^{\frac{1}{bd}} = (x^{\frac{1}{b}})^{\frac{1}{d}}, \quad x^{\frac{1}{1}} = x$$
$$(xy)^{\frac{1}{b}} = x^{\frac{1}{b}} y^{\frac{1}{b}}.$$

Proof. In each case, the proof is to raise both sides to an appropriate power, use equations already known for integer powers, and then take the root. For example, the first one follows from

$$((x^{\frac{1}{b}})^a)^b = ((x^{\frac{1}{b}})^{ab} = ((x^{\frac{1}{b}})^b)^a = x^a.$$

□

3.1.4 Non-negative Rational Exponents.

Having defined x^a and $x^{\frac{1}{b}}$ for $a \in \mathbb{N}$ and $b \in \mathbb{N}_+$ we combine these two constructions together as

$$x^{\frac{a}{b}} = \left(x^a\right)^{\frac{1}{b}}$$

Note that we get the fact that $x^{\frac{a}{b}}$ is a non-negative real for free due to Propositions 3.1.5 and 3.1.8. The only thing left to check is that this construction is well-defined with respect to the equivalence of rationals.

Proposition 3.1.11. *The exponential x^q, with x being a non-negative real and q a non-negative rational, is well-defined and satisfies the Basic Equations* (3.1) - (3.3).

Proof. To show that x^q is well-defined (with respect to the equivalence of rationals), we need to show that given any $\frac{a}{b} = \frac{c}{d}$, where $a, c \in \mathbb{N}$ and $b, d \in \mathbb{N}_+$, we have that $(x^a)^{\frac{1}{b}} = (x^c)^{\frac{1}{d}}$. Why is this? First, note that $x^{\frac{a}{b}} = x^{\frac{ak}{bk}}$, for $\frac{a}{b} \in Q$ and $k \in \mathbb{N}_+$. Indeed, by Proposition 3.1.9 and Corollary 3.1.10, we have:

$$x^{\frac{ak}{bk}} = (x^{ak})^{\frac{1}{bk}} = \left(\left((x^a)^k\right)^{\frac{1}{k}}\right)^{\frac{1}{b}} = x^{\frac{a}{b}}.$$

More generally, suppose we have that $\frac{a}{b}, \frac{c}{d} \in Q$ such that $\frac{a}{b} = \frac{c}{d}$. This obviously implies that $ad = bc$, and thus our previous computation yields the identity: $x^{\frac{a}{b}} = x^{\frac{ad}{bd}} = x^{\frac{bc}{bd}} = x^{\frac{c}{d}}$, as desired.

To see why the Basic Equations hold for non-negative rational exponents, this follows follows algebraically from the Basic Equations already established and Corollary 3.1.10. For example for the law of adding exponents, if $r = \frac{a}{b}$ and $s = \frac{c}{d}$, then

$$x^{r+s} = x^{\frac{ad+bc}{bd}} = \left(x^{\frac{1}{bd}}\right)^{ad+bc} = \left(x^{\frac{1}{bd}}\right)^{ad} \cdot \left(x^{\frac{1}{bd}}\right)^{bc} = x^{\frac{a}{b}} \cdot x^{\frac{c}{d}} = x^r \cdot x^s.$$

□

Lemma 3.1.12. *Fix $q \in Q_+$. Then $(—)^q$ preserves and reflects non-strict order on non-negative reals. Further, it is an unbounded map that preserves and reflects strict order on non-negative Dedekind reals.*

Proof. Preservation and reflection of strict (resp. non-strict) order on the positive Dedekind (resp. one-sided) reals is immediate from Lemma 3.1.7. Further, express q as $\frac{a}{b}$ for $a, b \in \mathbb{N}_+$. We know for any positive Dedekind $x \in (0, \infty)$ that there exists $s \in \mathbb{Q}$ whereby $1 < s$ and $x < s \leq s^a$. Since $s^a = (s^b)^{\frac{a}{b}} = (s^b)^q$, this proves that $(—)^q$ is unbounded. □

3.1.5 Signed Rational Exponents. In this subsection, we extend the previous definition of rational exponentiation to also include the non-positive rationals. Here we must restrict to the case where the base is Dedekind but not one-sided. This is because inverting reverses orientation — applying Definition 2.2.29, the inverse of a lower real x yields an upper real (and vice versa). Hence, much like subtraction, whilst negative exponents are well-defined on the Dedekinds, they are not well-defined on just the lower or upper reals alone. We further require the base to be *positive* Dedekind as well since inverses are only well-defined for non-zero Dedekinds.

Recall from Definition 2.2.29 that given any positive Dedekind $x \in (0, \infty)$, there exists a unique inverse $x^{-1} \in (0, \infty)$ such that $x^{-1} \cdot x = 1$.

Definition 3.1.13. Let $x \in (0, \infty)$, and $q \in \mathbb{Q}$. We define:

$$x^q = \begin{cases} x^q & \text{if } q \geq 0 \\ (x^{-q})^{-1} & \text{if } q \leq 0 \end{cases}$$

Using Lemma 3.1.12 we can see that $0 < x^q$. Further, we remark that this definition of non-positive exponentiation justifies our notation of denoting inverses as $(—)^{-1}$, as can be seen from the following lemma:

Lemma 3.1.14. *Fix $x \in (0, \infty)$. For any $q \in \mathbb{Q}$, we have that $x^{-q} = (x^q)^{-1} = (x^{-1})^q$.*

Proof. For the first identity, if $q \geq 0$ then $x^{-q} = (x^q)^{-1}$ by definition; while if $q \leq 0$ then $(x^q)^{-1} = \left((x^{-q})^{-1}\right)^{-1} = x^{-q}$.

As for the identity $(x^q)^{-1} = (x^{-1})^q$, if $q \geq 0$ then it follows from the Basic Equations because

$$x^q \cdot \left(x^{-1}\right)^q = \left(x \cdot x^{-1}\right)^q = 1.$$

Then for $q \leq 0$ we have

$$\left(x^{-1}\right)^q = \left(\left(x^{-1}\right)^{-q}\right)^{-1} = \left(\left(x^{-q}\right)^{-1}\right)^{-1} = x^{-q}.$$

□

Remark 3.1.15 (Gluing maps defined on subspaces of $\mathbb{Q}$ vs. $\mathbb{R}$)**.** There is a subtle geometricity issue hidden in our construction that bears highlighting. Definition 3.1.13 hinges upon a case-splitting: we gave two separate definitions of x^q based on whether $q \leq 0$ or $q \geq 0$, and (implicitly) claimed that this presents a geometric account of x^q for all $q \in \mathbb{Q}$. Why is this? The short answer: unlike the case for $\mathbb{R}$, we get geometricity of the case-splitting essentially for free since $<$ is decidable on $\mathbb{Q}$ (cf. Discussion 2.2.13).

We conclude by proving the Basic Equations (3.1) - (3.3). A common theme runs through the proofs: for signed rational exponentiation, we must now keep track of how non-negative and non-positive exponents interact with one another, forcing us to consider the various possible cases. Nonetheless, most of these can be handled similarly (modulo some technical adjustments) and so the case-splitting primarily serves as a form of bookkeeping as opposed to being a sign of some hidden complexity.

Proposition 3.1.16. *Let $x \in (0, \infty)$, and $q, q' \in \mathbb{R}$. Then $(x^q)^{q'} = x^{q \cdot q'}$.*

Proof. Strict order is decidable on $\mathbb{Q}$, and so given any $q \in \mathbb{Q}$, we can split our proof into the cases when q, q' have the same signs or opposite signs.

Case 1: $q, q' \geq 0$. By Proposition 3.1.11.
Case 2: $q, q' \leq 0$. This follows from Case 1 and Lemma 3.1.14, which yields:

$$(x^q)^{q'} = \left(\left((x^{-q})^{-1}\right)^{-q'}\right)^{-1} = \left(\left((x^{-q})^{-q'}\right)^{-1}\right)^{-1} = \left(\left(x^{q\cdot q'}\right)^{-1}\right)^{-1} = x^{q\cdot q'}$$

Case 3: $q \leq 0 \leq q'$. This also follows from Case 1 and Lemma 3.1.14, since:

$$(x^q)^{q'} = ((x^{-q})^{-1})^{q'} = \left((x^{-q})^{q'}\right)^{-1} = (x^{-q\cdot q'})^{-1} = \left(x^{q\cdot q'})^{-1}\right)^{-1} = x^{q\cdot q'}$$

Case 4: $q' \leq 0 \leq q$. By symmetry with Case 3. ☐

Proposition 3.1.17. *Let $x \in (0, \infty)$, and $q, q' \in \mathbb{Q}$. Then $x^{q+q'} = x^q \cdot x^{q'}$.*

Proof. Similar to Proposition 3.1.16, we split our proof into various cases, based on the sign of the rational exponents of the identity. By previous work, we have already shown the following case:

Basic Case: $q, q' \geq 0$ (and thus $q + q' \geq 0$): Immediate from Proposition 3.1.11.

We claim that all the possible (signed) combinations of q, q' and $q + q'$ ultimately reduce to this basic case after some elementary algebraic manipulations. If at least one of them, say q, is negative, then the equation to prove is equivalent to

$$x^{q'} = x^{q+q'} \cdot x^{-q},$$

once we multiply both sides by $x^{-q} = (x^q)^{-1}$. This is another instance of the identity in the statement, but with (q, q') replaced by $(-q, q + q')$, with strictly fewer of the three exponents negative. If in addition $q + q'$ and/or q' are also negative, then we iterate the process so that we eventually hit the Basic Case. ☐

Proposition 3.1.18. *Let $x, y \in (0, \infty)$, and $q \in \mathbb{Q}$. Then $(x \cdot y)^q = x^q \cdot y^q$.*

Proof. Immediate from Proposition 3.1.11. and definitions. ☐

3.2 Real Exponents (The General Case)

Moving on to real exponents, we have to be careful with monotonicity if we are to include one-sided reals. This is because any map must be monotone with respect to the specialisation order. Hence, if an argument is a one-sided real, then the result is numerically monotone with respect to that argument if it is the same orientation, antitone if opposite.

Fixing ζ, the map $(—)^\zeta$ is monotone or antitone in x according as $\zeta \geq 0$ or $\zeta \leq 0$ — this follows immediately from the fact that inverting reverses orientation. Fixing x, the map $x^{(—)}$ is monotone or antitone in ζ according as $x \geq 1$ or $x \leq 1$ — this is clearly seen in the case of rational exponents:

Proposition 3.2.1 (Monotone/Antitone behaviour of rational exponents)**.**

(i) Fix Dedekind real x such that $x > 1$. Then the map $x^{(—)}$ is strictly increasing on $\mathbb{Q}$.
(ii) Fix Dedekind real x such that $0 < x < 1$. Then the map $x^{(—)}$ is strictly decreasing on $\mathbb{Q}$.

(iii) Fix one-sided real x such that $x \geq 1$. Then the map $x^{(\text{—})}$ is non-strictly increasing on Q.
(iv) Fix one-sided real x such that $0 \leq x \leq 1$. Then the map $x^{(\text{—})}$ is non-strictly decreasing on Q.

Proof. Fix Dedekind x such that $x > 1$. Suppose we are given some $r, s \in \mathbb{Q}$ such that $r < s$. By the exponent laws for rational exponents, this implies that $x^s = x^{r+s-r} = x^r \cdot x^{s-r}$. By Lemma 3.1.12, we have that $1 < x^q$ for any positive rational $q > 0$. Since $0 < s - r \implies 1 < x^{s-r}$, this in turn implies that $x^s = x^r \cdot x^{s-r} > x^r$, proving that $x^{(\text{—})}$ is indeed strictly increasing on $\mathbb{Q}$. The case for $x^{(\text{—})}$ when we have Dedekind $0 < x < 1$ is analogous.

The same proof works for the one-sided case modulo the following adjustments: one, $x^{(\text{—})}$ is not defined for negative rational exponents, so we restrict the map to just the non-negative rationals Q; and two, Lemma 3.1.12 now holds that $(\text{—})^q$ only preserves *non-strict* order, so we only get weakly monotonic/antitonic results for $x^{(\text{—})}$ when x is one-sided. □

What are the implications of these varying monotonicity behaviours? For one-sided real arguments, this fragments the exponentiation into different cases based on the ranges of values and the one-sided orientations. We present the possibilities in the table below. Each table entry shows the type of x^ζ for given types of x and ζ. Some combinations are impossible, because the monotonicities for x and ζ conflict.

$x \backslash \zeta$	$\overrightarrow{[0,\infty]}$	$\overleftarrow{[0,\infty]}$	$\overrightarrow{[-\infty,0]}$	$\overleftarrow{[-\infty,0]}$
$\overrightarrow{[1,\infty]}$	$\overrightarrow{[1,\infty]}$			$\overleftarrow{[0,1]}$
$\overleftarrow{[1,\infty]}$		$\overleftarrow{[1,\infty]}$	$\overrightarrow{[0,1]}$	
$\overrightarrow{[0,1]}$		$\overrightarrow{[0,1]}$	$\overleftarrow{[1,\infty]}$	
$\overleftarrow{[0,1]}$	$\overleftarrow{[0,1]}$			$\overrightarrow{[1,\infty]}$

(3.4)

In Theorem 3.2.11 we shall prove the monotone cases for $x \geq 1$ and $\zeta \geq 0$, top-left in the table, where x, ζ and x^ζ all have the same orientation. Meanwhile, however, it seems easiest to start with the case where $x \geq 1$ is Dedekind, and ζ is signed: in Proposition 3.2.4, we lift ζ from rationals to one-sideds. In Theorem A, we extend this to get the case where ζ is Dedekind, and a gluing argument allows us to finally extend the construction to the whole space of positive Dedekinds. Further, since Theorem A also covers the case where x is rational, we later use that in Theorem 3.2.8 for cases where x is one-sided.

3.2.1 Dedekind Real Base. Once again, our plan of attack involves using the Lifting Lemma. Proposition 3.2.1, however, alerts us to the fact that the behaviour of $\{x^{q_n}\}_{n\in\mathbb{N}}$ differs depending on whether the base $0 < x < 1$ or $x > 1$. Given the monotonicity condition of the Lifting Lemma, this indicates a natural case-splitting in our analysis.

As such, to control the behaviour of rational exponentials, we first restrict to the case when $x \geq 1$ — this allows us to extend the range of ζ to the whole real line without worrying about monotonicity issues. We start by establishing the following two lemmas:

Lemma 3.2.2 (Bernoulli's Inequality)**.** *For any positive real x (Dedekind or one-sided), and any natural number $k \in \mathbb{N}$, we have the inequality:*

$$(1+x)^k \geq 1 + k \cdot x.$$

Proof. The proof is entirely algebraic, so it works identically regardless of whether x is Dedekind or one-sided. We proceed by induction. For our base case $k = 0$, we want to show:

$$(1+x)^0 \geq 1 + 0 \cdot x = 1.$$

But this is obvious since $(1+x)^0 = 1$. To prove the inductive hypothesis, suppose that the desired inequality holds for k. To show that it also holds for $k+1$, note that:

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\ &\geq (1+kx) \cdot (1+x) \\ &= 1 + (k+1) \cdot x + k \cdot x^2 \\ &\geq 1 + (k+1) \cdot x,\end{aligned}$$

where the first inequality is by the inductive hypothesis, and the last inequality by the fact that multiplying two non-negative reals yields another non-negative real. □

Lemma 3.2.3 (Continuity Lemma)**.** *Suppose $0 < q < q'$, for a pair of (positive) rationals $q, q' \in Q_+$. Let x be a Dedekind real such that $x \geq 1$. Then there exists a positive integer $k \in \mathbb{N}_+$ such that $q \leq q \cdot x^{\frac{1}{k}} < q'$.*

Proof. Denote $\delta := \frac{q'}{q} - 1$ (which is a positive rational), and so $q' = q(1+\delta)$. By Bernoulli's Inequality and the Archimedean property, there exists $k \in \mathbb{N}_+$ such that:

$$x < 1 + k \cdot \delta \leq (1+\delta)^k.$$

By Lemma 3.1.12, this implies $1 \leq x^{\frac{1}{k}} < 1 + \delta$, and so further multiplying through by q yields:

$$q \leq q \cdot x^{\frac{1}{k}} < q'.$$

□

Fixing a Dedekind $x \in [1, \infty)$, our definition of x^ζ (for arbitrary $\zeta \in \mathbb{R}$) rests on two levels of extensions. We first extend rational exponents to one-sided exponents, before combining the one-sided exponents to yield a Dedekind exponent. Note that in the previous section (e.g. Proposition 3.1.5), we fixed the exponent before applying lifting arguments to the base. In this setting, we work inversely: we fix the base before applying lifting arguments to the exponent.

Proposition 3.2.4. *For Dedekind $x \geq 1$, the exponentiation by arbitrary rationals can be extended to one-sided exponents, giving exponentiation maps*

$$[1,\infty) \times \overrightarrow{[-\infty,\infty]} \to \overrightarrow{[0,\infty]} \text{ and } [1,\infty) \times \overleftarrow{[-\infty,\infty]} \to \overleftarrow{[0,\infty]}.$$

Each map is unique subject to being monoid homomorphism for $\zeta \in \overrightarrow{[-\infty,\infty]}$ or $\zeta \in \overleftarrow{[-\infty,\infty]}$, and satisfies the Basic Equations (3.1) - (3.3).

Proof. Fix Dedekind $x \in [1, \infty)$ We prove the result for the lower case of ζ. The upper case is analogous.

Following Example 2.2.51, we check the two conditions of the Lifting Lemma 2.2.56 based on $\overrightarrow{[-\infty,\infty]} \cong \mathsf{RIdl}(\mathbb{Q} \cup \{-\infty\}, <)$ (the involvement of $-\infty$ is largely irrelevant, so we shall leave this case to the reader). Monotonicity amounts to holding for any $q, r \in \mathbb{Q}$, we have that $q < r \implies x^q \leq x^r$ — which is immediate from Proposition 3.2.1. For continuity: if $r < x^q$, with $r, q \in \mathbb{Q}$, we want rational $q' < q$ with $r < x^{q'}$. This is clear if $r \leq 0$, hence suppose instead that $0 < r$. By strict order $<$, there exists $r' \in \mathbb{Q}$ such that $0 < r < r' < x^q$. Applying the Continuity Lemma 3.2.3, we find k with $rx^{1/k} < r' < x^q$. Then $r < x^{q-1/k}$.

The basic equations follow from the case of rational exponents. □

Theorem A. We have an exponentiation map on the Dedekinds

$$\exp\colon (0,\infty) \times \mathbb{R} \to (0,\infty).$$

It satisfies the exponent laws, i.e. Basic Equations (3.1) and (3.2).

Proof. First, we claim the two maps of Proposition 3.2.4 combine to give an exponentiation map

$$\begin{aligned}[1,\infty) \times \mathbb{R} &\to (0,\infty)\\ (x,\zeta) &\mapsto (x^{L_\zeta}, x^{R_\zeta}) =: x^\zeta\end{aligned}$$

From the definition, we calculate that:

$$q < x^\zeta \leftrightarrow \exists q' \in \mathbb{Q}.(q' < \zeta \wedge q < x^{q'})$$

$$q > x^\zeta \leftrightarrow \exists q' \in \mathbb{Q}.(q' > \zeta \wedge q > x^{q'}).$$

As before, to show this map is well-defined, it remains for us to check inhabitedness, positivity, separatedness and locatedness. Inhabitedness and positivity are easy: we know there exist rationals $q_0, r_0 \in \mathbb{Q}$ such that $q_0 < \zeta < r_0$ (since ζ is an inhabited Dedekind) and so there exists $q, r \in \mathbb{Q}$ such that $q < x^{q_0}$ and $r > x^{r_0}$ (since $0 < x^{q_0}$ and x^{r_0} are inhabited as well).

For separatedness, suppose $x^{R_\zeta} < q < x^{L_\zeta}$. Then we have $x^{r_1} < q < x^{r_2}$ for some rationals r_1, r_2 with $R_\zeta < r_1$ and $r_2 < L_\zeta$. But ζ is separated, and so $r_2 < r_1$, which implies $x^{r_2} < x^{r_1}$ by Proposition 3.2.1, a contradiction.

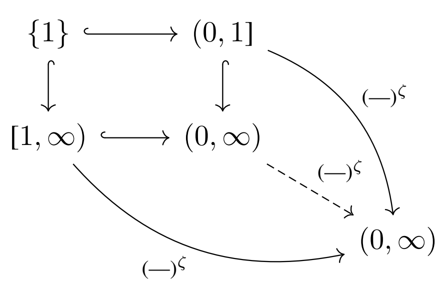
For locatedness, suppose we have rationals $q < r$. When $q \leq 0$, then $q < x^\zeta$ since x^ζ is positive, so let's assume $q > 0$. Leveraging previous results, we then define a series of parameters:

- Denote $r' := (q + r)/2$.
- By the Continuity Lemma, find k such that $r' x^{1/k} < r$.
- By Remark 2.2.9, find a rational s such that $s < \zeta < s + 1/k$.

Since x^s is a (located) Dedekind real, this means that $q < x^s$ or $x^s < r'$. If $q < x^s$ then $q < x^\zeta$, while if $x^s < r'$ then $x^\zeta < x^{s+1/k} = x^s \cdot x^{1/k} < r' \cdot x^{1/k} < r$.

The exponent laws for this map follow from those for the maps in Proposition 3.2.4.

Having defined x^ζ for $x \in [1,\infty)$, we can also define an exponentiation $(0,1] \times \mathbb{R} \to (0,\infty)$, by $x^\zeta := (x^{-1})^{-\zeta}$. Let us now fix $\zeta \in \mathbb{R}$. Since the two maps agree on $x = 1$, we can apply the Gluing Principle (Proposition 2.2.60) to glue them together and obtain the general exponentiation map via the pushout property:



Externalising yields the desired exponentiation map $(0,\infty) \times \mathbb{R} \to (0,\infty)$ (cf. Convention 2.1.34). The exponent laws follow immediately. ☐

Remark 3.2.5. The reader may have observed that we only proved the exponent laws in Theorem A — this is because the base product equation ($(x \cdot y)^\zeta = x^\zeta \cdot y^\zeta$) does not transfer directly over the gluing, and thus requires separate proof. We defer proof of the base product law to Subsection 3.2.2.

As an immediate corollary of the theorem, we generalise Proposition 3.2.1 to obtain the following monotonic/antitonic result for real exponentiation with respect to the *exponent*:

Corollary 3.2.6. *Let $\zeta, \zeta' \in \mathbb{R}$ such that $\zeta < \zeta'$. Then:*

(i) If $x \in (1, \infty)$ is a Dedekind real, then $x^\zeta < x^{\zeta'}$.
(ii) if $x \in (0, 1)$ is a Dedekind real, then $x^{\zeta'} < x^\zeta$.

Proof. Let $x \in (1, \infty)$, and $\zeta, \zeta' \in \mathbb{R}$ such that $\zeta < \zeta'$. By Exponent Law (3.1) established in Theorem A, we obtain:

$$x^{\zeta'} = x^{\zeta' - \zeta + \zeta} = x^{\zeta' - \zeta} \cdot x^\zeta.$$

We claim $x^{\zeta'-\zeta} > 1$. Why? By strict order $<$ on the reals, pick some $r \in Q_+$ such that $0 < r < \zeta' - \zeta$. Unpacking definitions, we know $q < x^r \implies q < x^{\zeta'-\zeta}$. By Proposition 3.2.1, we get $x^r > 1$, which in turn implies that $x^{\zeta'-\zeta} > 1$. In particular, we get:

$$1 < x^{\zeta'-\zeta} \implies x^\zeta < x^{\zeta'-\zeta} \cdot x^\zeta = x^{\zeta'}.$$

The case when $x \in (0, 1)$ is entirely analogous. □

3.2.2 One-Sided Real Base.

In contrast to the previous subsection, we now work with a one-sided real base and Dedekind exponent. Subtleties regarding negative exponents (as discussed at the start of this section) require some care, but they can be manoeuvred around sensibly.

Lemma 3.2.7. *The base product law $(s \cdot t)^\zeta = s^\zeta \cdot t^\zeta$ holds for $\zeta \in \mathbb{R}$ and* positive rationals $s, t \in Q_+$.

Proof. Similar to Propositions 3.1.17 and 3.1.16, we shall need to case split based on which side of 1 the values $s, t, s \cdot t$ lie. If all three are at least 1, then the equation holds from Proposition 3.2.4. If at least one of them, say s, is less than 1, so $s^\zeta = (s^{-1})^{-\zeta}$, then the equation is equivalent to $(s^{-1})^\zeta \cdot (s \cdot t)^\zeta = t^\zeta$, with $s^{-1} > 1$. We may have to apply similar transformations for t and st, but eventually we end up with an equation in which all three values are at least 1. □

To prove the base product law more generally, we shall work via the one-sided reals in our usual way.

Theorem 3.2.8. *For Dedekind exponent $\zeta \geq 0$ and one-sided base x, we can define exponentiation maps*

$$\overrightarrow{(0, \infty]} \times [0, \infty) \to \overrightarrow{(0, \infty]} \text{ and } \overleftarrow{[0, \infty)} \times [0, \infty) \to \overleftarrow{[0, \infty)}$$

such that, for x Dedekind, $(L_x)^\zeta = L_{x^\zeta}$ and $(R_x)^\zeta = R_{x^\zeta}$. The Basic Equations hold for these maps.

Proof. We prove the lower case. Following Remark 3.1.15, we know that a map from the rationals can be defined (geometrically) via case-splitting on $<$. Thus define:

$$Q_+ \times [0, \infty) \longrightarrow \overrightarrow{(0, \infty]}$$
$$(s, \zeta) \longmapsto \begin{cases} L_{s^\zeta}, & \text{if } s \geq 1 \\ (R_{(s^{-1})^\zeta})^{-1}, & \text{if } s < 1. \end{cases}$$

Note that $(R_{(s^{-1})^\zeta})^{-1}$ is indeed a lower real since $(—)^{-1}\colon \overrightarrow{(0,1]} \cong \overleftarrow{[1,\infty)}$ (cf. Definition 2.2.29).

We now apply the Lifting Lemma in the case of $\overrightarrow{(0,\infty]} \cong \mathsf{RIdl}(Q_+,<)$. Monotonicity for $s < s'$ essentially follows from Lemma 3.2.7, which yields:

$$\left(\frac{s'}{s}\right) > 1 \implies (s')^\zeta \cdot (s^{-1})^\zeta = \left(\frac{s'}{s}\right)^\zeta \geq 1 \implies (s')^\zeta \geq s^\zeta.$$

For continuity, suppose $q < s^\zeta$. If $1 \leq s$, then by definition of s^ζ we have $0 < q < s^r$ for some rational $r < \zeta$. Let $r = \frac{a}{b}$ for $a, b, \in \mathbb{N}_+$. Raising to the power of b, we know that $q < s^{\frac{a}{b}} \iff q^b < s^a$ by Lemma 3.1.12. Applying Lemma 3.1.3, there exists some positive rational $s' \in Q_+$ such that $q^b < (s')^a < s^a$, which (taking bth roots) yields the desired inequality $q < (s')^r < s^r < s^\zeta$. If instead $s < 1$, then we have $s^\zeta = (s^{-1})^{-\zeta}$, and so $(s^{-1})^\zeta < q^{-1}$. By definition of $(s^{-1})^\zeta$, there exists $r \in Q_+$ with $\zeta < r$ and $(s^{-1})^r < q^{-1}$. Applying Lemma 3.1.12 again, we know there exists some t such that $s^{-1} < t < (q^{-1})^{\frac{1}{r}}$, and so $t^r < q^{-1}$. Thus, since $s^{-1} < t \iff t^{-1} < s$, by Corollary 3.2.6 we get $q < (t^{-1})^r < (t^{-1})^\zeta < s^\zeta$.[69]

Finally, checking definitions, it is clear that $(L_x)^\zeta = L_{x^\zeta}$. The proof for upper reals is similar.[70]

By Theorem A and Lemma 3.2.7, the basic laws hold for rational bases, and we can lift those to the one-sided bases. □

Corollary 3.2.9. *The map* $\exp\colon (0,\infty) \times \mathbb{R} \to (0,\infty)$ *of Theorem A satisfies the base product law.*

Proof. Fixing ζ, the base product equation is between two maps from $(0,\infty)^2$ to $(0,\infty)$. Since $(0,\infty)$ is locally compact, hence exponentiable, this amounts to equality between two maps $1 \to (0,\infty)^{(0,\infty)^2}$, and that is a subspace of 1 — internally in $\mathcal{S}\mathbb{R}$. Hence the ζs for which that internal subspace is 1 form a subspace of $\mathbb{R}$. To prove that subspace is the whole of $\mathbb{R}$, we use the fact that $\mathbb{R}$ is the subspace join of $(-\infty, 0)$ and its closed complement $[0,\infty)$ (see, e.g. [Joh82], or, for a geometric treatment, [Vic07c]). It thus suffices to check the equation in the two cases $\zeta < 0$ and $0 \leq \zeta$.

If $0 \leq \zeta$, then the base product law follows from Theorem 3.2.8. If $\zeta < 0$, then we reduce to the former case by applying the inverse map $(—)^{-1}$:

$$(x \cdot y)^\zeta = \left((x \cdot y)^{-\zeta}\right)^{-1} = \left(x^{-\zeta} \cdot y^{-\zeta}\right)^{-1} = x^\zeta \cdot y^\zeta.$$

□

As an application of the base product law, we generalise Lemma 3.1.12, and establish monotonic/antitonic result for real exponentiation with respect to the *base*:

Corollary 3.2.10. *Let $x, y \in (0,\infty)$ be positive Dedekind reals. Then:*

(i) If ζ is a positive Dedekind real, then $x < y \implies x^\zeta < y^\zeta$.
(ii) If ζ is a negative Dedekind real, then $x < y \implies y^\zeta < x^\zeta$.

Proof. Suppose $\zeta \in (0,\infty)$. Let us first prove (i) when $x = 1$. In which case, pick some $q \in \mathbb{Q}$ such that $0 < q < \zeta$. By Corollary 3.2.6, deduce that $1 < y$ implies $1 = 1^\zeta < y^q < y^\zeta$, as desired. In the general case when $x < y$ for $x, y \in (0,\infty)$, observe that $x < y \implies 1 < y \cdot x^{-1}$. Apply Corollary 3.2.9 to our previous calculation to get $1 < (y \cdot x^{-1})^\zeta = y^\zeta \cdot x^{-\zeta}$, which in turn implies that $x^\zeta < y^\zeta$, proving (i). The case when ζ is negative follows from (i) and the fact that inverses reverese orientation. □

[69] Why $(t^{-1})^\zeta < s^\zeta$? This follows from Lemma 3.2.7 and the same argument as in Corollary 3.2.10.

[70] Note that the definition includes $0^0 = \inf_{0<s\in\mathbb{Q}} s^0 = 1$.

For completeness, we also deal with two entirely one-sided cases, as seen in Table (3.4). Proposition 3.1.11 defines x^q for $0 < x$ a one-sided real and $0 \leq q$ rational. Restricting to $1 \leq x$, we now lift q to one-sided ζ.

Theorem 3.2.11. *For one-sided $x \geq 1$, the exponentiation by non-negative rational exponents (Subsection 3.1.4) can be extended to one-sided exponents, giving exponentiation maps*

$$\overrightarrow{[1,\infty]} \times \overrightarrow{[0,\infty]} \to \overrightarrow{[1,\infty]} \text{ and } \overleftarrow{[1,\infty]} \times \overleftarrow{[0,\infty]} \to \overleftarrow{[1,\infty]}.$$

They satisfy the Basic Equations (3.1) - (3.3).

Proof. We prove the lower case. The upper case is very similar.

Fix $x \in \overrightarrow{[1,\infty]}$, and recall that Q denotes the non-negative rationals. Following Example 2.2.51, we apply the Lifting Lemma in the case of $\overrightarrow{[0,\infty]} \cong \mathsf{RIdl}(Q,<)$, with the special understanding that $0 < 0$.

Monotonicity follows from Proposition 3.2.1. For continuity, suppose $r < x^q$ with $0 < q$. We want to prove $r < x^{q'}$ for some q' with $0 < q' < q$. If $r < 1$ then we can just take $q' = \frac{q}{2}$, so suppose $1 \leq r$. Since $(—)^q$ preserves and reflects strict order, we may find some rational s such that $1 \leq r^{\frac{1}{q}} < s < x$, which in turn implies that $r < s^q < x^q$. Now using the fact that s, a rational, is Dedekind, we can apply the argument of Proposition 3.2.4 to find our q'.

The Basic Equations follow from the rational case. □

3.3 Logarithms

Let $b \in (1,\infty)$ be Dedekind. Proposition 3.2.4 and Theorem A then gives us three maps $b^{(—)}$ from $\mathbb{R}$ to $(0,\infty)$ for Dedekinds or, with adjustments for zero and infinities, for one-sided reals. The main result of this section is to prove that all three maps are invertible by constructing the relevant $\log_b$ maps. In the case of Dedekinds, this yields the following:

$$\begin{aligned} \log\colon (1,\infty)\times(0,\infty) &\longrightarrow \mathbb{R} \\ (\,b\,,\,y\,) &\longmapsto \log_b(y) \end{aligned}$$

providing a geometric account of the usual logarithmic map. We remark that the case-splitting indicated by the monotone/antitone behaviour of rational exponents x^q when $x < 1$ vs. $x > 1$ emerges here through the case-splitting of the logarithmic base b, which forces us to consider the case of $b \in (0,1)$ and $b \in (1,\infty)$ separately.

Before proceeding, we shall need the following lemma.

Lemma 3.3.1. *Let $b \in (1,\infty)$ be Dedekind, and $q, q' \in Q_+$ be positive rationals such that $0 < q < q'$. Then there exists rational r such that $q < b^r < q'$.*

Proof of Lemma. This is a sharper version of the Continuity Lemma 3.2.3, which only required the Dedekind base to satisfy $b \geq 1$. In contrast, this lemma requires the Dedekind base b to be *strictly* greater than 1 (it is clearly false when $b = 1$).

We start by defining a series of parameters:

- By Bernoulli's Inequality (Lemma 3.2.2), find a natural number t such that $q' < b^t$, and a negative integer s such that $b^s < q$ (i.e. $q^{-1} < b^{-s}$).

- By the Continuity Lemma, find $k \in \mathbb{N}_+$ such that $1 \leq b^{\frac{1}{k}} < 1 + \frac{q'-q}{b^t}$.

Observe that $s + \frac{i}{k} \leq t$, for all $i \in \mathbb{N}$ such that $0 \leq i \leq (t-s)k$, with equality exactly when $i = (t-s)k$. This yields:

$$b^s \cdot b^{\frac{(i+1)}{k}} - b^s \cdot b^{\frac{i}{k}} = b^{s+\frac{i}{k}}(b^{\frac{1}{k}} - 1) < b^t\left(\frac{q'-q}{b^t}\right) = q' - q.$$

By Corollary 3.2.6, $\left\{b^{s+\frac{i}{k}}\right\}_{0 \leq i \leq (t-s)k}$ is a monotonic sequence of *located* Dedekinds. In particular, this means $q < b^{s+\frac{i}{k}} \vee b^{s+\frac{i}{k}} < q'$ for all relevant i. Thus, by a reasoning similar to the proof of Lemma 3.1.3, one shows there indeed exists an $i \in \mathbb{N}_+$ so that $q < b^{s+\frac{i}{k}} < q'$. □

Theorem B. Fix $b \in (1, \infty)$. We can then define one-sided logarithm maps

$$\log_b \colon \overrightarrow{[0,\infty]} \to \overrightarrow{[-\infty,\infty]} \quad \text{and} \quad \log_b \colon \overleftarrow{[0,\infty]} \to \overleftarrow{[-\infty,\infty]}$$

inverse to the corresponding exponentiation maps $b^{(-)}$ on the one-sideds. These combine to yield an isomorphism on the Dedekinds

$$\log_b \colon (0,\infty) \xrightarrow{\sim} (-\infty,\infty)$$

Proof. The proof proceeds in stages.

Step 0: Set-up. Fix $b \in (1,\infty)$. In the case where y is a one-sided real, we define the $\log_b$ maps as:

$$q < \log_b(y) \leftrightarrow b^q < y$$

$$q > \log_b(y) \leftrightarrow b^q > y$$

with the understanding that when y is a lower real, we define $b^q < y$ to mean $b^q < r < y$ for some rational r (and similarly when y is upper). Note that this definition makes sense since b^q is Dedekind. Finally, we remark that we shall freely make use of Corollary 3.2.6 without explicit mention, in particular the fact that $q < r \implies b^q < b^r$ for any rationals q and r.

Step 1: $\log_b(y)$ *as a one-sided.* To verify downward closure and upper-roundedness for the lower reals, observe that this coincides with verifying the monotonicity and continuity conditions for the Lifting Lemma in our proof of Proposition 3.2.4. We remark that when $y = 0$, then $\log_b 0$ is the empty lower real, i.e. $-\infty$. The case for upper reals is similar.

Step 2: $\log_b(y)$ *as a Dedekind.* After Step 1, it remains to check inhabitedness, separatedness and locatedness. Inhabitedness can be derived from Lemma 3.3.1.

To show separatedness, suppose $q < \log_b(y) < r$. By definition, this means $b^q < y < b^r$. By decidability of $<$ on $\mathbb{Q}$, we know that $r \leq q$ or $q < r$. If $r \leq q$, then this yields $b^r \leq b^q < b^r$, a contradiction. Hence, this forces the inequality $q < r$.

For locatedness, suppose we have $q, r \in \mathbb{Q}$ such that $q < r$. As before, we know that $q < r$ yields the inequality $b^q < b^r$. Pick $u_1, u_2 \in \mathbb{Q}$ such that $b^q < u_1 < u_2 < b^r$. Since y is located, either $u_1 < y$ or $y < u_2$. Since $u_1 < y \implies b^q < y$ and $y < u_2 \implies y < b^r$, this implies $q < \log_b(y) \vee r > \log_b(y)$, i.e. $\log_b(y)$ is located.

Step 3: $\log_b$ *and* $b^{(—)}$ *are inverses.* We prove $\log_b$ and $b^{(—)}$ are inverses on the one-sideds, where

$$b^{(—)}\colon \overrightarrow{[-\infty,\infty]} \to \overrightarrow{[0,\infty]} \quad \text{and} \quad b^{(—)}\colon \overleftarrow{[-\infty,\infty]} \to \overleftarrow{[0,\infty]}.$$

The case for the Dedekinds follows immediately by Corollary 2.2.24.

Step 3a: Show that $b^{\log_b(y)} = y$. Now suppose $q < b^{\log_b(y)}$. Then by definition (cf. Theorem A), $\exists q' \in \mathbb{Q}$ such that $q' < \log_b(y)$ and $q < b^{q'}$. Since $q' < \log_b(y) \implies b^{q'} < y$, this assembles to yield the inequality $q < b^{q'} < y$, proving that $b^{\log_b(y)} \leq y$. Conversely, suppose $q < y$. If $q < 0$, then $q < b^{\log_b(y)}$ automatically. If $q \geq 0$, then pick $q', q'' \in Q_+$ such that $q < q' < q'' < y$. By Lemma 3.3.1, there exists $r \in \mathbb{Q}$ such that $q < q' < b^r < q'' < y$, hence $r < \log_b(y)$ and $q < b^{\log_b(y)}$, i.e. $y \leq b^{\log_b(y)}$. Since $y \leq b^{\log_b(y)}$ and $b^{\log_b(y)} \leq y$, this shows that $b^{\log_b(y)} = y$ for lower reals. The case for the upper reals is entirely analogous.

Step 3b: Show that $\log_b(b^\zeta) = \zeta$. Suppose $q < \log_b(b^\zeta)$. Then, unwinding definitions, there exists rationals $q', q'' \in \mathbb{Q}$ such that $b^q < q' < b^\zeta$ and $q'' < \zeta \wedge q' < b^{q''}$. In particular, this yields the inequality $b^q < b^{q''}$, which in turn implies $q < q'' < \zeta$, proving that $\log_b(b^\zeta) \leq \zeta$. Conversely, suppose that $q < \zeta$. Then, this gives $b^q < b^\zeta$, which by definition yields $q < \log_b(b^\zeta)$, proving that $\zeta \leq \log_b(b^\zeta)$. Putting everything together yields $\log_b(b^\zeta) = \zeta$. As before, the case for upper reals is entirely analogous. □

Observation 3.3.2. As a sanity check, note that Steps 3a and 3b in the previous proof verify standard logarithmic identities. In particular, for the map $\log_b\colon (0,\infty) \to (-\infty,\infty)$ on the Dedekinds, Step 3 essentially says: "Given any $y \in (0,\infty)$, $\log_b(y)$ is the *unique* Dedekind ζ such that $b^\zeta = y$ holds." In particular, suppose $y = b$. Then since $b^1 = b$, uniqueness of $\log_b(b)$ implies that $\log_b(b) = 1$, as expected. Similarly, suppose $y = 1$. Since $b^0 = 1$, this thus implies that $\log_b(1) = 0$, again exactly as expected.

Theorem B fixes $b \in (1,\infty)$ in order to define a map $\log_b(—)\colon (0,\infty) \to (-\infty,\infty)$ on the Dedekinds. Externalising this map yields the desired map

$$\log\colon (1,\infty) \times (0,\infty) \to \mathbb{R}.$$

Remark 3.3.3. One can also define a logarithm map with base $b \in (0,1)$ in terms of the previous $\log$ map, as follows:

$$\begin{aligned}\log\colon (0,1) \times (0,\infty) &\to \mathbb{R} \\ (\,b, y\,) &\longmapsto -\log(\,b^{-1}, y\,).\end{aligned}$$

3.4 Comparisons with other approaches and Applications

It is reasonable to ask: why did we choose to develop point-free real exponentiation in the way we did? For instance, why did we choose to work with Dedekind reals as opposed to Cauchy reals? Or why did we not choose to define exponentiation via power series — e.g. by first defining the functions

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for } x \in \mathbb{R}$$

$$\ln(x) := \int_1^x t^{-1}dt \qquad \text{for } x \in (0,\infty),$$

before defining $x^\zeta := \exp(\zeta \ln(x))$, for $x \in (0, \infty)$ and $\zeta \in \mathbb{R}$? The answer to both questions is that we felt that our chosen approach would be comparatively cleaner in the point-free setting. Indeed, whilst there does exist a point-free account of quotiented Cauchy reals [Vic98, Theorem 7.8] and of integration [Vic07a], both descriptions involve rather complicated and technical details. In contrast, by choosing to define real exponentiation by successively lifting it from the rational case to the reals, our construction explicitly (and geometrically) highlights how many familiar algebraic properties of real exponentiation are in fact inherited from the properties of $\mathbb{Q}$ as one might expect — something which would likely to have been obscured in the other two approaches.

One may also reasonably ask: why even construct a point-free account of real exponentiation in the first place? One simple answer is that point-free topology is a school of constructive mathematics with many attractive features, and so it's worth translating familiar ideas from real analysis into this setting and demonstrating how they (more or less) work in the ways we expect them to. From a methodological standpoint, it is also instructive to illustrate how point-wise reasoning works in a point-free setting.

A slightly deeper answer, however, comes from our interest in applying ideas from geometric logic to questions arising from number theory. As explained in e.g. Chapter 1, motivated by the goal of constructing the (point-free) space of completions of $\mathbb{Q}$, we would like to be able to first define the places of $\mathbb{Q}$, i.e. the equivalence classes of absolute values on $\mathbb{Q}$ modulo the equivalence relation $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_1^\alpha = |\cdot|_2$ for some $\alpha \in (0, 1]$. As should be obvious, this requires two important ingredients: (a) a geometric account of real exponentiation; (b) a geometric account of absolute values on $\mathbb{Q}$. Item (a) was precisely the work of this chapter. For item (b), we turn to the next chapter.

Chapter 4

The Topos of Absolute Values

Whereas one-sided reals played a primarily computational role in Chapter 3, in this chapter they acquire a conceptual significance. Having established a geometric account of real exponentiation, we now define the topos of absolute values (both upper and Dedekind-valued) and prove the corresponding Ostrowski-type theorems.

On a basic level, this chapter can be read as just a piece of constructive mathematics, where we sharpen the classical Ostrowski's theorem in various sensible ways (e.g. by phrasing it as a representation result instead of just a classification result, by working geometrically instead of classically etc.). However, our work also sets up two broader lines of analysis. Firstly, our investigations bring to light a subtle connection between multiplicative seminorms and upper reals, invisible from the classical perspective. This link between topology and algebra will later be leveraged in Chapter 5 to examine how the structural gap between trivially vs. non-trivially valued fields in Berkovich geometry may be eliminated via point-free techniques. Secondly, a major payoff for reworking these algebraic ideas geometrically is that point-free spaces can be regarded dually as toposes, allowing us to bring a deep collection of topos-theoretic tools to bear on the analysis of absolute values. This sets up Chapter 6, where descent techniques are used to reveal some striking differences between the Archimedean vs. non-Archimedean places of $\mathbb{Q}$.

4.1 Preliminaries

Classically, an absolute value on $\mathbb{Q}$ is defined as a map $|\cdot|\colon \mathbb{Q} \to [0,\infty)$. Reworking this geometrically presents us with several options: should the map be valued in Dedekinds or one-sided reals? Further, notice that since $|\cdot|\colon \mathbb{Q} \to [0,\infty)$ is determined by its values on the non-zero integers $\mathbb{Z}_{\neq 0}$, one may also define an absolute value as a map from $\mathbb{Z}$, or $\mathbb{Q}$. This selection of topological and algebraic options have a curious interaction, which we summarise in the following observation.

Observation 4.1.1.

(i) An absolute value on $\mathbb{Q}$ (or indeed any field) must be valued in Dedekinds, and not the one-sideds.

(ii) An absolute value on $\mathbb{Z}$ can be valued in the one-sided reals.

Proof/Discussion. We give two natural reasons for (i). First, one typically requires an absolute value to preserve invertibility, i.e. given $q, q' \in \mathbb{Q}$ such that $qq' = 1$, we expect

$$1 = |1| = |qq'| = |q| \cdot |q'|.$$

But notice if, e.g. u, v are upper reals such that $u \cdot v = 1$, then they must be Dedekind since invertibility enforces $u < q$ iff $q^{-1} < v$. The same argument also applies to the lower reals.

Second, notice that orientation issues prevent us from exponentiating $|\cdot|$ as a map from $\mathbb{Q}$ to the one-sideds. The reasons for this have already been discussed in Section 3.2. Namely, if x is one-sided, then x^α must have the same orientation (upper or lower) as α for continuity reasons. Since x^α is monotone when $x \geq 1$ and antitone when $x \leq 1$, then this forces a piecewise definition of $|\cdot|^\alpha$ on $\mathbb{Q}$ that *cannot* be glued back together.[71] Importantly, this means we are unable to extend the notion of places to upper-valued absolute values on $\mathbb{Q}$, leading us to a dead-end.

On the other hand, if we were to define absolute values on $\mathbb{Z}$, then we have more flexibility. Not only are the absolute values no longer required to preserve invertibility (since $\mathbb{Z}$ has no non-trivial multiplicative inverses), but we claim that the following definition of $|\cdot|^\alpha$ as:

$$|n|^\alpha = \begin{cases} 0 & \text{if } n = 0 \\ |n|^\alpha & \text{if otherwise,} \end{cases} \tag{4.1}$$

is geometric even if $|\cdot|$ is valued in the one-sideds. This follows from the fact that:

(a) $=$ is decidable on $\mathbb{Z}$ (and so the case-splitting between zero vs. non-zero elements is geometric);
(b) Either $|n| \geq 1$ or $|n| \leq 1$ for all non-zero $n \in \mathbb{Z}_{\neq 0}$ (and so we avoid the monotonicity issues that arose for absolute values on $\mathbb{Q}$).

Fact (a) is a direct application of Fact 2.2.4; Fact (b) requires more work, but will surface naturally in our proof of Ostrowski's Theorem. □

Observation 4.1.1 sets the scope for our present task: work towards an explicit description of the topos of absolute values, first by proving a modified version of Ostrowski's theorem for upper-valued absolute values on $\mathbb{Z}$

$$|\cdot|\colon \mathbb{Z} \to \overleftarrow{[0,\infty)},$$

before recovering the standard Ostrowski's Theorem for absolute values on $\mathbb{Q}$. Of course, the decision to use upper reals rather than Dedekinds may strike the reader as constructivist hair-splitting, but in fact it ties together two *a priori* unrelated mathematical threads:

(a) Vickers [Vic05]: To provide a geometric account of the completions of a (generalised) metric space, it suffices for the metric to be valued in non-negative upper reals (as opposed to the Dedekinds).
(b) Berkovich [Ber90]: For a suitable field K, every point x of the Berkovich affine line $\mathbb{A}^1_{\text{Berk}}$ corresponds to a nested descending sequence of closed discs in K:

$$D_1 \supseteq D_2 \supseteq \dots$$

Further discussion of this connection will be deferred to Chapter 5, but the reader familiar with Berkovich geometry should already notice the suggestive parallel between "a point of $\mathbb{A}^1_{\text{Berk}}$ = a nested sequence of discs" and "a point of the upper reals = a rounded ideal" (cf. Example 2.2.51).

[71] Notice: the Gluing Principle of Proposition 2.2.60 *only* applies to gluing Dedekind intervals, where orientation is no longer an issue.

4.1.1 Geometric Theories of Absolute Values.

Definition 4.1.2 (Absolute Values on $\mathbb{Z}$, valued in upper reals)**.** Take a signature Σ_{av} comprising

> **Sorts:** $\mathbb{Z}, Q_+$
> **Relations:** $R \subseteq \mathbb{Z} \times Q_+$ (write $|x| < q$ for $R(x,q)$)

where Q_+ is equipped with strict order $<$. Define $\overleftarrow{av}$ to be the theory over Σ_{av} along with the axioms:

1. $(\forall n \in \mathbb{Z}, \forall q \in Q_+).$ $\quad |n| < q \longleftrightarrow \exists q' \in Q_+.(q' < q \wedge |n| < q')$ $\quad (|n| \in \overleftarrow{[0,\infty]}\,)$
2. $(\forall n \in \mathbb{Z}).$ $\quad \exists q \in Q_+.|n| < q$ $\quad (|n| < \infty\,)$
3. $(\forall n, m \in \mathbb{Z}, \forall q, r \in Q_+).$ $\quad |n| < q \wedge |m| < r \longleftrightarrow |n \cdot m| < q \cdot r$ $\quad$ (Multiplicativity)
4. $(\forall q \in Q_+).$ $\quad |0| < q \longleftrightarrow 0 < q$ $\quad (|0| = 0)$
5. $(\forall q \in Q_+).$ $\quad |1| < q \longleftrightarrow 1 < q$ $\quad (|1| = 1)$
6. $(\forall q \in Q_+, \forall n \in \mathbb{Z}).$ $\quad |n| < q \longleftrightarrow |-n| < q$ $\quad (|n| = |-n|)$
7. $(\forall n, m \in \mathbb{Z}, \forall q, r \in Q_+).$ $\quad |n| < q \wedge |m| < r \longrightarrow |n + m| < q + r$ $\quad$ (Triangle Inequality)

Observation 4.1.3. The choice of axioms in $\overleftarrow{av}$ reflect the topological constraints the upper reals puts on the algebra. In particular, notice:

(i) Definition 4.1.2 does not contain an axiom for positive definiteness, one of the standard properties for an absolute value. In fact, this is impossible in the present set-up. To define positive definiteness, we would need add an axiom that essentially says:

$$(\forall n \in \mathbb{Z}, \forall q \in Q_+). \qquad |n| > 0 \longleftrightarrow n \neq 0.$$

However, since $|n|$ is an upper real, the formula $|n| > 0$ is not geometric, and so neither is the above sequent. Put otherwise, the topology of the upper reals forces us to work the *multiplicative seminorms* for $\mathbb{Z}$, as opposed to the usual norms.

