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# Sketches for Arithmetic Universes

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# Reasoning in point-free topology: examples

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Dedekind sections, e.g. $(L_x, R_x)$

Let $x, y \in \mathbb{R}$

Then $x+y \in \mathbb{R}$ where

$$L_{x+y} = \{q+r \mid q \in L_x, r \in L_y\}$$

$$R_{x+y} = \{q+r \mid q \in R_x, r \in R_y\}$$

# Reasoning in point-free topology: examples

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

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> $R_{x+y} = \{q+r \mid q \in R_x, r \in R_y\}$

- $x, y$ are points! Doesn't look point-free
- Conditions $s \in L$, $s \in R$ correspond to subbasic opens $(s, \infty)$, $(-\infty, s)$ of $\mathbb{R}$
- Calculate inverse images, e.g. $+^{-1}(s, \infty) = \bigvee_{q+r=s} (q, \infty) \times (r, \infty)$ - open

- Logical restrictions (geometricity) on how construction is done can guarantee this works
- Slogan Continuity is geometricity
- Geometric theory describes both points (models) and opens (propositions)

# Reasoning in point-free topology: examples

$[- \to -] : [SFP] \times [SFP] \to [SFP]^{\infty}$ "space" of SFP domains

Let $X, Y$ be (compact bases of) SFP domains

Then

⋮ (complicated construction)

$[X \to Y]$ is c.b. of SFP domain,
with right properties to be function space

cf. Abramsky "Domain theory in logical form"
Vickers "Topical categories of domains"

# Reasoning in point-free topology: examples

- "Space" is generalized a la Grothendieck - a topos
- Map is a geometric morphism
- Not easy to make this geometric - but underlying method already implicit in Abramsky

$[- \to -] : [SFP] \times [SFP] \to [SFP]$

Let $X, Y$ be (compact bases of) SFP domains
Then
(complicated construction)
$[X \to Y]$ is c.b. of SFP domain,
with right properties to be function space

- Geometric morphisms have continuity needed to solve domain equations $X \cong F(X)_\perp$

$$[SFP] \xrightarrow{F} [SFP] \xrightarrow{LIFT} [SFP_\perp] \longrightarrow [SFP]$$

Composite map has initial algebra

# Reasoning in point-free topology: examples

$\mathrm{Spec} : [BA] \longrightarrow \mathrm{Spaces}$

Let $B$ be a Boolean algebra

Then $\mathrm{Spec}(B)$ is point-free space of prime filters of $B$, presented by

- generators $(b)$ $(b \in B)$
- relations $(b_1 \wedge b_2) = (b_1) \wedge (b_2)$
  - $(1) = \top$
  - $(b_1 \vee b_2) = (b_1) \vee (b_2)$
  - $(0) = \bot$

meet, join in $B$

logical conjunction, disjunction

# Reasoning in point-free topology: examples

Spec : [BA] $\to$ Spaces

> Let B be a Boolean algebra
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>   $(0) = \bot$
>
> meet, join in B
>
> logical conjunction
>
> disjunction

- B a pt of space of Boolean algebras
- Out of that constructs another space
- Joyal-Tierney $\Rightarrow$ bundle

$$\begin{array}{c}[\mathrm{BA} + \text{prime filter}] \\ \downarrow \\ [\mathrm{BA}]\end{array}$$

forget prime filter

Spec(B) is fibre over B

- Geometricity $\Rightarrow$ construction is uniform
  Single construction on generic B
  also applies to specific B's

cf. Heunen-Landsman-Spitters spectral bundle for $C^*$-alg: commutative subalgebra $\mathcal{C} \longmapsto \mathrm{Spec}(\mathcal{C})$

# Grothendieck: topos = generalized topological space

of opens — of sheaves — classifying topos

Geometric theory $\mathbb{T}_0$ — Space — Frame — Topos — $\mathcal{S}[\mathbb{T}_0]$

$$
\begin{tikzcd}
\mathbb{T}_0 & \text{Space} \arrow[d, "\text{Map}"'] & \text{Frame} & \text{Topos} & \mathcal{S}[\mathbb{T}_0] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbb{T}_1 & {} & {} \arrow[u, "\text{Frame hom}"] & \text{Geometric morphism} & \mathcal{S}[\mathbb{T}_1] \arrow[u, bend left]
\end{tikzcd}
$$