(ii) In the classical setting, one typically derives the properties $|1| = 1$ and $|n| = |-n|$ from the fact that absolute values are multiplicative and positive definite.[72] In our setting, however, we no longer have positive definiteness and so these axioms must now be included explicitly.

Remark 4.1.4. Why not go one step further and work with the upper-valued absolute values on $\mathbb{N}$?[73] After all, aren't the absolute values on $\mathbb{Z}$ themselves determined by their values on the positive integers $\mathbb{N}_+$? A fuller answer will be presented in Discussion 4.2.8. For now, let us say that proving Ostrowski's Theorem without additive inverses results in various difficulties which inclines us to stick with $\mathbb{Z}$.

The standard examples of absolute values can all be reworked to obey Definition 4.1.2, which we sketch below. The syntactic details have been suppressed for readability, but one easily checks that the definitions only use arithmetic operations that are well-defined on the upper reals and satisfy the required axioms.

Example 4.1.5. Since $=$ is decidable on $\mathbb{Z}$, it suffices to define $|\cdot|$ for $n \neq 0$ since we already require that $|0| = 0$ by definition.

[72] How so? In the case of $|1| = 1$, multiplicativity gives us $|1| \cdot |1| = |1|$. This implies $|1| \cdot (|1| - 1) = 0$. Positive definiteness tells us that $|1| \neq 0$, and thus $|1| = 1$. A similar argument yields $|-1| = 1$, which in turn (by multiplicativity) implies $|n| = |-n|$.

[73] That is, replace the sort $\mathbb{Z}$ featured in Definition 4.1.2 with the natural numbers $\mathbb{N}$ and also eliminate the axiom $|n| = |-n|$.

(i) The *trivial absolute value* on $\mathbb{Z}$, denoted $|\cdot|_0$, is defined as

$$|n|_0 = 1, \qquad \text{for all } n \neq 0.$$

(ii) The *Euclidean absolute value* on $\mathbb{Z}$, denoted $|\cdot|_\infty$, is defined as the usual norm

$$|n|_\infty = n, \qquad \text{for all } n \in \mathbb{N}_+,$$

which is extended to the negative integers by the axiom $|n| = |-n|$ for all $n \in \mathbb{Z}$.

(iii) Fix some prime $p \in \mathbb{N}_+$. By unique prime factorisation, any non-zero integer $n \in \mathbb{Z}_{\neq 0}$ can be represented as $n = p^r z$, where $r \in \mathbb{N}$, $z \in \mathbb{Z}$ and $\gcd(p, z) = 1$. As such, define the *p-adic ordinal*

$$\operatorname{ord}_p(n) := \max\{r \in \mathbb{N} \mid p^r \text{ divides } n\}.$$

The canonical *p-adic absolute value* on $\mathbb{Z}$ is then defined as

$$|n|_p = p^{-\operatorname{ord}_p(n)} \qquad \text{for all } n \neq 0.$$

Next, let us recall from Example 2.2.37 the definition of a geometric field:

Definition 4.1.6. A ring R is called a *geometric field* if it satisfies the following:

(i) $0 \neq 1$

(ii) For any $x \in R$, either $x = 0$ or $\exists x^{-1} \in R$ such that $x \cdot x^{-1} = 1$.

Remark 4.1.7. Definition 4.1.6 is a good illustration of the difference between the classical vs. geometric perspective. Viewed classically, all the usual fields satisfy the stated properties trivially. Viewed geometrically, however, Definition 4.1.6 imposes very restrictive conditions on the topology. In particular, [Joh77a, Lemma 2.1] observes that $=$ has to be decidable on any geometric field. Recalling Discussion 2.2.13, this means that while $\mathbb{Q}$ is an example of a geometric field, topological fields like the Dedekinds $\mathbb{R}$ are not.

Observation 4.1.1 indicates that any absolute value defined on a geometric field must be valued in the Dedekinds; Remark 4.1.7 indicates that being a geometric field carries strong structural implications for the algebra. Together, they set up the following definition:[74]

Definition 4.1.8 (Absolute Values on Geometric Fields, valued in Dedekinds)**.** Let R be a geometric field. An absolute value on R is a map $|\cdot|\colon R \to [0, \infty)$ satisfying the following axioms:

(i) $|1| = 1$, $|0| = 0$;

(ii) $|xy| = |x| \cdot |y|$ for any $x, y \in R$;

(iii) $|x + y| \leq |x| + |y|$;

Observation 4.1.9. The richer structure of R yields two additional axioms not listed in Definition 4.1.8.

(iv) We also get positive definiteness for free. Why? Suppose $|x| = 0$. By Definition 4.1.6 (ii), we know either $x = 0$ or $\exists x^{-1}$ such that $x \cdot x^{-1} = 1$. Since the latter yields $1 = |1| = |x \cdot x^{-1}| = |x| \cdot |x^{-1}| = 0$, a contradiction, it must be that $x = 0$.

[74] Again, the full syntactic details have been suppressed for readability.

(v) Now that we have positive definiteness, the same argument as in Footnote 72 allows us to deduce that $|x| = |-x|$ for all $x \in R$.

Finally, in light of our discussion of Observation 4.1.1, we give the natural definition of the exponentiation of absolute values.

Definition 4.1.10. Let $|\cdot|$ be an absolute value (in the sense of either Definition 4.1.2 or 4.1.8), and let α be a real (Dedekind or upper, depending on context). We then define $|\cdot|^\alpha$ as follows:

$$|x|^\alpha = \begin{cases} 0 & \text{if } x = 0 \\ |x|^\alpha & \text{if otherwise.} \end{cases} \tag{4.2}$$

Remark 4.1.11. We emphasise that Observation 4.1.1 only gives that $|\cdot|^\alpha$ is a geometric construction; more work is needed to show that $|\cdot|^\alpha$ satisfies the required properties of an absolute value.

4.1.2 Archimedean vs. Non-Archimedean Absolute Values on $\mathbb{Z}$.

Convention 4.1.12 ("Absolute Value"). Until further notice, the term "absolute value" in this chapter should be taken to mean an absolute value on $\mathbb{Z}$ in the sense of Definition 4.1.2, i.e. an upper-valued multiplicative seminorm of $\mathbb{Z}$. We will be explicit about when we are considering absolute values on $\mathbb{Q}$.

We already saw how the upper reals impose unusual restrictions on the algebra. In fact, notice: while one can define upper real subspaces characterised by $x < 1$ and $1 \leq x$, there does not exist a subspace of upper reals such that $1 < x$.[75] This suggests the following definitions for the Archimedean and non-Archimedean absolute values:

Definition 4.1.13. As our setup:

- Let Σ_{av} be the signature of the theory $\overleftarrow{av}$ as in Definition 4.1.2.
- Denote $\mathbb{Z}_{\neq 0}$ to be the set of non-negative integers.
- Define the Archimedean and non-Archimedean axioms respectively as

(A) $(\forall q \in \mathbb{Q}, \forall n \in \mathbb{Z}_{\neq 0}). \quad |n| < q \to 1 < q.$

(NA) $\top \to \bigvee_{n \in \mathbb{Z}_{\neq 0}} |n| < 1.$

Then, we define …

(i) … the *theory of Archimedean absolute values on* $\mathbb{Z}$, denoted $\overleftarrow{av}_A$, to be the quotient theory of $\overleftarrow{av}$ with the added Axiom (A).

(ii) … the *theory of non-Archimedean absolute values on* $\mathbb{Z}$, denoted $\overleftarrow{av}_{NA}$, to be the quotient theory of $\overleftarrow{av}$ with the added Axiom (NA).

Remark 4.1.14. In English, Axiom (A) says $1 \geq |n|$ for all non-zero $n \in \mathbb{Z}_{\neq 0}$, and Axiom (NA) says there exists some non-zero $n \in \mathbb{Z}_{\neq 0}$ such that $|n| < 1$. Notice:

(i) (NA) defines an open subspace of $[\overleftarrow{av}]$ whereas (A) defines a closed subspace.

[75] Why? Again, this is because every subspace of upper reals is required to be closed under directed joins (Convention 2.2.17).

(ii) This is slightly different from the standard definitions — in particular, the trivial absolute value is now considered to be Archimedean, not non-Archimedean.

The following key result gives a geometric justification of the obvious: an absolute value is either Archimedean or non-Archimedean.

Proposition 4.1.15. *$[\overleftarrow{av}_A]$ and $[\overleftarrow{av}_{NA}]$ are complementary subspaces in $[\overleftarrow{av}]$.*

Proof. We organise the proof into stages.

Step 0: Plan of Attack. Let $\mathbb{T}$ be a geometric theory. Recall the following:

- Local operators of a topos form a Heyting algebra [Joh02a, Example A4.5.14].
- There is a bijection between subtoposes of $\mathcal{S}[\mathbb{T}]$ and local operators on $\mathcal{S}[\mathbb{T}]$ [Joh02a, Theorem A4.4.8].
- Subspaces of $[\mathbb{T}]$ (up to equivalence) correspond bijectively to the subtoposes of $\mathcal{S}[\mathbb{T}]$, which in turn correspond bijectively to the quotients of the theory $\mathbb{T}$ (Proposition 2.1.30).

Hence, in order to prove $[\overleftarrow{av}_A]$ and $[\overleftarrow{av}_{NA}]$ are complementary subspaces in $[\overleftarrow{av}]$, it suffices to show that no model of $\overleftarrow{av}$ is a model of the meet $\overleftarrow{av}_{NA} \wedge \overleftarrow{av}_A$, and all models of $\overleftarrow{av}$ satisfy the join $\overleftarrow{av}_{NA} \vee \overleftarrow{av}_A$.

Step 1: Verification. Applying Step 0, the Proposition follows from verifying:

Step 1a: No Model of $\overleftarrow{av}$ is a Model of $\overleftarrow{av}_{NA} \wedge \overleftarrow{av}_A$. Straightfoward, but we elaborate. Suppose there exists an absolute value $|\cdot|$ satisfying Axioms (A) and (NA). By Axiom (NA), there exists some $n \in \mathbb{Z}_{\neq 0}$ such that $|n| < 1$. But this means there exists some $s \in \mathbb{Q}$ such that $|n| < s < 1$, contradicting Axiom (A).

Step 1b: Models of $\overleftarrow{av}$ are also Models of $\overleftarrow{av}_{NA} \vee \overleftarrow{av}_A$. By [Vic07c, Theorem 20], one obtains the following general presentation result: given any pair of geometric theories $\mathbb{T}_1$ (whose axioms are of the form $a_i \vdash b_i$) and $\mathbb{T}_2$ (whose axioms are of the form $c_j \vdash d_j$) sharing the same signature Σ, their join $\mathbb{T}_1 \vee \mathbb{T}_2$ has axioms of the form $a_i \wedge c_j \vdash b_i \vee d_j$. In particular, this means that $\overleftarrow{av}_{NA} \vee \overleftarrow{av}_A$ essentially contains all the axioms of $\overleftarrow{av}$, in addition to the following axiom:

$$(\forall q \in \mathbb{Q}, \forall n \in \mathbb{Z}_{\neq 0}). \quad |n| < q \to \bigvee_{m \in \mathbb{Z}_{\neq 0}} |m| < 1 \vee 1 < q$$

Absolute values $|\cdot|$ on $\mathbb{Z}$ already satisfy the axioms of $\overleftarrow{av}$, so it remains to show that they also satisfy this new axiom. Suppose we have some $q \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\neq 0}$ such that $|n| < q$. By decidability of $<$ on $\mathbb{Q}$, either $1 < q$ or $q \leq 1$. If $1 < q$, then the sequent holds trivially. If $q \leq 1$, then $|n| < 1$, which implies $\bigvee_{m \in \mathbb{Z}_{\neq 0}} |m| < 1$, and so we are done. □

4.2 Ostrowski's Theorem for $\mathbb{Z}$

Ostrowski's Theorem is typically phrased as a classification result — it answers the question: *'What are all the non-trivial places of $\mathbb{Q}$?'*. Here, we sharpen this to a representation result for absolute values on $\mathbb{Z}$:

Theorem C (Ostrowski's Theorem for $\mathbb{Z}$)**.** As our setup, denote:

- $[\overleftarrow{av}]$:= The space of absolute values on $\mathbb{Z}$, valued in upper reals.
- $\mathrm{ISpec}(\mathbb{Z})$:= The space of prime ideals of $\mathbb{Z}$ (cf. Section 2.2.3).
- $\mathbb{Z}_{\neq 0}$:= The set of non-zero integers.
- $\overleftarrow{[-\infty, 1]}$:= The space of upper reals bounded above by 1.

Define

$$\mathfrak{P}_\Lambda := \{(\mathfrak{p}, \lambda) \in \mathrm{ISpec}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]} \mid \lambda < 0 \leftrightarrow \exists a \in \mathbb{Z}_{\neq 0}.(a \in \mathfrak{p})\}.$$

Then, the following spaces are equivalent:

$$[\overleftarrow{av}] \cong \mathfrak{P}_\Lambda.$$

Informally, Theorem C says: any absolute value $|\cdot|$ of $\mathbb{Z}$ can be canonically associated to a pair

$$(\mathfrak{p}, \lambda) \in \mathrm{ISpec}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]}$$

satisfying certain compatibility conditions.

Before we begin the proof, some preparatory remarks. One, the standard proofs of Ostrowski's Theorem (spelled out in, e.g. [Wae91b]) can be adapted to our setting, but they still only give us one direction of the isomorphism. Additional work is therefore needed to construct the second direction, and to show that the two directions are inverse to each other. Two, the decision to work with upper reals (as opposed to Dedekinds) makes the algebra rather delicate, for reasons already alluded to in Observation 4.1.1. Some care is needed in order to maintain geometricity throughout the proof, which (interestingly) results in a picture of $[\overleftarrow{av}]$ that is slightly different from the classical picture of the Berkovich spectrum $\mathcal{M}(\mathbb{Z})$ (see Example 5.1.9).

4.2.1 First Direction: Classification of Absolute Values. Recall: in order to define a map

$$[\overleftarrow{av}] \longrightarrow \mathfrak{P}_\Lambda,$$

it suffices to define a *geometric* construction

$$|\cdot| \longmapsto (\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|})$$

that transforms the generic point of $[\overleftarrow{av}]$ into another point of the space $(\mathfrak{P}_\Lambda)$. The following construction makes this explicit:

Construction 4.2.1. Suppose $|\cdot| \in [\overleftarrow{av}]$. For any $b \in \mathbb{Z}_{\neq 0}$ such that $b > 1$, define a logarithm map on the (inhabited) upper reals with base b

$$\log_b \colon \overleftarrow{[0, \infty)} \to \overleftarrow{[-\infty, \infty)}.$$

Then define

- $\mathfrak{p}_{|\cdot|} := \{n \in \mathbb{Z} \big| |n| < 1\}$
- $\lambda_{|\cdot|} := \inf\{\log_m |m| \big| \ m \in \mathbb{N}_+ \text{ is prime}\}$.

Discussion 4.2.2. A point-free account of $\log_b \colon \overleftarrow{[0, -\infty]} \to \overleftarrow{[-\infty, \infty]}$ with Dedekind base $b \in (1, \infty)$ was worked out in Theorem B. Since: (a) any integer $b > 1$ may be canonically viewed as a Dedekind; and (b) upper reals possess arbitrary $\inf$s (Observation 2.2.32), geometricity of Construction 4.2.1 is clear by inspection.

It remains to show that $(\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|})$ is indeed a point of $\mathfrak{P}_\Lambda$. We shall rely on two important tools — Lemmas 4.2.3 and 4.2.4 (Fundamental Lemma for Ostrowski) — to reveal the algebraic character of $|\cdot|$.

Lemma 4.2.3. *Let α, β be positive Dedekinds, and γ, γ' be non-negative upper reals such that*

$$\gamma^v \leq (\alpha v + \beta) \cdot (\gamma')^v$$

for all $v \in \mathbb{N}_+$. Then $\gamma \leq \gamma'$.

Proof. First, a basic but key observation: if $\gamma, \gamma' \in \overleftarrow{[0, \infty]}$ such that $\gamma \leq (1+\delta)\gamma'$ for all positive rationals δ, then this implies $\gamma \leq \gamma'$. It thus suffices to prove that the Lemma's hypothesis implies $\gamma \leq (1+\delta)\gamma'$, for all positive rationals δ.

Fix such a rational $\delta > 0$. Binomial expansion yields the inequality $(1+\delta)^v \geq 1 + v\delta + \frac{v(v-1)}{2}\delta^2$ for any integer $v \geq 2$. It is clear that for sufficiently large v, we get

$$v\delta > \beta \qquad \text{and} \qquad \frac{1}{2}(v-1)\delta^2 > \alpha,$$

and so

$$\gamma^v \leq (\alpha v + \beta) \cdot (\gamma')^v < (1+\delta)^v \cdot (\gamma')^v = \big((1+\delta)\gamma'\big)^v. \tag{4.3}$$

Since $(—)^v$ reflects non-strict order on the upper reals (cf. Lemma 3.1.7), Equation (4.3) implies $\gamma \leq \gamma'$. ☐

Lemma 4.2.4 (Fundamental Lemma for Ostrowski)**.** *Let $a, b > 1$ be any pair of integers greater than 1, and let $|\cdot|$ be any absolute value on $\mathbb{Z}$. Then:*

(i) $\log_a |a| \leq \max\{0, \log_b |b|\}$

(ii) $\max\{0, \log_b |b|\} = \max\{0, \log_a |a|\}$ *for any $a, b > 1$.*

In particular, we can associate a constant $M_{|\cdot|} := \max\{0, \log_b |b|\}$ to any absolute value $|\cdot|$ since by (ii) we know $M_{|\cdot|}$ is independent of our choice of $b > 1$.

Proof. (i): Given any pair of integers a, b such that $a, b > 1$, and given any $v \in \mathbb{N}_+$, we may expand a^v in powers of b as follows:

$$a^v = c_0 + c_1 b + ... + c_r b^r \tag{4.4}$$

where $0 \leq c_i < b$ for $0 \leq i \leq r$, and $c_r \neq 0$. It is obvious that

$$b^r \leq a^v,$$

which (taking $\log_b(—)$ on both sides) yields

$$r \leq \log_b a^v = v \log_b a. \tag{4.5}$$

Next, observe that for any $n \in \mathbb{N}_+$, the triangle inequality yields

$$|n| = |\underbrace{1 + 1 + \cdots + 1}_{n}| \leq \underbrace{|1| + |1| + \cdots + |1|}_{n} = n. \tag{4.6}$$

Hence, Equations (4.4) and (4.6) give

$$\begin{aligned} |a^v| &\leq |c_0| + |c_1||b| + ... + |c_n||b|^r \\ &\leq b(1 + |b| + \cdots + |b|^r) \leq b(r+1)B^r \end{aligned}$$

where $B = \max\{1, |b|\}$. By Equation (4.5), we get

$$|a^v| = |a|^v < b(v \log_b a + 1) \cdot (B^{\log_b a})^v.$$

Applying Lemma 4.2.3, this yields

$$|a| \leq B^{\log_b a},$$

which in turn yields

$$\begin{aligned}\log_a |a| &\leq \max\{0, \log_a |b|^{\log_b a}\} \\ &= \max\{0, \log_b a^{\log_a |b|}\} = \max\{0, \log_b |b|\}.\end{aligned}$$

To prove (ii), note that (i) yields for any pair of integers $a, b > 1$

$$\max\{0, \log_a |a|\} \leq \max\{0, \max\{0, \log_b |b|\}\} = \max\{0, \log_b |b|\},$$

and so by symmetry

$$\max\{0, \log_a |a|\} = \max\{0, \log_b |b|\}.$$

□

Interestingly, even though the constant $M_{|\cdot|}$ of Lemma 4.2.4 is defined for generic $|\cdot|$, it is still sensitive to important differences between the Archimedean vs. non-Archimedean case. The next two propositions develop this remark.

Proposition 4.2.5. *Let $|\cdot|$ be an Archimedean[76] absolute value on $\mathbb{Z}$. Then, $(\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|})$ is such that*

- $\mathfrak{p}_{|\cdot|} = (0)$
- $|\cdot| = |\cdot|_\infty^{\lambda_{|\cdot|}}$

In particular, $\lambda_{|\cdot|} \in \overleftarrow{[0, 1]}$.

Proof. Let $|\cdot|$ be an Archimedean absolute value, i.e.

$$1 \leq |b|, \qquad \forall b \in \mathbb{Z}_{\neq 0}. \tag{4.7}$$

It is clear this implies $\mathfrak{p}_{|\cdot|} = (0)$. Further, when $b > 1$, applying $\log_b(\text{—})$ to the Equation (4.7) gives

$$0 = \log_b 1 \leq \log_b |b|,$$

and so $\max\{0, \log_b |b|\} = \log_b |b|$. Thus, for any pair of integers $a, b > 1$, the Fundamental Lemma 4.2.4 implies

$$M_{|\cdot|} = \log_b |b| = \log_a |a|. \tag{4.8}$$

Equation (4.8) establishes that $M_{|\cdot|} = \log_a |a|$ for all integers $a > 1$, which thus implies $\lambda_{|\cdot|} = M_{|\cdot|}$. Since (a) $n = |n|_\infty$, for arbitrary $n \in \mathbb{N}$; and (b) absolute values on $\mathbb{Z}$ are determined by their values on integers $a > 1$,[77] deduce that $|\cdot| = |\cdot|_\infty^{\lambda_{|\cdot|}}$ as claimed. It remains to show that $\lambda_{|\cdot|} \in \overleftarrow{[0, 1]}$, but this is easy. By the triangle inequality and Equation (4.7), we know that $1 \leq |b| \leq b$, and so $0 \leq \log_b |b| \leq \log_b b$, or equivalently, $0 \leq \lambda_{|\cdot|} \leq 1$. □

[76] Recall that this includes the trivial absolute value as well in the present setting.

[77] Why? We already know that $|0| = 0$ and $|1| = 1$, and $|-n| = n$ for all $n \in \mathbb{Z}$.

Proposition 4.2.6. *Let $|\cdot|$ be a non-Archimedean absolute value on $\mathbb{Z}$. Then, $(\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|})$ is such that*

- $\mathfrak{p}_{|\cdot|} = (p)$ *for some prime $p \in \mathbb{N}_+$;*
- $\lambda_{|\cdot|} = \log_p |p|$.

In particular, $\lambda_{|\cdot|} \in \overleftarrow{[-\infty, 0)}$.

Proof. Let $|\cdot|$ be a non-Archimedean absolute value, i.e.

$$\exists b \in \mathbb{Z}_{\neq 0} \text{ such that } |b| < 1.$$

We claim that $\mathfrak{p}_{|\cdot|}$ is a non-trivial prime ideal in $\mathbb{Z}$ in the sense of Definition 2.2.40. The fact that $\mathfrak{p}$ is non-trivial and contains 0 is clear by construction. It remains to check:

- *Closure under multiplication.* This is immediate if we know that $|n| \leq 1$ for all $n \in \mathbb{Z}$. But this follows from the Fundamental Lemma 4.2.4, which implies

$$\log_n |n| \leq M_{|\cdot|} = \max\{0, \log_b |b|\} = 0,$$

  for all integers $n > 1$, and the fact that $|n| = |-n|$.
- *Closure under addition.* This is immediate if we know $|a + b| \leq \max\{|a|, |b|\}$, for all $a, b \in \mathbb{Z}$. But this follows from Lemma 4.2.3, since for all $v \in \mathbb{N}_+$, $|\binom{v}{i}| \leq 1$ and so binomial expansion yields

$$|a + b|^v \leq |a|^v + |a|^{v-1}|b| + \cdots + |b|^v \leq (v + 1) \cdot (\max\{|a|, |b|\})^v ,$$

  which (by the Lemma) implies $|a + b| \leq \max\{|a|, |b|\}$, as desired.
- *Primeness.* It is clear that $1 \notin \mathfrak{p}_{|\cdot|}$ since $|1| = 1$. Further, one easily checks that for any $a, b \in \mathfrak{p}_{|\cdot|}$, $|ab| = |a| \cdot |b| < 1$ implies $|a| < 1$ or $|b| < 1$.

As such, applying Lemma 2.2.42 gives $\mathfrak{p} = (p)$ for some positive prime $p \in \mathbb{N}_+$. In particular, this means that $|n| < 1$ iff n is a multiple of p, and so $\log_q |q| = 0$ for all primes $q \neq p$. Hence, this gives $\lambda_{|\cdot|} = \log_p |p|$. Further, since $|p| < 1 < p$, we deduce that $\lambda_{|\cdot|} \in \overleftarrow{[-\infty, 0)}$, as claimed. □

Discussion 4.2.7 (Topological Constraints by Upper Reals)**.**

(i) Recalling our discussion of Observation 4.1.1, notice:

- When $|\cdot|$ is non-Archimedean, Proposition 4.2.6 proves that $|n| \leq 1$ for all $n \in \mathbb{Z}$;
- When $|\cdot|$ is Archimedean, $|n| \geq 1$ for all $n \in \mathbb{Z}_{\neq 0}$ essentially by definition.

In other words, the orientation issues that arose when exponentiating upper-valued absolute values on $\mathbb{Q}$ are no longer a problem if we restrict to just the integers $\mathbb{Z}$.

(ii) Also notice: unlike the classical Ostrowski's Theorem, Proposition 4.2.6 does not directly prove that any non-Archimedean absolute value is equivalent to $|\cdot|_p$ for some prime $p \in \mathbb{N}_+$. How come? Suppose, given some non-Archimedean $|\cdot|$, we want a real $\lambda_{|\cdot|}$ such that $|\cdot| = |\cdot|_p^{\lambda_{|\cdot|}}$. Since $|p|_p = p^{-1} = \frac{1}{p}$, the relevant exponent would be $\lambda_{|\cdot|} = \log_{\frac{1}{p}} |p|$, which we know to be a positive *lower real* by Remark 3.3.3. Geometrically, it is more natural to use the signed upper real λ to give a uniform treatment.

Discussion 4.2.8. Let us return to Remark 4.1.4. Thus far, all the main definitions and constructions (i.e. Definition 4.1.13, $\mathfrak{P}_\Lambda$ and Construction 4.2.1) can be translated to the setting of upper-valued absolute values on $\mathbb{N}$. So where do things go wrong? Interestingly, although one can easily adapt the proof of Proposition 4.2.5 to classify Archimedean absolute values on $\mathbb{N}$, difficulties arise when trying to rework the proof of Proposition 4.2.6. In particular, although Remark 2.2.44 tells us that for any inhabited prime ideal $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{N}_+)$, we have $\mathfrak{p} = (p)$ for some unique prime $p \in \mathbb{N}_+$, this result explicitly requires the prime ideal of $\mathbb{N}_+$ to be closed under formal subtraction. As such, in order to prove the analogue of Proposition 4.2.6 for non-Archimedean absolute values on $\mathbb{N}$, we need to verify the following sequent:

$$\forall i, n, j \in \mathbb{N}_+ \,.\, (i + n = j) \wedge |i| < 1 \wedge |j| < 1 \longrightarrow |n| < 1$$

It is presently unclear which additional axioms would be needed by the absolute values of $\mathbb{N}$ in order to deduce this result, or if indeed such a result is even provable in this weaker setting.[78]

Summarising, we have:

Conclusion 4.2.9. *Construction 4.2.1 defines a map*

$$\begin{aligned} \widehat{f} \colon [\overleftarrow{av}] &\longrightarrow \mathfrak{P}_\Lambda \\ |\cdot| &\longmapsto (\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}) \end{aligned}$$

Proof. Most of the legwork has already been done in Propositions 4.2.5 and 4.2.6. It remains to justify that the naive gluing together of the Archimedean and non-Archimedean cases is in fact geometrically valid. We start by collecting the following data:

- We claim Construction 4.2.1 defines a map

$$\begin{aligned} f \colon [\overleftarrow{av}] &\longrightarrow \mathcal{P}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]} \\ |\cdot| &\longmapsto (\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}) \end{aligned}$$

  where $\mathcal{P}(\mathbb{Z})$ denotes the powerset of $\mathbb{Z}$. Why? By inspection, $\mathfrak{p}_{|\cdot|} = \{n \in \mathbb{Z} \,\big|\, |n| < 1\}$ is clearly a subset of $\mathbb{Z}$, and the fact that $\lambda_{|\cdot|} \in \overleftarrow{[-\infty, 1]}$ follows essentially from Equation (4.6) or the triangle inequality. In particular, we emphasise that this reasoning does not appeal to any kind of case-splitting — we are only working with a generic $|\cdot|$. The claim thus immediately follows.
- By Definition 2.2.41, $\mathrm{ISpec}(\mathbb{Z})$ is a subspace of $\mathcal{P}(\mathbb{Z})$. This induces a subspace inclusion

$$i \colon \mathfrak{P}_\Lambda \hookrightarrow \mathcal{P}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]}.$$

- Similarly, by Proposition 2.1.30, we hav the following inclusion maps

$$i_1 \colon [\overleftarrow{av}_A] \hookrightarrow [\overleftarrow{av}]$$

$$i_2 \colon [\overleftarrow{av}_{NA}] \hookrightarrow [\overleftarrow{av}]$$

- Since the proofs of Propositions 4.2.5 and 4.2.6 were geometric, they define maps

$$f_1 \colon [\overleftarrow{av}_A] \to \mathfrak{P}_\Lambda$$

$$f_2 \colon [\overleftarrow{av}_{NA}] \to \mathfrak{P}_\Lambda$$

[78] One possibility might be to require non-Archimedean absolute values on $\mathbb{N}$ to satisfy this Formal Subtraction sequent by fiat – there are, however, still some details that will need working out regarding its implications e.g. for Proposition 4.1.15.

By Proposition 4.1.15, we know that $[\overleftarrow{av}_A]$ and $[\overleftarrow{av}_{NA}]$ are complementary subspaces of $[\overleftarrow{av}]$. Further, by inspection, one easily verifies that $f \circ i_1 = f_1 \circ i$ and $f \circ i_2 = f_2 \circ i$. Hence, apply the Case-Splitting Lemma 2.2.63 to construct the pullback square

$$\begin{array}{ccc} [\overleftarrow{av}] & \xrightarrow{\widehat{f}} & \mathfrak{P}_\Lambda \\ {\scriptstyle\cong}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ [\overleftarrow{av}] & \xrightarrow{f} & \mathcal{P}(\mathbb{Z}) \times \overleftarrow{[-\infty, 1]} \end{array}$$

which gives our desired map $\widehat{f}$. □

4.2.2 Second Direction: $(\mathfrak{p}, \lambda)$ determines an Absolute Value.

We now work to define a map

$$\begin{aligned} \widehat{g} \colon \mathfrak{P}_\Lambda &\longrightarrow [\overleftarrow{av}] \\ (\mathfrak{p}, \lambda) &\longmapsto |\cdot|_{\mathfrak{p},\lambda} \end{aligned}$$

inverse to $\widehat{f}$. As before, it suffices to construct an absolute value $|\cdot|_{\mathfrak{p},\lambda}$ from a generic $(\mathfrak{p}, \lambda) \in (\mathfrak{P}, \Lambda)$.

However, the fact that we require this construction to be geometric creates subtleties. In particular, it is not decidable if $\mathfrak{p}$ is non-trivial or (0), nor is it decidable if $\lambda \sqsubseteq \lambda'$ for any $\lambda, \lambda' \in \overleftarrow{[-\infty, 1]}$. Hence, given $(\mathfrak{p}, \lambda) \in (\mathfrak{P}, \Lambda)$, our desired map $\widehat{g}$ cannot be directly defined using the following case-splittings:

- **Case 1:** $\lambda < 0$, **Case 2:** $0 \leq \lambda$; or
- **Case 1:** $\exists a \in \mathbb{Z}_{\neq 0}$ such that $a \in \mathfrak{p}$, **Case 2:** $\mathfrak{p} = (0)$.

In other words, the natural faultlines along which one might split $(\mathfrak{p}, \lambda) \in \mathfrak{P}_\Lambda$ into the Archimedean vs. non-Archimedean case are unavailable to us, at least without further work. Nonetheless, one can manoeuvre around this issue sensibly:

Construction 4.2.10. Suppose $(\mathfrak{p}, \lambda) \in \mathfrak{P}_\Lambda$. Just in case $\mathfrak{p}$ is non-trivial, Lemma 2.2.42 says $\mathfrak{p} = (p)$ for some prime $p \in \mathbb{N}_+$. Hence, define $|\cdot|_{\mathfrak{p},\lambda} \colon \mathbb{Z} \to \overleftarrow{[0, \infty)}$ as

$$|n|_{\mathfrak{p},\lambda} = \begin{cases} 0 & \text{if } n = 0 \\ \min\left\{1, \inf\{(p^{\mathrm{ord}_p(n)})^\lambda \mid p \text{ prime in } \mathfrak{p}\}\right\} \cdot \max\left\{1, n^\lambda\right\} & \text{if } n > 0 \\ |-n|_{\mathfrak{p},\lambda} & \text{if } n < 0 \end{cases}$$

where $\mathrm{ord}_p(n) := \max\{r \in \mathbb{N} \mid p^r \text{ divides } n\}$ is the p-adic ordinal defined in Example 4.1.5.

Discussion 4.2.11. Although Construction 4.2.10 consists of many different components, each component is geometric, and so the final construction is also geometric. To see this, note:

- The initial case split into $n = 0$ vs. $n > 0$ vs. $n < 0$ is permitted since $<$ is decidable on $\mathbb{Z}$.
- The p-adic ordinal $\mathrm{ord}_p(n)$ is also geometric. This essentially follows from the Euclidean Algorithm, which gives a constructive account of unique prime factorisation for any $n \in \mathbb{Z}_{\neq 0}$.

- The construction
$$\inf\{(p^{\mathrm{ord}_p(n)})^\lambda \,|\, p \,\text{prime in}\, \mathfrak{p}\}$$
is a geometric workaround the fact it is undecidable if $\mathfrak{p} = (0)$. Its geometricity comes from the fact that upper reals possess arbitrary $\inf$s (Observation 2.2.32). Informally (by which, we emphasise, we do *not* mean definitionally), the construction says that given some $n \in \mathbb{Z}_{\neq 0}$,
$$\inf\{(p^r)^\lambda \,|\, p \,\text{prime in}\, \mathfrak{p}\} = \begin{cases} (p^r)^\lambda & \text{if } \mathfrak{p} = (p) \text{ for some prime } p \in \mathbb{N}_+ \\ \infty & \text{if } \mathfrak{p} = (0) \end{cases}$$
where r, z are integers such that $n = zp^r$ and $\gcd(z, p) = 1$, just in case $\mathfrak{p}$ is non-trivial.
- Finally, recall from Section 2.2.2 that $\min$, $\max$ and multiplication are all well-defined operations on the non-negative upper reals.

Moreover, since continuous maps on the upper reals must respect the Scott topology, the exponentiation $x^{(-)}$ is monotonic in the upper real exponents for fixed $x \in [1, \infty)$. Combining this insight with the third item of Discussion 4.2.11, we obtain:

Observation 4.2.12. The non-Archimedean vs. Archimedean case-splitting is implicit in Construction 4.2.10. More explicitly, given $(\mathfrak{p}, \lambda) \in \mathfrak{P}_\Lambda$, we get

- **Case 1:** $\lambda < 0$. Then, for some prime p, $|n|_{\mathfrak{p},\lambda} = (p^{\mathrm{ord}_p(n)})^\lambda$ for all integers $n > 0$.
- **Case 2:** $0 \leq \lambda$. Then $|n|_{\mathfrak{p},\lambda} = n^\lambda$ for all integers $n > 0$.

Proof. Any positive integer $n \in \mathbb{N}_+$ is bounded below by 1, and can thus be viewed as a Dedekind in $[1, \infty)$. As such, $|\cdot|_{\mathfrak{p},\lambda}$ of Construction 4.2.10 yields the following:

- **Case 1:** $\lambda < 0$. Then by definition of $\mathfrak{P}_\Lambda$ and Lemma 2.2.42, we know $\exists p \in \mathbb{N}_+$ such that $\mathfrak{p} = (p)$. Now suppose $n > 0$. For readability, we denote $r := \mathrm{ord}_p(n)$. By monotonicity of exponentiation, we know that $n^\lambda \leq n^0 = 1$ and $(p^r)^\lambda \leq 1$, and so
$$|n|_{\mathfrak{p},\lambda} = \min\left\{1, (p^r)^\lambda\right\} \cdot \max\left\{1, n^\lambda\right\} = (p^r)^\lambda.$$
- **Case 2:** $0 \leq \lambda$. Then by definition of $\mathfrak{P}_\Lambda$, $\mathfrak{p} = (0)$ — in particular, it contains no primes $p \in \mathbb{N}_+$. Now suppose $n > 0$. By monotonicity of exponentiation, we know that $n^\lambda \geq 1$ and so
$$|n|_{\mathfrak{p},\lambda} = \min\left\{1, \infty\right\} \cdot \max\left\{1, n^\lambda\right\} = n^\lambda.$$

□

It remains to verify that $|\cdot|_{\mathfrak{p},\lambda}$ defined by Construction 4.2.10 satisfies all the required properties from Definition 4.1.2. Notice: by construction, we get the following axioms essentially for free:

- $|0|_{\mathfrak{p},\lambda} = 0$
- $|n|_{\mathfrak{p},\lambda} = |-n|_{\mathfrak{p},\lambda}$ for all $n \in \mathbb{Z}$
- $|n|_{\mathfrak{p},\lambda} \in \overleftarrow{[0, \infty)}$ for all $n \in \mathbb{Z}$

For the remaining axioms, Observation 4.2.12 suggests we split our analysis into the two obvious cases:

Proposition 4.2.13. *Suppose* $(\mathfrak{p}, \lambda) \in \mathfrak{P}_\Lambda$ *and* $\lambda < 0$. *Then* $|\cdot|_{\mathfrak{p},\lambda}$ *is an absolute value.*

Proof. We apply Observation 4.2.12 and the various properties of exponentiation (developed in Chapter 3) to check the remaining axioms:

- $|1|_{\mathfrak{p},\lambda} = 1$. This follows from $|1|_{\mathfrak{p},\lambda} = (p^0)^\lambda = 1^\lambda = 1$.
- *Multiplicativity*. Suppose $m, n \in \mathbb{Z}$. If either m or n are 0, then multiplicativity holds trivially (since $|0|_{\mathfrak{p},\lambda} = 0$), so assume that $m, n \neq 0$. In which case, unique prime factorisation yields

$$|m \cdot n|_{\mathfrak{p},\lambda} = (p^{r_1} \cdot p^{r_2})^\lambda = (p^{r_1})^\lambda \cdot (p^{r_2})^\lambda = |m|_{\mathfrak{p},\lambda} \cdot |n|_{\mathfrak{p},\lambda},$$

  for appropriate $r_1, r_2 \in \mathbb{N}$.
- *Triangle Inequality.* Similar to multiplicativity, the triangle inequality

$$|m + n|_{\mathfrak{p},\lambda} \leq |m|_{\mathfrak{p},\lambda} + |n|_{\mathfrak{p},\lambda}$$

  holds trivially if either m or n is 0. Hence, assume that $m, n \neq 0$. In which case, suppose $m = z_1 p^{r_1}$ and $n = z_2 p^{r_2}$, where $z_1, z_2 \in \mathbb{Z}_{\neq 0}$, $r_1, r_2 \in \mathbb{N}$ and $\gcd(z_1, p) = 1 = \gcd(z_2, p)$. Suppose $r_1 \leq r_2$, and so

$$m + n = (p^{r_1}) \cdot (z_1 + z_2 p^{r_2 - r_1}),$$

  where $z_1 + z_2 p^{r_2 - r_1} \in \mathbb{Z}$. Then note

$$|m + n|_{\mathfrak{p},\lambda} = |p^{r_1}|_{\mathfrak{p},\lambda} \cdot |(z_1 + z_2 p^{r_2 - r_1})|_{\mathfrak{p},\lambda} \leq (p^{r_1})^\lambda = |m|_{\mathfrak{p},\lambda}, \tag{4.9}$$

  since $|z_1 + z_2 p^{r_2 - r_1}|_{\mathfrak{p},\lambda} \leq 1$, essentially by construction of $|\cdot|_{\mathfrak{p},\lambda}$. By symmetry, in the case where $r_2 \leq r_2$, we get

$$|m + n|_{\mathfrak{p},\lambda} \leq |n|_{\mathfrak{p},\lambda}, \tag{4.10}$$

  and so, combining Equations (4.9) and (4.10), we obtain the ultrametric inequality

$$|m + n|_{\mathfrak{p},\lambda} \leq \max\{|m|_{\mathfrak{p},\lambda}, |n|_{\mathfrak{p},\lambda}\},$$

  which implies the (weaker) triangle inequality.

□

Proposition 4.2.14. *Suppose* $(\mathfrak{p}, \lambda) \in \mathfrak{P}_\Lambda$ *and* $0 \leq \lambda \leq 1$. *Then* $|\cdot|_{\mathfrak{p},\lambda}$ *is an absolute value.*

Proof. We shall make free use of the basic properties of exponentiation, Observation 4.2.12 as well as Construction 4.2.10's requirement that $|n|_{\mathfrak{p},\lambda} = |-n|_{\mathfrak{p},\lambda}$ for all integers $n < 0$. Ignoring the trivial cases, we need to check:

- $|1|_{\mathfrak{p},\lambda} = 1$. Immediate from the identity $1^\lambda = 1$.
- *Multiplicativity.* Suppose we have integers $m, n > 0$. In which case

$$|m \cdot n|_{\mathfrak{p},\lambda} = (m \cdot n)^\lambda = m^\lambda \cdot n^\lambda = |m|_{\mathfrak{p},\lambda} \cdot |n|_{\mathfrak{p},\lambda}.$$

  One easily checks that if either (or both) of m, n are negative, then we can reduce to the above case by taking negations. For instance, if $m < 0 < n$, then

$$|m \cdot n|_{\mathfrak{p},\lambda} = (-m \cdot n)^\lambda = (-m)^\lambda \cdot n^\lambda = |m|_{\mathfrak{p},\lambda} \cdot |n|_{\mathfrak{p},\lambda}.$$

- *Triangle Inequality.* Suppose $m, n > 0$. By Observation 4.2.12, we need to show that

$$(m+n)^\lambda \leq m^\lambda + n^\lambda. \tag{4.11}$$

Suppose we have positive rationals $0 < q, t \leq 1$. Since $t^{(-)}$ is antitonic with respect to rational exponents, this implies that $t^q \geq t$, and so

$$m^q t^q \geq m^q t, \qquad \text{for any positive integer } m > 0.$$

This in turn implies

$$\begin{aligned} m^q + n^q &= (m+n)^q \left(\frac{m}{m+n}\right)^q + (m+n)^q \left(\frac{n}{m+n}\right)^q \\ &\geq (m+n)^q \left(\left(\frac{m}{m+n}\right) + \left(\frac{n}{m+n}\right)\right) = (m+n)^q. \end{aligned} \tag{4.12}$$

It remains to lift Equation (4.12) to the upper reals[79]. In the language of Example 2.2.51, first represent $\overleftarrow{[0,1]} \cong \mathsf{RIdl}(Q_{(0,1]}, \prec)$ as the space of rounded ideals of $Q_{(0,1]} := \{q \in \mathbb{Q} \,|\, 0 < q \leq 1\}$. Next, note that, e.g. m^q is a Dedekind real, and that its right Dedekind section is equivalent to m^{I_q}, essentially by construction (cf. Proposition 3.2.4). Hence, Equation (4.12) gives

$$m^{I_q} + n^{I_q} \geq (m+n)^{I_q}, \tag{4.13}$$

for all $q \in Q_{(0,1]}$. Since $I_\lambda = \bigsqcup^{\uparrow}_{\lambda \prec q} I_q$, apply Fact 2.2.50 to Equation (4.13) to get

$$m^{I_\lambda} + n^{I_\lambda} \geq (m+n)^{I_\lambda}, \tag{4.14}$$

which, since $\overleftarrow{[0,1]} \cong \mathsf{RIdl}(Q_{(0,1]}, \prec)$, is equivalent to Equation (4.11).

We now check the rest of the (non-trivial) cases. If $m, n < 0$ are both negative, then the argument reduces to the above case since:

$$|m+n|_{\mathfrak{p},\lambda} = (-(m+n))^\lambda = (-m-n)^\lambda \leq (-m)^\lambda + (-n)^\lambda = |m|_{\mathfrak{p},\lambda} + |n|_{\mathfrak{p},\lambda}. \tag{4.15}$$

On the other hand, suppose only one of the two integers are negative, say (without loss of generality) $m < 0 < n$. Then we have three additional subcases to check:

Subcase 1: $0 < m + n$. Two basic observations:

(a) Let $a, b \in \mathbb{N}_+$ be any pair of positive integers. By Corollary 3.2.10, we know $a \leq b \implies a^q \leq b^q$, for any positive rational $q \in Q_+$. By a similar lifting argument as above, we can extend this to: $a \leq b \implies a^\lambda \leq b^\lambda$, for any non-negative upper real λ.

(b) By hypothesis:
   - $m + n$ and n are positive integers;
   - $m + n \leq n$ (since $m < 0$).

[79]Why not directly establish Equation (4.12) for upper real exponent $\lambda \in \overleftarrow{[0,1]}$? The short answer: because we can't. Notice the equation $m^\lambda + n^\lambda = (m+n)^\lambda \left(\frac{m}{m+n}\right)^\lambda + (m+n)^\lambda \left(\frac{n}{m+n}\right)^\lambda$ features both upper reals and lower reals being multiplied together, which is not well-defined (cf. discussion of Observation 4.1.1). However, such complications do not arise if we only work with rational exponents – essentially because $\mathbb{Q}$ does *not* carry an order topology, and so Equation (4.12) as stated follows.

The triangle inequality then follows from combining Observations (a) and (b), since:

$$|m+n|_{\mathfrak{p},\lambda} = (m+n)^{\lambda} \leq n^{\lambda} \leq n^{\lambda} + (-m)^{\lambda} = |n|_{\mathfrak{p},\lambda} + |m|_{\mathfrak{p},\lambda}.$$

Subcase 2: $m+n<0$. In which case, $-m-n>0$. By Subcase 1, deduce that

$$|-m-n|_{\mathfrak{p},\lambda} \leq |-m|_{\mathfrak{p},\lambda} + |-n|_{\mathfrak{p},\lambda}$$

which is equivalent to the desired inequality

$$|m+n|_{\mathfrak{p},\lambda} \leq |m|_{\mathfrak{p},\lambda} + |n|_{\mathfrak{p},\lambda}.$$

Subcase 3: $m+n=0$. In which case, we trivially get

$$0 = |m+n|_{\mathfrak{p},\lambda} \leq |m|_{\mathfrak{p},\lambda} + |n|_{\mathfrak{p},\lambda}.$$

□

As in the previous subsection, we glue the two cases together in the following:

Conclusion 4.2.15. *Construction 4.2.10 defines a map*

$$\begin{aligned} \widehat{g} \colon \mathfrak{P}_\Lambda &\longrightarrow [\overleftarrow{av}] \\ (\mathfrak{p},\lambda) &\longmapsto |\cdot|_{\mathfrak{p},\lambda} \end{aligned}$$

Proof. Begin by extracting the following two ingredients from our set-up:

- Define $\overleftarrow{av}^*$ to be the theory over the signature Σ_{av}, but only satisfying Axioms (1), (2), (5) and (6) of Definition 4.1.2. By inspection, it is clear that Construction 4.2.10 satisfies $\overleftarrow{av}^*$ *without* appealing to any case-splitting on $\lambda \in \overleftarrow{[-\infty,1]}$. Hence, Construction 4.2.10 defines a map

$$g \colon \mathfrak{P}_\Lambda \longrightarrow [\overleftarrow{av}^*]$$

- By Proposition 2.1.30, quotients of $\overleftarrow{av}^*$ correspond to subspaces of $[\overleftarrow{av}^*]$. This gives the following subspace inclusion

$$i \colon [\overleftarrow{av}] \hookrightarrow [\overleftarrow{av}^*]$$

Next, using methods from e.g. [Vic07c], it is a standard computation to verify that the open subspace of $\mathfrak{P}_\Lambda$ corresponding to $\lambda<0$ (Case 1) and the closed subspace corresponding to $0 \leq \lambda$ (Case 2) are in fact complements in $\mathfrak{P}_\Lambda$. Hence, just as in the case of Conclusion 4.2.9, apply the Case-Splitting Lemma 2.2.63 to construct the following pullback square:

$$\begin{array}{ccc} \mathfrak{P}_\Lambda & \xrightarrow{\widehat{g}} & [\overleftarrow{av}] \\ {\scriptstyle\cong}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ \mathfrak{P}_\Lambda & \xrightarrow[g]{} & [\overleftarrow{av}^*] \end{array}$$

which yields the desired map $\widehat{g}$. □

Discussion 4.2.16 (Undecidability and Geometricity)**.** Given our previous remarks about decidability issues, our proof of Conclusion 4.2.15 highlights an interesting subtlety. Namely, why is undecidability a barrier to geometricity when defining constructions, but not when proving properties? Examining the hypotheses of the Case-Splitting Lemma, the main crux of our proof, reveals an interesting fine print. The Lemma only justifies analysing a construction via case-splitting *once* the construction already exists (i.e. the map $X \xrightarrow{f} Z$); it does not justify defining *a new construction* via case-splitting, as one might have hoped.

4.2.3 Finish. We now complete the proof of Theorem C.

Proof of Theorem C. It suffices to show that the maps $\widehat{f}$ and $\widehat{g}$ from Conclusions 4.2.9 and 4.2.15 are inverse to each other.

Step 1: Verifying $\widehat{g} \circ \widehat{f} = \mathrm{id}_{[\overleftarrow{av}]}$. It suffices to show for any $|\cdot| \in [\overleftarrow{av}]$ that

$$|n| = |n|_{\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}}, \qquad \text{for any positive integer } n \in \mathbb{N}_+.$$

As before, we split our analysis into the Archimedean vs. non-Archimedean case.

- **Case 1:** $|\cdot|$ *is non-Archimedean.* By Proposition 4.2.6, there exists a unique prime $p \in \mathbb{N}_+$ such that $|p| < 1$ and $\lambda_{|\cdot|} = \log_p |p|$. Thus, as before, any $n \in \mathbb{N}_+$ will be represented as $n = zp^r$ for some $r, z \in \mathbb{N}$ where $\gcd(z, p) = 1$ and $r = \mathrm{ord}_p(n)$. We then obtain the following equalities:

  (a) $|n| = |p^r|$.
  [Why? By the proof of Proposition 4.2.6, we know that $|n| < 1$ iff $p|n$, and $|n| \leq 1$ for all $n \in \mathbb{N}_+$. Hence, since $\gcd(z, p) = 1$, this implies $|n| = |zp^r| = |z| \cdot |p^r| = |p^r|$.]

  (b) $|p^r| = (p^{\lambda_{|\cdot|}})^r$
  [Why? Multiplicativity of $|\cdot|$ gives $|p^r| = |p|^r$, and $\lambda_{|\cdot|} = \log_p |p|$ gives $|p|^r = (p^{\lambda_{|\cdot|}})^r$.]

  (c) $(p^{\lambda_{|\cdot|}})^r = (p^r)^{\lambda_{|\cdot|}}$
  [Why? This follows from unpacking definitions from Chapter 3. More explicitly, note:

  $$p^{\lambda_{|\cdot|}} < q' \leftrightarrow \exists q'' \in Q_+.(q'' > \lambda_{|\cdot|} \wedge q' > p^{q''})$$

  $$(p^{\lambda_{|\cdot|}})^r < q \leftrightarrow \exists q' \in Q_+. \left(p^{\lambda_{|\cdot|}} < q' \wedge (q')^r < q\right)$$

  $$(p^r)^{\lambda_{|\cdot|}} < q \leftrightarrow \exists q' \in Q_+. \left(q' > \lambda_{|\cdot|} \wedge q > (p^r)^{q'}\right)$$

  Suppose $(p^{\lambda_{|\cdot|}})^r < q$. Then there exists $q', q'' \in Q_+$ such that $q > (q')^r > (p^{q''})^r = (p^r)^{q''}$ and $q'' > \lambda_{|\cdot|}$, thus implying $(p^r)^{\lambda_{|\cdot|}} < q$. Conversely, suppose $(p^r)^{\lambda_{|\cdot|}} < q$. Then, there exists $q' \in Q_+$ such that $q' > \lambda_{|\cdot|}$ and $(p^{q'})^r = (p^r)^{q'} < q$. By roundedness, there exists $q'' \in Q_+$ such that $q' > q'' > \lambda_{|\cdot|}$. Since $p^{q'} > p^{q''}$, this implies $p^{\lambda_{|\cdot|}} < p^{q'}$ by definition. Put together, this means that there exists $q' \in Q_+$ such that $p^{\lambda_{|\cdot|}} < p^{q'}$ and $(p^{q'})^r < q$, thus implying $(p^{\lambda_{|\cdot|}})^r < q$. Hence, conclude that $(p^{\lambda_{|\cdot|}})^r < q \leftrightarrow (p^r)^{\lambda_{|\cdot|}} < q$, and we are done.]

  (d) $(p^r)^{\lambda_{|\cdot|}} = |n|_{\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}}$
  [Why? Immediate from Observation 4.2.12.]

  Assembled together, Equalities (a) - (d) yield the desired identity

  $$|n| = |n|_{\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}} \quad .$$

- **Case 2:** $|\cdot|$ *is Archimedean.* By Proposition 4.2.5, $|n| = n^{\lambda_{|\cdot|}}$ for any positive integer $n \in \mathbb{N}_+$. Observation 4.2.12 thus implies

  $$|n| = n^{\lambda_{|\cdot|}} = |n|_{\mathfrak{p}_{|\cdot|}, \lambda_{|\cdot|}}$$

Step 2: Verifying $\widehat{f} \circ \widehat{g} = \mathrm{id}_{(\mathfrak{P},\Lambda)}$. Suppose $(\mathfrak{p}, \lambda) \in (\mathfrak{P}, \Lambda)$. We need to show

$$\mathfrak{p} = \mathfrak{p}_{|\cdot|_{\mathfrak{p},\lambda}} \qquad \text{and} \qquad \lambda = \lambda_{|\cdot|_{\mathfrak{p},\lambda}}$$

- **Case 1:** $\lambda < 0$. If $\lambda < 0$, then $\mathfrak{p} = (p)$ for some prime $p \in \mathbb{N}_+$. Applying Observation 4.2.12, we know
$$|p|_{\mathfrak{p},\lambda} = p^\lambda < 1. \tag{4.16}$$
As such, since $p \in \mathfrak{p}_{|\cdot|_{\mathfrak{p},\lambda}}$ by Construction 4.2.1, and since $\mathfrak{p}_{|\cdot|_{\mathfrak{p},\lambda}}$ is a principal prime ideal by Proposition 4.2.6 and Lemma 2.2.42, this yields
$$\mathfrak{p} = (p) = \mathfrak{p}_{|\cdot|_{\mathfrak{p},\lambda}}.$$
Further, applying $\log_p(\text{—})$ to Equation (4.16) and applying Proposition 4.2.6 again yields
$$\lambda_{|\cdot|_{\mathfrak{p},\lambda}} = \log_p |p|_{\mathfrak{p},\lambda} = \lambda.$$

- **Case 2:** $0 \leq \lambda$. If $0 \leq \lambda$, then $\mathfrak{p} = (0)$. By Observation 4.2.12, we know that for any integer $n \in \mathbb{N}_+$
$$|n|_{\mathfrak{p},\lambda} = n^\lambda,$$
and so by Construction 4.2.1
$$\lambda_{|\cdot|_{\mathfrak{p},\lambda}} = \inf\{\log_m |m|_{\mathfrak{p},\lambda} \big| \ m \in \mathbb{N}_+ \text{ is prime}\} = \lambda.$$
Further, since $0 \leq \lambda$ implies that
$$1 = n^0 \leq n^\lambda = |n|_{\mathfrak{p},\lambda}, \qquad \text{for all } n \in \mathbb{N}_+,$$
this yields
$$\mathfrak{p}_{|\cdot|_{\mathfrak{p},\lambda}} = \{n \in \mathbb{Z} \big| |n|_{\mathfrak{p},\lambda} < 1\} = (0)$$
since $|n|_{\mathfrak{p},\lambda} = |-n|_{\mathfrak{p},\lambda}$ for all $n \in \mathbb{Z}$.

This completes the proof of the Theorem. □

4.3 Absolute Values on $\mathbb{Q}$

We now re-examine our definitions. Recall that absolute values on $\mathbb{Q}$ must be Dedekind-valued (Observation 4.1.1), and not upper-valued. This means we no longer face the same constraints in Definition 4.1.13 when defining Archimedean absolute values on $\mathbb{Q}$, and can therefore return to the more standard notions of Archimedean and non-Archimedean absolute values (cf. Definition 2.4.8):

Definition 4.3.1. Let $|\cdot|$ be an absolute value on $\mathbb{Q}$ (in the sense of Definition 4.1.8). Then, $|\cdot|$ is called …

(i) … *non-Archimedean* if $|\cdot|$ satisfies the *ultrametric inequality*, that is
$$|x - y| \leq \max\{|x|, |y|\}.$$

(ii) … *non-trivial non-Archimedean* if there exists $|n| < 1$ for some $n \in \mathbb{Z}_{\neq 0}$.

(iii) …*Archimedean* if there exists some $n \in \mathbb{Z}_{\neq 0}$ such that $|n| > 1$.

Remark 4.3.2. Notice now that the trivial absolute value on $\mathbb{Q}$ is considered to be non-Archimedean, rather than Archimedean. Notice also that the definitions of the p-adic absolute value $|\cdot|_p$ and the Euclidean absolute value $|\cdot|_\infty$ in Example 4.1.5 extend automatically to this setting via multiplicativity.

With a bit more work, we can also recover the classical Ostrowski's Theorem for $\mathbb{Q}$ in this geometric setting. First notice:

Observation 4.3.3. Let $|\cdot|_1, |\cdot|_2$ be two absolute values on $\mathbb{Q}$. Then $|\cdot|_1 = |\cdot|_2$ iff the right Dedekind sections of $|n|_1$ and $|n|_2$ agree for all $n \in \mathbb{Z}_{\neq 0}$.

Proof. By multiplicativity, absolute values $|\cdot|$ on $\mathbb{Q}$ are determined by their value on $\mathbb{Z}_{\neq 0}$, and so two absolute values are equal iff they agree on the non-zero integers. The rest follows from Corollary 2.2.24, which asserts that $x = y$ iff $R_x = R_y$, given any two Dedekinds x, y with corresponding right Dedekind sections R_x, R_y. □

This allows us to leverage our previous results, which were carried out on the level of upper reals. Applying Observation 4.3.3, we obtain the following obvious characterisations.

Proposition 4.3.4. *Denote $[av_{NA}; p]$ to be the space of non-trivial non-Archimedean absolute values of $\mathbb{Q}$ such that $|p| < 1$ for some prime p. Then,*

$$[av_{NA}; p] \cong (-\infty, 0) \cong (0, \infty).$$

In particular, for any $|\cdot| \in [av_{NA}; p]$, there exists $\alpha \in (0, \infty)$ such that $|\cdot| = |\cdot|_p^\alpha$.

Proof. To show the first isomorphism, define

$$\begin{aligned} F\colon [av_{NA}; p] &\longrightarrow (-\infty, 0) \\ |\cdot| &\longmapsto \log_p |p| \end{aligned}$$

and

$$\begin{aligned} G\colon (-\infty, 0) &\longrightarrow [av_{NA}; p] \\ \lambda &\longmapsto |\cdot|_\lambda \end{aligned}$$

where $|\cdot|_\lambda$ is defined by $|n|_\lambda = \left(p^{\mathrm{ord}_p(n)}\right)^\lambda$ for all $n \in \mathbb{Z}_{\neq 0}$. That F is well-defined follows from $|\cdot|$ satisfying positive definiteness and triangle inequality, which yields the inequality

$$0 < |p| < 1 < p,$$

and so we get $-\infty < \log_p |p| < 0$. That G is well-defined follows from:

- Our hypothesis that $\lambda \in (-\infty, 0)$ (and so $|p|_\lambda = p^\lambda < 1$);
- Observations 4.2.12 and 4.3.3, along with the proof of Proposition 4.2.13 (which shows $|\cdot|_\lambda$ in fact determines a non-Archimedean absolute value on $\mathbb{Q}$).

To show that F and G are inverse to each other, we need to show that $|\cdot| = |\cdot|_{\log_p |p|}$ and $\lambda = \log_p |p|_\lambda$. But this was already shown by Theorem C for upper-valued absolute values on $\mathbb{Z}$; the same result extends to this setting by Observation 4.3.3.

The fact that $(-\infty, 0) \cong (0, \infty)$ comes from taking negations. In particular, given $|\cdot| \in [av_{NA}; p]$, define $\alpha := -\log_p |p| = \log_{\frac{1}{p}} |p|$. Then, note for any $n \in \mathbb{Z}_{\neq 0}$, multiplicativity yields

$$|n| = |p|^r = \left(\frac{1}{p}\right)^{\alpha r} = p^{-\alpha r} = |n|_p^\alpha,$$

where $r := \mathrm{ord}_p(n)$, and so $|\cdot| = |\cdot|_p^\alpha$. □

Proposition 4.3.5. *Denote $[av_A]$ to be the space of Archimedean absolute values on $\mathbb{Q}$. Then,*

$$[av_A] \cong (0, 1].$$

In particular, for any $|\cdot| \in [av_A]$, there exists $\alpha \in (0, 1]$ such that $|\cdot| = |\cdot|_\infty^\alpha$.

Proof. The main subtlety here is that we're now working with *non-trivial* Archimedean absolute values. As in Proposition 4.3.4, we shall make free use of Observation 4.3.3. Define the maps

$$\begin{aligned} F\colon [av_A] &\longrightarrow (0, 1] \\ |\cdot| &\longmapsto \log_b |b|, \end{aligned}$$

where $b \in \mathbb{N}_+$ such that $|b| > 1$, and

$$\begin{aligned} G\colon (0, 1] &\longrightarrow [av_A] \\ \alpha &\longmapsto |\cdot|_\infty^\alpha. \end{aligned}$$

That F is well-defined follows from

- Fundamental Lemma 4.2.4, which essentially shows that $\log_b |b| = \log_a |a|$ for all $a, b > 1$ (and so F does not depend on choice of $b > 1$);
- Triangle Inequality, which gives $1 < |b| \leq b$ (and so $0 < \log_b |b| \leq 1$, implying $\alpha \in (0, 1]$).

That G is well-defined follows from Proposition 4.2.14 plus the fact that $0 < \alpha \iff 1 < b^\alpha$ for any integer $b > 1$. By Proposition 4.2.5 and Observation 4.3.3, any $|\cdot| \in [av_A]$ gives

$$|n| = n^\alpha = |n|_\infty^\alpha \qquad \text{for all } n \in \mathbb{Z}$$

where $\alpha := \log_b |b|$, which extends to the whole of $\mathbb{Q}$ to give $|\cdot| = |\cdot|_\infty^\alpha$. In addition, given any $\alpha \in (0, 1]$, one easily computes

$$\log_b |b|_\infty^\alpha = \alpha \cdot \log_b b = \alpha$$

In sum, this shows that F and G are indeed inverse. □

Theorem D (Ostrowski's Theorem for $\mathbb{Q}$)**.** Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$. Then, one of the following must hold:

(i) $|\cdot| = |\cdot|_\infty^\alpha$ for some $\alpha \in (0, 1]$; or
(ii) $|\cdot| = |\cdot|_p^\alpha$ for some $\alpha \in (0, \infty)$ and some prime $p \in \mathbb{N}_+$.

Proof. The main arguments have already been established — it remains to make explicit what we mean by "non-trivial" and justify the case-splitting. Working syntactically, define:

- $|\cdot|$ is an *Archimedean absolute value* on $\mathbb{Q}$ if it satisfies the axiom

$$\top \to \bigvee_{n \in \mathbb{Z}_{\neq 0}} |n| > 1,$$

  that is there exists some $n \in \mathbb{Z}_{\neq 0}$ such that $|n| > 1$.
- $|\cdot|$ is a *non-trivial non-Archimedean absolute value* on $\mathbb{Q}$ if it satisfies Axiom (NA) from Definition 4.1.13.
- $|\cdot|$ is a *trivial absolute value* on $\mathbb{Q}$ if it satisfies

$$(\forall q \in \mathbb{Q}, \forall n \in \mathbb{Z}_{\neq 0}). \quad |n| < q \leftrightarrow 1 < q,$$

  that is $|n| = 1$ for all $n \in \mathbb{Z}_{\neq 0}$. An absolute value $|\cdot|$ is *non-trivial* if it belongs to the open complement in $[av]$.

A similar argument as in Proposition 4.1.15 shows that non-trivial absolute values on $\mathbb{Q}$ split into two cases:

- **Case 1:** There exists $n \in \mathbb{Z}_{\neq 0}$ such that $|n| > 1$.
- **Case 2:** There exists $n \in \mathbb{Z}_{\neq 0}$ such that $|n| < 1$.

If $|\cdot|$ belongs to Case 1, then Proposition 4.3.5 gives us item (i) of the stated theorem. If $|\cdot|$ belongs to Case 2, then Proposition 4.2.6 associates to $|\cdot|$ a unique prime $p \in \mathbb{N}_+$ such that $|p| < 1$. Then, Proposition 4.3.4 gives us item (ii) of the theorem, and we are done. □

Discussion 4.3.6. While negation inverts orientation on one-sided reals (and so, a negated upper real becomes a lower real), a negated Dedekind real is still a Dedekind, so we avoid the same issues mentioned in Discussion 4.2.7. Further, unlike the one-sided reals, $-\infty, \infty$ are not Dedekinds, which is why $\lambda = \log_{\frac{1}{p}} |p| \in (0, \infty)$, as opposed to $\lambda \in (0, \infty]$. In other words, if we wish to view λ as a Dedekind, then we lose the ability to speak about the true seminorms, which correspond to $|\cdot|_p^\infty$ for prime $p \in \mathbb{N}_+$. This, combined with Observations 4.1.1 and 4.1.3, brings into focus the following connection between the algebra (absolute values) and topology (the reals):

- Multiplicative Seminorms $\leftrightsquigarrow$ Upper reals
- (Positive Definite) Norms $\leftrightsquigarrow$ Dedekind reals.

At this current juncture, the reader may reasonably wonder: does ISpec still play a role in the analysis of absolute values on $\mathbb{Q}$? After all, ISpec is not mentioned in the statement of Theorem D (unlike Theorem C), and only features implicitly in its proof. As such, having established (geometrically) that the equivalence classes of absolute values on $\mathbb{Q}$ were what we expected them to be, do we still need to carry around this unfamiliar notion of ISpec that we inherited from Chapter 2? Might we not e.g. use the Zariski spectrum $\mathrm{LSpec}(\mathbb{Z})$ instead to denote the non-Archimedean places of $\mathbb{Q}$?

The following observation highlights a surprising obstruction to this.

Observation 4.3.7 (coZariski vs. Zariski/Constructible Topology)**.** As our setup,

- Denote $[av_{NA}]$ to be the space of non-Archimedean absolute values on $\mathbb{Q}$.

- Denote $[av_{NA\neq 0}]$ to be the space of *non-trivial* non-Archimedean absolute values on $\mathbb{Q}$.

Then:

(i) There exists a map from $[av_{NA\neq 0}]$ to $\mathrm{LSpec}(\mathbb{Z}), \mathrm{ISpec}(\mathbb{Z})$ and $\mathrm{FSpec}(\mathbb{Z})$

(ii) There exists a map from $[av_{NA}]$ to $\mathrm{ISpec}(\mathbb{Z})$, but there *cannot* exist a map from $[av_{NA}]$ to $\mathrm{LSpec}(\mathbb{Z})$ or $\mathrm{FSpec}(\mathbb{Z})$.

Proof. We start by analysing the case of $\mathrm{ISpec}(\mathbb{Z})$, and define the following map:

$$\begin{aligned} \mathcal{I}\colon [av_{NA}] &\longrightarrow \mathrm{ISpec}(\mathbb{Z}) \\ |\cdot| &\longmapsto \{n \,\big|\, |n| < 1\} \end{aligned} \tag{4.17}$$

In particular, notice this transformation of points makes sense for *both* the trivial and the non-trivial non-Archimedean norms. If $|\cdot|$ is non-trivial, then the same argument as in Proposition 4.2.6 shows $\{n \,\big|\, |n| < 1\}$ is a non-zero prime ideal of $\mathbb{Z}$; if $|\cdot|$ is trivial, then we get $\{n \,\big|\, |n| < 1\} = (0)$ by definition.

On the other hand, working with $\mathrm{LSpec}(\mathbb{Z})$ or $\mathrm{FSpec}(\mathbb{Z})$ introduces additional difficulties. For instance, in the standard case of the Zariski spectrum, the map

$$\begin{aligned} L\colon [av_{NA}] &\longrightarrow \mathrm{LSpec}(\mathbb{Z}) \\ |\cdot| &\longmapsto \{n \,\big|\, |n| < 1\} \end{aligned} \tag{4.18}$$

is not well-defined because the points of $\mathrm{LSpec}(\mathbb{Z})$ are defined to be the *prime filters* of $\mathbb{Z}$, not its prime ideals.[80] The obvious fix

$$\begin{aligned} L'\colon [av_{NA}] &\longrightarrow \mathrm{LSpec}(\mathbb{Z}) \\ |\cdot| &\longmapsto \{n \,\big|\, |n| = 1\} \end{aligned} \tag{4.19}$$

also fails, because $|n| = 1$ defines a *closed* subspace in the Dedekinds, and so L' is not a geometric transformation (even if $\{n \,\big|\, |n| = 1\}$ is a prime filter).

Nevertheless, there is a workaround if we restrict to the space of non-trivial non-Archimedean norms. Applying Theorem D, we know for any $|\cdot| \in [av_{NA\neq 0}]$ there exists a prime p such that $|p| < 1$. This motivates the following definition:

$$\begin{aligned} \mathcal{L}\colon [av_{NA;\neq 0}] &\longrightarrow \mathrm{LSpec}(\mathbb{Z}) \\ |\cdot| &\longmapsto \{n \,\big|\, \big(\exists \,\text{prime}\, p \in \mathbb{N}_+ \wedge \exists q \in \mathbb{Q}\big).\, 0 < |p| < q < |n|\} \end{aligned} \tag{4.20}$$

It is clear by inspection that $\mathcal{L}(|\cdot|)$ is geometric, and one easily checks that

$$\mathcal{L}(|\cdot|) = \{n \,\big|\, |n| = 1\}$$

[since $|p| < 1 = |n|$ for any $n \in \mathbb{Z}$ such that $p \nmid n$], and thus $\mathcal{L}(|\cdot|)$ indeed defines a prime filter. However, it is also clear that $\mathcal{L}$ cannot be extended to include the trivial norm $|\cdot|_0$ since $\mathcal{L}(|\cdot|_0) = \emptyset$, which fails to even be a filter of $\mathbb{Z}$.

This raises a natural question: can there exist a geometric reformulation of L' over the whole of $[av_{NA}]$, including the trivial norm? In fact, there cannot. Why? Suppose there did exist a map

$$f\colon [av_{NA}] \to \mathrm{LSpec}(\mathbb{Z}).$$

Then, notice:

[80] Although, as already noted in Section 2.2.3, is is true that the global points of LSpec and ISpec coincide classically in $\mathbf{Set}$.

(a) The trivial filter $\mathcal{F}$ (containing all elements of $\mathbb{Z}$ except 0) is a top element $\top$ in $\mathrm{LSpec}(\mathbb{Z})$ and is therefore open. Hence, the inverse image of $\top$ under f must be open in $[av_{NA}]$ as well, and contain only $|\cdot|_0$.[81]

(b) The opens of $[av_{NA}]$ are generated by finite meets of subbasic opens of the form

$$U(q,r) := \{|\cdot| \in [av_{NA}] \,\big|\, |q| < r\}$$
$$V(q,r) := \{|\cdot| \in [av_{NA}] \,\big|\, |q| > r\}$$

Now consider the open $U := f^{-1}(\top)$, as indicated in item (a). By item (b), we know that U can be represented by a finite join of opens,

$$U = \bigvee_{k=1}^{n} U_k, \qquad n \in \mathbb{N},$$

each of which is of the form

$$U_k = \left(\bigwedge_{i\in I} U(q_i, r_i)\right) \wedge \left(\bigwedge_{j\in J} V(q_j, r_j)\right), \qquad \text{where } I, J \text{ are finite (possibly empty) sets.}$$

Since U consists of a single point $|\cdot|_0$, there exists a basic open U_k such that $U_k = U$. Denote $\mathcal{P}_{U_k}$ to be the set of all the primes dividing the numerator and denominator of the q_i's and q_j's in such a U_k. Notice that $\mathcal{P}_{U_k}$ is a finite set of primes, and thus U_k can only exclude finitely many p-adic norms, implying that $|\cdot|_p \in U_k$ for $p \notin U_k$. But this contradicts the fact that $U = \{|\cdot|_0\}$ has to exclude p-adic norms for *all* primes p, not just for a finite subset of primes.

The same issues show up in the case of $\mathrm{FSpec}(\mathbb{Z})$, whose points are complemented prime ideals, i.e. a pair (P, S) with P a prime ideal and S a prime filter that are complements to each other in $\mathbb{Z}$. □

Discussion 4.3.8. Two important takeaways from Observation 4.3.7:

(i) This result gives decisive evidence for working with $\mathrm{ISpec}(\mathbb{Z})$ over the other spectral spaces. In particular, it is natural to expect there to exist a quotient map

$$\mathrm{quot}\colon [av_{NA}] \longrightarrow [\mathrm{places}_{NA}]$$

that sends a non-Archimedean absolute value to its corresponding place. However, if $[\mathrm{places}_{NA}] = \mathrm{LSpec}(\mathbb{Z})$ or $\mathrm{FSpec}(\mathbb{Z})$, then Observation 4.3.7 tells us that no such map exists.

(ii) This result also touches on an interesting theme that underscores both Chapters 5 and 6: it is often easier to work with just the non-trivial absolute values. In particular, working with both the trivial and non-trivial absolute values often raises topological difficulties, which can be subtle.

[81] Why? Otherwise, f is a map that sends a non-trivial $|\cdot|$ to both the top element and something below the top element in $\mathrm{LSpec}(\mathbb{Z})$, which is not well-defined.

Chapter 5

Logical Berkovich Geometry

In this chapter, we extend our insights from Chapter 4 to explore how point-free techniques can sharpen our understanding of non-Archimedean geometry, even if we no longer work geometrically (= when we make classical assumptions). Our investigations in this chapter can be understood as being framed by the question: what is the relationship between topology and logic?

We motivate our study with the following summary theorem, which collects together various known characterisations of the Berkovich affine line.

Summary Theorem 5.0.1. *As our setup,*

- *Fix K to be an algebraically closed field complete with respect to a non-trivial non-Archimedean norm $|\cdot|$;*
- *Denote Γ to be the value group of K, which we shall assume to be contained in $\mathbb{R}$;*
- *Denote $\mathbb{A}^1_{\mathrm{Berk}}$ to be the Berkovich affine line.*

Then, $\mathbb{A}^1_{\mathrm{Berk}}$ *can be equivalently characterised as:*

(i) The set of multiplicative seminorms on $K[T]$ extending $|\cdot|$ on K, equipped with the Berkovich topology;

(ii) A space whose points are defined by a sequence of nested closed discs $D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots$ contained in K;

(iii) The space of types over K, concentrating on $\mathbb{A}^1_K$, that are "almost orthogonal to Γ";[82]

(iv) A profinite $\mathbb{R}$-tree.

Proof. (i) is the definition of $\mathbb{A}^1_{\mathrm{Berk}}$. (ii) can be proved similarly to [Ber90, Example 1.4.4]. (iii) is a special case of what was proved in [HL16, pp. 187-188]. (iv) follows from [BR10, Theorem 2.20] and the fact that $\mathbb{P}^1_{\mathrm{Berk}}$ is the 1-point compactification of $\mathbb{A}^1_{\mathrm{Berk}}$ at the level of topological spaces. □

For the number theorist, the different characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$ in Summary Theorem 5.0.1 reflect the variety of tools that have been used when studying Berkovich spaces. In more detail:

- The equivalence of items (i) and (ii), a foundational result in Berkovich geometry, sets up the classification of points of $\mathbb{A}^1_{\mathrm{Berk}}$.

[82] This is a technical definition which the non-logician may wish to treat as a black box — it will not be needed to understand the main results of this thesis. For the model theorist: K here is taken to be a model of the theory ACVF, which is a 3-sorted theory comprising VF as the value field sort, Γ as the value group sort, and κ as the residue field sort. Let $\mathbb{U}$ be the monster model. If $C \subset \mathbb{U}$ and p is a C-type, then we say p is *almost orthogonal to* Γ if for any realisation a of p we have that $\Gamma(C(a)) = \Gamma(C)$.

- The language of "almost orthogonal types" reflects the model-theoretic methods pioneered by Hrushovski and Loeser [HL16]. This perspective was particularly useful for establishing the topological "tameness" of Berkovich spaces under very mild hypotheses (see e.g. Theorem 5.1.6).
- Viewing $\mathbb{A}^1_{\text{Berk}}$ as a profinite $\mathbb{R}$-tree emphasises its semilattice structure: given any $x, y \in \mathbb{A}^1_{\text{Berk}}$ there exists a unique least upper bound $x \vee y \in \mathbb{A}^1_{\text{Berk}}$ [with respect to the partial order that $x \leq y$ iff $|f|_x \leq |f|_y$ for $f \in K[T]$]. A key insight of Baker and Rumely [BR10] was that this structure on $\mathbb{A}^1_{\text{Berk}}$ (in fact, on $\mathbb{P}^1_{\text{Berk}}$) could be used to define a Laplacian operator, laying the foundations for a non-Archimedean analogue of complex potential theory. Their work later found surprising applications in the analysis of preperiodic points of complex dynamical systems [BD11].[83]

For the topos theorist, however, Summary Theorem 5.0.1 is suggestive because it mirrors the different representations of a point-free space: as a universe of (algebraic) models axiomatised by a first-order theory, as a certain space of prime filters, or as a distributive lattice (cf. Summary Theorem 2.2.3 and Theorem C). One may therefore wonder if the listed characterisations of $\mathbb{A}^1_{\text{Berk}}$ reflect a constellation of perspectives on Berkovich spaces that move together in a tightly-connected way. It is this intuition that will guide us to the main result of this chapter, Theorem E, which generalises (and reformulates) the equivalence of items (i) and (ii) in Summary Theorem 5.0.1 by eliminating the hypothesis that K has to be non-trivially valued.[84]

5.1 Preliminaries in Berkovich Geometry

5.1.1 Motivation. The development of Berkovich geometry continues the theme of navigating the differences between the Archimedean vs. the non-Archimedean setting. In broad strokes: it is well-known that any complex algebraic variety[85] X can be canonically associated to a complex analytic space X^{an} via a (functorial) construction known as *complex analytification*. This opens up the study of complex algebraic varieties to powerful tools in complex analysis and differential geometry, prompting the natural question: can we play the same game for algebraic varieties over fields which are not $\mathbb{C}$? For instance, over $\mathbb{Q}$? Over the field of Laurent series $\mathbb{C}((t))$? The p-adic numbers $\mathbb{Q}_p$?

The general thrust of these questions is challenging, but over-simplistic. It is over-simplistic because the naive analytification of algebraic varieties over non-Archimedean fields loses significant information about the original variety, limiting its intended usefulness (for details, see Appendix A). Still, it is challenging because it brings into focus the main issue behind this lossy-ness: unlike the complex numbers $\mathbb{C}$, a non-Archimedean field K is totally disconnected. Once understood and made precise, this tells us where to start looking for a robust non-Archimedean analogue of complex analytification.

5.1.2 The Berkovich Perspective . . . The key premise of Berkovich geometry [Ber90] is that the naive analytification of non-Archimedean varieties is disconnected because it does not have enough points. The solution[86] then, by way of a construction known as *Berkovich analytification*, is to fill in those missing points before developing techniques to study these new analytic spaces.

[83] The fact that non-Archimedean analytic techniques, developed in analogy with the complex case, should find applications in the non-Archimedean setting is reasonable; what is perhaps less expected is that these non-Archimedean techniques should also find applications in the complex setting. [BD11] gives an example of this in complex dynamics, but non-Archimedean methods have also been useful when studying complex algebraic varieties. For details, we recommend [Pay15, §5].

[84] Technically, Theorem E works with multiplicative seminorms on the ring of convergent power series $K\{R^{-1}T\}$ and not those on $K[T]$, but in fact the result extends to the latter setting by Remark 5.1.11.

[85] More precisely, a scheme of (locally) finite type over $\mathbb{C}$. See Section A.1.

[86] There are also other solutions to this disconnectedness problem, e.g. Tate's rigid analytic geometry, which involves defining an appropriate Grothendieck topology that finitises the usual notion of a topology. See e.g. [Pay15, §1.5].

5.1.2.1 ...on algebraic varieties. Let $(K, |\cdot|)$ be a non-Archimedean field, and X be an affine variety over K, i.e. X is the zero locus in K^n of a finite set of polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Recall that the coordinate ring of X is defined as

$$K[X] := K[T_1, \ldots, T_n]/(f_1, \ldots, f_m) \tag{5.1}$$

One easily checks that every point $k \in X(K)$ gives rise to a multiplicative seminorm on $K[X]$

$$\begin{aligned} |\cdot|_k \colon K[X] &\longrightarrow \mathbb{R}_{\geq 0} \\ f &\longmapsto |f(k)|, \end{aligned} \tag{5.2}$$

otherwise known as the *evaluation seminorm at* k. Extending this insight, one can then define the analytification of X in terms of seminorms on its coordinate ring.

Definition 5.1.1 (Analytification of Affine Varieties)**.** Fix a non-Archimedean field $(K, |\cdot|)$, and let X be an affine variety over K with associated coordinate ring $K[X]$.

(i) Given a multiplicative seminorm on $K[X]$, which we denote

$$|\cdot|_x \colon K[X] \longrightarrow \mathbb{R}_{\geq 0} \tag{5.3}$$

we say that $|\cdot|_x$ *extends the given norm on* K if $|k|_x = |k|$ for all $k \in K$. Notice that this includes the evaluative seminorms.

(ii) The *analytification of* X, denoted X^{an}, is defined as the following point-set space:

- *Underlying set of* X^{an} = the set of all multiplicative seminorms on $K[X]$ extending the original norm $|\cdot|$ on the base field K;
- *Topology on* X^{an} = the weakest topology such that all maps of the form

$$\begin{aligned} \psi_f \colon X^{\text{an}} &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x. \end{aligned} \tag{5.4}$$

are continuous, for any $f \in K[X]$, which we shall call the *Berkovich topology*.[87] For clarity, we emphasise that ψ_f is a mapping on *all* multiplicative seminorms on $K[X]$ extending $|\cdot|$ — not just the evaluative seminorms from before.

Example 5.1.2. Given a non-Archimedean field $(K, |\cdot|)$, then the underlying set of the Berkovich Affine line $\mathbb{A}^1_{\text{Berk}}$ is the set of multiplicative seminorms

$$|\cdot|_x \colon K[T] \longrightarrow \mathbb{R}_{\geq 0} \tag{5.5}$$

extending the norm on K, as already seen in Summary Theorem 5.0.1.

Remark 5.1.3. The expert reader may have noticed Definition 5.1.1 technically only defines the underlying topological space of the Berkovich analytification. To be correct, let us mention that the full definition of Berkovich analytification also includes a structure sheaf of analytic functions on X^{an}, yielding a locally ringed space. For details, see [Ber90, §2.3, 3.1–4].

[87] This is also sometimes called the *Gel'fand topology*.

Discussion 5.1.4. Regarding the classical aspects of Definition 5.1.1:

(i) Despite its point-set formulation, the Berkovich analytification X^{an} is suggestive from the point-free perspective since we already have a point-free account of multiplicative seminorms on $\mathbb{Z}$ and $\mathbb{Q}$ (and $\mathbb{N}$) in Chapter 4, which gives natural indications on how to extend the same ideas to more general rings.

(ii) The Berkovich topology can be more explicitly characterised as the weakest topology such that for all $f \in K[X]$ and for all $\alpha \in \mathbb{R}$, the sets

$$U(f,\alpha) := \{|\cdot| \in X^{\mathrm{an}} \,\big|\, |f| < \alpha\}$$
$$V(f,\alpha) := \{|\cdot| \in X^{\mathrm{an}} \,\big|\, |f| > \alpha\} \tag{5.6}$$

are open in X^{an}. Notice then that Berkovich topology crucially depends on the fact that $|\cdot|$ is valued in the Dedekinds as opposed to say, the upper reals.[88]

In fact, Definition 5.1.1 can be extended to the more general case of K-schemes of (locally) finite type (though the details are more involved, see [Ber90, Ch. 2 - 3]). One important appeal of the Berkovich analytification is that it constructs well-behaved spaces that are sensitive to the topological character of the original variety.

Summary Theorem 5.1.5 ([Ber90, §3.4-3.5])**.** *Let X be a K-scheme of finite type. Then X^{an} is locally compact and locally path-connected. Furthermore, we also have the following GAGA-type results:*

(i) X is a connected scheme iff X^{an} is connected;
(ii) X is a separated scheme iff X^{an} is Hausdorff;
(iii) X is a proper scheme iff X^{an} is compact.

As a beautiful example of the interactions between (classical) logic and Berkovich geometry, let us also mention the following result by Hrushovski and Loeser.

Theorem 5.1.6 ([HL16, Theorem 14.4.1])**.** *Let X be a K-scheme of finite type. Then, its Berkovich analytification X^{an} is locally contractible.*

Prior to Theorem 5.1.6, local contractibility was only known in the case of smooth Berkovich analytic spaces [Ber99]; by contrast, the model-theoretic techniques developed by Hrushovski and Loeser [HL16] were sufficiently general to handle both the singular and non-singular cases.

5.1.2.2 … on Banach rings. The Berkovich analytification of algebraic varieties can also be understood via the language of Berkovich spectra, a similar construction in the setting of Banach rings[89].

Definition 5.1.7 (The Berkovich Spectrum)**.** Let $(\mathcal{A}, ||\cdot||)$ be a commutative Banach ring with identity.

(i) A *bounded* multiplicative seminorm on $\mathcal{A}$ is a multiplicative seminorm

$$|\cdot|_x \colon \mathcal{A} \longrightarrow \mathbb{R}_{\geq 0} \tag{5.7}$$

that satisfies the inequality $|f|_x \leq ||f||$ for all $f \in \mathcal{A}$.

[88] Why? Note that $V(f,\alpha)$ from Equation (5.6) would no longer be well-defined since an absolute value $|\cdot|\colon K[X] \to \overleftarrow{[0,\infty)}$ valued in the upper reals is unable to sense which values are smaller than $|f|$, only those larger than it.

[89] Recall: a Banach ring $(\mathcal{A}, ||\cdot||)$ is a normed ring that is complete with respect to $||\cdot||$.

(ii) The *Berkovich Spectrum* $\mathcal{M}(\mathcal{A})$ is the set of all bounded multiplicative seminorms on $\mathcal{A}$, equipped with the weakest topology such that the map

$$\begin{aligned} \psi_f \colon \mathcal{M}(\mathcal{A}) &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned} \tag{5.8}$$

is continuous for all $f \in \mathcal{A}$.

Convention 5.1.8 (On the Berkovich Spectrum)**.**

(i) In this section, all seminorms should be assumed to be multiplicative unless otherwise stated.
(ii) The given norm on an arbitrary Banach ring $\mathcal{A}$ will always be denoted as $||\cdot||$. By contrast, to emphasise that $\mathcal{M}(\mathcal{A})$ is a topological space, its points will be represented as $|\cdot|_x$, or even $x \in \mathcal{M}(\mathcal{A})$ when the contextual meaning is clear. For clarity, we again emphasise that $|\cdot|_x$ should not be assumed to be an evaluative seminorm on $\mathcal{A}$, unless explicitly stated.

We illustrate this construction by way of examples. A few orienting remarks are in order. First, one may have noticed there is nothing specifically non-Archimedean about Definition 5.1.7 — in fact, as we shall see in Example 5.1.9, the Berkovich spectrum of $\mathbb{Z}$ yields a space that naturally includes both Archimedean and non-Archimedean components. Another notable feature of Berkovich geometry is that the basic setup accommodates both the trivially and non-trivially valued fields — see Example 5.1.10. Interestingly, this flexibility appears to be abandoned/lost in many modern approaches to the subject (the curious reader may wish to have a look at [BR10; Ben19], both of which require the field to be non-trivially valued). Finally, the generality of Definition 5.1.7 also allows us to define the Berkovich spectrum of an important class of Banach rings known as Tate Algebras — its role in Berkovich geometry will be discussed in Example 5.1.12.

Example 5.1.9. The ring of integers equipped with the usual Euclidean norm, i.e. $(\mathbb{Z}, |\cdot|_\infty)$, is a Banach ring. The characterisation of its Berkovich spectrum $\mathcal{M}(\mathbb{Z})$ usually proceeds by a series of case-splittings.

- *Stage 1: Identify the degenerate seminorms.* Any point $x \in M(\mathbb{Z})$ corresponds to a seminorm $|\cdot|_x$ on $\mathbb{Z}$, which induces a prime ideal $\mathfrak{p}_x = \{|\cdot|_x = 0\} \subseteq \mathbb{Z}$. In the case where $\mathfrak{p}_x = p\mathbb{Z}$, deduce that $|\cdot|_x$ induces the unique trivial seminorm on $\mathbb{F}_p$,[90] and so

$$|\cdot|_x = |n|_{p,\infty} := \begin{cases} 0 & \text{if } p|n \\ 1 & \text{if otherwise.} \end{cases} \tag{5.9}$$

  [Notice that $|\cdot|_{p,\infty}$ is "degenerate" in the sense that it fails to be positive definite.] Otherwise, note that the induced prime ideal $\mathfrak{p}_x = (0)$ must be the zero ideal.
- *Stage 2: Classify the remaining seminorms.* Suppose $\mathfrak{p}_x = (0)$. Extend $|\cdot|_x$ to an absolute value on $\mathbb{Q}$ via multiplicativity. Then, appeal to Ostrowski's Theorem to deduce that $|\cdot|_x$ must be one of the following:

  Case 2a: $|\cdot|_x = |\cdot|_0$ is the trival norm on $\mathbb{Z}$.
  Case 2b: $|\cdot|_x = |\cdot|_\infty^\alpha$ for some $\alpha \in (0, 1]$, where $|\cdot|_\infty$ is the usual Euclidean norm.
  Case 2c: $|\cdot|_x = |\cdot|_p^\alpha$ for some $\alpha \in (0, \infty)$, where $|\cdot|_p$ is the standard p-adic norm.

Assembling this data together, one obtains the following picture:

[90] Why? Notice that $|\cdot|_x$ induces a multiplicative seminorm on $\mathbb{Z}/\mathfrak{p}_x = \mathbb{F}_p$ essentially by hypothesis. One then easily checks that the only multiplicative norm on $\mathbb{F}_p$ is the one which coincides with Equation (5.9).

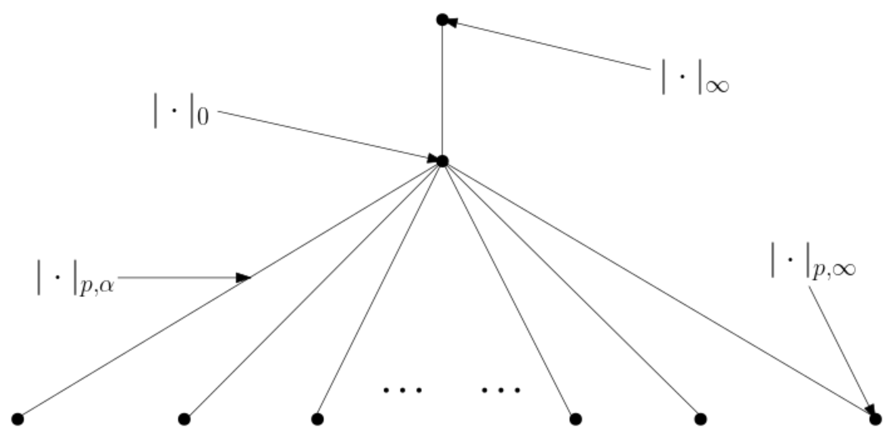


Figure 5.1: $\mathcal{M}(\mathbb{Z})$

Notice this is almost identical to our picture of the topos of upper-valued absolute values on $\mathbb{Z}$ (Theorem C) *except* for the fact that the intervals here are valued in Dedekinds as opposed to upper reals.

Example 5.1.10. Fix an algebraically-closed non-Archimedean field $(K, |\cdot|)$. We define the Banach ring $(\mathcal{A}, ||\cdot||)$ whereby:

- $\mathcal{A}$ is the ring of power series converging in radius R

$$\mathcal{A} = K\{R^{-1}T\} := \left\{ f = \sum_{i=0}^{\infty} c_i T^i \,\middle|\, c_i \in K, \lim_{i\to\infty} |c_i| R^i = 0 \right\} \tag{5.10}$$

- $||\cdot||$ is the so-called *Gauss norm*

$$||f|| := \sup_i |c_i| R^i, \quad \text{where } f \in \mathcal{A}. \tag{5.11}$$

The description of $\mathcal{M}(\mathcal{A})$ differs depending on whether K is trivially or non-trivially valued.

- *Case 1: K is trivially valued.* In which case,

$$K\{R^{-1}T\} = \begin{cases} K[[T]] & \text{if } R < 1 \\ K[T] & \text{if } R \geq 1 \end{cases} \tag{5.12}$$

  where $K[[T]]$ is the formal power series ring and $K[T]$ is the polynomial ring.[91] When $R < 1$, one checks that the the map $|\cdot|_x \mapsto |T|_x$ yields a homeomorphism $\mathcal{M}(\mathcal{A}) \cong [0, R]$; when $R \geq 1$, a similar (but more involved) argument shows $\mathcal{M}(\mathcal{A})$ has the structure of $\mathcal{M}(\mathbb{Z})$ (see Figure 5.1).
- *Case 2: K is non-trivially valued.* In which case, the characterisation of $\mathbb{A}^1_{\text{Berk}}$ in Summary Theorem 5.0.1 extends here as well: all points of $x \in \mathcal{M}(\mathcal{A})$ can be realised as

$$|\cdot|_x = \lim_{n\to\infty} |\cdot|_{D_{r_i}(k_i)} \tag{5.13}$$

[91] Why? Notice if $R < 1$, then $\lim_{i\to\infty} |c_i| R^i$ is always zero since $|c_i| = 0$ or 1; if $R \geq 1$ instead, then the sequence $\{c_i\}$ must eventually be 0.

for some nested descending sequence of discs

$$D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \dots \tag{5.14}$$

where $|\cdot|_{D_r(k)}$ is a multiplicative seminorm canonically associated to the closed disc

$$D_r(k) := \{b \in K \,\big|\, |b-k| \leq r\}. \tag{5.15}$$

This description of $\mathcal{M}(\mathcal{A})$ results in a complicated tree with infinite branching points:

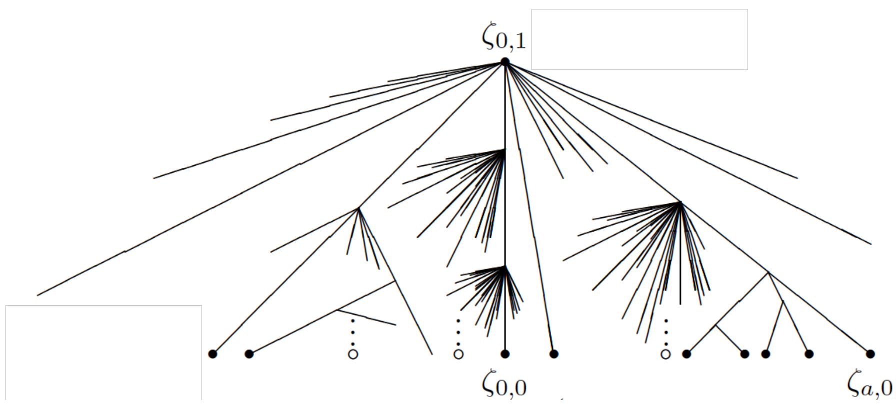


Figure 5.2: $\mathcal{M}(K\{R^{-1}T\})$ when $R = 1$, adapted from [BR10; Sil07]

The reader may wonder: where did we use the fact that K was non-trivially valued? Notice that the closed discs $D_{r_i}(k_i)$ in Equation (5.15) are defined as subsets of K. As such, in order for their radii to be well-defined, i.e. if $D_r(k) = D_{r'}(k')$ then $r = r'$, the base field K is forced to be non-trivially valued.[92] Otherwise, the proof as originally presented in [Ber90] no longer works.

Remark 5.1.11. Recall that the Berkovich Affine Line $\mathbb{A}^1_{\text{Berk}}$ is defined as the space of multiplicative seminorms on $K[T]$. As such, is $\mathbb{A}^1_{\text{Berk}}$ just another example of a Berkovich spectrum? The answer, perhaps surprisingly, is generally no. Recall: Berkovich spectra are defined for *Banach Rings*. When K is non-trivially valued, one can check that $K[T]$ fails to be complete with respect to $||\cdot||$, and is therefore not Banach. However, two important caveats:

(a) One can also check that $K\{R^{-1}T\}$ is in fact a Banach ring (with respect to $||\cdot||$), and that $\mathbb{A}^1_{\text{Berk}}$ can be represented as an infinite union of Berkovich spectra

$$\mathbb{A}^1_{\text{Berk}} \cong \bigcup_{R>0} \mathcal{M}(K\{R^{-1}T\}).$$

(b) When K is trivially valued, then $K[T]$ turns out to be complete with respect to $||\cdot||$ and thus defines a Banach ring; in which case, the two constructions do coincide. This gives another way of reading the difference between the trivially vs. non-trivially valued fields in the Berkovich setting.

[92] Why? If K is equipped with a trivial norm, then by definition $|k| = 1$ or 0 for all $k \in K$. In which case, note that, e.g. $D_{\frac{1}{2}}(k) = \{k\} = D_{\frac{1}{3}}(k)$ for any $k \in K$.

For details, see e.g. [Ber90, Example 1.4.4] or [BR10, Ch. 1-2].

Example 5.1.12. Extending Example 5.1.10, given $R_1, \ldots, R_n > 0$, define the ring

$$K\{R_1^{-1}T_1, \ldots, R_n^{-1}T_n\} := \left\{ f = \sum_{\mathsf{v} \in \mathbb{N}^n} a_{\mathsf{v}} T^{\mathsf{v}} \;\middle|\; a_{\mathsf{v}} \in K, \lim_{|\mathsf{v}| \to \infty} |a_{\mathsf{v}}| R^{\mathsf{v}} = 0 \right\}, \tag{5.16}$$

where $|\mathsf{v}| = \mathsf{v}_1 + \cdots + \mathsf{v}_n$ and $R^{\mathsf{v}} = R_1^{\mathsf{v}_1} \ldots R_n^{\mathsf{v}_n}$. To turn $K\{R_1^{-1}T_1, \ldots, R_n^{-1}T_n\}$ into a Banach ring, we equip it with the Gauss norm $||\cdot||$ where

$$||f|| := \sup_{\mathsf{v}} |a_{\mathsf{v}}| R^{\mathsf{v}}. \tag{5.17}$$

When $R_i = 1$ for all i, then $K\{R_1^{-1}T_1, \ldots, R_n^{-1}T_n\}$ is called the *Tate Algebra*.

These Banach rings play an important role in Berkovich geometry because they allow us to define K-*affinoid algebras*, which are the quotients of these power series rings. More precisely, a K-affinoid algebra $\mathcal{A}$ is a commutative Banach ring for which there exists an admissible epimorphism

$$K\{R_1^{-1}T_1, \ldots, R_n^{-1}T_n\} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}. \tag{5.18}$$

This should be understood as being the analogues of quotients of polynomial rings in classical scheme theory; in particular, one constructs a Berkovich K-analytic space by gluing together the Berkovich spectra $\mathcal{M}(\mathcal{A})$ of these K-affinoid algebras.