Map: Models of $\mathbb{T}_0 \longrightarrow$ Models of $\mathbb{T}_1$

Model of $\mathbb{T}_1$ in "$\mathcal{L}$-maths generated by model of $\mathbb{T}_0$"

$= \mathbf{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle$ classifying category

$\mathcal{L}$-functor: $\mathbf{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle \longleftarrow \mathbf{Cat}_{\mathcal{L}}\langle \mathbb{T}_1 \rangle$

## Map $\mathbb{T}_0 \to \mathbb{T}_1$

generic model

Let $M$ be a model of $\mathbb{T}_0$

⋮

$\mathcal{L}$-maths generated by $M$

⋮

Then $f(M) = \ldots\ldots$ is a model of $\mathbb{T}_1$

$\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle$

Model of $\mathbb{T}_1$ in $\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle$

$= \mathcal{L}$-functor $\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle \Leftarrow \mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T}_1 \rangle$

e.g. $\mathcal{L}$ = geometric

$\mathcal{L}$ parameter for logic

Geometric theory: Sorts $\sigma$

Predicates $P(x_i : \sigma_i)_i^n$

Functions $f(x_i : \sigma_i)_i^n : \sigma$

signature

In context $\vec{x} = (x_i : \sigma_i)_i^n$ derive terms $t$, and formulae $\phi, \psi$ using $\wedge \bigvee = \exists$

Theory also has axioms in form of sequents $\phi \vdash_{\vec{x}} \psi$

$\mathcal{L}$-categories: Grothendieck toposes

$\mathcal{L}$-functors: preserve colim, finite lim
– inverse image parts of geometric morphisms

$\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T} \rangle = \mathcal{S}[\mathbb{T}]$

## Bundle over $\mathbb{T}_0$

generic model

Let M be a model of $\mathbb{T}_0$

⋮

$\mathcal{L}$-maths generated by M

⋮

Then $f(M) = \ldots\ldots$ is a ~~model~~ space ~~of $\mathbb{T}_1$~~

$\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T}_0 \rangle$

Constructs $\mathbb{T}_1$ &

$$\begin{array}{c} \mathbb{T}_1 \\ \downarrow \\ \mathbb{T}_0 \end{array}$$

$f(M) =$ fibre over M

cf. Joyal-Tierney localic bundle theorem

## e.g. SFP domains $\quad\quad \mathbb{T}_{SFP}$

- Sorts $X$, $\mathcal{F}_{fn}X$
- Partial order $\sqsubseteq$ on $X$
- Operations & axioms to make $\mathcal{F}_{fn}X \cong$ Karatowski finite powerset of $X$
- Binary relation CUB on $\mathcal{F}_{fn}X$

$$\mathrm{CUB}(S,T) \vdash_{S,T} \forall s \in S.\, \forall t \in T.\, s \sqsubseteq t$$

$$\mathrm{CUB}(S,T) \wedge \forall s \in S.\, s \sqsubseteq u \vdash_{S,T,u} \exists t \in T.\, t \sqsubseteq u$$

$$\vdash_S \exists T.\, (S \subseteq T \wedge \forall U \subseteq_{fin} T.\, \exists V \subseteq_{fin} T.\, \mathrm{CUB}(U,V))$$

$$\mathrm{CUB}(S,T') \wedge \forall s \in S.\, \forall t \in T.\, s \sqsubseteq t \wedge \forall t' \in T'.\, \exists t \in T.\, t \sqsubseteq t' \vdash_{S,T',T} \mathrm{CUB}(S,T)$$

Model = compact base of SFP domain

## e.g. SFP domains

$\mathbb{T}_{SFP}$

$$\mathcal{S}[\mathbb{T}_{SFP}] \simeq [\mathrm{Poset}^{\mathrm{fin}}_{\mathrm{adj}}, \mathcal{S}]_0$$

But don't really need this!

object = finite cardinal with decidable partial order

morphism X to Y is adjunction $X \underset{\leftarrow}{\overset{\rightarrow}{\perp}} Y$

e.g.

Abramsky's construction of SFP function spaces gives geometric morphism

$$\mathcal{S}[\mathbb{T}_{SFP}] \times \mathcal{S}[\mathbb{T}_{SFP}] \longrightarrow \mathcal{S}[\mathbb{T}_{SFP}]$$

$$(X, Y) \longmapsto [X \to Y]$$

# Constructing classifying category $\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T} \rangle$

For finitary $\mathcal{L}$: universal algebra.