5.2 Berkovich's Classification Theorem

An organising theme of this section is the language of filters, which gives a transparent way of understanding how certain key notions in topology, logic and non-Archimedean geometry may interact.

We motivate our study by way of a biased historical overview. On the side of geometry, the fact that the points of a Berkovich spectrum[93] $\mathcal{M}(\mathcal{A})$ may be characterised as ultrafilters was already known by the 1990s [Ber90, Remark 2.5.21]; on the side of logic, the fact that the (complete) types over a model[94] may also be characterised as ultrafilters was well understood by the 1960s [Mor65], if not earlier. Yet it was only within the last 10 years that the two perspectives started to converge. Most notably, fixing a valued field K of rank 1, Hrushovski and Loeser [HL16, §14.1] showed that the Berkovich analytification of any quasi-projective variety V over K can be described using the language of definable types[95] (cf. item (iii)

[93] Here, we assume that $\mathcal{A}$ is strictly K-affinoid and K has non-trivial valuation.

[94] An informal picture for readers unfamiliar with model theory: if the *models* $M_{\mathbb{T}}$ of a theory $\mathbb{T}$ can be thought of as corresponding to point-set spaces, then the *complete types* over a model $M_{\mathbb{T}}$ correspond to the limit points of $M_{\mathbb{T}}$, which may or may not be realised in $M_{\mathbb{T}}$. This gives rise to a robust way of measuring the logical completeness of a model (which *a priori* does not rely on a metric). For a short accessible introduction, we recommend [Mal19].

[95] It is helpful to compare the distinction between complete types vs. definable types with the topos theorist's distinction between global points vs. generalised points. Recall from Discussion 2.2.19: a *complete type* $p(x)$ in variables $x = (x_1, \ldots, x_n)$ can be viewed as a Boolean homomorphism $\mathcal{L}_x \to \{0,1\}$ from the set of $\mathcal{L}$-formulas in x to the two-element Boolean algebra. Then, a $\emptyset$*-definable type* $p(x)$ is a special kind of complete type with a built-in extension: it is a function $d_p x \colon \mathcal{L}_{x,y_1,\ldots,} \to \mathcal{L}_{y_1,\ldots,}$ such that for any finite $y = (y_1, \ldots, y_n) \in Y_1 \times \ldots Y_n$, $d_p x$ restricts to a Boolean retraction $\mathcal{L}_{x,y} \to L_y$. Given such a retraction, and given any model $M_{\mathbb{T}}$ of $\mathbb{T}$, one obtains a type over $M_{\mathbb{T}}$, namely

$$p|M_{\mathbb{T}} := \{\phi(x, b_1, \ldots, b_n) \mid M_{\mathbb{T}} \models (d_p x)(\phi)(b_1, \ldots, b_n)\}.$$

of Summary Theorem 5.0.1). The power of this unique perspective may be measured by the fact that the authors were able to establish many deep results (e.g. Theorem 5.1.6) that were inaccessible to previous methods (at least, without relying on stronger hypotheses).

This sets up our present investigation. The understanding that models of a propositional geometric theory can be characterised as completely prime filters is well known to topos theorists, although the connections with model-theoretic types appear to be undeveloped (but see Discussion 2.2.19). In this section, we follow the model theorist's cue and use point-free techniques to study the points of the Berkovich spectra $\mathcal{M}(K\{R^{-1}T\})$ (cf. Example 5.1.10). The main surprise is that, unlike Berkovich's original classification result, the point-free approach works equally well for both the trivially and non-trivially valued fields. This indicates that the *algebraic* hypothesis of K being non-trivially valued is in fact a *point-set* hypothesis, and is not essential to the underlying mathematics.

5.2.1 Berkovich's Disc Theorem. We fix the following hypothesis for the rest of this section.

Hypothesis 5.2.1.

(i) K is an algebraically closed field, complete with respect to a non-Archimedean norm $|\cdot|$. We emphasise that $|\cdot|$ need not be non-trivial.
(ii) K is a geometric field (Definition 4.1.6) whose points can be regarded as a set.
(iii) R denotes any positive Dedekind real $R > 0$, and Q_+ denotes the set of positive rationals.
(iv) Define $K_R := \{k \in K \mid |k| \leq R\}$.
(v) Following Example 5.1.10, we define a Banach ring $(\mathcal{A}, ||\cdot||)$, where $\mathcal{A} = K\{R^{-1}T\}$ denotes the ring of power series converging on radius R, and $||\cdot||$ denotes the associated Gauss norm.

In order to classify the bounded multiplicative seminorms on $\mathcal{A}$, we first reduce our study to something algebraically simpler.

Definition 5.2.2 (Bounded K-Seminorms)**.**

(i) Define the space of linear polynomials $\mathcal{A}_{\text{Lin}}$ as

$$\mathcal{A}_{\text{Lin}} := \{aT - b \mid a, b \in K\} \cong K^2. \tag{5.19}$$

In particular, by setting $a = 0$, notice that $K \subset \mathcal{A}_{\text{Lin}}$.

(ii) We define a K*-seminorm* on $\mathcal{A}_{\text{Lin}}$ to be an upper-valued map

$$|\cdot|_x \colon \mathcal{A}_{\text{Lin}} \longrightarrow \overleftarrow{[0, \infty)} \tag{5.20}$$

such that

- *(Preserves constants)* $|a|_x$ = the right Dedekind section of $|a|$;
- *(Semi-multiplicative)* $|aT|_x = |a| \cdot |T|_x$;
- *(Ultrametric Inequality)* $|f + f'|_x \leq \max\{|f|_x, |f'|_x\}$;

for all $a \in K$, and $f, f' \in \mathcal{A}_{\text{Lin}}$.

(iii) We define the *Gauss Norm on* $\mathcal{A}_{\text{Lin}}$ as

$$||aT - b|| := \text{right Dedekind section of} \max\{|a|R, |b|\}$$

where $aT - b \in \mathcal{A}_{\text{Lin}}$. We call a K-seminorm $|\cdot|_x$ is called *bounded* if $|\cdot|_x \leq ||\cdot||$.

Remark 5.2.3. It is well-known [BR10, Lemma 1.1] that any bounded multiplicative seminorm $|\cdot|_x$ satisfies $|a|_x = |a|$ and $|f + g|_x \leq \max\{|f|, |g|\}$ for any $f, g \in \mathcal{A}$ — this justifies the axioms we chose in Definition 5.2.2(ii). Notice also that we did not require a K-seminorm to be multiplicative (only semi-multiplicative), but this is reasonable since $\mathcal{A}_{\mathrm{Lin}}$ is not closed under multiplication.

Reminder 5.2.4. To eliminate potential confusion:

- A *multiplicative seminorm* is defined on the whole ring $\mathcal{A}$ of convergent power series.
- A K*-seminorm*, which is not multiplicative, is only defined on the space of linear polynomials $\mathcal{A}_{\mathrm{Lin}}$.

We justify the reduction to linear polynomials in the following Preparation Lemma.

Lemma 5.2.5 (Preparation Lemma)**.**

(i) Let $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ be an arbitrary Banach ring with dense subring $\mathcal{B}'$. Then, any bounded multiplicative seminorm on $\mathcal{B}$ is determined by its values on $\mathcal{B}'$.
(ii) Any bounded multiplicative seminorm on $\mathcal{A}$ is determined by its values on linear polynomials $T - a$, where $a \in K_R$. The same also holds for bounded K-seminorms on $\mathcal{A}_{\mathrm{Lin}}$.

Proof. (i) Fix a bounded multiplicative seminorm $|\cdot|_x$ on $\mathcal{B}$, and suppose $f \in \mathcal{B}$. Next, for any positive rational $\epsilon > 0$ and any $g \in \mathcal{B}'$ such that $||f - g||_{\mathcal{B}} < \epsilon$, compute:

$$|f|_x \leq |g|_x + |f - g|_x \leq |g|_x + \epsilon,$$

$$|g|_x \leq |f|_x + |f - g|_x \leq |f|_x + \epsilon.$$

Hence, deduce that if $g \to f$ with respect to $||\cdot||_{\mathcal{B}}$, then $|g|_x \to |f|_x$. The claim then follows from $\mathcal{B}'$ being a dense subalgebra in $\mathcal{B}$.

(ii) We start with two basic observations that get us almost all the way:

(a) The polynomial ring $K[T]$ is a dense subalgebra in $\mathcal{A}$, since any $f \in \mathcal{A}$ can be expressed as

$$f = \sum_{i=0}^{\infty} a_i T^i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i T^i.$$

(b) Since K is algebraically closed, any polynomial $g \in K[T]$ can be expressed as

$$g = c \cdot \prod_{j=1}^{m} (T - b_j),$$

where $c, b_j \in K$, for all $1 \leq j \leq m$.

Applying item (i) of the Lemma, Observations (a) and (b) together imply that any bounded multiplicative seminorm $|\cdot|_x$ on $\mathcal{A}$ is determined by its values on linear polynomials $T - b$ with $b \in K$. In fact, a simple argument shows that we can restrict to just the linear polynomials $T - a$ where $a \in K_R$. Denote $h := T - b$ to be a linear polynomial where $b \in K$. By polynomial division, we can represent the polynomial T as

$$T = qh + z.$$

Taking the Gauss norm $||\cdot||$ on both sides of the equation, we obtain:

$$R = ||T|| = \max\{||qh||\,,\,||z||\},$$

and so deduce $z \in K_R$. Since it is clear that q must be a unit in K, we get

$$q^{-1}\cdot(T-z) = h = T-b,$$

and so our claim follows. The argument for the K-seminorm case is virtually identical. □

Discussion 5.2.6 (Comparison with the Weierstrass Preparation Theorem)**.** Our use of polynomial division in the proof of the Preparation Lemma 5.2.5 gives a prototype argument that can be generalised to rings of convergent power series. This leads to the well-known Weierstrass Preparation Theorem [Jona, Prop. 1.9.10], which informally says: convergent power series often look like polynomials when restricted to a closed disc. Or, more precisely: all non-zero $f \in \mathcal{A}$ with finite order $m \geq 0$ can be represented as

$$f = W \cdot u(T), \tag{5.21}$$

where $u(T)$ is a unit power series on the closed disc K_R, and W is a monic polynomial (the "Weierstrass polynomial") with both degree and order m. In fact, in the special case when K is non-trivially valued and $R = 1$, one can also use the Weierstrass Preparation Theorem to prove that bounded multiplicative seminorms on $K\{R^{-1}T\}$ are determined by their values on $T-a$ for $a \in K_R$ (see, e.g. [BR10]).

However, unlike our Preparation Lemma, this argument fails to extend to the general case. Why? Notice the Weierstrass Preparation Theorem only applies to non-zero $f \in \mathcal{A}$ when f has *finite order*. In particular, while all power series converging on a *rational*[96] *radius* R have finite order, this is no longer true when R is *irrational* and K is non-trivially valued. See for instance [Ben19, Exercise 3.5], which gives an example of a power series $f \in K\{R^{-1}T\}$ with infinitely many zeros in K_R in the irrational case.

We now introduce a special class of filters that highlights the topological structure of $\mathcal{M}(\mathcal{A})$.

Definition 5.2.7 (Formal Non-Archimedean Balls)**.** A *formal non-Archimedean ball* is an element $(k,q) \in K_R \times Q_+$. We represent this using the more suggestive notation $B_q(k)$, to emphasise that we should view this pair as denoting a disc of radius q centred at k. In particular, we write:

$$B_{q'}(k') \subseteq B_q(k) \text{ just in case } |k-k'| < q \wedge q' \leq q.$$

Observation 5.2.8. By decidability of $<$ on Q_+, one obtains the following sequents from the definition of $\subseteq$ from Definition 5.2.7:

(i) $|k-k'| < q \longrightarrow B_q(k') = B_q(k)$.
(ii) $q \in Q_+$ and $|k-k'| < q' \longrightarrow B_q(k') = B_q(k)$ or $q < q'$
(iii) $B_q(k) = B_{q'}(k') \longrightarrow q = q'$

Discussion 5.2.9 (Point-set discs vs. Formal balls)**.** Notice item (iii) of Observation 5.2.8 says that the radii of the formal balls are well-defined (essentially by construction), even when the norm $|\cdot|$ on K is trivial. This should be contrasted with the classical rigid discs from Example 5.1.10

$$D_r(k) := \{b \in K \,\big|\, |b-k| \leq r\}, \tag{5.22}$$

whose radii are well-defined only if $|\cdot|$ is non-trivial due to their point-set formulation (cf. Footnote 92).

[96]Warning: in the present context, R is said to be *rational* if $R = |k|$ for some $k \in K$ and *irrational* if otherwise. In particular, the claim that R is rational here does not mean $R \in \mathbb{Q}$.

Remark 5.2.10. The language of formal balls may strike the classical reader as a peculiar abstraction, but in fact they are sufficiently expressive to provide point-free accounts of standard completions of metric spaces [Vic09]. Once the analogy between the Berkovich construction and the completion of a space is made precise, these techniques can be adjusted accordingly to our present context.[97]

Definition 5.2.11 (Filters of Formal Non-Archimedean Balls)**.** A *filter* of formal non-Archimedean balls $\mathcal{F}$ is an inhabited[98] subset of $K_R \times Q_+$ satisfying the following conditions:

- *(Upward closed with respect to $\subseteq$)* If $B_{q'}(k') \subseteq B_q(k)$ and $B_{q'}(k') \in \mathcal{F}$, then $B_q(k) \in \mathcal{F}$.
- *(Closed under pairwise intersections)* If $B_q(k), B_{q'}(k') \in \mathcal{F}$, then there exists some $B_r(j) \in \mathcal{F}$ such that $B_r(j) \subseteq B_q(k)$ and $B_r(j) \subseteq B_{q'}(k')$.

Further, we call $\mathcal{F}$ an R*-good* filter if it also satisfies the following two conditions:

- For any $k \in K_R$, and $q \in Q_+$ such that $R < q$, $B_q(k) \in \mathcal{F}$.
- If $B_q(k) \in \mathcal{F}$, there exists $B_{q'}(k') \in \mathcal{F}$ such that $q' < q$.

Convention 5.2.12. In this chapter, unless stated otherwise:

- A ball $B_q(k)$ will always mean a formal non-Archimedean ball (cf. Definition 5.2.7).
- A filter $\mathcal{F}$ will always means a filter of formal non-Archimedean balls (cf. Definition 5.2.11).

Observation 5.2.13 (Radius of R-Good Filters)**.** Let $\mathcal{F}$ be an R-good filter. Define the *radius* $\mathsf{rad}_{\mathcal{F}}$ of $\mathcal{F}$ as:

$$\mathsf{rad}_{\mathcal{F}} := \{q \in Q_+ \mid B_q(k) \in \mathcal{F}\}.$$

Then, $\mathsf{rad}_{\mathcal{F}}$ defines an upper real in $\overleftarrow{[0, R]}$.

Proof. Roundedness and upward closure is immediate from the definition of $\mathcal{F}$ being an R-good filter, and so $\mathsf{rad}_{\mathcal{F}}$ defines an upper real. The fact that $\mathsf{rad}_{\mathcal{F}} \in \overleftarrow{[0, R]}$ follows from additionally noting:

- $\mathsf{rad}_{\mathcal{F}}$ is a subset of positive rationals. Hence, $0 \leq \mathsf{rad}_{\mathcal{F}}$.
- $q \in \mathsf{rad}_{\mathcal{F}}$ for all rationals $q > R$ — again because $\mathcal{F}$ is R-good. Hence, $\mathsf{rad}_{\mathcal{F}} \leq R$.

□

Construction 5.2.14. Suppose we have a bounded K-seminorm $|\cdot|_x$ on $\mathcal{A}_{\mathrm{Lin}}$. We then define the following collection of formal balls:

$$\mathcal{F}_x := \{B_q(k) \mid k \in K_R \text{ and } |T - k|_x < q\}$$

Claim 5.2.15. *$\mathcal{F}_x$ is an R-good filter.*

Proof. By Definition 5.2.11, we need to check that $\mathcal{F}_x$ is ...

[97] One important adjustment is that the *non-strict* order defined in Definition 5.2.7 is quite different from the *strict* order defined in [Vic05; Vic09]. Nonetheless, since the non-Archimedean spaces possess a rich supply of clopens, we expect (though have yet to check the details) that the two different orders to define the same Cauchy filters in our setting.

[98] The classical reader may substitute mentions of "inhabited" with "non-empty" without too much trouble.

- . . . *Upward closed.* Suppose $B_{q'}(k') \subseteq B_q(k)$ and $B_{q'}(k') \in \mathcal{F}_x$. Unpacking definitions, this means $|k - k'| < q$ and $q' \leq q$, as well as $|T - k'|_x < q'$. But since

$$|T - k|_x = |(T - k') + (k' - k)|_x \leq \max\{|T - k'|_x, |k - k'|\} < \max\{q', q\} = q,$$

  this implies $B_q(k) \in \mathcal{F}_x$.

- . . . *Closed under Pairwise Intersection.* We first claim $\mathcal{F}_x$ is totally ordered by $\subseteq$. Why? Given any $B_q(k), B_{q'}(k') \in \mathcal{F}_x$, we get $|T - k|_x < q$ and $|T - k|_x < q'$, and so

$$|k - k'| = |(T - k') - (T - k)|_x \leq \max\{|T - k'|_x, |T - k|_x\} < \max\{q', q\}.$$

  By decidability of $<$ on Q_+, this means either $B_q(k) \subseteq B_{q'}(k')$ or $B_{q'}(k') \subseteq B_q(k)$, as claimed. The fact that $\mathcal{F}_x$ is closed under pairwise intersection follows immediately.

- . . . *R-good.*
  - Suppose $B_q(k) \in \mathcal{F}_x$, and so $|T - k|_x < q$ by definition. Since $|\cdot|_x$ defines an upper real, there exists $q' \in Q_+$ such that $|T - k|_x < q' < q$, and so $B_{q'}(k) \in \mathcal{F}_x$.
  - Suppose $k \in K_R$ and $q \in Q_+$ such that $R < q$. This gives

$$|T - k|_x \leq \max\{|T|_x, |k|\} \leq \max\{||T||, |k|\} = R < q,$$

    since $k \in K_R$ implies $|k| < R$ by definition, and $|T|_x \leq ||T|| = R$. Hence, $B_q(k) \in \mathcal{F}_x$.

- . . . *Inhabited.* Immediate from R-goodness.

□

In the converse direction, we define the following:

Construction 5.2.16. For any formal non-Archimedean ball $B_q(k)$, we define $|\cdot|_{B_q(k)}$ as follows:

$$|T - a|_{B_q(k)} := \max\{|k - a|, q\}, \qquad \text{where } T - a \in \mathcal{A}_{\mathrm{Lin}}.$$

More generally, given an R-good filter $\mathcal{F}$, we define $|\cdot|_{\mathcal{F}}$ as

$$|T - a|_{\mathcal{F}} := \inf_{B_q(k) \in \mathcal{F}} |T - a|_{B_q(k)} = \inf_{B_q(k) \in \mathcal{F}} \max\{|k - a|, q\}$$

for any linear polynomial $T - a \in \mathcal{A}_{\mathrm{Lin}}$.

Remark 5.2.17. In classical Berkovich Geometry, given a suitable rigid disc $D_r(k)$ (as defined in Example 5.1.10), one defines [Ber90, Example 1.4.4] a multiplicative seminorm $|\cdot|_{D_r(k)}$ on $\mathcal{A}$ by setting

$$|f|_{D_r(k)} := \sup_i |c_i| r^i \qquad \text{where } f = \sum_{i=0}^{\infty} c_i (T - k)^i. \tag{5.23}$$

Notice that $|T - a|_{D_q(k)}$ is classically equivalent to $|T - a|_{B_q(k)}$ just in case $r = q$. Nonetheless, the definition as stated is problematic in our setting for two reasons:

(a) Recall: a $\sup$ of Dedekinds yields a *lower real*, whereas our K-seminorms are valued in upper reals, and there is no constructive way to turn a lower real L into its corresponding upper real U (see Discussion 5.2.21). On the other hand, $\max$ and $\inf$ *are* well-defined on the upper reals (cf. Definition 2.2.33), which explains the formulation of Construction 5.2.16.

(b) The rigid discs used in [Ber90] are required to have bounded radius $0 < r \leq R$, which ensures the convergence of $|f|_{D_r(k)}$ for any $f \in \mathcal{A}$. On the other hand, the radius of the formal balls $B_q(k)$ have no upper bound. Nonetheless, we avoid convergence issues since $|\cdot|_{B_q(k)}$ is restricted to just the linear polynomials.

Having established Construction 5.2.16, we perform the obligatory check:

Claim 5.2.18. *$|\cdot|_{\mathcal{F}}$ canonically defines a bounded K-seminorm on $\mathcal{A}_{\mathrm{Lin}}$.*

Proof. Following Remark 5.2.17, we extend $|\cdot|_{B_q(k)}$ to

$$\begin{aligned} |\cdot|_{B_q(k)} \colon \mathcal{A}_{\mathrm{Lin}} &\longrightarrow \mathbb{R}_{\geq 0} \\ aT - b &\longmapsto \max\{|ak - b|, |a| \cdot q\} \end{aligned} \tag{5.24}$$

It is easy to see that this recovers the original $|\cdot|_{B_q(k)}$ in Construction 5.2.16 when we restrict to $T - b$. In particular, note that this extension of $|\cdot|_{B_q(k)}$ is canonical in that it is essentially forced upon us by Definition 5.2.2. [Why? Suppose $|\cdot|_x$ is a K-seminorm such that $|T - c|_x =$ "the right Dedekind section of $|T - c|_{B_q(k)}$", for any $c \in K$. Note that K is a geometric field by Hypothesis 5.2.1, and thus it is decidable if $a \in K$ is a unit or $a = 0$. If a is a unit, then semi-multiplicativity then gives

$$|aT - b|_x = |a \cdot (T - b \cdot a^{-1})|_x = |a| \cdot |T - b \cdot a^{-1}|_{B_q(k)} = \max\{|ak - b|, |a| \cdot q\},$$

whereas if $a = 0$, then

$$|aT - b|_x = |b| = \max\{|ak - b|, |a| \cdot q\},$$

coinciding with the (right Dedekind section) of the extended $|\cdot|_{B_q(k)}$.]

As such, given any R-good filter $\mathcal{F}$, we extend $|\cdot|_{\mathcal{F}}$ to a map on $\mathcal{A}_{\mathrm{Lin}}$ by

$$|aT - b|_{\mathcal{F}} := \inf_{B_q(k) \in \mathcal{F}} |aT - b|_{B_q(k)}. \tag{5.25}$$

We claim that Equation (5.25) defines a bounded K-seminorm on $\mathcal{A}_{\mathrm{Lin}}$. This follows from noting:

- $|\cdot|_{\mathcal{F}}$ is valued in the upper reals, since $|f|_{\mathcal{F}}$ takes the infimum of an arbitrary set of Dedekinds.
- To verify $|\cdot|_{\mathcal{F}}$ satsifies the properties listed in Definition 5.2.2(ii), one first verifies their obvious analogues for $|\cdot|_{B_q(k)}$, before observing that they are preserved by taking inf's. To illustrate, let us check the ultrametric inequality. Suppose we have $aT - b, a'T - b' \in \mathcal{A}_{\mathrm{Lin}}$. Then, given any $B_q(k) \in \mathcal{F}$, compute:

$$\begin{aligned} |aT - b + a'T - b'|_{B_q(k)} &= \max\{|(a + a')k - (b + b')|\,,\, |a + a'| \cdot q\} \\ &\leq \max\left\{\max\{|ak - b|, |a'k - b'|\}, \max\{|a| \cdot q, |a'| \cdot q\}\right\} \\ &= \max\{|aT - b|_{B_q(k)}, |a'T - b'|_{B_q(k)}\} \end{aligned} \tag{5.26}$$

  where the middle inequality is by the ultrametric inequality satisfied by the original norm $|\cdot|$ on K. Since this inequality holds for all $B_q(k) \in \mathcal{F}$, taking the infimum on both sides of the Inequality (5.26) over all $B_q(k) \in \mathcal{F}$ gives the desired ultrametric inequality for $|\cdot|_{\mathcal{F}}$

$$|aT - b + a'T - b'|_{\mathcal{F}} \leq \max\{|aT - b|_{\mathcal{F}}\,,\, |a'T - b'|_{\mathcal{F}}\}. \tag{5.27}$$

- For any R-good filter $\mathcal{F}$,

$$|T-a|_{\mathcal{F}} = \inf_{B_q(k)\in\mathcal{F}} \max\{|k-a|, q\} \leq \inf_{B_q(k)\in\mathcal{F}} \max\{|a|, |k|, q\} \leq \max\{|a|, R\}, \tag{5.28}$$

where the final inequality is by R-goodness of $\mathcal{F}$ plus the fact that $k \in K_R$. The same argument extends to show that $|aT-b|_{\mathcal{F}} \leq \max\{|a|R, |b|\}$. Hence, conclude that $|\cdot|_{\mathcal{F}}$ is bounded by the Gauss norm $||\cdot||$.[99]

□

As the reader may have anticipated, the algebraic constructions (i.e. bounded multiplicative seminorms and K-seminorms) and the topological constructions (i.e. the R-good filters) defined in this section have a natural interaction. This is made precise in the following two theorems.

Theorem 5.2.19. *The space of bounded K-seminorms on $\mathcal{A}_{\mathrm{Lin}}$ is equivalent to the space of R-good filters.*

Proof. It suffices to show that Constructions 5.2.14 and 5.2.16 are inverse to each other. This amounts to checking:

First Direction: $|\cdot|_x = |\cdot|_{\mathcal{F}_x}$. Fix a bounded K-seminorm $|\cdot|_x$. By the Preparation Lemma 5.2.5, it suffices to check that $|\cdot|$ and $|\cdot|_{\mathcal{F}_x}$ agree on linear polynomials $T-a$ such that $a \in K_R$.

Suppose $|T-a|_x < q$ for some $q \in Q_+$. Then, $B_q(a) \in \mathcal{F}_x$ by construction. In particular, there exists $q' \in Q_+$ such that

$$|T-a|_x < q' < q \tag{5.29}$$

since $|T-a|_x$ defines a rounded upper real. Further, since

$$|T-a|_{B_{q'}(a)} = \max\{|a-a|, q'\} = q', \tag{5.30}$$

and since Equation (5.29) implies $B_{q'}(a) \in \mathcal{F}_x$, deduce that

$$|T-a|_{\mathcal{F}_x} = \inf_{B_q(k)\in\mathcal{F}_x} |T-a|_{B_q(k)} \leq |T-a|_{B_{q'}(a)} < q,$$

and so

$$|T-a|_{\mathcal{F}_x} \leq |T-a|_x. \tag{5.31}$$

Conversely, suppose $|T-a|_{\mathcal{F}_x} < q$ for some $q \in Q_+$. Again, since $|T-a|_{\mathcal{F}_x}$ defines an upper real, deduce there exists $B_{q'}(k) \in \mathcal{F}_x$ such that

$$|T-a|_{\mathcal{F}_x} \leq |T-a|_{B_{q'}(k)} = \max\{|k-a|, q'\} < q.$$

By definition, $B_{q'}(k) \in \mathcal{F}_x$ implies $|T-k|_x < q'$, and so

$$|T-a|_x = |(T-k)+(k-a)|_x \leq \max\{|T-k|_x, |k-a|\} < q,$$

which in turn implies

$$|T-a|_x \leq |T-a|_{\mathcal{F}_x}. \tag{5.32}$$

Put together, Equations (5.31) and (5.32) give $|\cdot|_x = |\cdot|_{\mathcal{F}_x}$, as claimed.

[99]Technically, $\max\{|a|, R\}$ defines a Dedekind real and not an upper real, but we can always take the corresponding right Dedekind section.

Second Direction: $\mathcal{F} = \mathcal{F}_{|\cdot|_{\mathcal{F}}}$. Fix an R-good filter $\mathcal{F}$. Suppose $B_q(k) \in \mathcal{F}$. Since the radius $\mathsf{rad}_{\mathcal{F}}$ of $\mathcal{F}$ defines an upper real, find a $q' \in Q_+$ such that $B_{q'}(k') \in \mathcal{F}$ and $\mathsf{rad}_{\mathcal{F}} < q' < q$. Without loss of generality, we may assume $k = k'$.[100] Since $|T - k|_{B_{q'}(k)} = q'$, deduce that

$$|T - k|_{\mathcal{F}} \leq |T - k|_{B_{q'}(k)} = q' < q.$$

In particular, this implies $B_q(k) \in \mathcal{F}_{|\cdot|_{\mathcal{F}}}$, and so

$$\mathcal{F} \subset \mathcal{F}_{|\cdot|_{\mathcal{F}}}. \tag{5.33}$$

Conversely, suppose $B_q(k) \in \mathcal{F}_{|\cdot|_{\mathcal{F}}}$. Unpacking definitions, deduce there exists $B_{q'}(j) \in \mathcal{F}$ such that

$$|T - k|_{\mathcal{F}} \leq |T - k|_{B_{q'}(j)} = \max\{|k - j|, q'\} < q. \tag{5.34}$$

To show that $B_{q'}(j) \subseteq B_q(k)$, we need to show that $|k - j| < q$ and $q' \leq q$. But this is clear from Equation (5.34). Since $B_{q'}(j) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed, conclude that $B_q(k) \in \mathcal{F}$, and so

$$\mathcal{F}_{|\cdot|_{\mathcal{F}}} \subset \mathcal{F}. \tag{5.35}$$

Combining Equations (5.33) and (5.35) gives $\mathcal{F} = \mathcal{F}_{|\cdot|_{\mathcal{F}}}$, finishing the proof. □

Theorem E. As our setup, denote:

- $\mathcal{M}(\mathcal{A})$ as the classical Berkovich spectrum (of Dedekind-valued multiplicative seminorms);
- $\overleftarrow{\mathcal{M}(\mathcal{A}_{\text{Lin}})}$ as the space of bounded K-seminorms on $\mathcal{A}_{\text{Lin}}$.

<u>Then</u>,

(i) $\mathcal{M}(\mathcal{A})$ embeds into the space of R-good filters.

(ii) $\mathcal{M}(\mathcal{A})$ is *classically* equivalent to $\overleftarrow{\mathcal{M}(\mathcal{A}_{\text{Lin}})}$. In particular, it is *classically* equivalent to the space of R-good filters.

Proof. It is clear that any $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ restricts to a bounded K-seminorm, which we denote

$$|\cdot|_x\big|_{\mathcal{A}_{\text{Lin}}} : \mathcal{A}_{\text{Lin}} \longrightarrow \overleftarrow{[0,\infty)}. \tag{5.36}$$

(i) then follows immediately from the Preparation Lemma 5.2.5 and Theorem 5.2.19.

For (ii), we start by declaring the following classical assumption, which we will use freely in the remainder of the proof:

($\star$) Any upper real γ_U besides ∞ or $-\infty$ can be canonically associated to a Dedekind real γ whose right Dedekind section corresponds to γ_U. Similarly, any non-infinite lower real can be canonically associated to its corresponding Dedekind real.

[100] Why? Since $\mathcal{F}$ is closed under pairwise intersection, there exists $B_r(j) \subseteq B_q(k) \cap B_{q'}(k')$, and so $B_{q'}(k') = B_{q'}(j) \subseteq B_q(j) = B_q(k)$.

The argument proceeds (again) by way of construction. By Equation (5.36), we already know how to send $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ to a bounded K-seminorm in $\overleftarrow{\mathcal{M}(\mathcal{A}_{\text{Lin}})}$. In the converse direction, notice Theorem 5.2.19 helpfully allows us to explicitly characterise the generic bounded K-seminorm as $|\cdot|_{\mathcal{F}} \in \overleftarrow{\mathcal{M}(\mathcal{A}_{\text{Lin}})}$, where $\mathcal{F}$ is the generic R-good filter. We extend this to a multiplicative seminorm on $\mathcal{A}$ in two steps. First, since any polynomial $f \in K[T]$ can be represented as

$$f = c \cdot \prod_{j=1}^{m} (T - b_j),$$

$|\cdot|_{\mathcal{F}}$ naturally extends to a map on $K[T]$, which (abusing notation) we also represent as:

$$\begin{aligned} |\cdot|_{\mathcal{F}} \colon K[T] &\longrightarrow [0,\infty) \\ f &\longmapsto |c| \cdot \prod_{j=1}^{m} |T - b_j|_{\mathcal{F}}. \end{aligned} \tag{5.37}$$

Second, since any power series $f \in \mathcal{A}$ can be represented as limit of polynomials

$$f = \sum_{i=0}^{\infty} a_i T^i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i T^i,$$

we define the obvious extension of $|\cdot|_{\mathcal{F}}$ to $\mathcal{A}$:

$$\begin{aligned} \widetilde{|\cdot|}_{\mathcal{F}} \colon \mathcal{A} &\longrightarrow [0,\infty) \\ f &\longmapsto \lim_{n\to\infty} |\sum_{i=0}^{n} a_i T^i|_{\mathcal{F}} \end{aligned} \tag{5.38}$$

A couple of orienting remarks:

(a) Notice the implicit use of Assumption ($\star$) throughout. The original definition of $|\cdot|_{\mathcal{F}}$ was valued in the upper reals, but we extended it to a Dedekind-valued map on $K[T]$ in Equation (5.37). Similarly, although the supremum of a set of Dedekinds defines a lower real (cf. Definition 2.2.33), we have defined $\widetilde{|\cdot|}_{\mathcal{F}}$ in Equation (5.38) as being Dedekind-valued.

(b) Suppose the construction $\widetilde{|\cdot|}_{\mathcal{F}}$ defines a bounded multiplicative seminorm on $\mathcal{A}$. Since Assumption ($\star$) allows us to treat K-seminorms as if they were valued in Dedekinds, it becomes a straightforward check to show that the constructions $|\cdot|_x\big|_{\mathcal{A}_{\text{Lin}}}$ and $|\cdot|_{\mathcal{F}}$ define continuous maps which are inverse to each other. In particular, observe that the identity

$$|\cdot|_{\mathcal{F}} = \widetilde{|\cdot|}_{\mathcal{F}}\big|_{\mathcal{A}_{\text{Lin}}} \tag{5.39}$$

is immediate by construction, whereas the identity

$$|\cdot|_x = \widetilde{|\cdot|_x\big|_{\mathcal{A}_{\text{Lin}}}} \tag{5.40}$$

follows from the Preparation Lemma 5.2.5 and checking the values on the linear polynomials.

As such, in order to prove the (classical) equivalence stated in the Theorem, it remains to verify that $\widetilde{|\cdot|}_{\mathcal{F}}$ is in fact a bounded multiplicative seminorm on $\mathcal{A}$. The check relies on standard arguments from non-Archimedean analysis; for details, see Appendix B. □

We conclude with some discussions on various aspects of the proof.

Discussion 5.2.20. Let us sketch the original argument in [Ber90, Example 1.4.4]:

- First, given a rigid disc $D_r(k)$ such that $0 < r \leq R$, define a multiplicative seminorm $|\cdot|_{D_r(k)}$ on $\mathcal{A}$.
- Next, given any nested sequence of discs $\mathsf{D} := D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \dots$, define the multiplicative seminorm $|\cdot|_{\mathsf{D}} := \inf_{\mathsf{D}} |\cdot|_{D_{r_i}(k_i)}$.
- Finally, given $|\cdot|_x \in \mathcal{M}(\mathcal{A})$, define the nested sequence $\mathsf{D}_x := \{D_{|T-k|_x}(k) \mid k \in K \text{ and } |k| \leq R\}$. Check that $|\cdot|_x$ and $|\cdot|_{\mathsf{D}_x}$ agree on linear polynomials, and conclude $|\cdot|_x = |\cdot|_{\mathsf{D}_x}$.

The parallels with the proof for Theorem 5.2.19 are clear. However, the argument must be adjusted and finitised appropriately in order to work in our context. Some important differences:

(i) *On "rational" discs.* Both R-good filters and the nested sequence of discs D_x give rise to approximation arguments, but their approximants differ in important ways. In particular, whereas the radius of a formal ball $B_q(k)$ is rational in the usual sense that $q \in Q_+$ is a positive rational number, the radius of the rigid disc $D_r(k) \in \mathsf{D}_x$ is called "rational" in the different sense that:

$$r \in \Gamma := \{|k| \in [0, \infty) \mid k \in K\},$$

i.e. r belongs to the value group Γ of K. The suggestive terminology ("rational") indicates an analogy between Q_+ and Γ, but it is important to remember that they are not the same — particularly when K is trivially-valued.

(ii) *On K-seminorms.* Whereas the original argument starts by defining a multiplicative seminorm on $\mathcal{A}$, before restricting it to the linear polynomials to perform certain checks, we instead defined a new algebraic structure (which we called K-seminorms) on the space of linear polynomials $\mathcal{A}_{\mathrm{Lin}}$.

(iii) *On the use of filters.* Note that while Berkovich's original argument shows that every $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ corresponds to a nested descending sequence of discs, this representation is not unique. In particular, two different sequences of discs may define the same multiplicative seminorm on $\mathcal{A}$. We resolve this issue by appealing to the more natural language of filters, which allows us to obtain a representation result: every $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ is canonically associated to an R-good filter $\mathcal{F}_x$. This should be compared with our Theorem C and its improvement of the original Ostrowski's Theorem.

(iv) *On the use of formal balls.* As already pointed out in Discussion 5.2.9, our result holds for both trivially and non-trivially valued K. This is in contrast to Berkovich's original argument, which only works for non-trivially valued K.

Items (i) and (ii) reflect our decision to work with the upper reals as opposed to the Dedekinds, and strike a careful balance: whilst the upper reals are particularly suited to analysing the filters of formal balls (cf. Observation 5.2.13), they also impose strong restrictions on the algebra which requires deliberate adjustments (cf. Remark 5.2.17). Items (iii) and (iv) give evidence that filters (as opposed to nested sequences of discs) and formal balls (as opposed to the classical rigid discs) are the correct language for studying $\mathcal{M}(\mathcal{A})$.

Discussion 5.2.21. Recall that Theorem E is classical because our proof relies on Assumption ($\star$), which informally says: any bounded one-sided real can be canonically associated to a Dedekind real. Some natural questions (and answers):

(i) *Why is Assumption* $(\star)$ *classical?* Consider the obvious argument: given an upper real R defined as the set of rationals strictly greater than 1, we define a lower real L as the set of rationals strictly less than 1. Hence, we conclude that (L, R) is the Dedekind real canonically associated to R.

This argument will strike most as reasonable, so where does it fail constructively? Discussion 2.2.22 reminds us that an upper real is blind to the rationals less than itself, which suggests potential issues when trying to prove that (L, R) satisfies separatedness. Stated more precisely, it turns out if any upper real R can be associated to a lower real L such that (L, R) defines a Dedekind real, then this implies that every proposition p is decidable — which holds classically, but *not* constructively (cf. Discussion 2.1.14).

To see why, let p be a proposition, and define the subset of rationals:

$$R := \{q \in \mathbb{Q} | \text{ either "}q > 1\text{" or "}p \text{ holds and } q > 0\text{"}\}$$

In other words, R as defined is a kind of schizophrenic upper real: since p holds iff $R < 1$, R may define the upper real 1 or the upper real 0, depending on the truth value of p. Now, given some lower real L, we define a new proposition p' iff $\frac{1}{2} < L$. Notice that if (L, R) is a Dedekind, then:

(a) $\top \to p \vee p'$;
[Why? Locatedness of (L, R) gives $\frac{1}{2} < L \vee R < 1$.]
(b) $p \wedge p' \to \bot$.
[Why? Since $p \to R < \frac{1}{2}$, this gives $p \wedge p' \to (\frac{1}{2} < L) \wedge (R < \frac{1}{2})$, contradicting separatedness of (L, R).]

In other words, p' is a Boolean complement of p, as claimed.

(ii) *Is Assumption* $(\star)$ *necessary?* If Assumption $(\star)$ is only used because the statement of Theorem E involves comparing *upper-valued* K-seminorms with *Dedekind-valued* multiplicative seminorms, why not prove an alternative result comparing upper-valued multiplicative seminorms on $\mathcal{A}$ with K-seminorms on $\mathcal{A}_{\mathrm{Lin}}$?

The short answer: because we cannot. In particular, recall that $\mathcal{A}$ is a ring with multiplicative units, which the multiplicative seminorms are expected to respect. Hence, the same reasoning as in Observation 4.1.1 forces any multiplicative seminorm on $\mathcal{A}$ to be Dedekind-valued. Notice, however, the same problem does not arise with the K-seminorms since the space of linear polynomials $\mathcal{A}_{\mathrm{Lin}}$ is *not* equipped with a multiplicative structure (cf. Remark 5.2.3).

Finally, for the constructivist reader's convenience, we sort out and summarise the classical vs. constructive/geometric aspects of our result.

Summary 5.2.22 (Classical vs. Geometric Assumptions)**.**

(i) Hypothesis 5.2.1 makes two assumptions on our base field K: one, that we can take its underlying set, and two that it is geometric (i.e. it is decidable if $k \in K$ is a unit or $k = 0$). Taken at face value, these are unwelcome restrictions — if one wished to be consistently geometric throughout, many natural fields of interest, e.g. the p-adic complex numbers $\mathbb{C}_p$, would be excluded. However, if the reader is willing to work classically, then this is no longer a problem (cf. also Remark 4.1.7).

(ii) Although the polynomial ring $K[T]$ and $\mathcal{A}_{\mathrm{Lin}}$ can be defined geometrically, it is presently unclear how to do the same for the convergent power series ring $K\{R^{-1}T\}$ — in particular, how to formulate the condition "$f \in \mathcal{A}$ converges on a radius R" geometrically. [101]

(iii) So long as Hypothesis 5.2.1 holds (in fact, we can eliminate the assumption that K is complete here), Theorem 5.2.19 is a fully geometric result. On the other hand, Theorem E is a classical result due to our use of Assumption ($\star$) (cf. Discussion 5.2.21).

5.2.2 Applications to the Trivial Case. As a slick application of Theorem E, we recover the familiar characterisations of $\mathcal{M}(\mathcal{A})$ when K is trivially valued (see Example 5.1.10). What's new here? For one, the proofs given here appear quite different from the standard arguments [Jonb], and are also shorter. More fundamentally, they give a very interesting indication of how Berkovich's characterisation of $\mathcal{M}(\mathcal{A})$ (via nested sequences of discs) is in fact more robust than previously thought.

Example 5.2.23 (Case: $R < 1$)**.** If $R < 1$ and K is trivially valued, then notice $K_R = \{0\}$. This indicates that the R-good filters are entirely determined by their radii, and so the space of R-good filters is equivalent to $\overleftarrow{[0, R]}$ (see Observation 5.2.13). Hence, applying Theorem E, we get that $\mathcal{M}(\mathcal{A})$ is classically equivalent to $\overleftarrow{[0, R]}$, essentially for free.

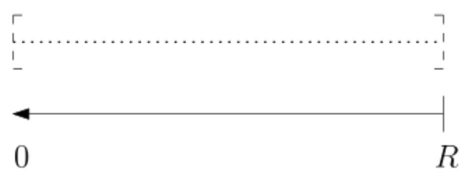


Figure 5.3: $\mathcal{M}(\mathcal{A})$, when K is trivially valued and $R < 1$

Example 5.2.24 (Case: $R \geq 1$)**.** If $R \geq 1$ and K is trivially valued, then notice $K_R = K$. Consider the following two subcases:

Subcase 1: *$\mathcal{F}$ is an R-good filter with radius* $\mathsf{rad}_{\mathcal{F}} \geq 1$. In which case, $B_q(k) \in \mathcal{F}$ for any $k \in K$ and any $q > \mathsf{rad}_{\mathcal{F}}$. [Why? Since $\mathcal{F}$ is R-good, there must exist some $B_q(k') \in \mathcal{F}$ for any $q > \mathsf{rad}_{\mathcal{F}}$. Since K is trivially valued, we get $|k - k'| \leq 1 \leq \mathsf{rad}_{\mathcal{F}} < q$ for any $k \in K$, which by Observation 5.2.8 implies $B_q(k') = B_q(k)$.]

Hence, an R-good filter (with radius $\mathsf{rad}_{\mathcal{F}} \geq 1$) is entirely determined by its radius, and so the space of such R-good filters form an interval $\overleftarrow{[1, R]}$.

Subcase 2: *$\mathcal{F}$ is an R-good filter with radius* $\mathsf{rad}_{\mathcal{F}} < 1$. In which case:

(a) If $B_q(k)$ such that $q > 1$, then $B_q(k) \in \mathcal{F}$.
[Why? Since $\mathsf{rad}_{\mathcal{F}} < 1$, there must exist $B_{q'}(k') \in \mathcal{F}$ such that $q' \leq 1 < q$. Since K is trivially valued, deduce that $B_{q'}(k') \subseteq B_q(k)$ and so $B_q(k) \in \mathcal{F}$ as $\mathcal{F}$ is upward closed.]

[101] We have yet to check the details, but here's a plausible constructive workaround for items (i) and (ii): view $K\{R^{-1}T\}$ as a completion of some polynomial ring $K_0[T]$ with respect to the Gauss norm $||\cdot||$, where K_0 is a geometric field that can be completed with respect to the non-Archimedean norm $|\cdot|$ to get K.

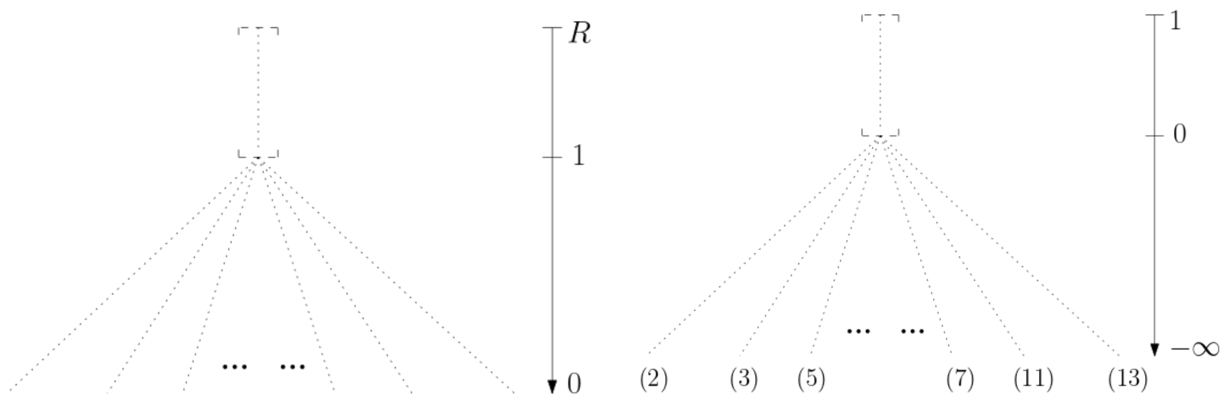
(b) If $B_q(0) \in \mathcal{F}$ and $q \leq 1$, then $k = 0$ for any $B_{q'}(k) \in \mathcal{F}$ such that $q' \leq 1$.

[Why? Take the intersection of $B_{q'}(k), B_q(0) \in \mathcal{F}$ to get $B_{q''}(j) \in \mathcal{F}$. Since $B_{q''}(j) \subseteq B_q(0)$, this forces $j = 0$ since otherwise $1 = |j - 0| < q \leq 1$, contradiction. Further, since this implies $B_{q''}(0) \subseteq B_{q'}(k)$, deduce that $k = 0$ for the same reason.]

(c) If $B_q(k), B_{q'}(k') \in \mathcal{F}$ such that $q, q' \leq 1$ and k, k' non-zero, then $k = k'$.

[Why? Take the intersection of $B_q(k), B_{q'}(k') \in \mathcal{F}$ to get $B_{q''}(k'') \in \mathcal{F}$. Notice that k'' is non-zero, since otherwise the same argument as in item (b) gives $k = 0 = k'$, contradiction. Further, since $B_{q''}(k'') \subseteq B_q(k)$, this forces $k'' = k$ since otherwise $1 = |k'' - k| < q \leq 1$, contradiction. The same argument shows that $k'' = k'$, and so we conclude $k' = k$.]

Summarising, an R-good filter $\mathcal{F}$ with radius $\mathtt{rad}_{\mathcal{F}} \geq 1$ is entirely determined by its radius (Subcase 1), whereas an R-good filter $\mathcal{F}$ with radius $\mathtt{rad}_{\mathcal{F}} < 1$ is determined by its radius plus its unique choice of $k \in K$ (Subcase 2). Applying Theorem E once more, this gives the following diagram on the left:



(a) $\mathcal{M}(\mathcal{A})$, when K is trivially valued and $R \geq 1$

(b) $[\overleftarrow{av}]$

Figure 5.4

This should of course be compared with the space $[\overleftarrow{av}] \cong \mathfrak{P}_\Lambda$ (cf. Theorem C), pictured on the right. In particular, notice the key branching points of the spaces are identical up to a $\log_R(\text{—})$ transformation.

Remark 5.2.25. Of course, Examples 5.2.23 and 5.2.24 present $\mathcal{M}(\mathcal{A})$ using upper reals instead of Dedekinds (in particular, they are not Hausdorff), but this can be resolved by applying Assumption ($\star$) once more. Notice also the case-split between zero and non-zero elements of K in Subcase 2 of Example 5.2.24 implicitly uses Hypothesis 5.2.1 that K is a geometric field.

5.3 An Algebraic Fork in the Road

Let us review our work. In principle, the extension of Berkovich's original result to Theorem E could have been discovered much earlier. And yet it was not — to our knowledge, the idea that one could modify the language of rigid discs to classify the points of $\mathcal{M}(K\{R^{-1}T\})$ *without* requiring K to be non-trivially valued was not suspected by the experts.[102] The reason for this seems to be that Theorem E, both in its

[102] Difficult, of course, to properly gauge what the experts may or may not have suspected, but this may be inferred from how the literature emphasises the necessity of being non-trivially normed. For instance, in Jonsson's lecture notes on Berkovich's

formulation and proof, belongs to the point-free perspective in an essential way. Of course, one certainly does not need to be a topos theorist in order to e.g. understand what a formal ball is, but there are specific intuitions from the point-free perspective that guided us to our result:

(a) The topos theorist is trained to recognise how the same idea may be expressed in different settings, and to ask about the connections.[103] In particular, recall: any essentially propositional theory corresponds to a space of completely prime filters. Given that the theory of multiplicative seminorms on $\mathbb{Z}$ is essentially propositional, plus the obvious parallels between $\mathcal{M}(\mathbb{Z})$ and $\mathcal{M}(K\{R^{-1}T\})$ when K is trivially normed and $R \geq 1$ (cf. Example 5.2.24), the topos theorist may guess that $\mathcal{M}(K\{R^{-1}T\})$ can also be described via completely prime filters.
(b) Once we defined the formal ball $B_q(k)$ and the correct inclusion relation $B_{q'}(k') \subseteq B_q(k)$, the rest of the argument essentially fell into place. But notice: the decision to use formal balls (as opposed to the classical rigid discs) reflects the point-free perspective that it is the *opens* that are the basic units for defining a space, and **not** the underlying *set of points*. This perspective was discussed extensively in Chapter 2.

Though a relatively simple result, the surprising aspects of Theorem E hints at the potential of using point-free ideas to clarify and investigate foundational issues in non-Archimedean geometry. Motivated by this, we conclude with a list of inter-related problems, geared towards testing this idea.

5.3.1 Trivially vs. Non-Trivially valued Fields. First, an obvious piece of mathematical due diligence. Many results in Berkovich geometry are sensitive to the case-split between trivially vs. non-trivially valued (non-Archimedean) fields. This motivates the following general exercise:

Problem 8. Pick an interesting result in Berkovich geometry that appears to rely on the base field K being non-trivially valued. Examine why. In particular, just as in Theorem E, can we eliminate this hypothesis by applying point-free techniques? If yes, what applications does this generalised result give us?

Discussion 5.3.1. Here's one place to start looking. In non-Archimedean geometry, there's a common strategy for proving results on a general closed disc D: first prove the result for *rational closed discs*, before extending the result to D by expressing it as a nested union of rational closed discs (see e.g. [Ben03]). This strategy obviously breaks down when K is trivially valued, but the language of upper reals may offer a workaround (cf. Discussion 5.2.20).

5.3.2 Overconvergent Lattices and Rigid Geometry. We now discuss a more solid lead. In unpublished work of Dudzik [Dud12] as well as Baker's Berkeley Lecture Notes [Bak12], the following notion was defined:

Definition 5.3.2. Consider a lower-bound distributive lattice L, with finite $\wedge$ and $\vee$ and a minimal element denoted $\perp$.

(i) For $x, x' \in L$, we say x *is inner in* x', written $x \triangleleft x'$, if for all $z \geq x'$, there exists w with $x \wedge w = \perp$ and $x' \vee w = z$.

classification of the points of the Berkovich Affine line $\mathbb{A}^1_{\mathrm{Berk}}$ over K, he remarks: "The second assumption [that K is non-trivially valued] is necessary [...] if the norm on K is trivial, then there are too few discs in $\mathbb{A}^1$." [Jon15, Proof of Theorem 3.10]

[103] For the insider: the topos theorist knows that there are many sketches of the same elephant [Joh02a; Joh02b].

(ii) We call L an *overconvergent lattice* if for all $x, y \in L$ where $x \wedge y = \bot$, there exists $x' \in L$ such that $x \lhd x'$ and $x' \wedge y = \bot$.

It was then showed that classical ideas of rigid analytic geometry have a tight connection to these so-called overconvergent lattices. We summarise some of their key results in the following:

Theorem 5.3.3. *As our setup,*

- *Let $\mathcal{A}$ be a (strict) affinoid algebra over a suitable*[104] *field K;*
- *Let $X = \mathrm{Sp}(\mathcal{A})$, i.e. the set of maximal ideals of $\mathcal{A}$ equipped with the canonical topology;*
- *Let L be the lattice of special subdomains of X;*
- *Let $\mathsf{P}(L)$ be the set of prime filters and $\mathsf{M}(L)$ the set of maximal filters.*

Then:

(i) The lattice L is overconvergent and its elements form a neighbourhood base.

(ii) There exists a canonical surjective map $\mathsf{P}(L) \to \mathsf{M}(L)$ sending a prime filter to the unique maximal filter containing it. When equipped with the quotient topology, $\mathsf{M}(L)$ is equivalent to the Berkovich analytification X^{an}.

(iii) $\mathsf{P}(L)$ is equivalent to Huber's adic space of continuous semivaluations on $\mathcal{A}$.

Closer examination of the mechanics underlying the proof of Theorem 5.3.3 seems warranted. Although the theorem is essentially a reworking of classical facts about rigid geometry [FP81; PS95], it uses some key notions from lattice theory in a very interesting way. For instance, locale theorists may recognise the family resemblance between overconvergent lattices and *normal* lattices, which suggests that the hypothesis of overconvergence was chosen precisely to guarantee that each prime filter is contained in a unique maximal filter[105] (see item (ii) of the Theorem). Some natural test problems:

Problem 9. Develop and make precise the connections between overconvergent lattices and normal lattices. For instance, quite intriguingly, normal lattices have independently shown up in topos-theoretic approaches to quantum theory (see e.g. [SVW14]) — are there productive parallels that can be drawn between this and the role of overconvergent latices in non-Archimedean geometry?

Problem 10. In his note [Dud12], Dudzik left unfinished the problem of applying overconvergent lattices to the classification of the points of $\mathbb{A}^1_{\mathrm{Berk}}$. A good exercise: finish this. In particular, our proof of Theorem E should be relevant. However, what do R-good filters have to do with the filters of overconvergent lattices?

Problem 11. A deeper challenge: inspired by Hrushovski-Loeser's work [HL16], can we use overconvergent lattices to prove tameness results about Berkovich spaces?

[104] By which we mean: complete, non-trivially valued and non-Archimedean. Compare this with [Ber90, Remark 2.5.21], which we briefly discussed at the start of Section 5.2.

[105] Normality and overconvergence appear to be essentially dual notions. Recall: for a (bounded) distributive lattice L and $a, a' \in L$, we say a' *is well inside* a, written $a' \leqslant a$ if there exists y such that $a \vee y = \top$ and $a' \wedge y = \bot$. Such a lattice L is said to be *normal* if whenever $a \vee b = \top$, there exists $a' \leqslant a$ with $a' \vee b = \top$. In particular, a (bounded) distributive lattice L is normal iff each prime ideal in L is contained in a unique maximal ideal [Joh82, §3.6-3.7]. Of course, this is a characterisation of normality for *bounded* distributive lattices, but a similar (dual) result appears under the weaker hypothesis of only being lower-bounded [Cor72].

5.3.3 Model-theoretic vs. Point-free perspectives. Interspersed throughout this chapter were various mentions of Hrushovski-Loeser's groundbreaking work [HL16], which applied model-theoretic tools to Berkovich geometry. For the topos theorist, a natural question is the following:

Problem 12. Leveraging point-free techniques, simplify and/or extend the framework of Hrushovski-Loeser spaces.

Discussion 5.3.4. A good place to start: where do overconvergent lattices feature in their framework? In addition, their technology of so-called pro-definable sets bears a strong resemblance to R-structures and rounded ideal completions (which featured in our use of upper reals). It would be interesting to see if this connection can be developed to indicate some natural simplifications. (Discussion 5.3.1 may be relevant.)

Discussion 5.3.5. As already noted in Discussion 2.2.20, the topos theorist and model theorist share rather different understandings of what constitutes logical complexity. For the model theorist, the presence of strict order in a theory is a sign of complexity [more precisely, that a theory is unstable] — examples include the theory of dense linear orders DLO, and (more relevantly) the theory of algebraically closed valued fields ACVF. In particular, many standard model-theoretic tools only work well for stable theories. As such, a great deal of work has to be done in order to extend these tools to the unstable setting of valued fields, before we can apply them to e.g. obtain tameness results in Berkovich geometry [HMH08; HL16].

On the other hand, strict order is *not* a sign of complexity for the topos theorist — the theory of the rationals and the Dedekinds both have strict order and are quite well-behaved [more precisely, they are essentially propositional]. One is therefore led to ask: are model-theoretic notions such as stable/meta-stable/unstable genuinely essential to our understanding of non-Archimedean geometry? Or is it sufficient to simply reformulate the geometry using point-free techniques (cf. Problem 11)?

Discussion 5.3.6. There have been recent ambitions to extend Hrushovski-Loeser's work to the setting of adic spaces (see e.g. [KY21]). However, Theorem 5.3.3 alerts us to an obvious obstacle. At least in the setting of K-affinoid algebras, the distinction between Berkovich spaces vs. adic spaces is analogous to the distinction between maximal filters vs. prime filters. Recalling Discussion 2.2.19, this raises questions about whether the model theorist's language of (complete) types is suited for analysing adic spaces since these should be properly understood as ultrafilters (and not prime filters). Our knowledge of localic spaces (especially their points) combined with Theorem 5.3.3 lead us to wonder if point-free methods are a more natural choice, although some care should be taken regarding the distinction between prime vs. completely prime filters.

Chapter 6

Topos of Places of $\mathbb{Q}$

In this chapter, building on work in Chapters 3 and 4, we apply descent techniques to study the places of $\mathbb{Q}$. This is where our earlier adherence to geometricity begins to pay off. Since point-free spaces can be dually regarded as toposes, committing to working geometrically now allows us to bring a deep collection of structure theorems from topos theory to bear on our investigations. The organising principle of the chapter is the following key question: considering the exponentiation action on the space of absolute values of $\mathbb{Q}$, what does its quotient space look like?

The picture we obtain is a surprising one. Although all non-Archimedean places of $\mathbb{Q}$ are singletons (Theorem F), this is emphatically not the case for the Archimedean place; instead, we found that it corresponds the space $\overleftarrow{[0,1]}$ of upper reals (Theorem G), a kind of blurred unit interval. This overturns a longstanding classical assumption in number theory that we first saw in Chapter 1, raising many urgent questions about the implications.

At this critical juncture, our current understanding is still incomplete, but some partial answers can be found in the final two sections. Setting aside the number-theoretic implications, Section 6.4 gives a purely topos-theoretic account of the differences between the Archimedean and non-Archimedean places of $\mathbb{Q}$: in our language, the topos corresponding to the Archimedean place witnesses non-trivial forking in its sheaves whereas the topos corresponding to a non-Archimedean place eliminates all forking structure. Section 6.5 brings into focus a key theme that has implicitly guided our investigations thus far: namely, the interactions between the connected vs the disconnected. This insight has a surprisingly far reach. As we shall discuss, thinking carefully about its placement in both topos theory and number theory forces a careful re-examination of definitions and our assumptions behind them, often from challenging angles.

6.1 Preliminaries on Descent

This section collects some standard results and folklore on descent in topos theory. Much of the material is standard (see, e.g. [JT84; Moe88]), except that we shall rework the key constructions in the language of point-free topology. We shall also be interested in both standard descent and lax descent, the latter of which appears to be less studied (but see [Pit86; Joh02b; Bun15]).

6.1.1 Standard Descent. Consider a discrete set X and a discrete group G acting upon it. One then typically defines the *quotient of X by the G-action* as the set of G-orbits on X, i.e.

$$X/G := \{\mathrm{Orb}(x) \,|\, x \in X\}\,,$$

where $\mathrm{Orb}(x) := \{y \mid \exists g \in G \text{ s.t. } g \cdot x = y\}$. Of course, an immediate translation of this construction to the point-free setting is obstructed by its point-set formulation — not only does X/G start with a *set of elements*, but it gives no information on how topology should interact with the quotienting since X and G are both discrete. Additional work is therefore needed to figure out the correct analogue of a quotient construction in the setting of toposes, which we outline below.

Convention 6.1.1. In this chapter, we shall use caligraphic letters $\mathcal{E}, \mathcal{E}' \ldots$ to denote generalised spaces, and $\mathcal{S}\mathcal{E}, \mathcal{S}\mathcal{E}' \ldots$ to denote their corresponding categories of sheaves (cf. Remark 2.1.22). If working localically, then we use standard capital letters $X, Y \ldots$ to denote the localic spaces and $\mathcal{S}X, \mathcal{S}Y$ to denote their corresponding categories of sheaves.

Definition 6.1.2. A *2-truncated simplicial space* $\mathcal{E}_\bullet$ is a diagram of spaces of the form:

$$\mathcal{E}_2 \;\overset{\widehat{d}_0,\ \widehat{d}_1,\ \widehat{d}_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}\; \mathcal{E}_1 \;\overset{d_0,\ s_0,\ d_1}{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}}\; \mathcal{E}_0 \tag{6.1}$$

Since the spaces in Diagram (6.1) need not be localic, we shall only require the maps to commute up to isomorphism. More explicitly, this means that the following simplicial identities hold:

$$d_0 \circ \widehat{d}_1 \cong d_0 \circ \widehat{d}_0, \qquad d_0 \circ \widehat{d}_2 \cong d_1 \circ \widehat{d}_0, \qquad d_1 \circ \widehat{d}_2 \cong d_1 \circ \widehat{d}_1. \tag{6.2}$$

$$d_0 \circ s_0 \cong \mathrm{id}, \qquad\qquad d_1 \circ s_0 \cong \mathrm{id}. \tag{6.3}$$

In the case where $\mathcal{E}_\bullet$ is a diagram in Loc, we shall ask that Equations (6.2) and (6.3) hold up to equality (since points of localic spaces do not possess automorphisms).

Construction 6.1.3 (Standard Descent)**.** Given any 2-truncated simplicial space $\mathcal{E}_\bullet$, we can construct its *universal descent cocone* in two main steps.

- **First**, taking the sheaves on the spaces of Diagram (6.1), we obtain the following diagram:

$$\mathcal{S}\mathcal{E}_2 \;\overset{\widehat{d}_0^*,\ \widehat{d}_1^*,\ \widehat{d}_2^*}{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}}\; \mathcal{S}\mathcal{E}_1 \;\overset{d_0^*,\ s_0^*,\ d_1^*}{\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}}\; \mathcal{S}\mathcal{E}_0 \tag{6.4}$$

  Note the reversal of arrows: since the objects of Diagram (6.4) now denote categories of sheaves (as opposed to spaces), the correct arrows between them are the inverse image functors [corresponding to the maps of Diagram (6.1)].
- **Next**, we define the following two pieces of data:

  1. The *descent category* Des, which is a category possessing

     Objects: (F, θ), where F is an object of $\mathcal{S}\mathcal{E}_0$ and $\theta : d_0^*(F) \xrightarrow{\sim} d_1^*(F)$, also known as the *descent data*, is an isomorphism satisfying the identities

     (i) *(Unit Condition)* $s_0^*(\theta) \cong \mathrm{id}$;
     (ii) *(Cocycle Condition)* $\widehat{d}_0^*(\theta) \circ \widehat{d}_2^*(\theta) \cong \widehat{d}_1^*(\theta)$.

Morphisms: $\alpha\colon (F,\theta) \to (F',\xi)$, where $u\colon F \to F'$ is a morphism in $\mathcal{S}\mathcal{E}_0$ that is compatible with the descent data, i.e. $d_1^*(u) \circ \theta = \xi \circ d_0^*(u)$.

2. p^* is the forgetful functor

$$\begin{aligned} p^*\colon \mathrm{Des} &\longrightarrow \mathcal{S}\mathcal{E}_0 \\ (F,\theta) &\longmapsto F \end{aligned}$$

The proof that Des is a topos is [Moe88, §3.3]. In particular, this means:

- Des may be regarded as a point-free space, which we shall denote as $[\mathbb{T}_{\mathrm{Des}}]$;
- p^* may be regarded as an inverse image functor between two toposes.

As such, the universal descent cocone can thus be represented as the following diagram of spaces:

$$p\colon \mathcal{E}_\bullet \longrightarrow [\mathbb{T}_{\mathrm{Des}}], \tag{6.5}$$

where $p \circ d_0 \cong p \circ d_1$.

Two main examples of Construction 6.1.3 will be important for this chapter.

Example 6.1.4 (Descent Topos of a Groupoid)**.** Let $G := (G_0, G_1)$ be a groupoid in Loc

$$G_1 \times_{G_0} G_1 \ \begin{smallmatrix}\xrightarrow{\pi_0}\\ \xrightarrow{\mathfrak{m}}\\ \xrightarrow{\pi_1}\end{smallmatrix}\ G_1 \ \begin{smallmatrix}\xrightarrow{d_0}\\ \xleftarrow{s}\\ \xrightarrow{d_1}\end{smallmatrix}\ G_0 \tag{6.6}$$

whereby:

- G_0 denotes the space of objects and G_1 the space of arrows.
- $d_0, d_1\colon G_1 \to G_0$ are the domain and codomain maps
- $s\colon G_0 \to G_1$ is the unit map
- $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \xrightarrow{d_0} G_0 \xleftarrow{d_1} G_1$, and $\mathfrak{m}\colon G_1 \times_{G_0} G_1 \to G_1$ is the multiplication or composition map, i.e. $\mathfrak{m}(g,f) = g \circ f$.

It is clear that Diagram (6.6) is a 2-truncated simplicial space. Hence, following Construction 6.1.3 we can construct its universal descent cocone, with descent category Des, which we represent as the following diagram of spaces

$$G \longrightarrow\!\!\!\!\!\rightarrow [\mathbb{T}_{\mathrm{Des}}]. \tag{6.7}$$

By construction, the objects of Des are the sheaves $F \in \mathcal{S}G_0$ equipped with an isomorphism $\theta\colon d_0^*(F) \xrightarrow{\sim} d_1^*(F)$ (respecting the descent conditions). As such, $[\mathbb{T}_{\mathrm{Des}}]$ can be morally regarded as the space G_0 quotiented by the action of G_1.

Example 6.1.5. Let $\Phi\colon \mathcal{E}' \to \mathcal{E}$ be a map of spaces. Via iterated pullbacks, construct the diagram

$$\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \ \begin{smallmatrix}\xrightarrow{\pi_{02}}\\ \xrightarrow{\pi_{01}}\\ \xrightarrow{\pi_{12}}\end{smallmatrix}\ \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \ \begin{smallmatrix}\xrightarrow{\pi_1}\\ \xleftarrow{\Delta}\\ \xrightarrow{\pi_0}\end{smallmatrix}\ \mathcal{E}' \xrightarrow{\ \Phi\ } \mathcal{E} \tag{6.8}$$

where

- $\Delta\colon \mathcal{E}' \to \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$ is the diagonal map sending $x \mapsto (x, x)$;
- π_{01}, π_{02} and π_{12} are the projection maps such that, e.g. π_{01} maps $(x_0, x_1, x_2) \mapsto (x_0, x_1)$.

It is clear that Diagram (6.8) is a 2-truncated simplicial space (in fact, a groupoid). Hence, following Construction 6.1.3 we can construct the descent category, which we denote as $\mathrm{Des}(\Phi)$, whose objects are pairs (F, θ) such that

- $F \in \mathcal{S}\mathcal{E}'$; and
- $\theta\colon \pi_0^*(F) \to \pi_1^*(F)$ is a morphism[106] in $\mathcal{S}(\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}')$ satisfying:

$$\begin{aligned} &\Delta^*(\theta) = \mathrm{id}; \\ &\pi_{02}^*(\theta) = \pi_{12}^*(\theta) \circ \pi_{01}^*(\theta). \end{aligned}$$

The analogy between Example 6.1.4 and the earlier example of X/G is clear, and illustrates how descent can be regarded as a quotient construction. Example 6.1.5 emphasises something different. Since $\mathrm{Des}(\Phi)$ can be constructed from any map $\Phi\colon \mathcal{E}' \to \mathcal{E}$ of spaces, the descent construction links topological properties of the map Φ to various structural consequences between the corresponding toposes $\mathrm{Des}(\Phi)$, $\mathcal{S}\mathcal{E}$ and $\mathcal{S}\mathcal{E}'$. The next series of definitions and results develop this remark.

Definition 6.1.6. We continue with the setup of Example 6.1.5. Note that the associated inverse image functor of Φ, i.e.

$$\Phi^*\colon \mathcal{S}\mathcal{E} \to \mathcal{S}\mathcal{E}', \tag{6.9}$$

induces a functor $\chi\colon \mathcal{S}\mathcal{E} \to \mathrm{Des}(\Phi)$ such that

$$\begin{array}{ccc} \mathcal{S}\mathcal{E} & \xrightarrow{\quad\chi\quad} & \mathrm{Des}(\Phi) \\ & {\scriptstyle\Phi^*}\searrow \quad \swarrow{\scriptstyle U} & \\ & \mathcal{S}\mathcal{E}' & \end{array} \tag{6.10}$$

commutes, where U is the forgetful functor $U(F, \theta) = F$. Depending on the context, we call Φ or Φ^* an *effective descent morphism* when χ is an equivalence.

This definition sets up a key result in Joyal-Tierney's groundbreaking monograph [JT84]:

Theorem 6.1.7 ([JT84, Theorem VIII.2.1]). *Open surjections of toposes are effective descent morphisms.*

The general definition of an open surjection of arbitrary toposes is technical and will not be needed here (details can be found in [Joh02b, C3.1]). For our purposes, it suffices to understand the corresponding notions on the level of localic spaces.

Definition 6.1.8 (see [Joh02b, Lemma C1.5.3]). Let $f\colon X \to X'$ be a map of localic spaces, and consider its corresponding frame homomorphism on its frame of opens $f^{-1}\colon \Omega_{X'} \to \Omega_X$. We call f …

(i) … a *surjection* if f^{-1} is 1-to-1.
(ii) … an *open map* if either one of the equivalent conditions hold:

[106] In fact, an isomorphism, where the inverse is given by swapping the projections — see e.g. Footnote 110.

(a) f^{-1} has a left adjoint satisfying the Frobenius reciprocity condition

$$f_!(U \wedge f^{-1}(V)) = f_!(U) \wedge V,$$

for all $U \in \Omega_X, V \in \Omega_{X'}$.

(b) f^{-1} preserves arbitrary meets and the Heyting implication.

Fact 6.1.9. Open surjections $\phi\colon X \to X'$ in Loc correspond to open surjections $\phi^*\colon \mathcal{S}X' \to \mathcal{S}X$ in $\mathfrak{Top}$.

Proof. Apply [Joh02b, Prop. C1.5.1 and Theorem C1.5.4]. □

Remark 6.1.10 (Étale = Fibrewise Discrete)**.** Following Joyal-Tierney [JT84, Chapter V], Definition 6.1.8 can be used to give a new characterisation of étale bundles: a map $f\colon Y \to X$ in Loc is *étale* if the maps f and its diagonal $\Delta\colon Y \to Y \times_X Y$ are both open. In particular, by [JT84, Theorem V.5.1], when $X = \{*\}$ the canonical projection $f\colon Y \to \{*\}$ is étale iff Y is a discrete space . This gives rise to a useful slogan: "étale bundles = fibrewise discrete bundles".

Combining Example 6.1.4 and Remark 6.1.10, one obtains a particularly nice description of the descent category associated to a suitable groupoid G.

Fact 6.1.11 ([Moe88, §4.2 and §5.2])**.** As our setup,

- Let G be an open localic groupoid (= d_0, d_1 are open maps);
- Define an *étale G-space* to be an étale bundle $E \xrightarrow{p} G_0$ equipped with an action $E \times_{G_0} G_1 \xrightarrow{\bullet} E$ satisfying the usual axioms (the pullback here is along $G_1 \xrightarrow{d_1} G_0$). Denote BG to be the *category of étale G-spaces*;
- Denote Des to be the descent category associated to G (cf. Example 6.1.4).

Then, Des $\simeq BG$.

Discussion 6.1.12 (Descent and automorphisms of points)**.** If $\Phi\colon \mathcal{E}' \to \mathcal{E}$ is of effective descent, then this is generally weaker than an equivalence of spaces but it still says something deep about their structural relationship. For instance, Theorem 6.1.7 was leveraged by Joyal and Tierney to prove a remarkable structure theorem [JT84, Theorem VIII.3.2]:

> Any topos is equivalent to BG, for some open localic groupoid G (cf. Example 6.1.4).

Notice that Joyal-Tierney's result says that *all* toposes are of the form BG. As remarked by Johnstone [Joh02b, C5.1], this resonates with an informal picture, dating back to Grothendieck's work on étale cohomology of schemes, that a topos is "a space whose points have enough internal structure to allow them to possess non-trivial automorphisms". As an illustration, consider a connected atomic topos $\mathcal{S}\mathcal{E}$ with a global point $p\colon \text{Set} \to \mathcal{S}\mathcal{E}$. One can can then apply Joyal-Tierney's construction to show that $\mathcal{S}\mathcal{E} \simeq BG$, where G is the localic group of automorphisms of the point $p \in \mathcal{E}$ [Joh02b, Remark C5.2.14(c)].

6.1.2 Working Internally and Base-Changes. The descent construction can also be relativised. Recalling the language of Convention 2.1.7, consider a localic groupoid G over $\mathcal{S}$, where $\mathcal{S}$ is *any* elementary topos with nno. In which case, the BG construction from Fact 6.1.11 still works except it now yields a bounded topos over $\mathcal{S}$ (instead of over Set, as was assumed before) — let us make this difference explicit by denoting the construction as $B(\mathcal{S}, G)$.

The flexibility of this construction sets up an important base-change argument. Denote $\mathbf{Loc}(\mathcal{E})$ to be the category of locales over $\mathcal{E}$ [technically, the category of internal locales in the topos $\mathcal{S}\mathcal{E}$]. If $\Phi\colon \mathcal{E}' \to \mathcal{E}$ is a map of (generalised) spaces, then Φ induces an adjunction

$$\mathbf{Loc}(\mathcal{E}) \underset{\Phi_!}{\overset{\Phi^{\#}}{\rightleftarrows}} \mathbf{Loc}(\mathcal{E}') \;, \qquad \Phi_! \dashv \Phi^{\#}.$$

The reader may find the following informal picture helpful: given a locale L over $\mathcal{E}$, one can pullback L along $\Phi\colon \mathcal{E}' \to \mathcal{E}$ to obtain a new locale $\Phi^{\#}(L)$ internal to $\mathcal{E}'$.[107] In fact, given a localic groupoid G over $\mathcal{E}$, one obtains a new localic groupoid $\Phi^{\#}(G)$ over $\mathcal{E}'$ via a similar argument. A key result in [Moe88] guarantees that, under mild hypotheses, this pullback construction of localic groupoids interacts well with the $B(\mathcal{S}, G)$ construction ("$B(\mathcal{S}, G)$ is stable under base-change"):

Theorem 6.1.13 (Moerdijk's Stability Theorem [Moe88, Theorem 6.7])**.** *As our setup,*

- *Let $\Phi\colon \mathcal{E}' \to \mathcal{E}$ be a map of (generalised) spaces;*
- *Let G be an open localic groupoid over $\mathcal{E}$.*

Then, the canonical geometric morphism

$$B(\mathcal{S}\mathcal{E}', \Phi^{\#}(G)) \xrightarrow{\sim} \mathcal{S}\mathcal{E}' \times_{\mathcal{S}\mathcal{E}} B(\mathcal{S}\mathcal{E}, G)$$

is an equivalence of toposes.

6.1.3 Lax Descent. There is also an important weakening of Construction 6.1.3 known as lax descent.

Construction 6.1.14 (Lax Descent)**.** Given a 2-truncated simplicial space $\mathcal{E}_\bullet$, the *lax descent category* of $\mathcal{E}_\bullet$, which we denote LDes, is the same as the standard descent category Des of Construction 6.1.3 except we omit the requirement that descent data θ be an isomorphism.

Discussion 6.1.15 (Standard vs. Lax Descent)**.** To understand the distinction between standard vs. lax descent, we shall need to understand the significance of requiring the descent data to be an isomorphism. We give two ways of reading this: one more category-theoretic, the other more algebraic.

(i) Let $\mathcal{C}$ be a category, and consider the following diagram in $\mathcal{C}$:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{6.11}$$

[107] Some technical legwork is required in order to demonstrate the existence of such pullbacks — in particular, we do not get $\Phi^{\#}(L)$ for free simply by applying the inverse image functor $\Phi^*\colon \mathcal{S}\mathcal{E} \to \mathcal{S}\mathcal{E}'$ to L. Rather, one has to construct $\Phi^{\#}(L)$ explicitly, typically by relying on the technology of frame presentations [Moe88, §1.6] or GRD-systems [Vic04, §5].

As is well-known, the *coequaliser* of (f, g) (if it exists) gives the universal solution to the problem of finding a morphism $h\colon B \to C$ such that $hf = hg$. Less familiar is a weaker construction known as the *coinserter* of (f, g), which gives the universal solution to the problem of finding a morphism $h\colon B \to C$ together with a 2-cell $hf \to hg$. Notice, in particular, the 2-cell provided by the universal property of the coinserter is not required to be invertible, unlike the coequaliser.

Given the difference in universal properties, both constructions give rise to rather different kinds of colimits, which occasionally coincide but generally do not. In particular, in our setting, the standard descent category Des can be regarded as a (pseudo-)coequaliser in $\mathfrak{Top}$ subject to specific descent conditions and the lax descent category LDes as a (pseudo-)coinserter subject to the same conditions.

(ii) The following example by Johnstone [Joh02a, Example B3.4.14] is instructive. Consider an internal category $\mathbb{M}$ in a topos $\mathcal{S}$, which we represent as a 2-truncated simplicial bounded $\mathcal{S}$-topos:

$$\mathcal{S}/\mathbb{M}_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{S}/\mathbb{M}_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathcal{S}/\mathbb{M}_0 \tag{6.12}$$

where the arrows commute in the obvious way. Then, the lax descent category of Diagram (6.12) is the diagram topos $[\mathbb{M}, \mathcal{S}]$ whereas the standard descent category is the diagram topos $[\mathbb{G}, \mathcal{S}]$ where $\mathbb{G}$ is the groupoid reflection of $\mathbb{M}$, i.e. the category obtained from $\mathbb{M}$ by freely adjoining inverses for all morphisms of $\mathbb{M}$. The reason for this difference is because in the standard descent, we require the descent data to be invertible.

In particular, suppose $\mathcal{S} = \mathrm{Set}$, and so we may regard $\mathbb{M}_1$ is a discrete monoid acting on a set $\mathbb{M}_0$. Then, the lax descent topos gives the quotient of $\mathbb{M}_0$ by the monoidal action $\mathbb{M}_1$ whereas the standard descent topos gives the quotient of $\mathbb{M}_0$ by its group completion. This association of standard descent with group completion is suggestive, particularly because group completion signals a potential loss of information[108], which alerts us to the same possibility when using standard descent.

A natural question to ask: is LDes a topos just as in Construction 6.1.3? The answer is yes, but the result seems to only exist as folklore. For completeness, we give a quick sketch of the proof:

Theorem 6.1.16. *Let $\mathcal{E}_\bullet$ be a 2-truncated simplicial space. Then, its lax descent category is a topos.*

Proof (Sketch). We follow the hints provided by [Joh02b, Remark B3.4.10 and Example B3.4.14 (a)]. The argument is essentially the same as Moerdijk's proof that the 2-category $\mathfrak{BTop}/\mathcal{S}$ of bounded $\mathcal{S}$-toposes has all coequalisers [Moe88, Theorem 2.1], except we now work with coinserters instead (cf. Discussion 6.1.15).

We review our setup. Given a 2-truncated simplicial topos

$$\mathcal{E}_2 \;\substack{\xrightarrow{\widehat{d}_0}\\ \xrightarrow[\widehat{d}_1]{}\\ \xrightarrow[\widehat{d}_2]{}}\; \mathcal{E}_1 \;\substack{\xrightarrow{d_0}\\ \xleftarrow[s_0]{}\\ \xrightarrow[d_1]{}}\; \mathcal{E}_0 \qquad ,$$

108 To illustrate this, consider the example of the monoid $(\mathbb{N} \cup \{\infty\}, +)$ equipped with standard addition. This monoid is clearly non-trivial, yet its group completion is trivial: $n + \infty = \infty$ and thus $n = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$. This general style of argument is called an Eilenberg swindle, whose basic moral is: group completions typically trivialise whenever we introduce infinities. Let us also mention that there are other ways in which group completion may result in a loss of information (besides trivialisation) — e.g. although $[X] = [Y]$ in the Grothendieck semiring of complex varieties $SK_0(\mathcal{V}\mathrm{ar}_{\mathbb{C}})$ implies that X and Y are piecewise isomorphic as varieties (i.e. there exists a finite partition of both varieties such that each partition subvariety of X is isomorphic to some partition subvariety of Y, and vice versa) [CNS18, Corollary 1.4.9], this is no longer true in its group completion, otherwise known as the Grothendieck ring of varieties $K_0(\mathcal{V}\mathrm{ar}_{\mathbb{C}})$ [Bor18].

construct the inserter of the associated inverse image functors of

$$d_0, d_1 \colon \mathcal{E}_1 \to \mathcal{E}_0,$$

that is the category whose objects are pairs (F, θ) where $F \in \mathcal{S}\mathcal{E}_0$ and $\theta \colon d_0^*(F) \to d_1^*(F)$ is a (not-necessarily-invertible) morphism in $\mathcal{S}\mathcal{E}_1$. In particular, we may consider its full subcategory consisting of (F, θ) that also satisfy the required unit and cocycle conditions, which is our lax descent category $\mathcal{D}'$.

Following Giraud's Theorem 2.1.6, it suffices to prove that $\mathcal{D}'$ is an ∞-pretopos with a set of generators. The fact that $\mathcal{D}'$ is a ∞-pretopos follows from $\mathcal{S}\mathcal{E}_0$ satisfying the required conditions (since $\mathcal{S}\mathcal{E}_0$ is a topos) and using the fact that d_0^* and d_1^* preserve colimits and finite limits. For instance, suppose

$$(E, \theta) \rightarrowtail (X, \mu) \times (X, \mu)$$

is an equivalence relation in $\mathcal{D}'$. Then $E \rightarrowtail X \times X$ is an (effective) equivalence relation in $\mathcal{S}\mathcal{E}_0$, and μ induces a morphism

$$\mu/E \colon d_0^*(X/E) \cong d_0^*(X)/d_0^*(E) \longrightarrow d_1^*(X)/d_1^*(E) \cong d_1^*(X/E),$$

which is easily verified to respect the unit and cocycle conditions. Hence, $(X/E, \mu/E) \in \mathcal{D}'$, proving that equivalence relations are effective in $\mathcal{D}'$.

The main difficulty is proving that $\mathcal{D}'$ has a set of generators, but we can adapt Moerdijk's construction. Choose a site representation of $\mathcal{S}\mathcal{E}_0 \simeq \mathbf{Sh}(\mathcal{C}, J)$ and $\mathcal{S}\mathcal{E}_1 \simeq \mathbf{Sh}(\mathcal{B}, K)$. Then, construct an increasing sequence of (small) full subcategories of $\mathcal{S}\mathcal{E}_0$, $\{\mathcal{C}_n\}_{n \in \mathbb{N}}$, whereby

$\mathcal{C}_0 = \mathcal{C}$
$\mathcal{C}_{n+1} =$ objects of the form $\coprod_{i \in I} C_i$, where $C_i \in \mathcal{C}_n$ and I is the index set of some cover of $\mathcal{B}$.

Define $\mathcal{C}_\infty$ to be the full subcategory whose objects are of the form $\coprod_{n \in \mathbb{N}} C_n$ (for $C_n \in \mathcal{C}_n$), and define $\widehat{\mathcal{C}}$ to be the category whose objects are quotients of objects in $\mathcal{C}_\infty$. An involved check then verifies that $\mathcal{D}'$ is indeed generated by the set of objects $(F, \theta) \in \mathcal{D}'$, where $F \in \widehat{\mathcal{C}}$ and $\theta \colon d_0^*(F) \to d_1^*(F)$. □

6.2 Non-Archimedean Places

Building on work in Chapter 4, we now begin our analysis of the non-Archimedean places. For clarity, let us state upfront: unless stated otherwise, the absolute values in the present chapter will always be the standard Dedekind-valued absolute values on $\mathbb{Q}$. We start with a basic, but important, observation.

Observation 6.2.1. Let $|\cdot|$ be a non-Archimedean absolute value on $\mathbb{Q}$, and $\alpha \in (0, \infty)$ a Dedekind. Then $|\cdot|^\alpha$ is a non-Archimedean absolute value.

Proof. Immediate from Ostrowski's Theorem (Theorem D). However, one can also prove this directly by verifying (geometrically) that $|\cdot|^\alpha$ satisfies all the relevant axioms of a non-Archimedean absolute value. In fact, once we know that an absolute value is non-Archimedean iff it satisfies the ultrametric inequality (a fact implicit in our proof of Proposition 4.2.6), the check is a routine algebra exercise. □

Observation 6.2.1 suggests the following reformulation of a non-Archimedean place. By Ostrowski's Theorem, given any pair of non-Archimedean absolute values, say $|\cdot|_1, |\cdot|_2$, we now know that $|\cdot|_1 \sim |\cdot|_2$ iff there exists some $\alpha \in (0, \infty)$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ (and not just when $\alpha \in (0, 1]$). Since each non-Archimedean place is uniquely associated to a prime p, this suggests a natural reduction of our problem: instead of considering all places of $\mathbb{Q}$ at once, we can start by first "localising" and working prime by prime, before recovering the "global" picture.

6.2.1 Local: At single $\mathfrak{p}$. Denote $\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$ as *the space of non-trivial prime ideals of* $\mathbb{Z}$, i.e. prime ideals that possess a non-zero integer. Throughout this subsection, we fix a single $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$. By Lemma 2.2.42, there exists a prime $p \in \mathbb{N}_+$ such that $\mathfrak{p} = (p)$. As such, let us construct the following diagram of (point-free) spaces

$$(0,\infty) \times [av_{NA}; p] \underset{\mathrm{ex}}{\overset{\pi}{\rightrightarrows}} [av_{NA}; p] \tag{6.13}$$

whereby:

- $[av_{NA}; p]$ is the space of non-Archimedean absolute values such that $|p| < 1$;
- π is the projection map sending $(\alpha, |\cdot|) \mapsto |\cdot|$;
- ex sends $(\alpha, |\cdot|) \mapsto |\cdot|^\alpha$.

Note that ex is a well-defined map by Observation 6.2.1 and strict monotonicity of positive Dedekind exponentiation (which gives $|p| < 1 \implies |p|^\alpha < 1^\alpha = 1$). By Proposition 4.3.4, we know that $[av_{NA}; p] \cong (0,\infty)$. Hence, we may reformulate Diagram (6.13) as

$$(0,\infty) \times (0,\infty) \underset{\mathcal{M}}{\overset{\pi}{\rightrightarrows}} (0,\infty) \tag{6.14}$$

where $\mathcal{M}$ is the multiplication map sending $(\alpha, \beta) \mapsto \alpha \cdot \beta$. The β should be understood as representing $|\cdot| \in [av_{NA}; p]$ via the relation $|\cdot| = |\cdot|_p^\beta$ (justified by Ostrowski's Theorem), whereas the multiplication action should be understood as corresponding to $|\cdot|^\alpha = \left(|\cdot|_p^\beta\right)^\alpha = |\cdot|_p^{\alpha\cdot\beta}$.

We now pose the main question underpinning this section:

Question 13. Denote $\mathcal{D}$ to be the topos corresponding to a single non-Archimedean place of $\mathbb{Q}$. What is $\mathcal{D}$?

Classically, a place of $\mathbb{Q}$ is defined as *just* an equivalence class of absolute values. As such, a reasonable first approximation is to define $\mathcal{D}$ as the coequaliser[109] of Diagram (6.14), especially since $\mathfrak{Top}$ contains all coequalisers [Moe88, §2.1]. However, examining the exponentiation of absolute values more carefully, it becomes clear that ex should be properly understood as an algebraic action: indeed, note that $|\cdot|^1 = |\cdot|$ and $(|\cdot|^\alpha)^\lambda = |\cdot|^{\alpha\cdot\lambda}$, for any $|\cdot| \in [av_{NA}; p]$ and any $\alpha, \lambda \in (0,\infty)$.

As such, let us reformulate Diagram (6.14) as follows:

Construction 6.2.2. Define the localic groupoid $\mathbf{G} := (G_0, G_1)$ as

$$(G_1 \times_{G_0} G_1) \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} (0,\infty)\times(0,\infty) \underset{\mathcal{M}}{\overset{\pi}{\rightrightarrows}} (0,\infty), \qquad (G_1 \times_{G_0} G_1) \xrightarrow{\ \mathfrak{m}\ } (0,\infty)\times(0,\infty) \xleftarrow{\ s\ } (0,\infty) \tag{6.15}$$

where

- $G_0 := (0,\infty)$ and $G_1 := (0,\infty) \times (0,\infty)$;
- π and $\mathcal{M}$ correspond to the projection and multiplication maps from Diagram (6.14);

[109] More correctly, to define $\mathcal{D}$ as the category of sheaves on the space that is the coequaliser of Diagram (6.14).

- $\mathfrak{m}$ corresponds to the obvious multiplication map, s corresponds to the unit map sending $\beta \mapsto (1, \beta)$, and π_0, π_1 are the obvious projection maps.

Construction 6.2.2 allows us to give the following definition:

Definition 6.2.3. Following Example 6.1.4, define $\mathcal{D}$ to be the descent category corresponding to the universal descent cocone of Diagram (6.15).

On the topos-theoretic side, Definition 6.2.3 gives a well-defined answer to Question 13: by Construction 6.1.3, we know that the universal descent cocone of Diagram (6.15) exists, and its corresponding category is in fact a topos. On the number-theoretic side, the point-free perspective is illuminating. Since $\mathcal{D}$ is a topos, denote $[\mathbb{T}_{\mathcal{D}}]$ to be its corresponding space of points. Recall from Example 6.1.4 that $[\mathbb{T}_{\mathcal{D}}]$ can be regarded as "the quotient space of G_0 by the G_1-action". As such, since:

(a) $G_0 := (0, \infty)$ represents the space of all non-Archimedean absolute values associated to prime p; and
(b) Two (non-trivial) non-Archimedean absolute values are equivalent iff they are both associated to the same prime iff they are related by a G_1-action,

this justifies our definition of $\mathcal{D}$ as a non-Archimedean place of $\mathbb{Q}$.

In what follows, we work to improve our understanding of $[\mathbb{T}_{\mathcal{D}}]$: what *kind* of (quotient) space does $[\mathbb{T}_{\mathcal{D}}]$ look like? To start, one may first observe G_1 represents a free transitive action of $(0, \infty)$ on G_0, and deduce that there exists a single G_1-orbit on G_0. Recall also Discussion 6.1.12, which suggested an informal picture of a topos as a generalised space whose points may possess non-trivial automorphisms. Put together, the following guess is reasonable:

Guess 6.2.4. $[\mathbb{T}_{\mathcal{D}}]$ is the singleton space $\{*\}$, with $(0, \infty)$ as the group of automorphisms acting on $\{*\}$.

Very interestingly, Guess 6.2.4 turns out to be wrong. The fundamental reason behind this has to do with the misplaced expectation that $[\mathbb{T}_{\mathcal{D}}]$ possesses non-trivial automorphisms. In particular, non-triviality of the G_1-action does not imply non-triviality of the resulting quotient space — e.g. as already pointed out by Bunge in [Bun90], $BG \simeq \mathrm{Set}$ for any connected topological group G. A similar issue arises in our setting:

Theorem F. $\mathcal{D} \simeq \mathrm{Set}$. Or, equivalently, $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$.

In other words, the quotient space $[\mathbb{T}_{\mathcal{D}}]$ is trivial: it is the singleton $\{*\}$ with *no* non-trivial automorphisms. Comparing our groupoid **G** with Bunge's example, one might then suspect that the connectedness of $(0, \infty)$ is the main culprit behind the trivialisation, but this is again a red herring. In fact, Theorem F follows from a more general result:

Theorem 6.2.5. *Consider the following groupoid* **H** *in* Loc

$$(G \times M) \times_M (G \times M) \;\substack{\xrightarrow{\pi_0} \\ \xrightarrow[\mathfrak{m}]{} \\ \xrightarrow[\pi_1]{}}\; G \times M \;\substack{\xrightarrow{\pi} \\ \xleftarrow[s]{} \\ \xrightarrow[\mathcal{M}]{}}\; M \tag{6.16}$$

where

- *The unique map $\rho\colon M \to \{*\}$ is an open surjection.*
- *π is the projection map sending $(g,m) \mapsto m$;*
- *G induces a free transitive action on M. More explicitly, denoting $\mathcal{M}(g,m) = g \cdot m$,*

    (a) Given any $m \in M$ and $h, h' \in G$, we have $h \cdot m = h' \cdot m$ implies $h = h'$;
    (b) Given any $m_0, m_1 \in M$, there exists $g \in G$ such that $g \cdot m_0 = m_1$.

- *s is the unit map sending $m \mapsto (\mathrm{id}_G, m)$, where $\mathrm{id}_G \in G$ represents the unit of the G-action.*

Then:

(i) $B\boldsymbol{H}$, the category of étale $\boldsymbol{H}$-spaces (Fact 6.1.11), is equivalent to the descent category of the simplicial topos associated to Diagram (6.16)*;*
(ii) $B\boldsymbol{H} \simeq \mathrm{Set}$.

Proof. The proof proceeds in stages.

Step 0: Setup. Construction 6.1.3 gives us an explicit description of the universal descent cocone of the simplicial topos associated to Diagram (6.16). This comprises two pieces of data:

(a) The descent category Des, whereby:

> An **object** in Des is a pair (F, θ), where F is an object of $\mathcal{S}M$ and $\theta\colon \pi^*(F) \xrightarrow{\sim} \mathcal{M}^*(F)$ is an isomorphism in $\mathcal{S}G \times \mathcal{S}M$ such that $s^*(\theta) \cong \mathrm{id}$ and $\pi_0^*(\theta) \circ \pi_1^*(\theta) \cong \mathfrak{m}^*(\theta)$;
> A **morphism** $(F, \theta) \to (F', \xi)$ in Des corresponds to a morphism $F \xrightarrow{u} F'$ in $\mathcal{S}M$ such that $\mathcal{M}^*(u) \circ \theta = \xi \circ \pi^*(u)$.

(b) The (inverse image of the) geometric morphism $p^*\colon \mathrm{Des} \to \mathcal{S}M$.

Our analysis of Des rests on identifying precisely which sheaves over M (i.e. objects of topos $\mathcal{S}M$) are able to support the descent data satisfying the properties required by Construction 6.1.3.

Step 1: A toy example. Following Example 6.1.5, construct the following diagram in Loc by taking iterated pullbacks of $\rho\colon M \to \{*\}$:

$$M \times M \times M \overset{\pi_{01}}{\underset{\pi_{12}}{\xrightarrow{\pi_{02}}}} M \times M \overset{\pi_1}{\underset{\pi_0}{\xleftarrow{\Delta}}} M \xrightarrow{\rho} \{*\} \tag{6.17}$$

where

- $\Delta\colon M \to M \times M$ is the diagonal map sending $m \mapsto (m,m)$;
- $\pi_{0,1}, \pi_{02}$ and π_{12} are the projection maps such that, e.g. π_{01} maps $(m_0, m_1, m_2) \mapsto (m_0, m_1)$.

In particular, Diagram (6.17) defines a groupoid[110], which we shall denote as $\mathbf{M}$.

Next, taking the category of sheaves on Diagram (6.17), construct the corresponding descent category $\mathrm{Des}(\rho)$. For explicitness, we record that the objects of $\mathrm{Des}(\rho)$ are pairs (F', θ') where

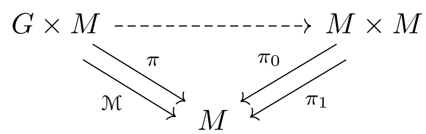
[110] Why? Define $(—)^{-1}\colon M \times M \to M \times M$ as sending $(m_0, m_1) \mapsto (m_1, m_0)$, and check that it gives the required inverse map.

- $F' \in \mathcal{S}M$; and
- $\theta' \colon \pi_0^*(F') \to \pi_1^*(F')$ is a morphism in $\mathcal{S}M \times \mathcal{S}M$ such that

$$\Delta^*(\theta') \cong \mathrm{id};$$
$$\pi_{02}^*(\theta') \cong \pi_{12}^*(\theta') \circ \pi_{01}^*(\theta').$$

By hypothesis, $M \xrightarrow{\rho} \{*\}$ is an open surjection in Loc, and thus so is the corresponding inverse image functor $\rho^* \colon \mathrm{Set} \to \mathcal{S}M$ (Fact 6.1.9). Applying Theorem 6.1.7, this means that ρ is an effective descent morphism, i.e. $\mathrm{Des}(\rho) \simeq \mathrm{Set}$.

*Step 2: **H** and **M** are isomorphic.* Consider the following diagram in Loc



(6.18)

Next, define the map

$$\langle \pi, \mathcal{M} \rangle \colon G \times M \longrightarrow M \times M \qquad (6.19)$$
$$(g, m) \longmapsto (m, g \cdot m).$$

This sets up the key claim of our proof.

Claim 6.2.6. *$\langle \pi, \mathcal{M} \rangle$ induces an isomorphism of groupoids **H** $\cong$ **M**.*

Proof. The proof proceeds by recording a series of basic observations about our setup.

(a) $\langle \pi, \mathcal{M} \rangle$ makes the inner and outer triangles of Diagram (6.18) commute.

[Why? A quick diagram chase shows that

$$\pi(g, m) = m = \pi_0 \circ \langle \pi, \mathcal{M} \rangle (g, m)$$

$$\mathcal{M}(g, m) = g \cdot m = \pi_1 \circ \langle \pi, \mathcal{M} \rangle (g, m).]$$

(b) $\langle \pi, \mathcal{M} \rangle$ defines an isomorphism between $G \times M \xrightarrow{\sim} M \times M$.

[Why? The standard argument works, so long as we are careful to reason point-wise. More explicitly, define

$$\langle \pi, \mathcal{M} \rangle^{-1} \colon M \times M \longrightarrow G \times M \qquad (6.20)$$
$$(m_0, m_1) \longmapsto (g_{m_0, m_1}, m_0)$$

where we denote $g_{m_0,m_1} \in G$ to be the point of G such that $g_{m_0,m_1} \cdot m_0 = m_1$, given to us by transitivity of the G-action. Then, the the two maps are clearly inverse to each other, since

$$\langle \pi, \mathcal{M} \rangle \circ \langle \pi, \mathcal{M} \rangle^{-1}(m_0, m_1) = (m_0, g_{m_0,m_1} \cdot m_0) = (m_0, m_1)$$

$$\langle \pi, \mathcal{M} \rangle^{-1} \circ \langle \pi, \mathcal{M} \rangle (g, m_0) = (g_{m_0, g \cdot m_0}, m_0) = (g, m_0)$$

where the fact that $g_{m_0, g \cdot m_0} = g$ follows from the hypothesis that the G-action is free.]

(c) $\langle \pi, \mathcal{M} \rangle$ induces an isomorphism between $(G \times M) \times_M (G \times M)$ and $M \times M \times M$.