- cartesian theory of $\mathcal{L}$-categories
- $\mathbb{T}$ provides generators & relations
- universal property:
  model of $\mathbb{T}$ ~ $\mathcal{L}$-functor from $\mathrm{Cat}_{\mathcal{L}}\langle \mathbb{T} \rangle$

For geometric logic:
available infinities supplied by base topos $\mathcal{S}$ (up to equivalence)
"of sets"
Classifying topos $\mathcal{S}[\mathbb{T}]$ has right property
(for $\mathcal{S}$-toposes)
- constructed ad hoc (sheaves)

# Arithmetic universes instead of Groth. toposes

Pretopos – finite limits + all well behaved
coequalizers of equivalence relations
finite coproducts

$\mathcal{S}$

+ set-indexed coproducts
+ smallness conditions

Giraud's theorem

Grothendieck toposes

bounded $\mathcal{S}$-toposes

extrinsic $\infty$s from $\mathcal{S}$

+ parametrized list objects

$$1 \xrightarrow{\varepsilon} \mathrm{List}(A) \xleftarrow{\mathrm{cons}} A \times \mathrm{List}(A)$$

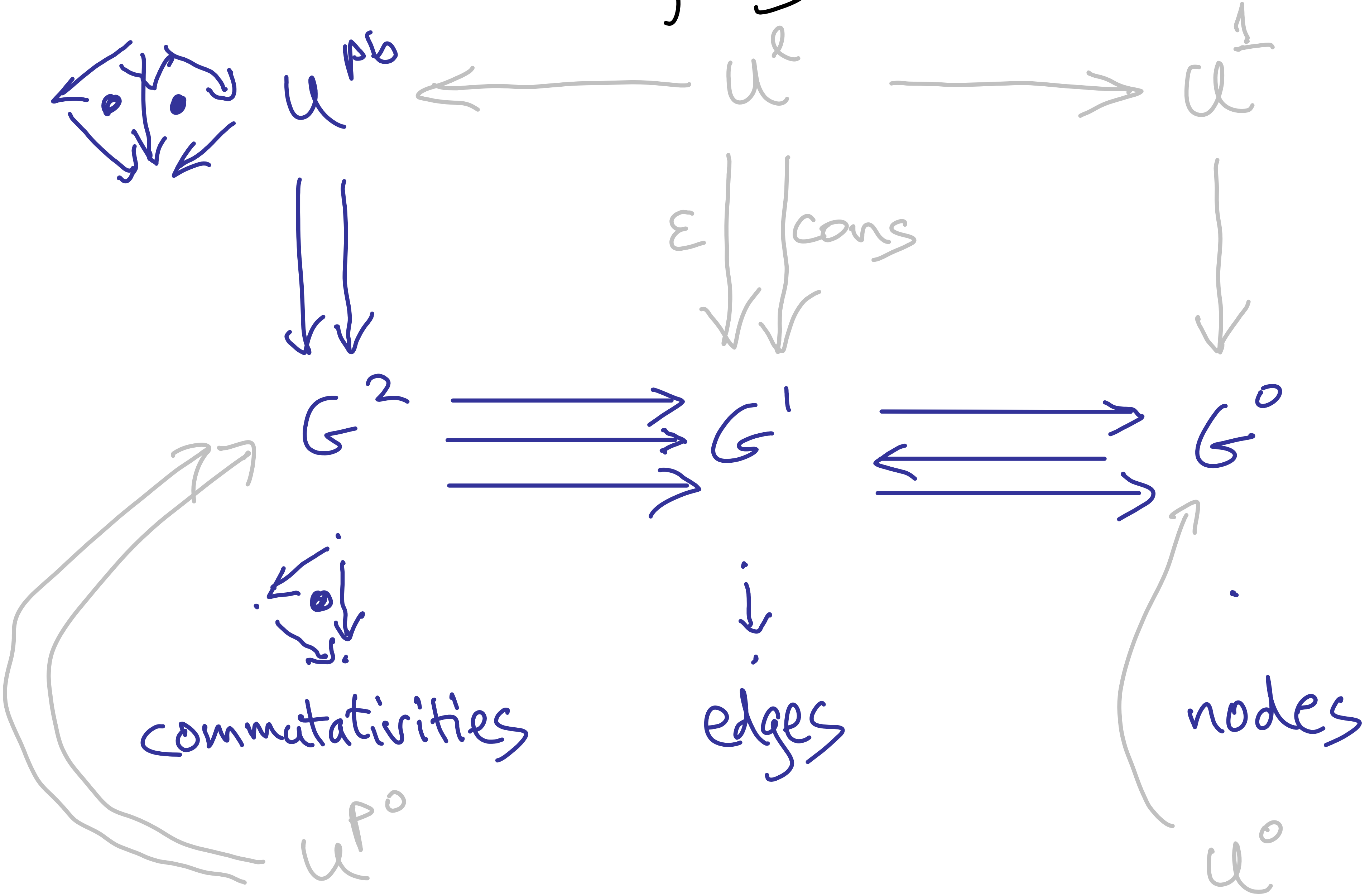
Arithmetic universes (AUs)