[Why? First, recall that $(G \times M) \times_M (G \times M)$ and $M \times M \times M$ are constructed via pullbacks of the domain and codomain maps of **H** and **M** respectively:

$$\begin{array}{ccc} (G \times M) \times_M (G \times M) & \xrightarrow{\pi_1} & G \times M \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \mathcal{M}} \\ G \times M & \xrightarrow{\quad \pi \quad} & M \end{array} \qquad \begin{array}{ccc} M \times M \times M & \xrightarrow{\pi_{12}} & M \times M \\ {\scriptstyle \pi_{01}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M \times M & \xrightarrow{\quad \pi_0 \quad} & M \end{array} \tag{6.21}$$

Consider the following composition of maps

$$\begin{array}{ccc} (G \times M) \times_M (G \times M) & \xrightarrow{\langle \pi, \mathcal{M} \rangle \circ \pi_1} & M \times M \\ {\scriptstyle \langle \pi, \mathcal{M} \rangle \circ \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M \times M & \xrightarrow{\qquad \pi_0 \qquad} & M \end{array} \tag{6.22}$$

One easily checks that the diagram commutes, since:

$$\begin{aligned} \pi_0 \circ \langle \pi, \mathcal{M} \rangle \circ \pi_0(g_1, g_0 \cdot m, g_0, m) &= \pi \circ \pi_0(g_1, g_0 \cdot m, g_0, m) && \text{[By item (a)]} \\ &= \mathcal{M} \circ \pi_1(g_1, g_0 \cdot m, g_0, m) && \text{[By Diagram (6.21)]} \\ &= \pi_1 \circ \langle \pi, \mathcal{M} \rangle \circ \pi_0(g_1, g_0 \cdot m, g_0, m, && \text{[By item (a)]} \end{aligned}$$

and so by the universal pullback property of $M \times M \times M$, we obtain a map

$$i \colon (G \times M) \times_M (G \times M) \longrightarrow M \times M \times M \tag{6.23}$$

such that $\pi_{01} \circ i = \langle \pi, \mathcal{M} \rangle \circ \pi_0$ and $\pi_{12} \circ i = \langle \pi, \mathcal{M} \rangle \circ \pi_1$. A similar argument yields a map

$$i^{-1} \colon M \times M \times M \longrightarrow (G \times M) \times_M (G \times M) \tag{6.24}$$

such that $\pi_0 \circ i^{-1} = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{01}$ and $\pi_1 \circ i^{-1} = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{12}$.

By item (b), $\langle \pi, \mathcal{M} \rangle$ and $\langle \pi, \mathcal{M} \rangle^{-1}$ are inverse to each other. Hence, one computes that

$$\pi_0 \circ i^{-1} \circ i = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{01} \circ i = \langle \pi, \mathcal{M} \rangle^{-1} \circ \langle \pi, \mathcal{M} \rangle \circ \pi_0 = \pi_0, \tag{6.25}$$

and that $\pi_1 \circ i^{-1} \circ i = \pi_1$. Since π_0, π_1 are jointly monic, this implies $i^{-1} \circ i = \mathrm{id}$. An analogous argument gives $i \circ i^{-1} = \mathrm{id}$. This shows that $\langle \pi, \mathcal{M} \rangle$ induces an isomorphism of spaces between $(G \times M) \times_M (G \times M)$ and $M \times M \times M$ via the pullback property.]

(d) $\langle \pi, \mathcal{M} \rangle$ induces an isomorphism between all the structure maps of **H** and **M**.

[Why? For the domain and codomain maps of **H** and **M**, this was items (a) and (b). A similar diagram-chase shows that $\langle \pi, \mathcal{M} \rangle$ and $\langle \pi, \mathcal{M} \rangle^{-1}$ commute with the unit maps:

$$\langle \pi, \mathcal{M} \rangle \circ s(m) = (m, m) = \Delta(m)$$

$$\langle \pi, \mathcal{M} \rangle^{-1} \circ \Delta(m) = (g_{m,m}, m) = (\mathrm{id}_G, m) = s(m),$$

where the second equation once again follows from the freeness of the G-action.

For the isomorphism between the projection maps

$$G \times M \xleftarrow{\pi_0} (G \times M) \times_M (G \times M) \xrightarrow{\pi_1} G \times M \tag{6.26}$$

$$M \times M \xleftarrow{\pi_{01}} M \times M \times M \xrightarrow{\pi_{12}} M \times M, \tag{6.27}$$

this was already established in item (c). Note that our computations made use of both $\langle \pi, \mathcal{M} \rangle$ and the induced map i (cf. Equation (6.23)). Similarly, to establish an isomorphism between the composition/multiplication maps of **H** and **M**, it suffices to show that the following two diagrams commute:

$$\begin{array}{ccc} (G \times M) \times_M (G \times M) & \xrightarrow{i} & M \times M \times M \\ {\scriptstyle \mathfrak{m}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ G \times M & \xrightarrow{\langle \pi, \mathcal{M} \rangle} & M \times M \end{array} \qquad \begin{array}{ccc} M \times M \times M & \xrightarrow{i^{-1}} & (G \times M) \times_M (G \times M) \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathfrak{m}} \\ M \times M & \xrightarrow{\langle \pi, \mathcal{M} \rangle^{-1}} & G \times M \end{array} \tag{6.28}$$

and that they compose horizontally to recover both $\mathfrak{m}$ (composing from the left diagram to the right) as well as Δ (composing from the right diagram to the left). Both claims are easily verified by a diagram-chase, which we leave to the reader.]

By items (a) - (d), conclude that $\langle \pi, \mathcal{M} \rangle$ induces an isomorphism of groupoids $\mathbf{H} \cong \mathbf{M}$, as claimed. □

*Step 3: **H** and **M** are open groupoids.* Fact 6.1.11 tells us that $\mathrm{Des} \simeq \mathrm{B}\mathbf{H}$ if **H** is an open groupoid, i.e. if its domain and codomain maps

$$\pi, \mathcal{M} \colon G \times M \rightrightarrows M \tag{6.29}$$

are open maps. By Claim 6.2.6, we know that Diagram (6.29) is equivalent to the diagram

$$\pi_0, \pi_1 \colon M \times M \rightrightarrows M, \tag{6.30}$$

and so it suffices to prove that the projection maps are open. But this is straightforward. First note that these projection maps can be obtained via the kernel pair of $\rho \colon M \to \{*\}$, as depicted:

$$\begin{array}{ccc} M \times M & \xrightarrow{\pi_1} & M \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow{\rho} & \{*\} \end{array}$$

Next, recall that ρ is an open map by hypothesis. Since open maps are stable under pullback [JT84, §V.4], we deduce that π_0, π_1 must also be open.

Step 4: Reduction to Step 1. By construction, both Des and $\mathrm{Des}(\rho)$ select certain sheaves of $\mathcal{S}M$ compatible with their respective definitions for descent data. This suggests the following heuristic: if the *descent data* of Des and $\mathrm{Des}(\rho)$ are equivalent (in some appropriate sense), then this should imply $\mathrm{Des} \simeq \mathrm{Des}(\rho)$.

To prove this, one can apply Claim 6.2.6 and explicitly check that the isomorphism $\mathbf{H} \cong \mathbf{M}$ induces the desired equivalence of descent data. Let us however give a more conceptual argument. By Step 3, both **H** and **M** are open groupoids. Applying Fact 6.1.11, we obtain the equivalences $\mathrm{Des} \simeq \mathrm{B}\mathbf{H}$ and $\mathrm{Des}(\rho) \simeq \mathrm{B}\mathbf{M}$. Finally, since $\langle \pi, \mathcal{M} \rangle \colon \mathbf{H} \xrightarrow{\sim} \mathbf{M}$ induces an isomorphism of groupoids, it is also an open, fully faithful and essentially surjective groupoid map. We can therefore apply [Moe88, Summary Theorem 5.15] to deduce

$$\mathrm{Des} \simeq \mathrm{B}\mathbf{H} \simeq \mathrm{B}\mathbf{M} \simeq \mathrm{Des}(\rho).$$

Step 5: Finish. By Step 1, we obtain the equivalence

$$\mathrm{Des}(\rho) \simeq \mathrm{Set}.$$

By Steps 2-4, we obtain the characterisation

$$\mathrm{Des} \simeq \mathrm{B}\mathbf{H} \simeq \mathrm{Des}(\rho),$$

and so putting everything together gives

$$\mathrm{B}\mathbf{H} \simeq \mathrm{Des} \simeq \mathrm{Set}.$$

□

Remark 6.2.7 (On the hypothesis "open")**.** The locale theorist may ask: why did Theorem 6.2.5 require $M \xrightarrow{\rho} \{*\}$ to be an *open* surjection in its hypotheses? After all, as Borceux proves in [Bor94, Example 1.6.5c], the unique map $L \to \{*\}$ can be shown to be open for any given locale L. This result appears to indicate that the additional hypothesis of openness is unnecessary.

The answer can be found in Borceux's working. Recall that Ω, i.e. the frame of opens on $\{*\}$, corresponds to the frame of truth values. In his argument, Borceux represents this as the classical frame of Boolean truth values $\Omega = \{\bot, \top\}$, which is constructively inequivalent to the geometric frame of truth values (see Footnote 23 and Discussion 2.1.14). In other words, Borceux's argument only holds true classically; it is constructively *false* that $L \to \{*\}$ is open for a general locale L (or indeed that all projection maps are open). Hence, this gap between constructive vs. classical validity justifies our original hypothesis.

As a special case of Theorem 6.2.5, we obtain:

Theorem F. $\mathcal{D} \simeq \mathrm{Set}$. Or, equivalently, $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$.

Proof. Recall that $\mathcal{D}$ is the descent category associated to the groupoid **G** defined in Construction 6.2.2. Examining the hypotheses of Theorem 6.2.5, it thus suffices to show that:

(a) The $\mathcal{M}$-action of **G** is both free and transitive;
(b) The unique map $!\colon (0,\infty) \to \{*\}$ is an open surjection.

(a) is easy. As for (b), the fact that $!\colon (0,\infty) \to \{*\}$ is an open map[111] follows from [Vic09, Corollary 6.2] and the fact that $(0,\infty)$ is a generalised metric space. To show that ! is a surjection, it suffices to show it has a right-sided inverse (since this shows that ! is an epi in Loc). Let us define one such possible map:

$$\begin{aligned} !^{-1}\colon \{*\} &\longrightarrow (0,\infty) \\ \{*\} &\longmapsto 1. \end{aligned} \tag{6.31}$$

□

[111]Note: in the language of [Vic09], a localic space Y such that the unique map $!\colon Y \to \{*\}$ is an open map is called *overt*.

6.2.2 Global: Over all $\mathfrak{p}$. Theorem F sets the obvious expectation that the (point-free) space of non-Archimedean [hereafter: non-Arch] places ought to correspond to the usual space of primes of $\mathbb{Z}$. However, there are a few issues to be mindful of.

(a) If the space of non-Arch places corresponds to the space of primes of $\mathbb{Z}$, which topology should it have? The Zariski? coZariski? The Constructible Topology?
(b) How do we justify extending Theorem F to give a global characterisation of *all* non-trivial non-Arch places — beyond the fact that it corresponds to the classical picture?
(c) Relatedly, Theorem F only characterises the non-trivial non-Arch places. How does the extension in Item (b) account for the trivial place?

Viewed classically, where we have the option of dealing with the algebra and the topology separately, these questions are much less subtle — e.g. one may simply appeal to Ostrowski's Theorem to justify viewing the non-Arch places as the set of primes, and leave the reader to pick their favourite topology. In the context of point-free topology, however, where we must deal with the algebra and topology *simultaneously*, we have no such option, and shall need to be more deliberate in our approach.

In regards to items (a) and (c), Observation 4.3.7 certainly gives compelling evidence that the space of non-Archimedean places ought to be isomorphic to $\mathrm{ISpec}(\mathbb{Z})$. However, at present, we still do not have a geometric justification for this. Nonetheless, in regards to item (b), we *can* indeed construct the entire space of non-trivial non-Arch places by way of a base-change argument.

Let us review our previous work. Given some fixed $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$, we defined a groupoid $\mathbf{G}$ that expressed how the exponentiation acts on the space $[av_{NA}; p]$. We then defined the non-Archimedean place as corresponding to the quotient space $[\mathbb{T}_{\mathcal{D}}]$ of this action, before deducing that $\mathcal{S}[\mathbb{T}_{\mathcal{D}}] \simeq \mathrm{B}\mathbf{G} \simeq \mathrm{Set}$, or equivalently $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$ (Theorem F). However, notice the groupoid $\mathbf{G}$ was defined for an arbitrary fixed non-trivial $\mathfrak{p} = (p)$. Put otherwise, we were (implicitly) working internally within the topos $\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)})$ (cf. Convention 2.1.34). As such, in order to characterise the entire space of non-trivial non-Arch places, we shall need to externalise the descent topos construction.

The first step to doing this is to assemble the following pullback diagram

$$\begin{array}{ccc} v^{\#}(\mathbf{G}) & \longrightarrow & \mathbf{G} \\ \downarrow & & \downarrow \\ \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} & \xrightarrow{v} & \{*\} \end{array} \tag{6.32}$$

where we explicitly denote $v\colon \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} \to \{*\}$ as the unique terminal map in Loc. We then define the topos of non-trivial non-Arch places as $\mathcal{S}[\mathrm{places}_{NA\neq 0}] := \mathrm{B}(\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}), v^{\#}(\mathbf{G}))$, which (of course) we also regard as the category of sheaves on some space $[\mathrm{places}_{NA\neq 0}]$. The following theorem verifies that there are no surprises here, and that we obtain the expected characterisation of $[\mathrm{places}_{NA\neq 0}]$.

Theorem 6.2.8. $[\mathrm{places}_{NA\neq 0}] \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$.

Proof. Let us regard G as a localic groupoid internal to the topos Set. By Step 3 of the proof of Theorem 6.2.5, we know that G is an open groupoid. Hence, apply Moerdijk's Stability Theorem 6.1.13 to obtain the following equivalence of toposes:

$$\mathrm{B}(\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}), v^{\#}(\mathbf{G})) \simeq \mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}) \times_{\mathrm{Set}} \mathrm{B}(\mathrm{Set}, \mathbf{G}) \tag{6.33}$$

By Theorem F , we know that $\mathrm{B}(\mathrm{Set}, \mathbf{G}) \simeq \mathrm{Set}$. As such, translating Equation (6.33) into the language of point-free spaces gives

$$[\text{places}_{NA\neq 0}] \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} \times \{*\} \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}, \tag{6.34}$$

as claimed. □

6.3 The Archimedean Place

Recall Proposition 4.3.5's characterisation of the space of (non-trivial) Archimedean absolute values as $[av_A] \cong (0, 1]$. Playing the same game as we did for the non-Archimedean place, this suggests the following reformulation of the algebraic action of exponentiation on $[av_A]$:

Construction 6.3.1. Define the following diagram in Loc:

$$M_1 \times_{(0,1]} M_1 \underset{\mathfrak{m}}{\overset{\pi_1,\ \pi_2}{\longrightarrow}} (0,1] \times (0,1] \underset{s}{\longleftarrow} \overset{\pi,\ \mathcal{M}}{} (0,1] \tag{6.35}$$

where

- $M_1 := (0, 1] \times (0, 1]$;
- π corresponds to the projection map sending $(\alpha, \beta) \mapsto \beta$ and $\mathcal{M}$ corresponds to the multiplication map sending $(\alpha, \beta) \mapsto \alpha \cdot \beta$.
- $\mathfrak{m}$ corresponds to the obvious multiplication map, s corresponds to the unit map sending $\beta \mapsto (1, \beta)$, and π_1, π_2 are the obvious projection maps.

Remark 6.3.2. How does Diagram (6.35) represent exponentiation on $[av_A]$? Given Ostrowski's Theorem, $\beta \in (0, 1]$ should be understood as representing $|\cdot| \in [av_A]$ via the relation $|\cdot| = |\cdot|_\infty^\beta$, whereas the $\mathcal{M}$-action mapping $(\alpha, \beta) \mapsto \alpha \cdot \beta$ should be understood as representing $|\cdot|^\alpha = \left(|\cdot|_\infty^\beta\right)^\alpha = |\cdot|_\infty^{\alpha\cdot\beta}$.

The set-up is entirely analogous to Construction 6.2.2 except for one key difference: Diagram (6.35) is not a groupoid since the $\mathcal{M}$-action is not invertible. This is a crucial detail. By our work on Ostrowski's Theorem (Chapter 4), we know there exists a natural monoid action of $(0, 1]$ on all absolute values via exponentiation. When restricted to the non-Archimedean case, Observation 6.2.1 tells us that this can be extended to a group action of $(0, \infty)$ on $[av_{NA}]$, but the same argument cannot be extended to the Archimedean case.[112] In fact, when $|\cdot|$ is Archimedean and $\alpha \in (1, \infty)$, the triangle inequality clearly fails for $|\cdot|^\alpha$ since $(1 + 1)^\alpha > 1 + 1$.

This realisation points to new emerging subtleties. We would like to define the Archimedean place as corresponding to some colimit of Diagram (6.35) — but which one? Discussion 6.1.15 alerts us to the fact that the standard descent construction freely inverts the $\mathcal{M}$-action in the colimit, signalling a potential loss of information (at least in the discrete setting). Since we want to quotient $(0, 1]$ by a non-invertible monoid action [as opposed to its group completion], we are naturally led to use the lax descent construction instead.

[112] Why? Note: the argument in Observation 6.2.1 makes crucial use of the fact that non-Archimedean absolute values satisfy the ultrametric inequality, which the Archimedean absolute values do not satisfy.

Definition 6.3.3. Denote $\mathcal{D}'$ to be the topos corresponding to the Archimedean place. Following Construction 6.1.14, define $\mathcal{D}'$ to be the lax descent category of Diagram (6.35). As before, we denote $[\mathbb{T}_{\mathcal{D}'}]$ to be the space of points of $\mathcal{D}'$.

The justification for Definition 6.3.3 is entirely analogous to the previous section. Theorem 6.1.16 gives that $\mathcal{D}'$ is indeed a topos, and Diagram (6.35) indicates that $[\mathbb{T}_{\mathcal{D}'}]$ can be regarded as the quotient space of $(0, 1]$ by the $(0, 1]$-action, just as in the case of the (single) non-Archimedean place. It is then natural to wonder if we get the same result as before, i.e. if $[\mathbb{T}_{\mathcal{D}'}]$ corresponds to the singleton space $\{*\}$. It does not. In fact, we get the following surprising result:

Theorem G. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$, or equivalently, $[\mathbb{T}_{\mathcal{D}'}] \cong \overleftarrow{[0,1]}$.

Aside from the obvious difference with the non-Archimedean case, why else might Theorem G be surprising? One answer is its number-theoretic implications, which we will postpone to Section 6.5 for proper discussion. Here we give two other observations to round out our perspective:

(a) The appearance of the upper reals is unexpected.[113] Informally, Theorem G says that if we quotient the real interval $(0, 1]$ by the multiplicative action of the monoid $(0, 1]$, then we essentially kill off all the left Dedekind sections of the reals in $(0, 1]$ — something which is *a priori* not obvious.
(b) Although we were careful to exclude the trivial place — note that we considered $(0, 1]$ instead of the closed interval $[0, 1]$ — the fact that $[\mathbb{T}_{\mathcal{D}'}] \cong \overleftarrow{[0,1]}$ suggests that the (non-trivial) Archimedean place and trivial place cannot be definably separated.[114] This raises interesting questions on how we should understand the generic Archimedean completion, especially since $\mathbb{Q}$ and $\mathbb{R}$ are clearly not homeomorphic.

Before proceeding, a few words about strategy. The basic plan of attack for proving Theorem G is simple: construct two functors

$$\mathfrak{J}\colon \mathcal{D}' \rightleftarrows \mathcal{S}\overleftarrow{[0,1]} \colon \mathfrak{K}$$

and verify that $\mathfrak{J}$ and $\mathfrak{K}$ are inverse to each other. The mathematical devil, unsurprisingly, lies in the details. Nonetheless, though the constructions are involved, they are (implicitly) guided by a key topological insight regarding the Archimedean vs. non-Archimedean case: *$\mathcal{D}'$ witnesses non-trivial forking in the connected components of its sheaves whereas $\mathcal{D}$ does not.* Many of the arguments developed in this section can be understood as adjusting for this difference. To improve readability, we postpone further discussion of this so-called forking phenomena till Section 6.4, and focus on establishing the key moves of the proof first.

6.3.1 First Direction.

We start by working to construct the functor:

$$\begin{aligned} \mathfrak{J}\colon \mathcal{D}' &\longrightarrow \mathcal{S}\overleftarrow{[0,1]} \\ (F,\theta) &\mapsto \quad ? \end{aligned}$$

Throughout this subsection, fix the following setup:

[113] Although, in hindsight, perhaps less surprising once we step away from classical number theory and examine the lax descent construction by itself: the quotient converts actions by the monoid into 2-cells, which introduces the one-sidedness.

[114] Why? Recall that any subspace of the upper reals must be closed under arbitrary joins (cf. Convention 2.2.17).

Setup 6.3.4. *Let $(F, \theta) \in \mathcal{D}'$. Leveraging previous work, we obtain the following explicit presentation:*

(i) F can be equivalently characterised as:

- *F is a sheaf over $(0, 1]$;*
- *$F\colon (0, 1] \to [\mathbb{O}]$ is a map to the object classifier;*
- *F corresponds to an étale bundle $f\colon Y \to (0, 1]$, where F can be viewed as a fibrewise definition of the bundle space of f, i.e. $F(\gamma) = f^{-1}(\gamma)$ for any $\gamma \in (0, 1]$ (cf. Theorem 2.1.3). In particular, note by Remark 6.1.10 that each fibre $F(\gamma)$ defines a set.*

Throughout this section, we shall move freely between these different characterisations of F, depending on convenience.

(ii) Unpacking Construction 6.1.14 in the language of item (i), the pullback of f along π and $\mathcal{M}$ gives

$$\begin{array}{ccc} \pi^*(Y) & \xrightarrow{\ \phi\ } & Y \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle f} \\ (0,1] \times (0,1] & \xrightarrow{\ \pi\ } & (0,1] \end{array} \qquad \begin{array}{ccc} \mathcal{M}^*(Y) & \xrightarrow{\ \phi'\ } & Y \\ \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle f} \\ (0,1] \times (0,1] & \xrightarrow{\ \mathcal{M}\ } & (0,1] \end{array}$$

which can be represented as

$$\pi^*(Y) = Y \times (0, 1]$$
$$\mathcal{M}^*(Y) = \{(y, \alpha, \beta) \in Y \times (0, 1] \times (0, 1] \,\big|\, f(y) = \alpha\beta)\}$$

and where δ sends $(y, \beta) \mapsto (\beta, f(y))$ and ϕ sending $(y, \beta) \mapsto y$, whereas δ' maps $(y, \alpha, \beta) \mapsto (\alpha, \beta)$ and ϕ' maps $(y, \alpha, \beta) \mapsto y$.

(iii) Correspondingly, since the data $\theta\colon \pi^*(Y) \to \mathcal{M}^*(Y)$ defines a bundle map over $(0, 1] \times (0, 1]$, it is required to make the following diagram commute:

$$\begin{array}{ccc} \pi^*(Y) & \xrightarrow{\qquad \theta \qquad} & \mathcal{M}^*(Y) \\ & {\scriptstyle \delta}\searrow \quad \swarrow{\scriptstyle \delta'} & \\ & (0,1] \times (0,1] & \end{array}$$

As such, we can express θ coordinate-wise as the following:

$$\begin{aligned} \theta\colon Y \times (0, 1] &\longrightarrow \mathcal{M}^*(Y) \\ (y, \beta) &\longmapsto (\theta_0(y, \beta), \beta, f(y)) \end{aligned}$$

In particular, notice: $f(\theta_0(y, \beta)) = f(y) \cdot \beta$.

Remark 6.3.5. Notice the lax descent data was defined as $\theta\colon \pi^*(Y) \to \mathcal{M}^*(Y)$ as opposed to going the opposite direction $\theta\colon \mathcal{M}^*(Y) \to \pi^*(Y)$. In principle, one could have defined the lax descent data going in the latter direction, which would yield in a different lax descent topos[115]; however, we have chosen the former since we feel it is more natural to regard the projection map π as corresponding to the domain map d_0 of Construction 6.1.14 as opposed to the codomain map d_1.

[115]Note: in the case of standard descent, the choice of direction does not affect the resulting descent topos since the descent data is required to be an isomorphism.

Constructing $\mathfrak{J}(F,\theta)$. We now work to show how to construct a new sheaf $\overline{F} \in \mathcal{S}\overleftarrow{[0,1]}$ from the original $(F,\theta) \in \mathcal{D}'$. First, consider the canonical map

$$\Psi : \mathbb{Q}_{(0,1]} \longrightarrow (0,1] \tag{6.36}$$

which sends a (discrete) point of $\mathbb{Q}_{(0,1]} := \{q \in \mathbb{Q} | 0 < q \leq 1\}$ to its canonical representative in $(0,1]$. Regarding F as an étale bundle $f\colon Y \to (0,1]$, we can pullback f along Ψ to obtain:

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ {\scriptstyle f_{\mathrm{res}}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{Q}_{(0,1]} & \xrightarrow{\Psi} & (0,1] \end{array}$$

Morally, $f_{\mathrm{res}} : Z \to \mathbb{Q}_{(0,1]}$ — or equivalently the associated sheaf $F_{\mathrm{res}} : \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$ — can be thought of as the restriction of $F\colon (0,1] \to [\mathbb{O}]$ to the space $\mathbb{Q}_{(0,1]}$.[116] In particular, if F_{res} satisfies the continuity conditions stipulated by the Lifting Lemma 2.2.55, then Observation 2.2.58 tells us that it canonically defines a sheaf on $\overleftarrow{[0,1]}$. We verify this is indeed the case by proving a more general claim.

Claim 6.3.6 (Key Claim)**.** *Given* $(F,\theta) \in \mathcal{D}'$*, the descent data* θ *induces a function on sets*[117]

$$\theta_{\gamma'\gamma}\colon F(\gamma') \to F(\gamma),$$

for any $\gamma, \gamma' \in (0,1]$ *such that* $\gamma' \geq \gamma$*, satisfying the following conditions:*

(i) For all $\gamma \in (0,1]$*,* $\theta_{\gamma\gamma} = \mathrm{id}$ *for all* $\gamma \in (0,1]$*;*

(ii) If $\gamma, \gamma', \gamma'' \in (0,1]$ *such that* $\gamma'' \geq \gamma' \geq \gamma$*, then* $\theta_{\gamma''\gamma} = \theta_{\gamma'\gamma} \circ \theta_{\gamma''\gamma'}$*;*

(iii) For any $\gamma \in (0,1]$*, denote*

$$I_\gamma := \{q | \gamma < q < 1\} \cup \{1\}$$

to be its associated rounded ideal in $\mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$*, as defined in Example 2.2.51(ii).*

Then, the induced map

$$\theta_\gamma\colon \operatorname*{colim}_{q \in I_\gamma} F(q) \to F(\gamma)$$

is an isomorphism.

Proof. The proof involves performing various technical checks, but they all follow the same basic strategy: examine how the descent data θ imposes specific conditions on F, before leveraging them to deduce Conditions (i) - (iii).

[116]Warning: $\mathbb{Q}_{(0,1]}$ cannot be thought of as a naive subspace of $(0,1]$ since $\mathbb{Q}_{(0,1]}$ is discrete space and thus its topology is not the subspace topology inherited from $(0,1]$.

[117]Recall from Remark 6.1.10 that étale bundles are fibrewise discrete; hence, any morphism between their fibres (e.g. $\theta_{\gamma'\gamma}$) is a function on sets.

Step 0: Setup. We start by reformulating the action of the descent data. Given $\gamma, \gamma' \in (0,1]$ such that $\gamma' \geq \gamma$ and $z \in F(\gamma')$, define the following function of sets:

$$\begin{aligned} \theta_{\gamma'\gamma} \colon F(\gamma') &\longrightarrow F(\gamma) \\ z &\longmapsto \theta_0\left(z, \frac{\gamma}{\gamma'}\right) \end{aligned} \tag{6.37}$$

where $\theta_0 \colon Y \times (0,1] \to Y$ is the first coordinate map of θ as in Setup 6.3.4. We record two quick observations regarding $\theta_{\gamma'\gamma}$:

- $\theta_{\gamma'\gamma}$ is well-defined since $z \in F(\gamma') \subset Y$ and $\frac{\gamma}{\gamma'} \in (0,1]$.
- Item (iii) of Setup 6.3.4 tells us that the following identity holds:

$$f(\theta_0(z, \frac{\gamma}{\gamma'})) = f(z) \cdot \frac{\gamma}{\gamma'}.$$

  This equation makes precise how the multiplicative action on the base space $(0,1]$ (i.e. mapping $\gamma' \mapsto \gamma$) lifts to an action on the bundle space Y (i.e. mapping $F(\gamma') \to F(\gamma)$).

Step 1: Verifying Conditions (i) and (ii). Given this definition of $\theta_{\gamma'\gamma}$, Conditions (i) and (ii) essentially follows from the unit and cocycle condition on the descent data. [Why? For (i), notice the unit condition gives $\theta_0(y, \frac{\gamma}{\gamma}) = \theta_0(y, 1) = y$. Similarly for (ii), suppose $\gamma'' \geq \gamma' \geq \gamma$ in $(0,1]$, and denote $y'' \in Y$ such that $f(y'') = \gamma''$. The cocycle condition yields $\theta_0(y'', \frac{\gamma}{\gamma''}) = \theta_0(\theta_0(y'', \frac{\gamma'}{\gamma''}), \frac{\gamma}{\gamma'})$, which by Equation (6.37) gives $\theta_{\gamma''\gamma} = \theta_{\gamma'\gamma} \circ \theta_{\gamma''\gamma'}$.]

Step 2: Reformulating Condition (iii). Notice: $\operatorname*{colim}_{q \in I_\gamma} F(q)$ is a filtered colimit in Set, and so admits the canonical description

$$\operatorname*{colim}_{q \in I_\gamma} F(q) = \coprod_{q \in I_\gamma} F(q) / \sim$$

as a coproduct quotiented by the equivalence relation

$$(x, F(q)) \sim (y, F(q')) \leftrightarrow \exists r \in I_\gamma. \left(q \prec r \wedge q' \prec r \wedge \theta_{qr}(x) = \theta_{q'r}(y)\right),$$

where "$(x, F(q))$" denotes $x \in F(q)$ and "$(y, F(q'))$" denotes $y \in F(q')$.

As such, recall Condition (iii) involves verifying that the induced map $\theta_\gamma \colon \operatorname*{colim}_{q \in I_\gamma} F(q) \longrightarrow F(\gamma)$ is an isomorphism (in fact, a bijection of sets) for any $\gamma \in (0,1]$. Applying our above description of $\operatorname*{colim}_{q \in I_\gamma} F(q)$, explicitly this means verifying the following two sequents:

(a) $x \in F(\gamma) \longrightarrow \exists q \in I_\gamma. \left(\exists y \in F(q).(x = \theta_{q\gamma}(y)\right)$
(b) $y, z \in F(q), \theta_{q\gamma}(y) = \theta_{q\gamma}(z) \longrightarrow \exists r \in I_\gamma.(q \prec r \wedge \theta_{qr}(y) = \theta_{qr}(z))$

which correspond to verifying surjectivity and injectivity of θ_γ respectively.

Step 3: Verifying surjectivity. We structure the proof of Sequent (a) into two stages.

Step 3a: Setup. Suppose $x \in F(\gamma)$. Since $f\colon Y \to (0,1]$ is a local homeomorphism, there exists some open $U \subset Y$ such that $x \in U$ as well as a partial section $u\colon f(U) \to U$ where u induces an isomorphism $U \cong f(U)$. In particular, note $u(\gamma) = x$. Finally, since $(0,1]$ has a base of rational-ended open intervals, we may assume without loss of generality that $f(U)$ is of that form. More explicitly, we assume $f(U)$ is of the form (α, β) or $(\alpha, 1]$ where $0 \leq \alpha < \beta \leq 1$ for $\alpha, \beta \in \mathbb{Q}$.

Step 3b: Exploiting topology and the unit condition. Denote

$$X := f(U) \cap [\gamma, 1]$$

where $[\gamma, 1] := \{\gamma' \in (0,1] \,|\, \gamma \leq \gamma'\}$ denotes the obvious closed interval.[118] We then use the partial section $u\colon f(U) \to U$ to define the following map:

$$\begin{aligned} \Theta\colon X &\longrightarrow F(\gamma) \\ a &\longmapsto \theta_{a\gamma}(u(a)) \end{aligned}$$

Notice that X is inhabited, since $\gamma \in f(U) \cap [\gamma, 1]$ by construction. In fact, since $f(U) = (\alpha, \beta)$ or $f(U) = (\alpha, 1]$ by Step 1, it follows X can be characterised as one of the following (connected) subspaces of $(0,1]$:

- Case #1: $X = [\gamma, 1]$;
- Case #2: $X = [\gamma, \beta)$.

This description of X has two important implications. First, since X is a connected space and $F(\gamma)$ is a discrete space, the image of $\Theta(X)$ is constant. In particular, since

$$\Theta(\gamma) = \theta_{\gamma\gamma}(u(\gamma)) = \theta_0(x, 1) = x,$$

where the final equality is by the unit condition, this implies $\Theta(a) = x$ for all $a \in X$. Second, note that in both cases, there exists some rational $q \in X$ such that $q \in I_\gamma$. [Why? Case #2 is obvious. For Case #1, let $q = 1$. In particular, notice this works even when $X = \{1\}$ since $\prec$ on $\mathbb{Q}_{(0,1]}$ was defined to allow $1 \prec 1$.] Thus for $u(q) \in F(q)$, we obtain the identity $\Theta(q) = \theta_{q\gamma}(u(q)) = x$, proving Sequent (a).

Step 4: Verifying injectivity. The proof of Sequent (b) also proceeds in stages.

Step 4a: Setup. As our hypothesis, suppose $y, z \in F(q)$ such that $\theta_{q\gamma}(y) = \theta_{q\gamma}(z)$. For explicitness, denote $x := \theta_{q\gamma}(y) = \theta_{q\gamma}(z)$, which we observe to be an element of $F(\gamma)$. Next, define two maps $v, v'\colon (0, q] \to Y$ whereby $v(a) := \theta_{qa}(y)$ and $v'(a) := \theta_{qa}(z)$ respectively. In particular, notice:

- The images of v and v' coincide on γ, since

  $$v(\gamma) = \theta_{q\gamma}(y) = x = \theta_{q\gamma}(z) = v'(\gamma).$$

- v and v' are (partial) section maps of f since, e.g.:

  $$f \circ v(a) = f \circ (\theta_{qa}(y)) = a,$$

  for any $a \in (0, q]$.

Finally, just as in Step 3a, let $U \subset Y$ be an open subspace such that $x \in U$ equipped with a section $u\colon f(U) \xrightarrow{\sim} U$.[119]

[118] Notice this definition allows for the degenerate case $\gamma = 1$, in which case $[\gamma, 1] = \{1\}$.
[119] Unlike Step 3a, we do not require $f(U)$ to be connected.

Step 4b: Refinement of open subspaces. We work to identify an open subspace of $(0,1]$ on which all the section maps u, v, v' all agree. Notice: for our chosen $q \in I_\gamma$, either $\gamma < q$ or $q = 1$ (or both). As such, define the following subspace

$$V := \begin{cases} f(U) \cap (0,q), & \text{if } \gamma < q \\ f(U), & \text{if } q = 1 \end{cases} \tag{6.38}$$

In either case, one easily checks:

- V is an open subspace of $(0,1]$, and $\gamma \in V$;
- Both u and v are well-defined sections of f on the whole of V. In particular, the following diagram commutes:[120]

$$\begin{array}{ccc} V & \xhookrightarrow{u} & U \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & (0,1] \end{array} \tag{6.39}$$

Extending this, construct the obvious pullback:

$$\begin{array}{ccc} V_u & \longrightarrow & V \\ {\scriptstyle v_u}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u} \\ V & \xrightarrow{v} & Y \end{array} \tag{6.40}$$

In particular, observe the following:

- $\gamma \in V_u$.

  [Why? Recall $u(\gamma) = x = v(\gamma)$ by Step 4a.]
- $V_u \subset V$ defines an open subspace on which both v and u agree on.

  [Why? The fact that u, v agree on V_u follows from construction. The fact that V_u is an open subspace follows from noting that V itself is an open subspace, that $u\colon V \to Y$ is homeomorphic onto its image, and the general fact that pullbacks preserve open inclusions [Joh02b, pp. 504].]

Repeat the process to obtain an open inclusion $V'_u \hookrightarrow V$ of an open subspace V'_u on which u and v' agree, and also $\gamma \in V'_u$. Finally, repeat once more to obtain an open subspace $P \subset (0,1]$ on which u, v, v' all agree, and also $\gamma \in P$. By the same argument as in Step 3a, we assume without loss of generality that P is an open interval of the form (α, β) or $(\alpha, 1]$ where $\alpha, \beta \in \mathbb{Q}$ such that $0 \leq \alpha < \beta \leq q$.

Step 4c: Finish. By Step 4b, since $\gamma \in P$, we know that $\gamma \in (\alpha, \beta)$ or $(\alpha, 1]$ for appropriate rationals α, β. It is therefore clear there exists $r \in I_\gamma$ such that $q \prec r$ and $\theta_{qr}(y) = v(r) = v'(r) = \theta_{qr}(z)$. In particular, if $\gamma \in (\alpha, 1]$ then let $r = 1$, and if $\gamma \in (\alpha, \beta)$, then pick some rational r where $\alpha < \gamma < r < \beta$. ☐

As an immediate corollary of Claim 6.3.6, we get:

Corollary 6.3.7. *Any* $(F, \theta) \in \mathcal{D}'$ *defines a sheaf* $\overline{F} \in \mathcal{S}\overleftarrow{[0,1]}$.

[120] To ease notation, we will not use "$u|_V$" (resp. "$f|_U$") to express the restriction of u to V (resp. the restriction of f to U).

Proof. Represent $F \in \mathcal{S}(0,1]$ as a map $F\colon (0,1] \to [\mathbb{O}]$ to the object classifier, and consider its restriction $F_{\text{res}}\colon \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$. By Claim 6.3.6, F_{res} satisfies the continuity conditions required by item (iii) of Observation 2.2.58, and so (F,θ) defines a sheaf $\overline{F}$ on $\overleftarrow{[0,1]}$. $\square$

Remark 6.3.8. Let us flag a possible source of confusion. Recall the following argument employed in Steps 3a and 4a: "given a point $x \in F(\gamma)$, since $f\colon Y \to (0,1]$ is a local homeomorphism, we can pick an open $U \subset Y$ such that $x \in U \cong f(U)$". The cautious reader may ask: why is this argument geometrically justified? After all, given $x \in F(\gamma)$, there may in principle exist numerous opens $U \subset Y$ such that $x \in U \cong f(U)$. Hence, by picking a single open U, are we not implicitly invoking choice?

This touches upon a common misconception regarding "choice" vs. "existence" in constructive mathematics. To paraphrase Bauer [Bau17, §1.3], if we know (constructively) that "there exists an x satisfying property $\phi(x)$", then picking such an x is *not* an application of choice but rather an elimination of an existential quantifier. In our setting, recall from Remark 6.1.10 that local homeomorphisms can be characterised as maps $f\colon Y \to X$ such that both f and the diagonal $\Delta\colon Y \to Y \times_X Y$ are open. From this, one can deduce (constructively) that Y has a base of opens of the form U equipped with a unique section u. Hence, for any point $y \in Y$, one concludes that there exists an open $U \ni y$ with a section such that $u\colon f(U) \xrightarrow{\sim} U$.

Main Construction. We can now define our functor $\mathfrak{J}$. On the level of objects, we map:

$$\begin{aligned} \mathfrak{J}\colon \mathcal{D}' &\longrightarrow \mathcal{S}\overleftarrow{[0,1]} \\ (F,\theta) &\mapsto \quad \overline{F} \end{aligned}$$

where $\overline{F}$ is the sheaf over $\overleftarrow{[0,1]}$ associated to (F,θ) by Corollary 6.3.7.

On the level of morphisms, suppose we are given a morphism $u\colon (F,\theta) \to (G,\xi)$ of the lax descent category $\mathcal{D}'$. Let us pause to work out some of the details explicitly:

Observation 6.3.9.

(i) A $\mathcal{D}'$-morphism $u\colon (F,\theta) \to (G,\xi)$ is an $\mathcal{S}(0,1]$-morphism $u\colon F \to G$ (satisfying certain compatibility conditions). Representing $F, G\colon (0,1] \to [\mathbb{O}]$ as maps to the object classifier, the morphism u can be explicitly formulated as a (geometric) natural transformation, calculated point-wise:

$$\begin{array}{ccc} F(\gamma) & \xrightarrow{u_\gamma} & G(\gamma) \\ {\scriptstyle \theta_{\gamma'\gamma}}\uparrow & & \uparrow{\scriptstyle \xi_{\gamma'\gamma}} \\ F(\gamma') & \xrightarrow{u_{\gamma'}} & G(\gamma') \end{array}$$

where $\theta_{\gamma'\gamma}$ and $\xi_{\gamma'\gamma}$ are the maps induced by the respective descent data θ and ξ in the sense of Claim 6.3.6.

(ii) By Example 2.2.51 (ii), any upper real $\gamma \in \overleftarrow{[0,1]}$ corresponds to a rounded ideal $I_\gamma \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$. Applying Equation (2.5) from the proof of the Lifting Lemma 2.2.55, this gives the explicit characterisation $\overline{F}(I_\gamma) := \operatorname*{colim}_{q \in I_\gamma} F(q)$ and $\overline{G}(I_\gamma) := \operatorname*{colim}_{q \in I_\gamma} G(q)$.

Leveraging Observation 6.3.9, we now give a point-wise definition of $\mathfrak{J}(u)\colon \overline{F} \to \overline{G}$. Given an upper real $I_\gamma \in \overleftarrow{[0,1]}$, construct the following vertical composition of maps

$$
\begin{array}{ccc}
 & \underset{q\in I_\gamma}{\operatorname{colim}}\, G(q) & \\
\nearrow & & \nwarrow \\
G(q') & \xrightarrow{\quad \xi_{q'q} \quad} & G(q) \\
{\scriptstyle u_{q'}}\uparrow & & {\scriptstyle u_q}\uparrow \\
F(q') & \xrightarrow{\quad \theta_{q'q} \quad} & F(q)
\end{array}
\tag{6.41}
$$

where

- $q', q' \in I_\gamma$;
- The commutative square is the natural transformation square associated to $u\colon F \to G$;
- The upper triangle is the cocone associated to $\overline{G}(I_\gamma) = \underset{q\in I_\gamma}{\operatorname{colim}}\, G(q)$.

It is clear Diagram (6.41) defines a cocone over the diagram

$$\left\{ F(q') \xrightarrow{\theta_{q'q}} F(q) \right\}_{q,q' \in I_\gamma}$$

and thus by the universal property of colimits, this induces a map

$$\mathfrak{J}(u)(I_\gamma)\colon \overline{F}(I_\gamma) \to \overline{G}(I_\gamma)$$

which we define to be the image of morphism u under $\mathfrak{J}$. Since Diagram (6.41) is essentially defined via vertical composition, an easy check shows that $\mathfrak{J}$ as defined is indeed functorial.

6.3.2 Second Direction. We now construct functor inverse to $\mathfrak{J}$, i.e.

$$
\begin{aligned}
\mathfrak{K}\colon \mathcal{S}\overleftarrow{[0,1]} &\longrightarrow \mathcal{D}' \\
F &\mapsto ?
\end{aligned}
$$

Start by considering the following diagram of spaces:

$$(0,1] \times (0,1] \underset{\mathcal{M}}{\overset{\pi}{\rightrightarrows}} (0,1] \xrightarrow{\ r\ } \overleftarrow{[0,1]}$$

where $r\colon [0,1] \longrightarrow \overleftarrow{[0,1]}$ sends a Dedekind $\gamma \in (0,1]$ to its right Dedekind section. On the level of sheaves, the arrows reverse and the associated inverse image functors yield the following diagram

$$\mathcal{S}(0,1] \times \mathcal{S}(0,1] \underset{\mathcal{M}^*}{\overset{\pi^*}{\leftleftarrows}} \mathcal{S}(0,1] \xleftarrow{\ r^*\ } \mathcal{S}\overleftarrow{[0,1]}. \tag{6.42}$$

Observation 6.3.10. Let F be a sheaf over the space of upper reals $\overleftarrow{[0,1]}$. Examining Diagram (6.42), we record the following observations:

(i) *On* r^*. The functor r^* sends F to a sheaf $\widehat{F}$ over $(0,1]$.

(ii) *Pullback via* π^* *and* $\mathcal{M}^*$. Represent a sheaf $\widehat{F} \in \mathcal{S}(0,1]$ as a map $\widehat{F}\colon (0,1] \to [\mathbb{O}]$.

Then:

- $\pi^*(\widehat{F})\colon (0,1] \times (0,1] \to [\mathbb{O}]$ is a map that sends $(\alpha,\beta) \mapsto \widehat{F}(\alpha)$;
- $\mathcal{M}^*(\widehat{F})\colon (0,1] \times (0,1] \to [\mathbb{O}]$ is a map that sends $(\alpha,\beta) \mapsto \widehat{F}(\alpha\cdot\beta)$.

(iii) *Defining the cocone.* Recall: for any $\gamma \in (0,1]$, we denote

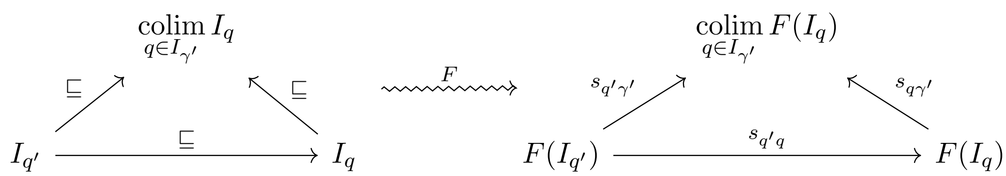
$$I_\gamma := \{q \in \mathbb{Q}_{(0,1]} \mid \gamma < q \text{ or } q = 1\} \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$$

to be its associated rounded ideal. In particular, note that $r(\gamma) = I_\gamma$ (cf. Convention 2.2.52).

Suppose we are given a sheaf $F\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$. Then for any $\gamma, \gamma', \in (0,1]$ such that $I_{\gamma'} \sqsubseteq I_\gamma$, functoriality of F yields a morphism

$$F(I_{\gamma'}) \xrightarrow{s_{\gamma'\gamma}} F(I_\gamma).$$

For later quotation, we explain how $s_{\gamma'\gamma}$ can be viewed as a map of cocones. First, apply F to the cocone diagram of $I_{\gamma'} = \operatorname*{colim}_{q \in I_{\gamma'}} I_q$ to get:



where the presentation of the RHS diagram is justified by the fact that

$$F(I_{\gamma'}) = F(\operatorname*{colim}_{q \in I_{\gamma'}} I_q) = \operatorname*{colim}_{q \in I_{\gamma'}} F(I_q),$$

since maps preserve filtered colimits (Lemma 2.2.53). Next, note that both $F(I_\gamma)$ and $F(I_{\gamma'})$ can be viewed as a cocone over the diagram:

$$\left\{ F(I_{q'}) \xrightarrow{s_{q'q}} F(I_q) \right\}_{q',q \in I_{\gamma'}} \tag{6.43}$$

By the universal colimit property, we know there exists a unique cocone map $F(I_{\gamma'}) \to F(I_\gamma)$ over Diagram (6.43). A routine check shows that this cocone map is in fact $s_{\gamma'\gamma}\colon F(I_{\gamma'}) \to F(I_\gamma)$.

(iv) *Defining descent data.* As before, suppose $F \in \mathcal{S}\overleftarrow{[0,1]}$. Note: for any $\alpha, \beta \in (0,1]$, we have $\alpha\cdot\beta \le \alpha$, or equivalently $I_\alpha \sqsubseteq I_{\alpha\cdot\beta}$. By Observation (iii), this gives the specialisation morphism

$$s_{\alpha(\alpha\cdot\beta)}\colon F(I_\alpha) \longrightarrow F(I_{\alpha\cdot\beta}). \tag{6.44}$$

Applying the functor $r^*\colon \mathcal{S}\overleftarrow{[0,1]} \to \mathcal{S}(0,1]$ to Equation (6.44), we get the morphism

$$\widehat{\theta}_{\alpha(\alpha\cdot\beta)}\colon \widehat{F}(\alpha) \longrightarrow \widehat{F}(\alpha\cdot\beta)$$

where $\widehat{\theta}_{\alpha(\alpha\cdot\beta)} := r^*\big(s_{\alpha(\alpha\cdot\beta)}\big)$. By Observation (ii), this data can be reformulated as defining an $\mathcal{S}(0,1] \times \mathcal{S}(0,1]$-morphism

$$\widehat{\theta}\colon \pi^*(\widehat{F}) \longrightarrow \mathcal{M}^*(\widehat{F}).$$

(v) $\widehat{\theta}$ *satisfies the unit condition.* This essentially follows from the fact that $\alpha \cdot 1 = \alpha$. [Why? Suppose, as before, $F\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$. Note that $\alpha \cdot 1 = \alpha$ implies that

$$F(I_\alpha) \xrightarrow{s_{\alpha(\alpha\cdot 1)}} F(I_{\alpha\cdot 1})$$

is the identity morphism, by functoriality. Applying r^* gives that

$$\widehat{\theta}_{\alpha(\alpha\cdot 1)}\colon \widehat{F}(\alpha) \to \widehat{F}(\alpha \cdot 1)$$

is also the identity morphism, again by functoriality.]

(vi) $\widehat{\theta}$ *satisfies the cocycle condition.* This essentially follows from the fact that multiplication is associative. [Why? Suppose we are given $\alpha, \beta, \beta' \in (0,1]$. Associativity of multiplication implies that $\alpha \cdot (\beta\beta') = \alpha\beta \cdot (\beta')$, and so

$$I_{\alpha\cdot(\beta\beta')} = I_{\alpha\beta\cdot(\beta')}.$$

Since the specialisation order $\sqsubseteq$ on $\overleftarrow{[0,1]}$ defines a unique morphism between its points, this means that the morphism

$$I_\alpha \sqsubseteq I_{\alpha\cdot(\beta\beta')}$$

is equivalent to the morphism

$$I_\alpha \sqsubseteq I_{\alpha\cdot\beta} \sqsubseteq I_{\alpha\beta\cdot\beta'}.$$

Appealing to functoriality once more, this means that the induced morphism:

$$\widehat{\theta}_{\alpha(\alpha\cdot\beta\beta')}\colon \widehat{F}(\alpha) \longrightarrow \widehat{F}(\alpha\cdot\beta\beta')$$

$$\widehat{\theta}_{\alpha(\alpha\beta)} \circ \widehat{\theta}_{\alpha\beta(\alpha\beta\cdot\beta')}\colon \widehat{F}(\alpha) \longrightarrow \widehat{F}(\alpha\cdot\beta) \longrightarrow \widehat{F}(\alpha\beta\cdot\beta')$$

are isomorphic, proving the cocycle condition.]

We now define our functor $\mathfrak{K}$. On the level of objects, we map:

$$\begin{aligned} \mathfrak{K}\colon \mathcal{S}\overleftarrow{[0,1]} &\longrightarrow \mathcal{D}' \\ F &\mapsto (\widehat{F}, \widehat{\theta}) \end{aligned}$$

where $\widehat{F}$ and $\widehat{\theta}$ are defined as in items (i) and (iv) of Observation 6.3.10. Note that this assignment well-defined: $\widehat{F} = r^*(F)$ is clearly a sheaf over $(0,1]$, and we've already checked that $\widehat{\theta}$ satisfies the required conditions for descent data (see items (v) and (vi) of Observation 6.3.10).

Now suppose we are given an $\mathcal{S}\overleftarrow{[0,1]}$-morphism $u\colon F \to G$ where $F, G\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$. Then, recalling Diagram (6.42), we define

$$\mathfrak{K}(u) := r^*(u)\colon \widehat{F} \to \widehat{G}.$$

We get functoriality of $\mathfrak{K}$ for free since $r^*\colon \mathcal{S}\overleftarrow{[0,1]} \to \mathcal{S}(0,1]$ is a functor. It only remains to show that $\mathfrak{K}(u)$ does in fact define a $\mathcal{D}'$-morphism, which we verify in the following claim.

Claim 6.3.11. *As our set-up, let:*

- $u\colon F \to G$ *be a morphism in* $F, G \in \mathcal{S}\overleftarrow{[0,1]}$*;*
- $(\widehat{F}, \widehat{\theta}_F), (\widehat{G}, \widehat{\theta}_G) \in \mathcal{D}'$ *be obtained by applying* $\mathfrak{K}$ *to sheaves* $F, G \in \mathcal{S}\overleftarrow{[0,1]}$

Then the following identity holds:

$$\mathcal{M}^*(\mathfrak{K}(u)) \circ \widehat{\theta}_F = \widehat{\theta}_G \circ \pi^*(\mathfrak{K}(u)).$$

Proof. By items (ii) and (iv) of Observation 6.3.10, the identity is equivalent to saying the following square commutes:

$$\begin{array}{ccc} \widehat{F}(\alpha) & \xrightarrow{\widehat{\theta}_F} & \widehat{F}(\alpha\cdot\beta) \\ {\scriptstyle \mathfrak{K}(u)}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{K}(u)} \\ \widehat{G}(\alpha) & \xrightarrow{\widehat{\theta}_G} & \widehat{G}(\alpha\cdot\beta) \end{array}$$

for generic point $(\alpha, \beta) \in (0, 1] \times (0, 1]$. In fact, something more general holds. Tracing through the details of Observation 6.3.10 (iv), one easily checks that for any $\gamma, \gamma' \in (0, 1]$ such that $\gamma \leq \gamma'$, we get

$$\begin{array}{ccc} \widehat{F}(\gamma') & \xrightarrow{\widehat{\theta}_F} & \widehat{F}(\gamma) \\ {\scriptstyle \mathfrak{K}(u)}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{K}(u)} \\ \widehat{G}(\gamma') & \xrightarrow{\widehat{\theta}_G} & \widehat{G}(\gamma) \end{array}$$

In the language of Observation 6.3.9 (i), this shows that $\mathfrak{K}(u)$ is a geometric natural transformation. □

6.3.3 Assemble and Finish. With the main constructions completed, all the gears line up and we now prove the main result of the section.

Theorem G. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$.

Proof. Let $\mathfrak{J} \colon \mathcal{D}' \to \overleftarrow{[0,1]}$ and $\mathfrak{K} \colon \mathcal{S}\overleftarrow{[0,1]} \to \mathcal{D}'$ be the two functors as defined in Sections 6.3.1 and 6.3.2. It remains to show that $\mathfrak{J}, \mathfrak{K}$ are inverse to each other. This means performing the following two checks:

Step 1: Verifying $\mathfrak{K} \circ \mathfrak{J} \cong \mathrm{id}_{\mathcal{D}'}$. Let $(F, \theta) \in \mathcal{D}'$. We know from Claim 6.3.6 (ii) that there exists a map θ_γ inducing an isomorphism

$$\theta_\gamma \colon \operatorname*{colim}_{q \in I_\gamma} F(q) \xrightarrow{\sim} F(\gamma).$$

In fact, in our present setting, the map θ_γ induces much more:

Claim 6.3.12. *Let $(\widehat{\overline{F}}, \widehat{\theta}) := \mathfrak{K} \circ \mathfrak{J}(F, \theta)$. Then, θ_γ induces an isomorphism $(F, \theta) \cong (\widehat{\overline{F}}, \widehat{\theta})$.*

Proof of Claim. This amounts to checking the following:

(a) *$\widehat{\overline{F}} \cong F$ as sheaves.* Recall from Observation 6.3.10 (iii) that $r \colon (0, 1] \to \overleftarrow{[0,1]}$ sends $\gamma \in (0, 1]$ to its associated rounded ideal $I_\gamma \in \overleftarrow{[0,1]}$. Unwinding definitions, we obtain the identity

$$\widehat{\overline{F}}(\gamma) = \overline{F}\big(r(\gamma)\big) = \overline{F}(I_\gamma) = \operatorname*{colim}_{q \in I_\gamma} F(q).$$

Since $\operatorname*{colim}_{q \in I_\gamma} F(q) \cong F(\gamma)$, conclude that $\widehat{\overline{F}}(\gamma) \cong F(\gamma)$ for any $\gamma \in (0, 1]$.

(b) $\widehat{\theta} \cong \theta$ *as morphisms.* The analysis proceeds by successive refinements of our original claim.

The first refinement. By Observation 6.3.10 (ii), note that $\widehat{\theta}\colon \pi^*(\widehat{\overline{F}}) \to \mathcal{M}^*(\widehat{\overline{F}})$ can be reformulated as $\widehat{\theta}\colon \widehat{\overline{F}}(\alpha) \to \widehat{\overline{F}}(\alpha \cdot \beta)$, for generic $(\alpha, \beta) \in (0, 1] \times (0, 1]$. The same reasoning can also be applied to reformulate $\theta\colon \pi^*(F) \to \mathcal{M}^*(F)$ as $\theta\colon F(\alpha) \to F(\alpha \cdot \beta)$. As such, the claim that θ_γ induces an isomorphism between $\widehat{\theta}$ and θ as morphisms of sheaves is equivalent to saying that the following diagram commutes[121]:

$$\begin{array}{ccc} \widehat{\overline{F}}(\alpha) & \xrightarrow{\theta_\alpha} & F(\alpha) \\ {\scriptstyle\widehat{\theta}}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ \widehat{\overline{F}}(\alpha \cdot \beta) & \xrightarrow{\theta_{\alpha\cdot\beta}} & F(\alpha \cdot \beta) \end{array} \tag{6.45}$$

for generic $(\alpha, \beta) \in (0, 1] \times (0, 1]$.

The second refinement. Since maps preserve filtered colimits, we know that

$$\widehat{\overline{F}}(\gamma) = \operatorname*{colim}_{q \in I_\gamma} F(q) = \operatorname*{colim}_{q \in I_\gamma} F(\operatorname*{colim}_{q' \in I_q} q') = \operatorname*{colim}_{q \in I_\gamma} \overline{F}(I_q)$$

for any $\gamma \in (0, 1]$. Hence, we may reformulate Diagram (6.45) as:

$$\begin{array}{ccc} \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) = \operatorname*{colim}_{q \in I_\alpha} F(q) & \xrightarrow{\theta_\alpha} & F(\alpha) \\ {\scriptstyle\widehat{\theta}}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ \operatorname*{colim}_{q \in I_{\alpha\cdot\beta}} \overline{F}(I_q) = \operatorname*{colim}_{q \in I_{\alpha\cdot\beta}} F(q) & \xrightarrow{\theta_{\alpha\cdot\beta}} & F(\alpha \cdot \beta) \end{array} \tag{6.46}$$

The re-appearance of colimits is suggestive. In particular, we make the following key observation. Recalling the morphisms defined in Observation 6.3.10 (iii), suppose all four corners of Diagram (6.46) can be regarded as cocones over the diagram

$$\left\{ s_{q'q}\colon \overline{F}(I_{q'}) \to \overline{F}(I_q) \right\}_{q', q \in I_\alpha} \tag{6.47}$$

By the universal colimit property, we know there exists a unique cocone map

$$\operatorname*{colim}_{q \in I_\alpha} F(q) \to F(\alpha \cdot \beta).$$

Hence, if $\theta \circ \theta_\alpha$ and $\theta_{\alpha\cdot\beta} \circ \widehat{\theta}$ both define cocone maps $\operatorname*{colim}_{q \in I_\alpha} F(q) \to F(\alpha \cdot \beta)$ over Diagram (6.47), then they must both be equivalent, i.e. Diagram (6.45) commutes.

The third and final refinement. We now work to clarify: how might $F(\alpha \cdot \beta)$ be regarded as a cocone over Diagram (6.47)? We start with $\operatorname{colim}_{q \in I_\alpha} \overline{F}(I_q)$. By items (iii) and (iv) of Observation 6.3.10,

[121] A remark on notation: We have been careful to denote the multiplication $\mathcal{M}(\alpha, \beta) = \alpha \cdot \beta$ as opposed to just $\mathcal{M}(\alpha, \beta) = \alpha\beta$. One reason for this is to emphasise that an algebraic action has taken place. Another reason is to reduce potential confusion between the morphism $\theta_{(\alpha\cdot\beta)}$ in Diagram (6.45) and the morphism $\theta_{\alpha\beta}\colon F(\alpha) \to F(\beta)$ as defined in Equation (6.37).

we get the cocone

$$\begin{array}{ccc} & \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) & \\ {\scriptstyle s_{q'\alpha}} \nearrow & & \nwarrow {\scriptstyle s_{q\alpha}} \\ \overline{F}(I_{q'}) & \xrightarrow{\quad s_{q'q} \quad} & \overline{F}(I_q) \end{array} \tag{6.48}$$

along with the fact that

$$\widehat{\theta}\colon \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) \longrightarrow \operatorname*{colim}_{q \in I_{\alpha \cdot \beta}} \overline{F}(I_q)$$

corresponds to the induced cocone map

$$s_{\alpha(\alpha\cdot\beta)}\colon \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) \longrightarrow \operatorname*{colim}_{q \in I_{\alpha \cdot \beta}} \overline{F}(I_q) \tag{6.49}$$

As such, composing Diagram (6.48) with $\theta_{\alpha\cdot\beta} \circ \widehat{\theta}$ in the obvious way, we get the following representation of $F(\alpha \cdot \beta)$ as a cocone over Diagram (6.47):

$$\begin{array}{ccc} & F(\alpha \cdot \beta) & \\ {\scriptstyle \theta_{\alpha\cdot\beta} \circ s_{q'(\alpha\cdot\beta)}} \nearrow & & \nwarrow {\scriptstyle \theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)}} \\ \overline{F}(I_{q'}) & \xrightarrow{\quad s_{q'q} \quad} & \overline{F}(I_q) \end{array} \tag{6.50}$$

By a similar argument, the map $\theta \circ \theta_\alpha$ induces the cocone:

$$\begin{array}{ccc} & F(\alpha \cdot \beta) & \\ {\scriptstyle \theta \circ \theta_\alpha \circ s_{q'\alpha}} \nearrow & & \nwarrow {\scriptstyle \theta \circ \theta_\alpha \circ s_{q\alpha}} \\ \overline{F}(I_{q'}) & \xrightarrow{\quad s_{q'q} \quad} & \overline{F}(I_q) \end{array} \tag{6.51}$$

Our present task now reduces to understanding how Diagrams (6.50) and (6.51) define the same cocone diagram. More explicitly, we wish to prove the identity

$$\theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)} = \theta \circ \theta_\alpha \circ s_{q\alpha} \tag{6.52}$$

for any $q \in I_\alpha$. To do this, let us review our proof of Key Claim 6.3.6. Read in our present context, it defines an isomorphism $\theta_\gamma\colon \operatorname*{colim}_{q \in I_\gamma} F(q) \to F(\gamma)$ for any $\gamma \in (0,1]$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname*{colim}_{q \in I_\gamma} \overline{F}(I_q) = \operatorname*{colim}_{q \in I_\gamma} F(q) & \xrightarrow{\theta_\gamma} & F(\gamma) \\ {\scriptstyle s_{q\gamma}} \uparrow & \nearrow {\scriptstyle \theta_{q\gamma}} & \\ \overline{F}(I_q) = F(q) & & \end{array} \tag{6.53}$$

As an easy consequence, we deduce that

$$\theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)} = \theta_{q(\alpha\cdot\beta)}. \tag{6.54}$$

Further, by examining definitions, it is also clear that the descent data

$$\theta\colon F(\alpha) \to F(\alpha \cdot \beta)$$

coincides with the induced morphism

$$\theta_{\alpha(\alpha\cdot\beta)}\colon F(\alpha) \to F(\alpha \cdot \beta)$$

as defined in Key Claim 6.3.6. As such, compute that:

$$\begin{aligned} \theta \circ \theta_\alpha \circ s_{q\alpha} &= \theta_{\alpha(\alpha\cdot\beta)} \circ \theta_\alpha \circ s_{q\alpha} && [\text{since } \theta = \theta_{\alpha(\alpha\cdot\beta)}] \\ &= \theta_{\alpha(\alpha\cdot\beta)} \circ \theta_{q\alpha} && [\text{by Diagram (6.53)}] \\ &= \theta_{q(\alpha\cdot\beta)} && [\text{by Key Claim 6.3.6 (ii)}] \end{aligned} \tag{6.55}$$

By Equations (6.54) and (6.55), we deduce Equation (6.52), proving our claim that $\widehat{\theta} \cong \theta$.

□

To complete Step 1, we need to check one final claim:

Claim 6.3.13. *Let $u\colon (F,\theta) \to (G,\xi)$ be a $\mathcal{D}'$-morphism. Then, $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$.*

Proof of Claim. The argument is similar to our proof that $\widehat{\theta} \cong \theta$: reformulate the morphisms as suitable maps of cocones, before checking the equivalence in the new language. We give a quick sketch and leave the details to the reader. Start by noting that $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$ follows from proving that the following diagram commutes for any $\gamma \in (0,1]$:

$$\begin{array}{ccc} F(\gamma) & \xrightarrow{\quad u \quad} & G(\gamma) \\ {\scriptstyle \theta_\gamma}\uparrow & & {\scriptstyle \xi_\gamma}\uparrow \\ \operatorname*{colim}_{q\in I_\gamma} F(q) & \xrightarrow{\mathfrak{K}\circ\mathfrak{J}(u)} & \operatorname*{colim}_{q\in I_\gamma} G(q) \end{array} \tag{6.56}$$

Next, by unwinding definitions and applying Claim 6.3.6 if necessary, verify that θ_γ and $\mathfrak{K} \circ \mathfrak{J}(u)$ define cocone maps over the diagram

$$\left\{ F(q') \xrightarrow{\theta_{q'q}} F(q) \right\}_{q,q' \in I_\gamma}, \tag{6.57}$$

before checking that $u \circ \theta_\gamma$ and $\xi_\gamma \circ \mathfrak{K} \circ \mathfrak{J}(u)$ define the same cocone representation of $G(\gamma)$ over Diagram (6.57). This implies Diagram (6.56) commutes, and so $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$, as claimed. □

Summarising, since

- $\mathfrak{K} \circ \mathfrak{J}(F,\theta) \cong (F,\theta)$ for any $(F,\theta) \in \mathcal{D}'$, by Claim 6.3.12
- $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$ for any $\mathcal{D}'$-morphism $u : (F,\theta) \to (G,\xi)$, by Claim 6.3.13

we conclude that $\mathfrak{K} \circ \mathfrak{J} \cong \mathrm{id}_{\mathcal{D}'}$, finishing Step 1.

Step 2: Verifying $\mathfrak{J} \circ \mathfrak{K} \cong \mathrm{id}_{\mathcal{S}\overleftarrow{[0,1]}}$. As our setup:

- Let F be a sheaf over $\overleftarrow{[0,1]}$;
- Denote $(\widehat{F}, \widehat{\theta}) := \mathfrak{K}(F)$ and $\overline{\widehat{F}} := \mathfrak{J} \circ \mathfrak{K}(F)$;
- Let $u\colon F \to G$ be a morphism of sheaves over $\overleftarrow{[0,1]}$.

To prove that $\mathfrak{J} \circ \mathfrak{K} \cong \mathrm{id}_{\mathcal{S}\overleftarrow{[0,1]}}$, we need to check two things:

(a) $\overline{\widehat{F}} \cong F$ *as sheaves.* For any upper real $I_\gamma \in \overleftarrow{[0,1]}$, one easily verifies that

$$\overline{\widehat{F}}(I_\gamma) = \operatorname*{colim}_{q \in I_\gamma} \widehat{F}(q) = \operatorname*{colim}_{q \in I_\gamma} F(r(q)) = \operatorname*{colim}_{q \in I_\gamma} F(I_q) = F(I_\gamma).$$

(b) $\mathfrak{J} \circ \mathfrak{K}(u) \cong u$ *as morphisms.* For any upper real $I_\gamma \in \overleftarrow{[0,1]}$, it is clear $u\colon F \to G$ can be defined level-wise as a map of cocones

$$\begin{array}{ccc} \operatorname*{colim}_{q\in I_\gamma} F(I_q) & \overset{u}{\dashrightarrow} & \operatorname*{colim}_{q\in I_\gamma} G(I_q) \\ {\scriptstyle s_{q\gamma}}\uparrow & \nearrow_{u\circ s_{q\gamma}} & \\ F(I_q) & & \end{array} \tag{6.58}$$

over the diagram

$$\left\{ F(I_{q'}) \xrightarrow{s_{q'q}} F(I_q) \right\}_{q,q' \in I_\gamma}, \tag{6.59}$$

where $s_{q'q}$ and $s_{q\gamma}$ are the morphisms from Observation 6.3.10 (iii). Unwinding definitions, it is clear $\mathfrak{J} \circ \mathfrak{K}(u)$ also defines a map between the cocones of Diagram (6.58). By the universal colimit property, we conclude that $\mathfrak{J} \circ \mathfrak{K}(u) \cong u$.

This completes Step 2, and we are done. □

Finally, given the association between group completion and standard descent (cf. Discussion 6.1.15), one is naturally led to ask if taking the standard descent in this context results in a loss of information. The following observation confirms this:

Observation 6.3.14. Denote Z to be the space corresponding to the standard descent topos of Construction 6.3.1. Then $Z = \{*\}$.

Proof. By construction, Z is the coequaliser of $\pi, \mathcal{M}\colon (0,1] \times (0,1] \to (0,1]$ regarded as a diagram of spaces (subject, of course, to the descent conditions). In particular, denote:

- $p\colon (0,1] \to Z$ to be the (universal) quotient map;
- Z' to be the image of $(0,1] \times (0,1]$ under the map $p \circ \pi$ (or equivalently, $p \circ \mathcal{M}$), along with the obvious inclusion map $i\colon Z' \hookrightarrow Z$.

Further, notice that:

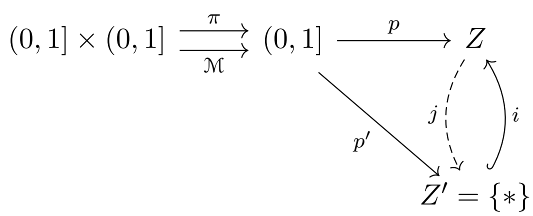
- $Z' = \{*\}$.

  [Why? Let $(\alpha, \beta), (\alpha', \beta') \in (0, 1] \times (0, 1]$ be any two (pairs of) points in $(0, 1] \times (0, 1]$. Notice that there exists $\gamma \in (0, 1]$ such that $\gamma \cdot \beta' = \beta$ or $\gamma \cdot \beta = \beta'$. If $\gamma \cdot \beta' = \beta$, one computes

  $$p \circ \pi(\alpha, \beta) = p(\beta) = p(\gamma \cdot \beta') = p \circ \mathcal{M}(\gamma, \beta') = p \circ \pi(\alpha', \beta').$$

  The argument when $\gamma \cdot \beta = \beta'$ is entirely symmetric, and so deduce that $Z' = \{*\}$.]
- There exists a unique map $p' \colon (0, 1] \to \{*\}$ into the singleton space.

We can therefore exploit the universal property of coequalisers to assemble the data into the following diagram:



(6.60)

It remains to show that $Z \cong \{*\}$. The fact that $j \circ i = \mathrm{id}_{\{*\}}$ is obvious. For the converse direction, compute for any $\beta \in (0, 1]$:

$$i \circ j \circ p(\beta) = i \circ p'(\beta) = i \circ p' \circ \pi(\beta, \beta) = p \circ \pi(\beta, \beta) = p(\beta),$$

where $i \circ p' \circ \pi = p \circ \pi$ by definition of Z'. Since p is an epi, this implies $i \circ j = \mathrm{id}_Z$. □

6.4 Discussion: Non-Trivial Forking of Sheaves

In this section, we work to sift out the differences between the Archimedean vs. non-Archimedean case. By Theorems F and G, we already know that $\mathcal{D} \simeq \mathrm{Set}$ whereas $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0, 1]}$. Motivated by this, we ask:

Question 14. What kinds of sheaves are eliminated by standard vs. lax descent data? Alternatively, how wild or complicated are the sheaves of $\mathcal{D}'$ compared to those of $\mathcal{D}$?

The following basic observation tells us where to start looking.

Observation 6.4.1. As our setup, let:

- X be a locally connected localic space;
- F be a sheaf on X, which we represent as an étale bundle $f \colon Y \to X$.

Then, Y is locally connected. In particular, there exists a pairwise disjoint decomposition of Y into (a set of) connected open subspaces:

$$Y = \coprod_{i \in I} U_i. \tag{6.61}$$

Proof. By definition, since X is locally connected, every open subspace of X is expressible as a union of connected open subspaces. Since f is a local homeomorphism, this gives Y an open cover of locally connected subspaces, and so Y is locally connected as well. In particular, applying [Joh02b, Lemma C.1.5.8], one obtains a pairwise disjoint decomposition of Y into open connected components. □

In particular, recall that:

- Any $(F, \theta) \in \mathcal{D}$ defines a sheaf F on $(0, \infty)$;
- Any $(F', \theta') \in \mathcal{D}'$ defines a sheaf F' on $(0, 1]$.

Since $(0, \infty)$ and $(0, 1]$ are both locally connected localic spaces, Observation 6.4.1 suggests that analysis of sheaves in $\mathcal{D}$ or $\mathcal{D}'$ ought to be reducible to analysis of their connected components. Leveraging this insight, we establish the next series of observations.

Observation 6.4.2. As our setup, let:

- $\mathcal{D}$ be the topos of a single non-trivial non-Archimedean place, as in Theorem F;
- $(F, \theta) \in \mathcal{D}$, where F corresponds to an étale bundle $f \colon Y \to (0, \infty)$;
- $\mathrm{id}_{(0,\infty)} \colon (0, \infty) \to (0, \infty)$

Then, the étale bundle f can be represented as the following disjoint (set-indexed) coproduct

$$f \cong \coprod_I \mathrm{id}_{(0,\infty)}.$$

Proof. By Theorem F, the (inverse image functor of the) unique geometric morphism

$$\gamma^* \colon \mathrm{Set} \to \mathcal{D}$$

induces an equivalence of categories. The following observations clarify our setup:

(a) Since γ^* induces an equivalence of categories, it must be essentially surjective, and so there exists $I \in \mathrm{Set}$ such that $\gamma^*(I) \cong f$.[122]

(b) Represent a singleton $\{*\} \in \mathrm{Set}$ as an étale bundle $\mathrm{id}_{\{*\}} \colon \{*\} \to \{*\}$. Then, any set S can be represented as the disjoint coproduct $S \cong \coprod_S \mathrm{id}_{\{*\}}$;

(c) It is clear that $\gamma^*(\mathrm{id}_{\{*\}}) \cong \mathrm{id}_{(0,\infty)}$, since pulling back along $(0, \infty) \to \{*\}$ gives

$$\begin{array}{ccc} (0,\infty) & \longrightarrow & \{*\} \\ {\scriptstyle \mathrm{id}_{(0,\infty)}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\{*\}}} \\ (0,\infty) & \longrightarrow & \{*\} \end{array}$$

Since γ^* preserves arbitrary colimits, Observations (a) - (c) give

$$f \cong \gamma^*(I) = \gamma^*(\coprod_I \mathrm{id}_{\{*\}}) = \coprod_I \gamma^*(\mathrm{id}_{\{*\}}) \cong \coprod_I \mathrm{id}_{(0,\infty)}.$$

□

[122] In particular, we may regard γ^* as pulling back the set I (viewed as a bundle over $\{*\}$) along the unique map $(0, \infty) \to \{*\}$ to obtain an étale bundle $f \colon Y \to (0, \infty)$.

Now suppose $(F, \theta) \in \mathcal{D}$ is a connected sheaf, i.e. F corresponds to an étale bundle $f: Y \to (0, \infty)$ where Y is connected. Observation 6.4.2 then forces $Y \cong (0, \infty)$, as illustrated in Figure 6.1.

Figure 6.1: A connected sheaf F of $\mathcal{D}$

Since we know $\mathcal{D}' \not\simeq \mathrm{Set}$ by Theorem G, one naturally suspects that the connected sheaves of $\mathcal{D}'$ are no longer quite as simple. The first indications of this can already be seen in the following example:

Example 6.4.3. Define $(F, \theta) \in \mathcal{D}'$ where

- F corresponds to the inclusion map $f: (0, \alpha) \to (0, 1]$, regarded as an étale bundle over $(0, 1]$;
- θ is the descent data whose first coordinate map is defined as

$$\begin{aligned} \theta_0 : (0, \alpha) \times (0, 1] &\longrightarrow (0, \alpha) \\ (y, \beta) &\longmapsto (y \cdot \beta) \end{aligned} \tag{6.62}$$

[Why is this sufficient? Recall from Setup 6.3.4 that descent data θ is determined by the first coordinate map θ_0. One then easily verifies that our θ_0 satisfies the unit and cocycle conditions.]

In particular, it is clear by inspection that $f((0, \alpha)) \cong (0, \alpha)$.

Example 6.4.3 signals an interesting difference with $\mathcal{D}$: the connected sheaves of $\mathcal{D}'$ need not be homeomorphic to the base space $(0, 1]$. In fact, they can turn out to be much more complicated:

Example 6.4.4 (Tuning Fork Sheaf)**.** Following Example 6.4.3, define $(F, \theta), (F', \theta') \in \mathcal{D}'$ whereby:

- F corresponds to the inclusion map $f: (0, \frac{1}{2}) \hookrightarrow (0, 1]$; and
- F' corresponds to the identity map $f': (0, 1] \to (0, 1]$,
- The corresponding descent data θ and θ' both act by multiplication, analogous to Equation (6.62).

Now observe: the inclusion map $(0, \frac{1}{2}) \hookrightarrow (0, 1]$ also induces a bundle map between f and f'. Since $\mathcal{D}'$ is a topos and toposes possess all pushouts, the cokernel pair of this bundle map exists, which we illustrate in Figure 6.2. For obvious reasons, we shall call this pushout sheaf the *Tuning Fork Sheaf*. In particular, since the pushout construction glues two connected spaces along a common subspace, one easily checks that the Tuning Fork sheaf is itself connected.[123]

[123]More rigorously, denote $g: Z \to (0, 1]$ to be the étale bundle corresponding to the Tuning Fork Sheaf. To show that Z is connected, we need to show that any map $h: Z \to S$ to a discrete space S is constant. As such, define two global sections $p_1, p_2 : (0, 1] \to Z$, one which maps the subspace $[\frac{1}{2}, 1]$ to the lower branch of Z, while the other maps it to the upper branch. Since $(0, 1]$ is connected, we know that $h \circ p_1$ and $h \circ p_2$ are both constant. Now let $\gamma \in [\frac{1}{2}, 1]$ and $\gamma' \in (0, \frac{1}{2})$. Since $p_1(\gamma') = p_2(\gamma')$, conclude that h is constant by observing: $h \circ p_1(\gamma) = h \circ p_1(\gamma') = h \circ p_2(\gamma') = h \circ p_2(\gamma)$.

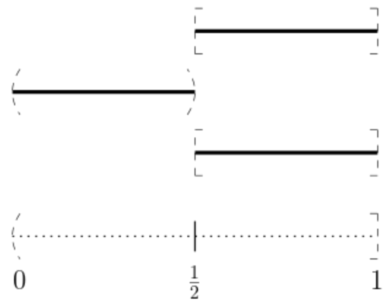
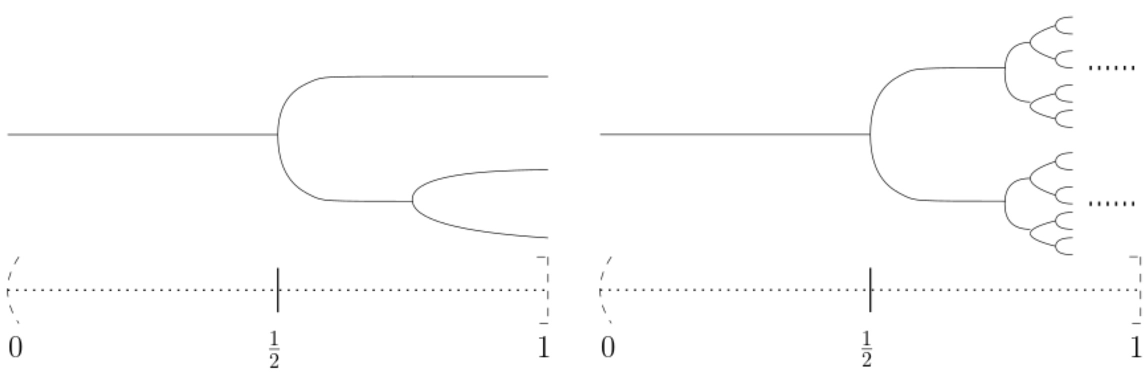


Figure 6.2: The Tuning Fork Sheaf of $\mathcal{D}'$

Discussion 6.4.5. The construction in Example 6.4.4 is fairly flexible, and can be used to construct a wide variety of forking structures in the [connected components of the] sheaves of $\mathcal{D}'$. This gives a new way to read the difference between standard descent vs. lax descent. In the non-Archimedean case, the rich sheafy structure over the original base space is completely flattened by the descent data: as shown in Observation 6.4.2 and illustrated by Figure 6.1, the connected sheaves are forced to be homeomorphic to $(0, \infty)$. In the Archimedean case, where the lax descent is comparatively weaker, this is no longer true. Example 6.4.4 gives an example of non-trivial forking persisting in the connected sheaves of $\mathcal{D}'$.

Discussion 6.4.5 gives an insight into the difference between the non-Archimedean vs. Archimedean case by identifying the kinds of sheaves present in $\mathcal{D}'$ (but absent in $\mathcal{D}$). For the rest of this section, we round out our understanding by identifying the kinds of sheaves which do *not* exist in $\mathcal{D}'$.

Example 6.4.6. Developing Discussion 6.4.5, note that there was nothing special about our choice of inclusion map $(0, \frac{1}{2}) \hookrightarrow (0, 1]$ in Example 6.4.4. In fact, one can iterate the argument to obtain the sheaf as illustrated in Figure 6.3a. However, a warning: there are limits to how far we can push this. For instance, we cannot iterate the forking construction for each branch of the 'fork' indefinitely, as illustrated in Figure 6.3b. Why? Note that the bundle space over 1 in Figure 6.3b gives the Cantor set, which is profinite. Since étale bundles must be fibrewise discrete, this means that Figure 6.3b no longer defines a sheaf over $(0, 1]$.

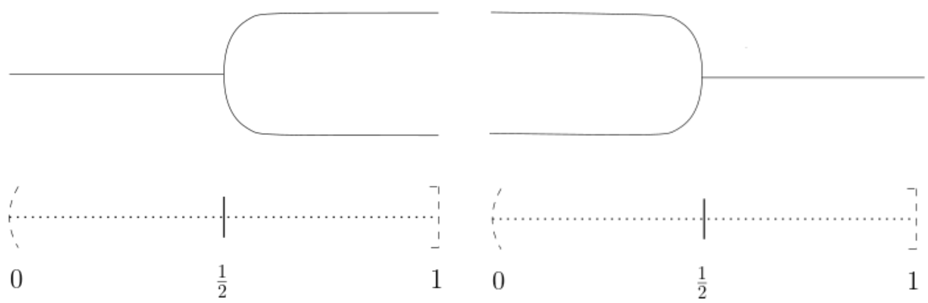


(a) Iterated Forking

(b) A Forbidden Configuration

Figure 6.3

Let us sharpen our language regarding this forking phenomena. Given an arbitrary sheaf $F \in \mathcal{S}(0,1]$, say that F witnesses *upper bound forking* if there exists a connected component (cf. Observation 6.4.1) with two branches on the right of the branching point and one on its left (as illustrated in Figure 6.4a). Analogously, say that F witnesses *lower bound forking* if there exists two branches on the left of the branching point and one on its right (as illustrated in Figure 6.4b). In principle, there may be multiple instances of forking (see, e.g. Figure 6.3a), but we shall always assume that the branches of the fork do not 'join' back up.



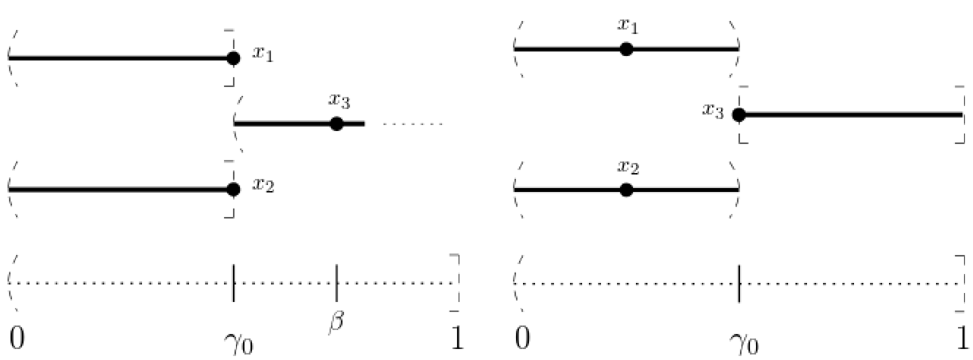
(a) Upper Bound Forking (b) Lower Bound Forking

Figure 6.4: Two Types of Forking with Branching Point at $\frac{1}{2}$

Claim 6.4.7. *For any* $(F, \theta) \in \mathcal{D}'$, $F \in \mathcal{S}(0,1]$ *does not witness lower bound forking.*

Proof. The argument proceeds in stages.

Step 0: Setup. Let $(F, \theta) \in \mathcal{D}'$. Suppose, for contradiction, that F witnesses an instance of lower bound forking. In principle, there may be multiple instances of forking in Y_i, but let us first assume for simplicity there only exists a single instance of lower bound forking — say in component Y_i at some $\gamma_0 \in (0,1)$. We can give the following explicit representation of Y_i: regard the obvious inclusions $(\gamma_0, 1] \hookrightarrow (0,1]$ and $(0,1] \hookrightarrow (0,1]$ as étale bundles over $(0,1]$, before obtaining Y_i as the cokernel pair of $(\gamma_0, 1] \hookrightarrow (0,1]$. See Figure 6.5a below for an illustration.



(a) The Connected Component Y_i (b) Another Forbidden Configuration

Figure 6.5

Step 1: A remark on the fibre at γ_0. Note that the forking begins *at* the branching point γ_0 and not after. More explicitly, our construction gives $Y_i \cap F(\alpha) = 2$ for all $\alpha \in (0, \gamma_0]$, where 2 denotes the discrete space of two points. In fact, this is canonical. How so? One may reasonably wonder if we can construct a sheaf that witnesses an instance of lower bound forking at γ_0 such that $Y_i \cap F(\gamma_0) = \{*\}$, as illustrated in Figure 6.5b. However, this is equivalent to taking the cokernel pair of $[\gamma_0, 1] \hookrightarrow (0, 1]$, which no longer defines an étale bundle — in particular, note that any open containing x_3 in the bundle space of Figure 6.5b will always contain parts of the two branches, obstructing local homeomorphism. In other words, despite the family resemblance with Figure 6.5a, the bundle space of Figure 6.5b does not depict a sheaf in $\mathcal{S}(0, 1]$.

Step 2: Analysis of Descent Data. To orient ourselves, we ask: how does the descent data θ of (F, θ) interact with the forking structure of Y_i? Recall from Key Claim 6.3.6 that θ induces maps on the fibres of F

$$\theta_{\gamma'\gamma} \colon F(\gamma') \to F(\gamma),$$

for any pair of Dedekinds $\gamma', \gamma \in (0, 1]$ such that $\gamma \leq \gamma'$. In particular, note: since θ satisfies the unit condition, we get the identity $\theta_{\gamma\gamma}(x) = x$, given any $\gamma \in (0, 1]$ and any $x \in F(\gamma)$. Read in our context, this allows us to make a series of useful deductions.

Step 2a: θ restricts nicely to Y_i. Let $\beta \in (\gamma_0, 1]$. By Step 1, there exists distinct points $x_1, x_2, x_3 \in Y_i$ such that $x_1, x_2 \in F(\gamma_0)$ and $x_3 \in F(\beta)$, as depicted in Figure 6.5a. Given that $\theta_{\beta\beta}(x_3) = x_3 \in Y_i$, and that Y_i is a connected component disjoint from other components of F, deduce that $\theta_{\beta\gamma}(x_3) \in Y_i$ for any $\gamma \in (0, \beta]$. In particular, this implies that either $\theta_{\beta\gamma_0}(x_3) = x_1$ or $\theta_{\beta\gamma_0}(x_3) = x_2$.

Step 2b: No jumps. Given Step 2a, assume without loss of generality that $\theta_{\beta\gamma_0}(x_3) = x_1$. Then, for any $y \in F(\gamma) \cap Y_i$ whereby $\gamma \in (\gamma_0, \beta]$, we claim that $\theta_{\gamma\gamma_0}(y) = x_1$. Why? First note that $F(\gamma) \cap Y_i = \{*\}$, essentially by construction. As such, since Step 2a gives $\theta_{\beta\gamma}(x_3) \in Y_i$, deduce that $\theta_{\beta\gamma}(x_3) = y$. Then, apply Key Claim 6.3.6 (ii) to get

$$x_1 = \theta_{\beta\gamma_0}(x_3) = \theta_{\gamma\gamma_0} \circ \theta_{\beta\gamma}(x_3) = \theta_{\gamma\gamma_0}(y).$$

Step 2c: A contradiction. Recall that the proof of Key Claim 6.3.6 (iii) involved verifying two sequents, which we reproduce below for the reader's convenience:

(a) $x \in F(\gamma) \longrightarrow \exists q \in I_\gamma.\big(\exists y \in F(q).(x = \theta_{q\gamma}(y)\big)$
(b) $y, z \in F(q), \theta_{q\gamma}(y) = \theta_{q\gamma}(z) \longrightarrow \exists r \in I_\gamma.(q \prec r \wedge \theta_{qr}(y) = \theta_{qr}(z))$

Applied to our setting, Sequent (a) says: given $x_2 \in F(\gamma_0)$, which lives on the lower branch of Figure 6.5a, there exists[124] some $q > \gamma_0$, and some $y \in F(q)$ such that $x_2 = \theta_{q\gamma_0}(y)$. But Step 2b forces the identity $\theta_{q\gamma_0}(y) = x_1 \neq x_2$, giving a contradiction.

[124]How did we get $q > \gamma_0$ from $q \in I_{\gamma_0}$? In the language of Example 2.2.51, $q \in I_\gamma$ for $\gamma \in (0, 1]$ iff $\gamma < q$ or $\gamma = 1 = q$. Since $\gamma_0 \in (0, 1)$ by hypothesis, the stated characterisation follows.

Step 3: Extend and Finish. The same argument extends naturally to sheaves witnessing more than just a single instance of lower bound forking. We give an informal sketch. Suppose $(F,\theta) \in \mathcal{D}'$ and F witnesses an instance of lower bound forking (of possibly many instances) in some connected component Y_i. Then, find a sufficiently small neighbourhood of Y_i such that only a single instance of lower bound forking is witnessed. Adapt Step 2 accordingly to obtain the same contradiction, and conclude that such an instance of lower bound forking cannot occur. This completes the proof of the Claim. ☐

Reviewing our work in this section, we present the following summary answer to Question 14.

Conclusion 6.4.8.

(i) Standard descent eliminates all forms of forking in the sheaves of $\mathcal{D}$.

(ii) Although upper bound forking persists in the sheaves of $\mathcal{D}'$*, lax descent eliminates lower bound forking.*

(iii) Lax descent also 'stretches' the sheaves of $\mathcal{S}(0,1]$ *downwards. More precisely, if* $(F,\theta) \in \mathcal{D}'$ *and* F *corresponds to an étale bundle* $f\colon Y \to (0,1]$*, then* $f(Y)$ *must be a downward-closed interval in* $(0,1]$.

Proof. (i) is by Observation 6.4.2, (ii) is Example 6.4.4 and Claim 6.4.7. (iii) is almost immediate by inspection, but we elaborate for clarity. Suppose $\gamma \in f(Y)$, i.e. there exists some $y \in Y$ such that $f(y) = \gamma$. Recall from Setup 6.3.4 that the (lax) descent data gives $\theta_0(y,\beta) \in Y$ such that

$$f(\theta_0(y,\beta)) = f(y) \cdot \beta,$$

for any $\beta \in (0,1]$. Now suppose $\gamma' \in (0,\gamma]$. Since $\gamma^{-1} \cdot \gamma' \in (0,1]$, the lax descent data thus gives us a $y' \in Y$ such that

$$f(y') = f(y) \cdot \gamma^{-1} \cdot \gamma' = \gamma \cdot \gamma^{-1} \cdot \gamma' = \gamma',$$

i.e. that $\gamma' \in f(Y)$, proving downward closure. Notice: in contrast to the standard descent case, we do not get upward closure of $f(Y)$ since we only have a (non-invertible) monoidal action induced by $(0,1]$. ☐

6.5 A Strange Woods

Our understanding of the mechanics underlying the various Local-Global principles has started to shift. Most notably, Theorem G overturns a longstanding assumption in classical number theory that the real place corresponds to a single point at infinity; our result indicates that it instead resembles a kind of blurred unit interval, namely $\overleftarrow{[0,1]}$. In addition, other (subtler) surprises have also emerged in the analysis — e.g. the various ways in which the topos BG may be trivial even when the groupoid G is not (cf. the discussion around Guess 6.2.4), as well as the importance of the coZariski topology (cf. Observation 4.3.7) even though one typically uses the Zariski topology instead in classical algebraic geometry.

All this, in one way or another, arose as a consequence of working geometrically. By pulling the mathematics away from the underlying set theory, this revealed a deep nerve connecting the topology and the algebra, with unexpected nuances. In this primarily expository section, we begin to work through some of the implications of this new picture. Our discussions can be understood as reframing the classical (algebraic) question of "How do we justify viewing the real place as a prime?" as part of a broader (topological) question: "How should the connected and the disconnected interact?"

6.5.1 Limitations of Classifying Toposes.

An important clue in our investigation of the non-Archimedean places was the following example by Bunge [Bun90], which we now discuss more fully:

Example 6.5.1. Let $G := (G_0, G_1)$ be a connected localic group. Then $BG \simeq \mathrm{Set}$.

Proof. Recall that an object of BG is an étale G-space, i.e. an étale bundle $E \xrightarrow{p} G_0$ equipped with a G_1-action $G_1 \times_{G_0} E \xrightarrow{\bullet} E$ satisfying the usual axioms. Since G is a group, this means that $G_0 = \{*\}$, and so deduce from Remark 6.1.10 that the bundle space of any étale G-space $E \xrightarrow{p} G_0$ is also discrete. Next, given any $x \in G_0$, and any $e \in p^{-1}(x)$ in the fibre over x, this defines a natural map:

$$\begin{aligned}(\text{—}) \bullet e \colon G_1 &\longrightarrow E \\ g &\longmapsto g \bullet e\end{aligned}$$

Notice this map must be constant since G_1 is a connected space by hypothesis; in fact, $g \bullet e = e$ for all $g \in G_1$ since the G_1-action forces the identity $s(x) \bullet e = e$ where $s \colon G_0 \to G_1$ is the unit map. As such, since the objects of BG are just sets (equipped with trivial G_1-action), this gives $BG \simeq \mathrm{Set}$. ☐

The following comments give some context as to why Example 6.5.1 is interesting.

Discussion 6.5.2. Example 6.5.1 gave us our first indication that the topos-theoretic characterisation of a single non-Archimedean place may in fact be a trivialisation result — contrary to the expectations of Guess 6.2.4. Although the eventual proof of Theorem F did not require the hypothesis that the $(0, \infty)$-action of the groupoid in Construction 6.2.2 is connected, it is in fact possible to extend the argument of Example 6.5.1 to give an alternate (though much more involved) proof of the theorem.[125]

Discussion 6.5.3 (Connected vs. Fibrewise Discrete)**.** The core mechanism of Example 6.5.1's argument rests on two general facts:

(a) All sheaves over localic spaces can be characterised as fibrewise discrete bundles (Remark 6.1.10).
(b) All maps from connected spaces into discrete sets must be constant.

Put together, this suggests that the present issues with connectedness is not a bug but rather a feature of toposes (since toposes are, after all, defined to be categories of sheaves).

Discussion 6.5.4. The trivialisation result is also striking because it contravenes a basic expectation from the discrete setting. By Diaconescu's Theorem, we know that the presheaf topos $BG \simeq [G, \mathrm{Set}]$ classifies all G-torsors for any discrete group G. Yet when G is a connected group, the fact that $BG \simeq \mathrm{Set}$ implies that for each topological space X there exists essentially only one geometric moprhism $\mathcal{S}X \to BG$, even though for suitable X and G there may exist many non-isomorphic G-torsors.

[125] In fact, this was how our original proof of Theorem F went before being shortened to its present form. The main idea was to define a functor $G \colon \mathrm{Set} \to \mathcal{D}$ and prove that G was both fully faithful and essentially surjective. The argument that G was a fully faithful functor is similar to the argument of Example 6.5.1; the key difficulty was showing that G was essentially surjective. The crux move was to note that for any pair $(F, \theta) \in \mathcal{D}$, F is a sheaf over a locally connected space $(0, \infty)$; hence, applying Observation 6.4.1, analysis of F reduces to analysis of its connected components. After which, one then shows that the descent restrictions force the connected components of F to all be homeomorphic to $(0, \infty)$ (cf. Observation 6.4.2). After which, essential surjectivity follows by a straightfoward (if involved) book-keeping argument.

To clarify Discussion 6.5.4, one may ask: is the failure to classify G-torsors due to the nature of the BG construction, or due to the very nature of toposes themselves (as suggested by Discussion 6.5.3)? Put another way, even in cases where BG does not classify G-torsors (e.g. when G is a connected localic group), can we find some other topos $\mathcal{S}\mathcal{E}$ that does? The following observation by Lurie gives a general instance where this cannot happen.

Observation 6.5.5 (Lurie's Observation [Lur14])**.** Let $G := (G_1, \{*\})$ be a localic group such that there exists a non-constant continuous map $\nu\colon \mathbb{R} \to G_1$. Then, there cannot exist a topos $\mathcal{S}\mathcal{E}$ that classifies G-torsors over localic spaces, i.e. there does not exist a topos $\mathcal{S}\mathcal{E}$ such that

$$\mathbf{Geom}(\mathcal{S}X, \mathcal{S}\mathcal{E}) \simeq \mathrm{Tor}_G(X),$$

where X is a localic space and $\mathrm{Tor}_G(X)$ denotes the category of G-torsors over X.

Proof. Since $\mathbb{R}$ is connected, [Joh02b, Lemma C.1.5.7] gives that the unique projection map $p\colon \mathbb{R} \to \{*\}$ induces a fully faithful embedding on the level of their sheaf toposes:

$$p^*\colon \mathrm{Set} \hookrightarrow \mathcal{S}\mathbb{R} \tag{6.63}$$

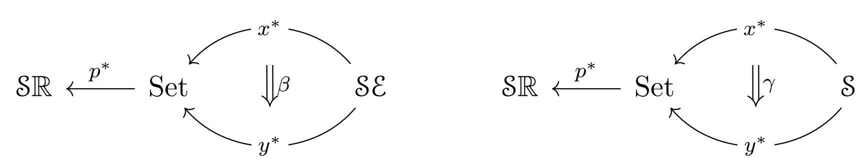
(We remark that the cited Lemma implicitly relies on the reasoning mentioned in Discussion 6.5.3 in its proof.) Lurie's Observation then follows from the following two claims:

Claim 6.5.6. *For any topos $\mathcal{S}\mathcal{E}$, p^* induces a fully faithful embedding*

$$\boldsymbol{Geom}(\mathrm{Set}, \mathcal{S}\mathcal{E}) \hookrightarrow \boldsymbol{Geom}(\mathcal{S}\mathbb{R}, \mathcal{S}\mathcal{E}). \tag{6.64}$$

Proof of Claim. Straightforward, but we check the details:

- *Faithfulness.* Let $x, y\colon \mathrm{Set} \to \mathcal{S}\mathcal{E}$ be two geometric morphisms, and consider 2-cells $\beta, \gamma\colon x^* \Rightarrow y^*$ between their inverse image functors. Then, denote $p^*\beta$ and $p^*\gamma$ to be the whiskering[126] of β and γ with p^* respectively, as depicted:


(6.65)

Now suppose that $p^*\beta = p^*\gamma$. This means, for any $X \in \mathcal{S}\mathcal{E}$, the components of the natural transformations at X agree, i.e. $p^*\beta_X = p^*\gamma_X$. But since p^* is a faithful, we have an injective map:

$$\mathrm{Hom}_{\mathrm{Set}}(x^*(X), y^*(X)) \longleftrightarrow \mathrm{Hom}_{\mathcal{S}\mathbb{R}}(p^*x^*(X), p^*y^*(X)),$$

and so this implies that $\beta_X = \gamma_X$. Since $X \in \mathcal{S}\mathcal{E}$ was arbitrarily chosen, conclude that $\beta \cong \gamma$, verifying faithfulness of the induced map of Equation (6.64).