intrinsic infinities
e.g. $\mathbb{N} = \mathrm{List}(1)$

# Hopes

- Finitary formalism for some geometric theories
- Dependent type theory of (generalized) spaces
- Use methods of classifying toposes in base-independent way
- Computer support for that
- Logic internalizable in itself (cf. Joyal applying AUs to Gödel's theorem)

# Sketches instead of logical theories



# Sketch homomorphism $T_0 \xrightarrow{\sigma} T_1$

Obvious definition:

$\text{nodes}_0 \longrightarrow \text{nodes}_1$

$\text{edges}_0 \longrightarrow \text{edges}_1$

etc

} preserving structure

Model reduction takes

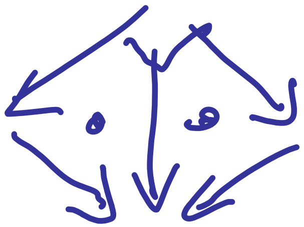
models of $T_1 \longrightarrow$ models of $T_0$

but not general enough –

too syntax bound

# Strict or non-strict?

Non-strict

 Usual semantics of sketches

is {a / the canonical} pullback of $\searrow\swarrow$ ?

strict – needed in universal algebra

Aim: Every non-strict model uniquely $\cong$ strict one

Context = sketch build up in steps

Each pb (etc) object is fresh.

# Simple extensions $\Pi_1$ extends $\Pi_0$

| | Data already in $\Pi_0$ | Delta add to make $\Pi_1$ |
|---|---|---|
| Primitive node | – | $\cdot$ |
| Primitive edge | $\cdot \quad \cdot$ | $\cdot \to \cdot$ |
| Commutativity | $\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd}$ | $\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] \arrow[r, phantom, "\bullet"] & & \cdot \end{tikzcd}$ |
| Universals: Pullback etc. | $\begin{tikzcd} \cdot \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd}$ | $\begin{tikzcd} & \cdot \arrow[dl] \arrow[d] \arrow[dr] & \\ \cdot \arrow[dr] & \cdot \arrow[d] & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd}$ $\in u^{pb}$ |

Extension $\Pi_0 \subset \Pi_1$ - chain of simple extensions

Context - extension of empty sketch $\mathbb{1}$

Aim Category Con

Object = context $\mathbb{T}$

Map

Morphism $\mathbb{T}_0 \longrightarrow \mathbb{T}_1$

$\mathcal{L} = AU$

= strict AU-functor

$AU\langle \mathbb{T}_0 \rangle \longleftarrow AU\langle \mathbb{T}_1 \rangle$

$cat_{\mathcal{L}}\langle \mathbb{T}_0 \rangle \leftarrow cat_{\mathcal{L}}\langle \mathbb{T}_1 \rangle$

= strict model of $\mathbb{T}_1$ in $AU\langle \mathbb{T}_0 \rangle$

Describe finitely: finite theory $\mathbb{T}_1$ modelled by stuff finitely derivable from finite theory $\mathbb{T}_0$

# Equivalence extensions $T_0 \in T_1$ {Finitely derivable stuff}

s.t. $\mathrm{Au}\langle T_0 \rangle \longrightarrow \mathrm{Au}\langle T_1 \rangle$ iso

| | Data already in $T_0$ | Delta add to make $T_1$ |
|---|---|---|
| ~~Primitive node~~ | — | $\bullet$ |
| ~~Primitive edge~~ | $\bullet \qquad \bullet$ | $\bullet \longrightarrow \bullet$ |
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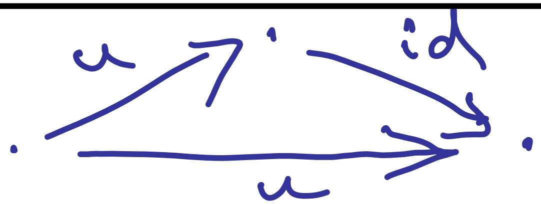
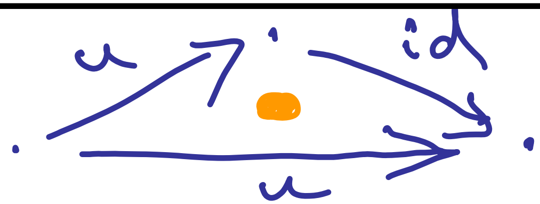
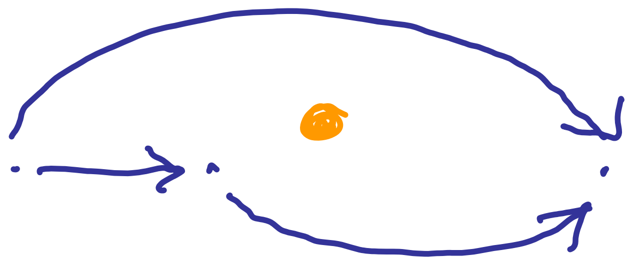
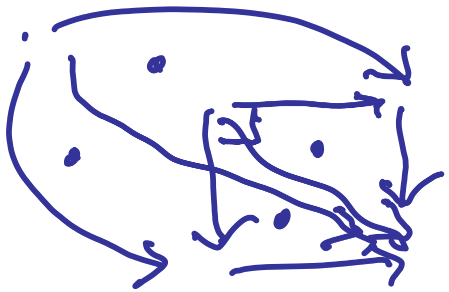
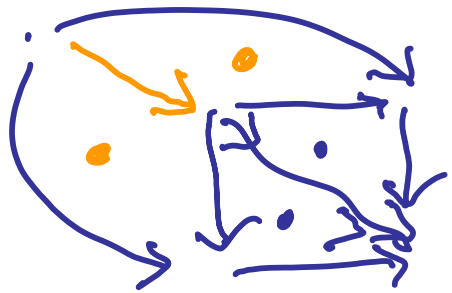
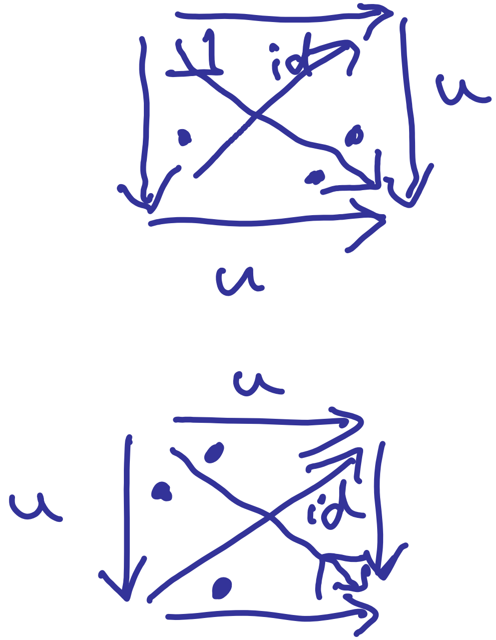
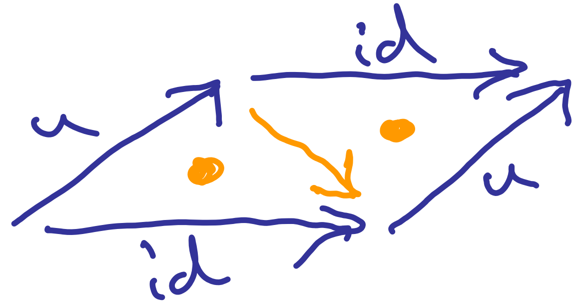
+ more