[126]Recall: whiskering is the horizontal composition of a 2-cell with an identity 2-cell between a 1-morphism.

- *Fullness.* Let $x, y\colon \text{Set} \to \mathcal{SE}$ be two geometric morphisms. Now, compose them with the geometric morphism associated to p, and consider a 2-cell $\alpha\colon p^*x^* \Rightarrow p^*y^*$ between their inverse image functors:

$$\mathcal{SR} \underset{p^*y^*}{\overset{p^*x^*}{\leftleftarrows}} \mathcal{SE} \qquad (\Downarrow \alpha)$$

Then, for objects $X, Y \in \mathcal{SE}$, with a morphism $f : X \to Y$ we have the following naturality square:

$$\begin{array}{ccc} p^*x^*(X) & \xrightarrow{p^*x^*(f)} & p^*x^*(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ p^*y^*(X) & \xrightarrow{p^*y^*(f)} & p^*y^*(Y) \end{array} \tag{6.66}$$

Restricting to the components of α, fullness of p^* guarantees the existence of morphisms

$$\alpha'_X\colon x^*(X) \to y^*(X)$$
$$\alpha'_Y\colon x^*(Y) \to y^*(Y),$$

such that $p^*(\alpha'_X) = \alpha_X$ and $p^*(\alpha'_Y) = \alpha_Y$, which we assemble into the following diagram

$$\begin{array}{ccc} x^*(X) & \xrightarrow{x^*(f)} & x^*(Y) \\ \downarrow{\scriptstyle \alpha'_X} & & \downarrow{\scriptstyle \alpha'_Y} \\ y^*(X) & \xrightarrow{y^*(f)} & y^*(Y) \end{array} . \tag{6.67}$$

Notice: Diagram (6.67) commutes since Diagram (6.66) commutes by hypothesis and p^* is faithful. In other words, Diagram (6.67) defines a naturality square corresponding to some 2-cell $\alpha'\colon x^* \Rightarrow y^*$ such that $p^*\alpha' = \alpha$.

□

Claim 6.5.7. *There does not exist a fully faithful embedding*

$$\text{Tor}_G(\{*\}) \hookrightarrow \text{Tor}_G(\mathbb{R}) \tag{6.68}$$

Proof of Claim. Consider the generic G-torsor, denoted $\mathbf{T} = (G_1 \to \{*\}, \mathfrak{m})$, where $\mathfrak{m}\colon G_1 \times G_1 \xrightarrow{\bullet} G_1$ is the multiplication map. Pulling back $\mathbf{T}$ along $p\colon \mathbb{R} \to *$ yields a G-torsor over $\mathbb{R}$,[127] as depicted:

$$\begin{array}{ccc} G_1 \times \mathbb{R} \times G_1 & \xrightarrow{\pi_{02}} & G_1 \times G_1 \\ {\scriptstyle \pi_{01}}\downarrow\downarrow{\scriptstyle \langle \mathfrak{m}\circ\pi_{02}, \pi_1\rangle} & & {\scriptstyle \pi_0}\downarrow\downarrow{\scriptstyle \mathfrak{m}} \\ G_1 \times \mathbb{R} & \xrightarrow{\pi_0} & G_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{p} & * \end{array} \tag{6.69}$$

[127] See the proof of [Bun90, Proposition 4.4] for details on why pullback sends G-torsors to G-torsors.

where π_{01} maps $(g,\gamma,g') \mapsto (g,\gamma)$ and $\langle \mathfrak{m} \circ \pi_{02}, \pi_1 \rangle$ maps $(g,\gamma,g') \mapsto (g \cdot g', \gamma)$. Denote this new G-torsor over $\mathbb{R}$ as $p^*(\mathbf{T})$.

To prove the claim, we construct a G-torsor endomorphism of $p^*(\mathbf{T})$ that cannot be determined by any G-torsor endomorphism of $\mathbf{T}$. Recall: to define a G-torsor map $p^*(\mathbf{T}) \to p^*(\mathbf{T})$, we need a localic map $f\colon G_1 \times \mathbb{R} \to G_1 \times \mathbb{R}$ such that the following two squares commute:

$$\begin{array}{ccc} G_1 \times \mathbb{R} \times G_1 & \xrightarrow{f \times \pi_2} & G_1 \times \mathbb{R} \times G_1 \\ {\scriptstyle \langle \mathfrak{m}\circ\pi_{02},\pi_1\rangle}\downarrow & & \downarrow{\scriptstyle \langle \mathfrak{m}\circ\pi_{02},\pi_1\rangle} \\ G_1 \times \mathbb{R} & \xrightarrow{\quad f \quad} & G_1 \times \mathbb{R} \end{array} \qquad \begin{array}{ccc} G_1 \times \mathbb{R} & \xrightarrow{f} & G_1 \times \mathbb{R} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} \end{array} \tag{6.70}$$

As before, it suffices to define f coordinate-wise as $f := \langle f_0, f_1 \rangle$ where

$$f_0\colon G_1 \times \mathbb{R} \to G_1$$

$$f_1\colon G_1 \times \mathbb{R} \to \mathbb{R}$$

Now, recall our hypothesis that there exists a non-constant map $\nu\colon \mathbb{R} \to G_1$. As such, define the maps:

$$f_0(g,\gamma) := \nu(\gamma) \cdot g$$

$$f_1(g,\gamma) := \pi_1(g,\gamma) = \gamma$$

An easy diagram-chase verifies that these coordinate maps define a map f that makes Diagram (6.70) commute, and thus give a G-torsor map.

Next, reviewing definitions once more, notice any G-torsor map $\mathbf{T} \to \mathbf{T}$ of the generic torsor $\mathbf{T}$ is determined by a map $f'\colon G_1 \to G_1$ that makes the following diagram commute

$$\begin{array}{ccc} G_1 \times G_1 & \xrightarrow{f' \times \pi_1} & G_1 \times G_1 \\ \downarrow{\scriptstyle \mathfrak{m}} & & \downarrow{\scriptstyle \mathfrak{m}} \\ G_1 & \xrightarrow{\quad f' \quad} & G_1 \end{array} \quad , \tag{6.71}$$

In particular, the diagram requires that $f'(1 \cdot g) = f'(1) \cdot g$ for any $g \in G_1$, and so f' is determined by multiplication of a constant $f'(1) \in G_1$. But since ν is a non-constant map by hypothesis, deduce that the G-torsor endomorphism $f = \langle f_0, f_1 \rangle$ on $p^*(\mathbf{T})$ cannot be determined by any G-torsor endomorphism on $\mathbf{T}$. In other words, there does not exist a fully faithful embedding of $\mathrm{Tor}_G(\{*\})$ into $\mathrm{Tor}_G(\mathbb{R})$, as claimed. □

With the two claims established, this sets up the final move. Suppose there exists a topos $\mathcal{SE}$ such that

$$\mathbf{Geom}(\mathcal{S}X, \mathcal{SE}) \simeq \mathrm{Tor}_G(X),$$

for any localic space X. Then, by Claim 6.5.6, this equivalence yields a fully faithful embedding

$$\mathrm{Tor}_G(\{*\}) \simeq \mathbf{Geom}(\mathrm{Set}, \mathcal{SE}) \hookrightarrow \mathbf{Geom}(\mathcal{S}\mathbb{R}, \mathcal{SE}) \simeq \mathrm{Tor}_G(\mathbb{R}),$$

contradicting Claim 6.5.7. □

Lurie's Observation 6.5.5 vindicates Bunge's original suspicion in [Bun90] that "toposes are not the right kind of structures to consider when dealing with G-bundles for a general G". This gives us a sharper understanding of the topos' limitations, and raises some challenging questions about its intended role in modern applications.

Discussion 6.5.8 (Caramello's Bridge Technique)**.** Going back to [Car10], an attractive proposal was developed by Caramello on how toposes might play a unifying role in mathematics. This programme is motivated a basic observation: even if two geometric theories $\mathbb{T}, \mathbb{T}'$ look very different (e.g. they may have different signatures, they may not be bi-interpretable etc.), their classifying toposes can still be equivalent, as depicted:

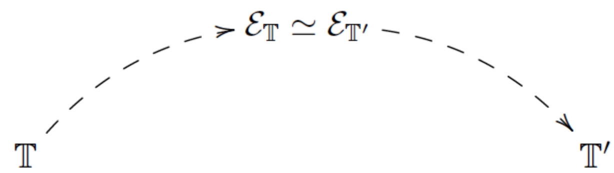


Figure 6.6: Classifying Toposes as "Bridges" [Car17]

This suggests that the topos plays a natural role in understanding the structural relationships between theories $\mathbb{T}$ and $\mathbb{T}'$. Extending this idea significantly, Caramello argues that the topos provides an abstract framework for the analysis of much more general structural connections in mathematics — using her language, the topos serves as a "bridge" to transfer information between different mathematical contexts.

This interesting programme has strong ambitions: throughout her work, Caramello has proposed potential applications of the Bridge technique to the Langlands correspondence, to mirror symmetry, to the study of motives, and to the AdS/CFT correspondence [CL16; Car21]. However, Example 6.5.1 and Lurie's Observation 6.5.5 raise questions about whether this overstates the unifying power of toposes. We now know that toposes sometimes lose important (cohomological) information when attempting to classify G-torsors. As such, given the potential lossy-ness of toposes, one is naturally led to ask: why are toposes the right framework for analysing a given structural connection? If we wish to use toposes as "bridges" to transfer information, how do we know that relevant information isn't being lost in the process?

Discussion 6.5.9 (∞-toposes)**.** Very interestingly, Lurie [Lur14] points out that the argument of Observation 6.5.5 can be extended to show that ∞-toposes also do not classify G-torsors for all topological groups G either. This is *a priori* surprising: one may have expected that the generality of ∞-toposes would resolve the previous issues faced by the standard topos.[128] In any case, Lurie's remark suggests that new ideas are needed if we wish to deal with G-torsors for general topological/localic G.

These discussions set up the following test problem:

Problem 15. What generalised space classifies G-torsors for all topological/localic groupoids G? What is its relationship to geometric logic?

Let's step back for a moment. As already highlighted by Discussion 6.5.3, the loss of information when analysing G-torsors for general topological G stems from the way in which connected spaces and fibrewise discrete bundles interact: this was made clear in Example 6.5.1, but also showed up implicitly in the proof of Lurie's Observation 6.5.5 (specifically Claim 6.5.6). This gives us our first clue: if Problem 15 requires

[128] Although, in hindsight, perhaps less surprising once we re-examine the motivations behind higher topos theory. Recall from [Lur09] that higher topos theory aims to develop a categorical framework that provides a concrete interpretation of higher cohomology classes, analogous to how G-torsors give a concrete description of first cohomology classes. Looked at this way, it becomes apparent that the programme is guided by a rather different objective from providing a categorical framework that classifies all G-torsors when G is topological/localic.

us to define a category whose objects are a kind of generalised sheaves, then this generalisation should also generalise the notion of being étale (or, equivalently, the notion of being fibrewise discrete); otherwise we are likely to run into the same problems again as we did before (cf. Discussion 6.5.9).

Discussion 6.5.10 (Weakly Étale Maps)**.** Consider a map of localic spaces

$$f\colon Y \to X,$$

and recall from Remark 6.1.10 that f is étale iff both f and its diagonal $\Delta\colon Y \to Y \times_X Y$ are open maps.

The beauty of this characterisation lies in a subtle point: it shows that even an ostensibly topological idea such as the local homeomorphism can be reformulated algebraically. To understand this, one should read this characterisation of étaleness in the original context of Joyal-Tierney's monograph [JT84], which regards a *frame*[129] as a commutative ring-like object in the category of sup-lattices $s\ell$: here the supremum plays the role of addition, the infimum plays the role of the product. One can then review Definition 6.1.8 to check that open maps between localic spaces are in fact defined in terms of their corresponding frame homomorphisms.

This algebraic perspective developed in [JT84] has turned out to be enormously productive, allowing the authors to bring a rich set of ideas from commutative algebra (e.g. R-modules, descent theorems) to bear on the analysis of point-free spaces. Extending this, let us recall another idea from commutative algebra: a morphism $A \to B$ of commutative rings is called *weakly étale* if both $A \to B$ and the multplication morphism $B \otimes_A B \to B$ are flat. This algebraic notion recently found remarkable applications in the work of Bhatt-Scholze on the pro-étale site [BS15], with subsequent extensions to condensed mathematics [Sch19b; Sch19a]. Inspired by this, we define the obvious translation to the localic setting:

Definition. *For any map $f\colon Y \to X$ for localic spaces, we call f ...*

(i) ... flat *if Ω_Y is a flat Ω_X-module (in the sense of [JT84, §II.4]) via the restriction of scalars along the frame homomorphism $\Omega_f\colon \Omega_X \to \Omega_Y$.*
(ii) ... weakly étale *if both $f\colon Y \to X$ and its diagonal $\Delta\colon Y \to Y \times_X Y$ are flat.*

This is an interesting definition, and appears to be new. Much more work will be needed in order to see if this leads to the right answer to Problem 15.[130] As a warm-up exercise, we pose the following test question: do the analogous relationships between étale, ind-étale and weakly étale (as described in [BS15, §2]) translate well to the localic setting? Making precise these connections will give us a better sense of what kind of leverage this new definition may give us.

Discussion 6.5.11. In terms of the current literature, Noohi's foundational work on the theory of topological stacks [Noo05] seems relevant to Problem 15, although the connections with geometric logic are presently unclear. Townsend's perspective [Tow15] that geometric morphisms can be viewed as so-called stably Frobenius adjunctions may also give some important clues.

[129] Warning: Joyal-Tierney use the term "locale" to describe what we call a frame, and the term "space" to describe what we call a localic space.

[130] A side-note/speculation: the intuition from classical algebraic geometry that flat families of varieties are families that vary "almost continuously" may give some useful ideas on how to relate this to geometricity, which (as already discussed in Chapter 2) corresponds to a generalised notion of continuity.

6.5.2 Local-Global Questions. An interesting asymmetry has emerged in the present state of arithmetic geometry. Whenever one wants to import analytical methods from the Archimedean setting (e.g. complex analytification of varieties) to the non-Archimedean setting, the barriers to this translation are clear: non-Archimedean fields are totally disconnected, and so e.g. the naive analytification of non-Archimedean varieties is of limited usefulness, unlike the complex case (see Appendix A). Nonetheless, there are various well-established solutions to remedy this problem of disconnectedness — e.g. we might modify the classical notion of topology (as in Tate's rigid analytic geometry, see [Pay15]) or perhaps we might "fill in the gaps" induced by the disconnected base field (as in Berkovich geometry, see Chapter 5).

Moving in the opposite direction, however, seems more difficult: how should we incorporate the analytic world into an algebraic framework? The classical position, apparently going back to Hasse and/or Artin [Wei05], is to regard the real place as a formal prime at infinity. Chapter 2 already discusses why this solution is dissatisfying; what is interesting is how the literature has repeatedly framed this as an algebraic problem, to be resolved once we find the correct generalisation of commutative rings (see e.g. [Dur07; Har07]).

Our results in this chapter suggests a different picture. If the real place corresponds to $\overleftarrow{[0,1]}$ (cf. Theorem G), then this indicates that completing respect to this place does not just give a single completion of $\mathbb{Q}$ but rather a parametrised family of completions. This is in contrast to completing with respect to the non-Archimedean places, where we expect to obtain the usual p-adics $\mathbb{Q}_p$ (or something essentially equivalent to $\mathbb{Q}_p$) since the non-Archimedean places are just singletons (cf. Theorem F). More work, of course, will be needed to make this precise, but notice this picture already tells us something interesting about the connected and the disconnected.

Observation 6.5.12. As our setup,

- Let $|\cdot|_\infty$ be the standard Euclidean norm on $\mathbb{Q}$ and $|\cdot|_0$ be the trivial norm.
- Let $\alpha \in [0,1]$.[131]

Then, one easily verifies that $|\cdot|_\infty^\alpha = |\cdot|_\infty$ when $\alpha = 1$ and that $|\cdot|_\infty^\alpha = |\cdot|_0$ when $\alpha = 0$. In particular, note that the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty^\alpha$ is $\mathbb{R}$ when $\alpha = 1$ and $\mathbb{Q}$ when $\alpha = 0$.

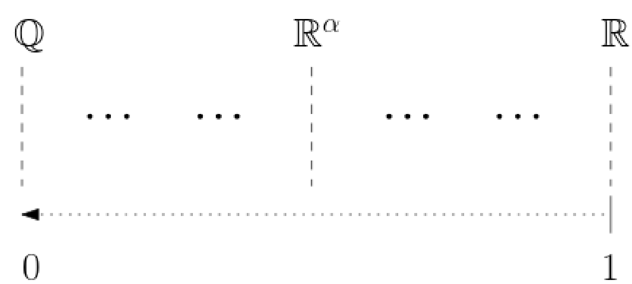


Figure 6.7: A parametrised family of completions over the Archimedean place

[131] Why not simply let α be an upper real from $\overleftarrow{[0,1]}$ instead of a Dedekind in $[0,1]$? In the former case, notice $|\cdot|^\alpha$ would be an upper-valued absolute value on $\mathbb{Q}$, creating the same issues surrounding multiplicative inverses that were already raised by Observation 4.1.1. The actual generic completion with respect to the Archimedean place will have to be approached differently from how Observation 6.5.12 is stated; nonetheless, the observation still gives us an interesting insight, even if slightly classical.

Discussion 6.5.13. Recall the following question from Discussion 2.4.13: if the usual finite primes measure the divisibility of an integer, what exactly does the infinite prime (or the real place) measure? Observation 6.5.12 gives some interesting clues. It is obvious that $\mathbb{Q}$ is disconnected whereas $\mathbb{R}$ is connected. Further, notice: although completions of $\mathbb{Q}$ with respect to $|\cdot|_\infty^\alpha$ are all connected (in fact, homeomorphic to $\mathbb{R}$) so long as $\alpha \in (0,1]$, the nearby points of these completions become increasingly "spaced out" as $\alpha \to 0$, where the completion finally becomes disconnected in the limit. Combined with Theorem G, this suggests the real place may be understood as measuring the "disconnectedness"[132] of the Archimedean completions.[133]

Discussion 6.5.14. A tentative picture is starting to emerge. The question of reconciling the p-adics with the reals (cf. Question 1) appears to be more closely bound up with the question of reconciling the connected with the disconnected than previously thought. Following Berkovich [Ber90], we already have a robust strategy for transforming disconnected structures defined over non-Archimedean fields to connected ones, essentially by "filling in the gaps" by way of adding more points. Conversely, in order to relate analytic structures over $\mathbb{R}$ to the (disconnected) arithmetic setting, Theorem G and Observation 6.5.12 suggest that we should look to somehow parametrise certain families of analytic structures $\{\mathcal{M}_q\}$, perhaps over $\mathbb{Q}_{(0,1]}$,[134] before examining the behaviour of $\{\mathcal{M}_q\}$ as we scale $q \to 0$ in the limit.

We are now at a strange mathematical juncture. Certain long-held assumptions have been subverted by new findings, calling for a significant reorientation in our approach to understanding Local-Global questions. Helpfully, certain key themes have emerged in the analysis — in particular, the subtle interplay between the connected and disconnected — which sets some basic expectations going forward. Nonetheless, our current picture is still a tentative one. New ideas will be needed to develop the suggestions mentioned in Discussion 6.5.14 and we also don't know the current extent of our blindness. As such, to guide the development of our understanding, we include below a varied list of test problems.

6.5.2.1 Compactifying and Living Below $\mathrm{Spec}(\mathbb{Z})$. Although we have characterised the trivial place and non-Archimedean places of $\mathbb{Q}$ as singletons, and the Archimedean place as $\overleftarrow{[0,1]}$, this only gives a piecewise account of the space of places of $\mathbb{Q}$. Taking seriously our objective of treating the places of $\mathbb{Q}$ as an actual space (as opposed to an indexing set), we are led to restate the following problems from the Introduction.

Problem 16. Characterise the (entire) point-free space of places of $\mathbb{Q}$.

Discussion 6.5.15. This problem is more interesting than its statement may indicate. Even after proving that the space of non-Archimedean places is equivalent to $\mathrm{ISpec}(\mathbb{Z})$ (which seems likely), it is still not obvious how the Archimedean and the non-Archimedean places fit together. On this, it is worth recalling a classical result by Artin-Whaples [AW45] that $\mathbb{Q}$ (in fact, all global fields) satisfies a product formula:

$$\prod_{v \in \Lambda_{\mathbb{Q}}} |x|_v = 1, \qquad \text{for all } x \neq 0 \tag{6.72}$$

where v ranges over all the places of $\mathbb{Q}$, including the Archimedean. Of course, the product formula may be normalised to feature the standard p-adic and Euclidean norms (which the reader can check satisfies

[132] Quotes are placed because these completions are still technically connected (since they are homeomorphic to $\mathbb{R}$), so we shall need a subtler notion. This will be discussed in Section 6.5.2.3 on q-liquidity.

[133] Although, one should still bear in mind the issues raised in Footnote 131.

[134] Or perhaps over $\overleftarrow{[0,1]}$ itself, but working with rationals may be easier.

Equation (6.72)), but in fact any normalisation will do. In particular, notice: if we exponentiate $|\cdot|_\infty$ by $\alpha \in [0,1]$, then all the p-adic norms $|\cdot|_p$ must also be exponentiated by α in order for the product formula to hold. In other words, the normalisation of the product formula across all places depends on just the normalisation at the Archimedean component; this gives another way of reading how the Archimedean place may function as a parameter space. More suggestively, if the normalisation $|\cdot|_\infty^\alpha$ "fixes" the α for the rest of the places via the product formula, then (recalling Convention 2.1.34) this suggests that the Archimedean place lives *below* $\mathrm{Spec}(\mathbb{Z})$.

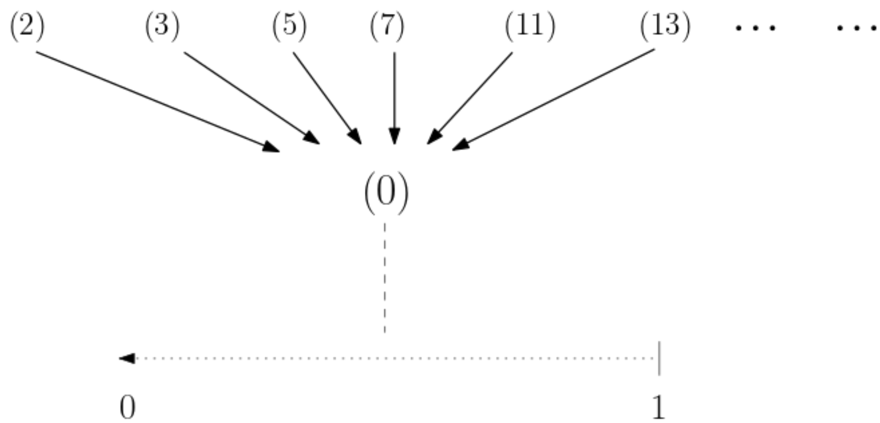


Figure 6.8: A candidate picture for the space of places

This candidate picture, intriguingly, appears to pull together various different threads in arithmetic geometry:

(a) The (point-free) space of places gives a new perspective on the Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ (see discussion in Chapter 1);
(b) The perspective that the Archimedean place lives below $\mathrm{Spec}(\mathbb{Z})$ will be suggestive to those familiar with $\mathbb{F}_1$-geometry (see e.g. [PL11]);
(c) In our setting, the Archimedean place "adds" numerous extra points to $\mathrm{Spec}(\mathbb{Z})$, which form a non-Hausdorff space $\overleftarrow{[0,1]}$ equipped with a specialisation order. One may therefore wonder: are these extra points analogous to the Type V points of adic spaces (see e.g. [Wei19, §1.7])? Dudzik's work [Dud12], as discussed in Section 5.3 may be relevant here. It would also be interesting to see if the language of vertical/horizontal specialisation (see [Mor19, §I.3.1]) finds a useful translation to our setting.

At this preliminary stage, we make no assertions about the correct characterisation of the space of places or its potential connections with other branches of arithmetic geometry. For now, we simply record some basic observations for later investigation, whilst keeping an eye out for useful clues on how to develop our perspective into a more robust algebro-geometric framework.[135]

Problem 17. Characterise and analyse the space of completions of $\mathbb{Q}$.

[135] As an illustration of this, A. Connes encouraged us to think about how the structure sheaf of $\mathbb{S}$-algebras on the (classical) Arakelov compactification, as introduced in [CC16], can be extended via a one parameter deformation by replacing the usual norm $||\cdot||$ with a real parameter $\alpha \in (0,1]$, corresponding to $||\cdot||^\alpha$ – for details, see [CC20, Prop. 4.1].

Discussion 6.5.16. Of particular interest to us are the following: (a) the generic Archimedean completion, especially since $\mathbb{Q}$ and $\mathbb{R}$ are not homeomorphic and since working the upper reals introduces technical subtleties; (b) how the trivial and non-trivial completions of $\mathbb{Q}$ interact; (c) the role of local compactness, since the non-trivial completions are locally compact (as is desired by the number theorist — see Discussion 2.4.12 on the adeles, and also [AW45, Axiom 2]) but not the trivial completion $\mathbb{Q}$.

It is also worth revisiting the function field analogy, and ask:

Problem 18. In what sense is the space of places a (smooth) compactification of $\mathrm{Spec}(\mathbb{Z})$?

Discussion 6.5.17 (On the hypothesis of "smooth")**.** Since the compactification of $\mathrm{Spec}(\mathbb{Z})$ is led by analogy with the affine curve case, we should be alert to the technicality that there are often different compactifications of the same affine curve. Most obviously: it is well-known that smooth compactifications of affine curves (over perfect fields, e.g. $\mathbb{Q}$) always exist and are in fact unique, whereas the usual compactification of an affine curve (= taking its projective closure) may yield a singular curve. In addition, one should also recall that smoothness is a standard hypothesis when doing intersection theory. Hence, taking seriously the function field analogy, one is naturally led to ask: how may we regard the canonical compactification of $\mathrm{Spec}(\mathbb{Z})$ as a *smooth* compactification?

This is a challenging question. Reviewing Arakelov's work on intersection theory [Ara74], it is unclear how the standard Arakelov compactification can be regarded as being "smooth" at $p = \infty$ (in fact, it is arguably highly singular since the element ∞ is formally adjoined to $\mathrm{Spec}(\mathbb{Z})$). On the other hand, since working geometrically means working continuously, this seems a more reasonable question to ask about the (point-free) space of places of $\mathbb{Q}$. In particular, notice that our candidate picture in Discussion 6.5.15 already gives a different perspective on how the Archimedean and non-Archimedean places interact. Another approach is to consider alternative hypotheses to smoothness — e.g. normality, since all normal algebraic curves are already smooth.

6.5.2.2 Revisiting Homotopy Theory. One of our original motivations behind this research programme was to explore if generic reasoning could be applied to investigate Local-Global Principles in Arithmetic Geometry. In particular, we asked in Section 2.5 if we could we reformulate statements of the form

"ϕ holds over $\mathbb{Q}$ iff ϕ holds over *all* (non-trivial) completions of $\mathbb{Q}$"

as

"ϕ holds over $\mathbb{Q}$ iff ϕ holds for the *generic* (non-trivial) completion of $\mathbb{Q}$" ?

Unfortunately, since the Archimedean place is equivalent to $\overleftarrow{[0,1]}$, this suggests that the non-trivial Archimedean completion(s) cannot be definably separated from the trivial completion. In other words, barring unexpected surprises, the evidence suggests that we are unable to reason geometrically about the generic *non-trivial* completion of $\mathbb{Q}$, only the generic completion of $\mathbb{Q}$.

Nonetheless, this only appears to be an immediate problem for Local-Global Principles so long as

"Global = $\mathbb{Q}$", and "Local = Non-Trivial Completions of $\mathbb{Q}$".

However, we already know that there are other interesting (and deep) variants of the same basic idea. For instance, one may recall our discussion of the Arithmetic Square from Chapter 2:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \widehat{\mathbb{Z}}_p \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p \end{array} \quad , \tag{6.73}$$

In particular, we gave the following interpretation: since the Arithmetic Square is a pullback, it tells us how $\mathbb{Z}$ can be reconstructed from a rational piece and infinitely many p-adic pieces (subject to coherence conditions). In other words, we have:

"Global = $\mathbb{Z}$", and "Local = $\mathbb{Q}$ and $\widehat{\mathbb{Z}}_p$ for all primes p".

Notice that $\mathbb{Q}$ now features as a local piece rather than a global piece. Further, the fact that $\mathbb{Q}$ may not be definably separable from $\mathbb{R}$ now becomes quite interesting, because it dovetails with an open problem discussed in Chapter 2, which we restate here:

Problem 19. Notice that the Arithmetic Square in Diagram (6.73) features only the finite adele ring $\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}$ and not the complete adele ring $\mathbb{A}_{\mathbb{Q}}$. In light of Theorem G, formulate a plausible conjecture on how to augment the Arithmetic Square to include $\mathbb{R}$. (Footnote 131 may be relevant — especially since working with the upper reals inclines us to look at the absolute values on $\mathbb{Z}$ rather than $\mathbb{Q}$.)

6.5.2.3 Interactions with q-liquidity. Independently, recent work of Clausen-Scholze on condensed mathematics [Sch19a; Sch19b] have also explored the interactions between topology and algebra, albeit from a very different angle. Many very interesting things can be said about how the condensed perspective and point-free perspective may interact (cf. Discussion 6.5.10), but here we shall focus specifically on how the condensed formalism engages with the p-adics vs. the reals.

Denote $\mathrm{Cond}(\mathrm{Ab})$ to be the category of condensed abelian groups, and compare the following two structure theorems:

(a) [Sch19b, Theorems 5.8 and 6.2]: The inclusion of the category of solid abelian groups into $\mathrm{Cond}(\mathrm{Ab})$ admits a left adjoint $M \mapsto M^{\blacksquare}$, known as *solidification*. In particular, there exists a unique tensor product on solid abelian groups making the solidification functor $M \mapsto M^{\blacksquare}$ symmetric monoidal (i.e. compatible with the tensor product).

(b) [Sch19a, Theorem 6.5]: Fix any $0 < q \leq 1$. Then, the inclusion of the category of q-liquid $\mathbb{R}$-vector spaces[136] into $\mathrm{Cond}(\mathrm{Ab})$ admits a left adjoint, known as *q-liquidification*, which is the unique colimit-preserving extension of

$$\mathbb{Z}[S] \mapsto \mathcal{M}_{<q}(S) = \varinjlim_{q' < q} \mathcal{M}_{q'}(S).$$

In particular, there exists a unique tensor product of q-liquid $\mathbb{R}$-vector spaces making q-liquidification symmetric monoidal.

[136] This is originally refered to as the category of *p-liquid $\mathbb{R}$-vector spaces*, but we have opted for the term q-liquid since the primes are already denoted as p.

For details, see the cited lecture notes by Scholze. Notice that both structure theorems work to construct a canonical tensor product for two different classes of condensed abelian groups. In fact, this is by necessity: whereas solid abelian groups contain all the usual algebraic suspects (e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$ etc.), it does *not* contain the (condensed) reals and so a different approach to tensoring condensed $\mathbb{R}$-vector spaces is required.

Examined closely, the statement of these results reveal an interesting disanalogy: whereas defining a canonical tensor product for structures over $\mathbb{R}$ requires a form of scaling based on choice of $0 < q \leq 1$, no such scaling is necessary to define a tensor product for the solid abelian groups.[137] Read in the context of this chapter, this disanalogy mirrors what was revealed by investigating the places of $\mathbb{Q}$ geometrically, where we found the Archimedean place to be equivalent to $\overleftarrow{[0,1]}$ while the non-Archimedean places were shown to be just singletons. This convergence in perspectives hints that both methods are detecting something deeper, which would be good to make precise.

For concreteness, we start by linking this objective with two previous test problems:

Problem 20. The mechanics underlying the proof of [Sch19a, Theorem 6.5] are still slightly mysterious, but have recently received substantial clarification after a key step was successfully formalised in LEAN — see Scholze's recent blogposts [Sch20; Sch21]. In particular, this formalisation has brought into focus a core mechanism of the argument: namely, reducing a non-convex problem over the reals to a convex problem over the integers. Reviewing the setup also reveals a close connection between q-liquidity and non-local convexity.

Some natural questions: what might these insights tell us about the generic completion of $\mathbb{Q}$ over the Archimedean place (Problem 17)? About augmenting the Arithmetic Square with $\mathbb{R}$ (Problem 19)?

In addition, building on Discussion 6.5.14 (but also the final comments in [Sch19a, Lecture XIV]), there is another interesting aspect worth meditating on:

Problem 21. Let $\{M_q\}_{q \in \mathbb{Q}_{(0,1]}}$ be a parametrised family of analytic structures on $\mathbb{R}$ — what happens in the limit as $q \to 0$? Construct and study various natural examples of this phenomena, and work to develop a general explanation. What can this scaling phenomena tell us about relating the connected with the disconnected? What kinds of key structural properties gets preserved/distorted as we move from $q = 1$ to $q = 0$? Do these point out certain topological/geometric features that carry arithmetic significance? Following e.g. [SS88], is it helpful to frame these properties as a kind of Zero-One Law?

[137] Although, it is worth noting that one can still define a notion of q-liquid $\mathbb{Q}_p$-vector spaces, except now we allow q to range over $0 < q < \infty$ [Sch20, Remark 5.5].

Appendix A

Non-Archimedean Analytic Geometry

A.1 Motivations from Complex Algebraic Geometry

In complex algebraic geometry, one typically begins with the study of algebraic subsets of $\mathbb{C}^n$, i.e. the common zeroes in $\mathbb{C}^n$ of a collection of polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$. This definition naturally extends to the more general notion of a *scheme of finite type over* $\mathbb{C}$, i.e. a locally-ringed space X covered by open affine subsets $Y_i = \operatorname{Spec}(A_i)$, where each $A_i \cong \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$.

One important advantage of working over $\mathbb{C}$ is that each complex scheme X of finite type can be canonically associated to a complex analytic space X^{an} as follows. First, regard the polynomials $f_1, \dots, f_m$ associated to each Y_i as analytic functions on $\mathbb{C}^n$, so that their set of common zeroes defines a complex analytic subspace $(Y_i)^{\mathrm{an}} \subset \mathbb{C}^n$. Next, since the scheme X is obtained by gluing together the open sets $\{Y_i\}$, use the same glueing data to glue $(Y_i)^{\mathrm{an}}$ into a complex analytic space X^{an}, which we call the *complex analytification* of X.

This simple translation[138] opens up the study of complex algebraic varieties to powerful tools of complex analysis and differential geometry (what Hartshorne calls "transcendental methods" [Har77, Appendix B]). This leads to the natural question: can we play the same game for algebraic varieties over fields which are not $\mathbb{C}$? For instance, over $\mathbb{Q}$? Over the field of Laurent series $\mathbb{C}((t))$? The p-adic numbers $\mathbb{Q}_p$?

It was mentioned in Chapter 5 that the general thrust of these questions were suggestive, but over-simplistic, due to the obvious topological differences between $\mathbb{C}$ and non-Archimedean settings. The following discussion develops this remark.

A.2 Non-Archimedean Topology and Analysis

Following Section A.1, one may wish to define a non-Archimedean analogue of a complex analytic space as follows:

Naive Definition A.2.1. Let $(K, |\cdot|)$ be a non-Archimedean field. Then define ...

(i) ... a *K-analytic function* to be a function $f \colon U \to K$ on an open $U \subseteq K^n$ that can be locally expressed as a convergent power series (in analogy with complex analytic functions).

[138] Warning: while complex algebraic varieties can be regarded as complex analytic spaces, it is not generally true that complex analytic spaces arise as analytifications of complex algebraic varieties. For more details, see [Har77, Appendix B.2].

(ii) …a *K-analytic space* to be a locally ringed space in K-algebras that are locally isomorphic to a pair of the form $(U, \mathcal{O}_U)$ with U an open set in K^n and $\mathcal{O}_U$ the sheaf of K-analytic functions on U (in analogy with complex analytic spaces).

Mimicking the argument in the complex setting, one can then associate to any K-scheme of finite type X a K-analytic space. Unfortunately, while this naive analytification of X is well-defined, its analytic-geometric structure breaks down because of the disconnectedness of K.

Observation A.2.2. For any non-Archimedean field K, define an open disc in K as

$$D_\rho(x) := \{y \in K \,\big|\, |x - y| < \rho\}. \tag{A.1}$$

where ρ is a positive Dedekind real. Then,

(i) If $x' \in D_\rho(x)$, then $D_\rho(x) = D_\rho(x')$;
(ii) $D_\rho(x)$ is closed in the metric topology;
(iii) K is totally disconnected.

Proof. (i) is obvious from the ultrametric inequality satisfied by K. For (ii), suppose $z \in \mathrm{cl}(D_\rho(x))$ belongs to the closure of $D_\rho(x)$. By definition, this means

$$D_r(z) \cap D_\rho(x) \neq \emptyset, \qquad \forall r > 0, \tag{A.2}$$

and so, in particular $D_\rho(z) \cap D_\rho(x) \neq \emptyset$. Apply (i) to deduce that

$$D_\rho(z) = D_\rho(x), \tag{A.3}$$

and so $z \in D_\rho(x)$. In other words, $\mathrm{cl}(D_\rho(x)) \subseteq D_\rho(x)$ and so $D_\rho(x)$ is closed. Finally, since the metric topology of K is generated by these open discs (which we have just shown to be clopen), K is therefore totally disconnected. □

In particular, Observation A.2.2 implies that X is totally disconnected, resulting in too many K-analytic functions. To illustrate, consider the following example adapted from [Pay15]. Given any polynomial f and g, define the piecewise function:

$$\chi(x) = \begin{cases} f(x) & \text{if } x \in D_1(0) \\ g(x) & \text{if otherwise.} \end{cases} \tag{A.4}$$

Notice that χ is a K-analytic function according to Definition A.2.1. Notice also that the polynomial f does not impose any geometric constraints on g — restricting χ to $D_1(0)$ does not give us any knowledge about the shape of χ outside of $D_1(0)$. This is in stark contrast with the complex setting, where we have the principle of analytic continuation.

More pressingly, the existence of too many K-analytic functions leads to many non-isomorphic K-varieties having isomorphic K-analytifications. This gives strong evidence that the naive K-analytification presented here is a very loss-y process. To illustrate, consider Serre's remarkable classification of compact K-analytic spaces when K is a non-Archimedean local field:

Theorem A.2.3 (Serre [Ser65])**.** *As our setup,*

- *Let K be a local non-Archimedean field.*

- *Denote q to be the cardinality of its residue field;*
- *Let M be a compact K-analytic space of dimension d;*
- *Define $\overline{D}_1(0) := \{y \in K \mid |x-y| \leq 1\}$ to be the closed unit disc in K.*

Then, there exists an integer $m \in \{1, \ldots, q-1\}$ such that M is isomorphic to m disjoint copies of $\overline{D}_1(0)^d$.

Appendix B

Constructing multiplicative seminorms from K-seminorms

We now provide the rest of the details of the proof of Theorem E. For clarity, let us review the key steps of the argument. Recall the main goal: we wish to show that $\mathcal{M}(\mathcal{A})$ is classically equivalent to $\overleftarrow{\mathcal{M}(\mathcal{A}_{\mathrm{Lin}})}$. After declaring our reliance on Assumption $(\star)$, the argument proceeds by construction: given $|\cdot|_x \in \mathcal{M}(\mathcal{A})$, one defines $|\cdot|_x\big|_{\mathcal{A}_{\mathrm{Lin}}}$ on $\mathcal{A}_{\mathrm{Lin}}$ by taking the obvious restriction, whereas given $|\cdot|_{\mathcal{F}} \in \overleftarrow{\mathcal{M}(\mathcal{A}_{\mathrm{Lin}})}$, we start by first extending $|\cdot|_{\mathcal{F}}$ to a bounded multiplicative seminorm on $K[T]$, before finally defining $\widetilde{|\cdot|}_{\mathcal{F}}$ on $\mathcal{A}$. It is fairly obvious that if both constructions are well-defined, then they are inverse to each other because they are both determined by their values on the linear polynomials (which would complete the proof). In addition, it is clear that the restriction $|\cdot|_x\big|_{\mathcal{A}_{\mathrm{Lin}}}$ defines a bounded K-seminorm, but more effort is needed to check that $\widetilde{|\cdot|}_{\mathcal{F}}$ satisfies the requires properties.

We organise the argument into the following two claims.

Claim B.0.1. *The extension map* $|\cdot|_{\mathcal{F}}\colon K[T] \to [0,\infty)$ *(given by Equation* (5.37)*) defines a bounded multiplicative seminorm on* $K[T]$*, satisfying the ultrametric inequality.*

Proof of Claim. Our argument relies on the explicit characterisation of $|\cdot|_{\mathcal{F}}$ to perform the required checks.

Step 1: Working "level-wise". Fix a ball $B_q(k) \in \mathcal{F}$, and define the obvious extension of $|\cdot|_{B_q(k)}$ from $\mathcal{A}_{\mathrm{Lin}}$ to $K[T]$:

$$\begin{aligned} |\cdot|_{B_q(k)}\colon K[T] &\longrightarrow [0,\infty) \\ f &\longmapsto |c|\cdot\prod_{j=1}^{m}|T-b_j|_{B_q(k)} \end{aligned} \tag{B.1}$$

Following the cue of Remark 5.2.17, we may also express f as a finite power series centred at k:

$$f = \sum_{i=0}^{m} c_i (T-k)^i, \tag{B.2}$$

and define a new map:

$$\widehat{|\cdot|_{B_q(k)}} \colon K[T] \longrightarrow [0,\infty) \qquad (B.3)$$
$$f \longmapsto \sup_i |c_i| q^i$$

We claim that $\widehat{|\cdot|_{B_q(k)}}$ defines a multiplicative seminorm (though not necessarily bounded, since q is an arbitrary positive rational). This follows from noting:

- $\widehat{|0|_{B_q(k)}} = 0$ *and* $\widehat{|1|_{B_q(k)}} = 1$. Obvious.
- $\widehat{|\cdot|_{B_q(k)}}$ *satisfies the ultrametric inequality.* Straightforward, but we elaborate for completeness. Given $f, f' \in K[T]$, assume WLOG that $\deg(f) = m \geq m' = \deg(f')$. Then, add their corresponding finite power series (as in Equation (B.2)) and compute:

$$\begin{aligned}
\widehat{|f+f'|_{B_q(k)}} &= \widehat{|\sum_{i=0}^{m} c_i(T-k)^i + \sum_{i=0}^{m'} c'_i(T-k)^i|_{B_q(k)}} \\
&= \widehat{|\sum_{i=0}^{m} (c_i + c'_i)(T-k)^i|_{B_q(k)}} && \text{[writing } c'_i = 0 \text{ for all } i > m'] \\
&= \sup_i |c_i + c'_i| q^i && \text{[by Definition of } \widehat{|\cdot|_{B_q(k)}}] \\
&\leq \sup_i \max\{|c_i|\,,\, |c'_i|\} \cdot q^i && \text{[since } |c_i + c'_i| \leq \max\{|c_i|, |c'_i|\}] \\
&= \max\{\widehat{|f|_{B_q(k)}}\,,\, \widehat{|f'|_{B_q(k)}}\} && \text{[since max and sup commute].}
\end{aligned}$$

- $\widehat{|\cdot|_{B_q(k)}}$ *is multiplicative.* Similar argument as above: given polynomials f, f', multiply their corresponding finite power series and use ultrametric inequality.

Finally, we claim that:

$$|\cdot|_{B_q(k)} = \widehat{|\cdot|_{B_q(k)}}. \qquad (B.4)$$

Why? Since $|\cdot|_{B_q(k)}$ is multiplicative and K is algebraically closed, it suffices to show that they agree on the level of linear polynomials. But this is clear since

$$|T-a|_{B_q(k)} = \max\{|k-a|, q\} = \widehat{|T-a|_{B_q(k)}}, \qquad \text{for any } T-a. \qquad (B.5)$$

In particular, Equation (B.4) allows us to conclude that $|\cdot|_{B_q(k)}$ is in fact a multiplicative seminorm satisfying the ultrametric inequality.

Step 2: Lifting to $\mathcal{F}$. The argument is similar to the proof of Claim 5.2.18. To show that $|\cdot|_{\mathcal{F}}$ is a muliplicative seminorm on $K[T]$ satisfying the ultrametric inequality, this amounts to checking a list of properties. But by Step 1, we know that these properties already hold for $|\cdot|_{B_q(k)}$, for all $B_q(k) \in \mathcal{F}$. Hence, since $|\cdot|_{\mathcal{F}} = \inf_{B_q(k)} |\cdot|_{B_q(k)}$, observe that these properties are respected by taking inf's, and conclude that they hold for $|\cdot|_{\mathcal{F}}$ as well.

To illustrate, suppose $f, f' \in K[T]$. By Step 1, we know that the ultrametric inequality holds for *all* $|\cdot|_{B_q(k)}$ where $B_q(k) \in \mathcal{F}$:

$$|f+f'|_{B_q(k)} \leq \max\{|f|_{B_q(k)}, |f'|_{B_q(k)}\} \qquad (B.6)$$

Then, since taking $\inf$'s respects weak inequalities, conclude that

$$|f + f'|_{\mathcal{F}} \leq \max\{|f|_{\mathcal{F}}, |f'|_{\mathcal{F}}\}. \tag{B.7}$$

Step 3: $|\cdot|_{\mathcal{F}}$ is bounded. We do not get boundedness of $|\cdot|_{\mathcal{F}}$ from Step 1, so this must be checked separately. But since $\mathcal{F}$ is R-good, this implies

$$|T - a|_{\mathcal{F}} \leq \max\{|a|, R\} = ||T - a||, \tag{B.8}$$

where $||\cdot||$ is the Gauss norm restricted to $K[T]$.[139] Since all polynomials factor into linear polynomials, and since both $||\cdot||$ and $|\cdot|_{\mathcal{F}}$ are multiplicative seminorms (by Step 2), we conclude that

$$|f|_{\mathcal{F}} \leq ||f||, \qquad f \in K[T]. \tag{B.9}$$

This finishes the proof of our claim. □

Claim B.0.2. *The construction $\widetilde{|\cdot|_{\mathcal{F}}}$ defines a bounded multiplicative seminorm on $\mathcal{A}$.*

Proof of Claim. The fact that $\widetilde{|0|_{\mathcal{F}}} = 0$ and $\widetilde{|1|_{\mathcal{F}}} = 1$ is obvious by construction. As for the other properties:

- $\widetilde{|\cdot|_{\mathcal{F}}}$ *is a well-defined map.* Since $\widetilde{|\cdot|_{\mathcal{F}}}$ takes values in $[0, \infty)$, we need to show that the limit of Equation (5.38) exists for $f \in \mathcal{A}$. For explicitness, let $f = \sum_{i=0}^{\infty} a_i T^i$. By Claim B.0.1, we know that $|\cdot|_{\mathcal{F}}$ is bounded, and also satisfies the ultrametric inequality on $K[T]$. Hence, for any natural numbers $M \leq N$, we obtain the inequality
$$\Big| |\sum_{i=0}^{N} a_i T^i|_{\mathcal{F}} - |\sum_{i=0}^{M} a_i T^i|_{\mathcal{F}} \Big| \leq \max_{M \leq i \leq N}\{|a_i| R^i\}. \tag{B.10}$$
Since $f \in \mathcal{A}$, we know $|a_i|R^i \to 0$ by definition. Hence, $\{|\sum_{i=0}^{n} a_i T^i|_{\mathcal{F}}\}_{n \in \mathbb{N}}$ is a Cauchy sequence and thus converges to a limit in $[0, \infty)$.
- *Bounded.* Let $f \in \mathcal{A}$ where $f = \sum_{i=0}^{\infty} a_i T^i$. Since
$$\lim_{n \to \infty} ||\sum_{i=0}^{n} a_i T_i|| = ||f||, \tag{B.11}$$
and since
$$|\sum_{i=0}^{n} a_i T_i|_{\mathcal{F}} \leq ||\sum_{i=0}^{n} a_i T_i||, \qquad \text{for all } n, \tag{B.12}$$
by Claim B.0.1, conclude that $\widetilde{|\cdot|_{\mathcal{F}}} \leq ||\cdot||$.
- *Ultrametric Inequality.* This also follows from $|\cdot|_{\mathcal{F}}$ satisfying the ultrametric inequality. Indeed, given $f, f' \in \mathcal{A}$, compute:
$$\begin{aligned} \widetilde{|f + f'|_{\mathcal{F}}} &= \lim_{n \to \infty} |\sum_{i=0}^{n} a_i T^i + \sum_{i=0}^{n} b_i T^i|_{\mathcal{F}} \\ &\leq \lim_{n \to \infty} \max\{|\sum_{i=0}^{n} a_i T^i|_{\mathcal{F}}, |\sum_{i=0}^{n} b_i T^i|_{\mathcal{F}}\} \\ &= \max\{\widetilde{|f|_{\mathcal{F}}}, \widetilde{|f'|_{\mathcal{F}}}\} \end{aligned}$$
with representations $f = \sum_{i=0}^{\infty} a_i T^i$ and $f' = \sum_{i=0}^{\infty} b_i T^i$.

[139] Why? See proof of Claim 5.2.18.

- *Multiplicativity.* We first claim that $f \in \mathcal{A}$ converges absolutely with respect to $\widetilde{|\cdot|}_{\mathcal{F}}$. It suffices to do the check for the Gauss norm $||\cdot||$ since $\widetilde{|\cdot|}_{\mathcal{F}} \leq ||\cdot||$ on $\mathcal{A}$. But this is clear by definition since

  $$||f|| = \left\|\sum_{i=0}^{\infty} a_i T^i\right\| = \max_i |a_i| R^i = \left\|\sum_{i=0}^{\infty} |a_i| T^i\right\|.$$

  As such, multiplicativity of $\widetilde{|\cdot|}_{\mathcal{F}}$ essentially follows from Mertens' Theorem for Cauchy products, which states: given real sequences $\{a_i\}$ and $\{b_i\}$, if $\sum_{i=0}^{\infty} a_i$ converges to A and $\sum_{i=0}^{\infty} b_i$ converges to B, and at least one of them converges absolutely, then their Cauchy product converges to $A \cdot B$.

This completes the proof of the claim. □

Remark B.0.3. Readers familiar with Berkovich geometry may recognise parallels between our proof of Claim B.0.2 and the standard proof of the homeomorphism

$$\mathbb{A}^1_{\mathrm{Berk}} \cong \bigcup_{R>0} \mathcal{M}(K\{R^{-1}T\}).$$

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