# Equivalence extensions $\Pi_0 \in \Pi_1$ ... contd.

| | Data already in $\Pi_0$ | Delta add to make $\Pi_1$ |
|---|---|---|
| Composition |  |  |
| Equality of morphisms - e.g. |  |  |
| |  |  |
| Fillers for universals, & their uniqueness |  |  |

# Equivalence extensions $\Pi_0 \in \Pi_1$ ... contd.

| | Data already in $\Pi_0$ | Delta add to make $\Pi_1$ |
|---|---|---|
| Categorical rules for balance, stability, exactness. e.g. balance: mono epi $\Rightarrow$ iso |   |   |

Equivalence ext$^n$ = chain of these simple steps

## Map $\mathbb{T}_0 \to \mathbb{T}_1$

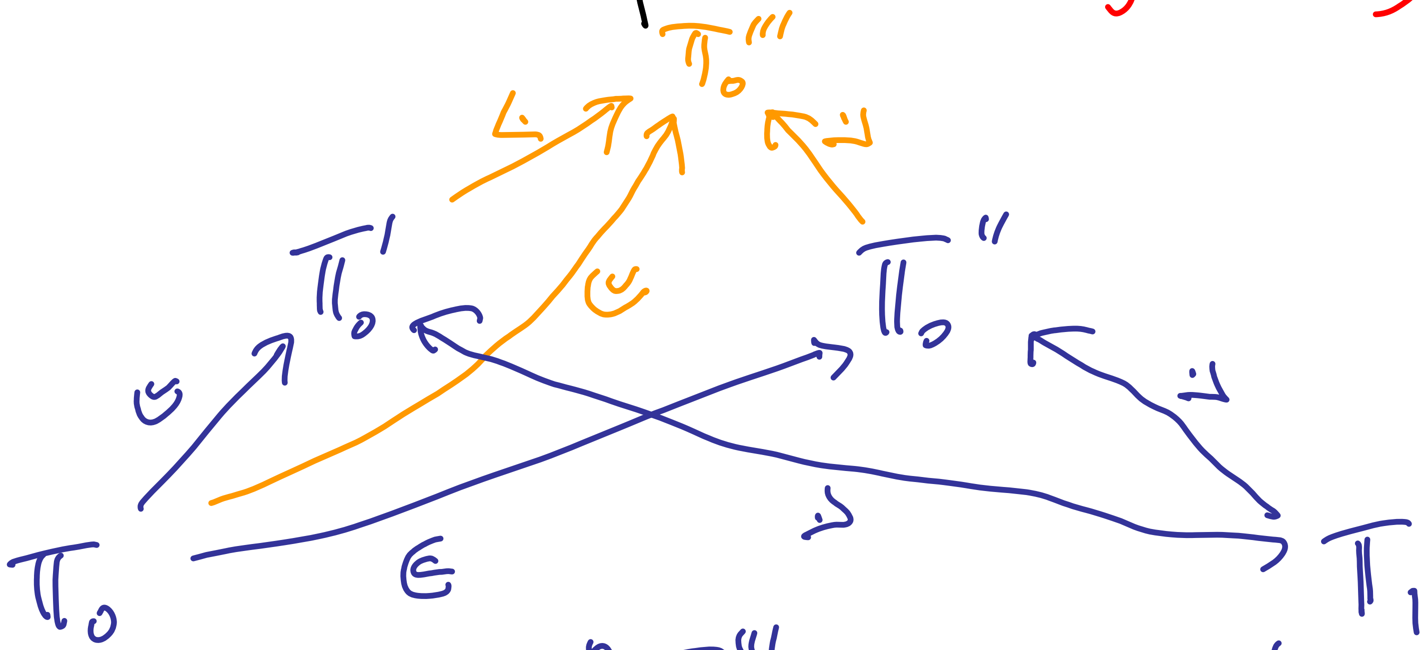
$$
\begin{tikzcd}
 & \mathbb{T}_0' \arrow[dl, "\text{equivalence extension}"'] \arrow[dr, "\text{sketch homomorphism}"] & \\
\mathbb{T}_0 & & \mathbb{T}_1
\end{tikzcd}
$$

$\mathbb{T}_1$ modelled by stuff derivable from $\mathbb{T}_0$

These are the 1-cells of Con

# When are maps equal?

"Objectively equal"



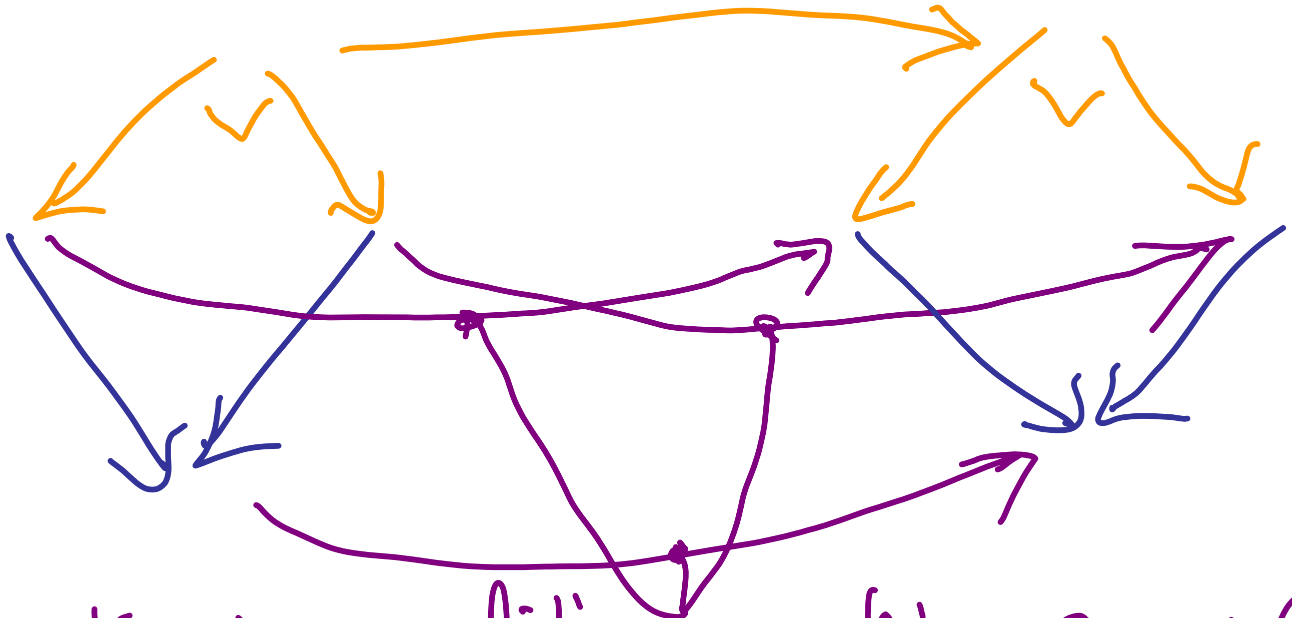

- Work in equiv. ext$^n$ $\Pi_0'''$ refining $\Pi_0'$ and $\Pi_0''$
- Compare images of $\Pi_1$ there?
- Do nodes have equal images? Object equalities
- Modulo those, are edges provably equal?
- May need $\Pi_0''' \in \Pi_0''''$ to get proof.

Object equalities - certain edges

- Any identity edge (on a single node)
- The same construction applied to equal data.

e.g. pullbacks

Then fillin edge is object equality



3 object equalities, making squares commute

# Main Theorem

Let $\mathrm{Con}_{\to}$ be opposite of cat. of contexts & sketch homomorphisms.

Then
- $\mathrm{Con}$ is universal over $\mathrm{Con}_{\to}$ subject to
  - object equalities being equalities
  - equivalence extensions being invertible
- $\mathrm{Con} \longrightarrow \mathrm{AU}_s^{op}$ $\qquad T \longmapsto \mathrm{Au}\langle T \rangle$

  is full and faithful
- Each object, morphism or equality in $\mathrm{Con}$ is a finite structure

Con is finitary

# Other structure in Con

- Con is a 2-category.

$$\Pi_0 \overset{\Downarrow}{\rightrightarrows} \Pi_1 \quad \text{is} \quad \Pi_0 \to \Pi_1^{\to}$$

Two copies of $\Pi_1$, with edges etc. to make a homomorphism between them

Product, Inserter, Equifier

- Con has finite PIE limits in strict sense

Power & Robinson, not Elephant

- Con has strict pullbacks of projection maps

reindex extensions

## Hopes

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- Dependent type theory of (generalized) spaces
- Use methods of classifying toposes in base-independent way
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