Arithmetic universes as generalized spaces

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Preprint on web "An induction principle for consequence in arithmetic universes"

- Vision: generalized spaces using AUs instead of Grothendieck toposes
- Problem: AUs not cartesian closed
- We proved: induction principle for $\phi(n) \vdash_{n:N} \psi(n)$

Geometric logic

First order, many sorted, positive, infinitary

Signature Σ: Sorts, functions, predicates

Formulae ϕ: use $\top, \wedge, \bot, \bigvee, =, \exists$

disjunctions can be infinite

Formulae in context $(\vec{x}.\ \phi)$

finite list of sorted variables ← All free variables are in $\vec{x}$

Sequent $\phi \vdash_{\vec{x}} \psi$ $(\vec{x}.\phi)$, $(\vec{x}.\psi)$ both formulae in context

$(\forall\vec{x})(\phi\to\psi)$

Theory $\mathbb{T}$ over Σ: set of sequents

axioms

Example: $\mathbb{R}$

Dedekind section

	Propositional	Predicate
Sorts:	none	none declared – but $\mathbb{Q}$ can be constructed geometrically
Predicates:	L_q, R_q $(q\in\mathbb{Q})$ "$q<x$" "$x<q$"	$L, R \subseteq \mathbb{Q}$
Axioms:	$L_q \dashv\vdash \bigvee_{q<q'} L_{q'}$	$L(q) \dashv\vdash_{q:\mathbb{Q}} \exists q':\mathbb{Q}(L(q')\wedge q<q')$
	$\top \vdash \bigvee_q L_q$	$\top \vdash \exists q:\mathbb{Q}\ L(q)$
	(similarly for R)	
Locatedness	$L_q\wedge R_q \vdash \bot$	$L(q)\wedge R(q) \vdash_{q:\mathbb{Q}} \bot$
weak	$\top \vdash L_q\vee R_r\ (q<r)$	$q<r \vdash_{q,r:\mathbb{Q}} L(q)\vee R(r)$
strong ↓	$\top \vdash \bigvee_{r-q<\varepsilon} L_q\wedge R_r\ (\varepsilon>0)$	$\varepsilon>0 \vdash_{\varepsilon:\mathbb{Q}} \exists q,r:\mathbb{Q}(L(q)\wedge R(r)\wedge r-q<\varepsilon)$

Weak locatedness $\Rightarrow$ strong

WL SL

WL: $q<r \vdash_{q,r:\mathbb{Q}} L(q)\vee R(r)$

SL: $\varepsilon>0 \vdash_{\varepsilon:\mathbb{Q}} \exists q,r:\mathbb{Q}(L(q)\wedge R(r)\wedge r-q<\varepsilon)$

Given $\varepsilon>0$

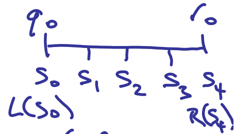
By induction:

$$\forall n:\mathbb{N}.\ (\exists q_0,r_0:\mathbb{Q}.\ (L(q_0)\wedge R(r_0)\wedge r_0-q_0<2^n\varepsilon) \to \exists q,r:\mathbb{Q}.\ (L(q)\wedge R(r)\wedge r-q<\varepsilon))$$

Base case $n=0$ immediate.

Suppose $L(q_0)\wedge R(r_0)\wedge r_0-q_0<2^{n+1}\varepsilon$

WL: $(L(s_1)\vee R(s_2))\wedge(L(s_2)\vee R(s_3))$



Cases	Define q_1	r_1
$R(s_2)$	s_0	s_2
$L(s_2)$	s_2	s_4
$L(s_1)\wedge R(s_3)$	s_1	s_3

$r_1-q_1=\frac{r_0-q_0}{2}<2^n\varepsilon$

$L(q_1), R(r_1)$

$\therefore$ use induction

Arithmetic universe = list-arithmetic pretopos

Joyal

Maietti - see also Cockett

enough for finitary algebra

For every A : list (A) has

$$1 \xrightarrow{[\,]} list(A)$$

$$A \times List(A) \xrightarrow{cons} List(A)$$

cons$(a, [a_1, \dots, a_n])$ $= [a, a_1, \dots, a_n]$

$$\begin{tikzcd}
 & List(A)\times B \arrow[d, "rec(y_0,g)"] & A\times List(A)\times B \arrow[l, "cons\times B"'] \arrow[d, "A\times rec(y_0,g)"] \\
B \arrow[ur, "\langle B,[\,]\circ !\rangle"] \arrow[r, "y_0"'] & Y & A\times Y \arrow[l, "g"]
\end{tikzcd}$$

$\forall y_0 g\ \exists!\ rec(y_0,g)$

Write $g(a,y) = a\cdot y$. Then $rec(y_0,g)([a_1,\dots,a_n],b)$ $= a_1 \cdot \cdots \cdot a_n \cdot y_0(b)$

AU has –

enough logic to present theory of $\mathbb{R}$

- finite limits
- finite colimits

} interact sensibly

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$
- free algebras

\+ more

Theory of AUs is cartesian

∴ can present with generators & relations

e.g. AU freely generated by a Dedekind section

similar to $Sh(\mathbb{R})$

Arithmetic space X = AU $\mathbb{A}X$

map = AU functor in reverse

cf. locale maps, geometric morphisms

Strictness

- AUs have canonical structure
- strict AU functors preserve it on the nose
- (non-strict) .. .. up to iso

categories

AU_s

AU

Universal algebra uses strict AU functors

(= homomorphisms for cartesian theory of AUs)

We had to use non-strict

AS = AU^{op}

e.g. $\mathbb{A}[u:U]$ characterized by –

$AU_s(\mathbb{A}[u:U], \mathbb{B}) \cong$ cat of pairs (F,u),

$\simeq$

$AU(\mathbb{A}[u:U], \mathbb{B})$

$F \in AU(\mathbb{A}, \mathbb{B})$

$u: 1 \to F(U)$

non-strict

Use comma categories as AUs

- Construct free AU over $\mathbb{A}$ qua cat
- Adjoin coherent isos between new & old AU structures

Locatedness

"AU freely generated by Dedekind section"

– which locatedness axiom?

$q<r \vdash_{q,r:\mathbb{Q}} L(q) \vee R(r)$ WL

$\varepsilon > 0 \vdash_{\varepsilon:\mathbb{Q}} \exists q,r:\mathbb{Q}(L(q) \wedge R(r) \wedge r-q < \varepsilon)$ SL

Equivalence proof

$\forall n:\mathbb{N}.\,(\exists q_0,r_0:\mathbb{Q}.\,(L(q_0)\wedge R(r_0)\wedge r_0-q_0<2^n\varepsilon)$
$\to \exists q,r:\mathbb{Q}.\,(L(q)\wedge R(r)\wedge r-q<\varepsilon))$

relies on cartesian closedness to interpret → as formula connective

BUT AUs are not cartesian closed in general!

Are axioms equivalent in AUs?

Induction in AUs $\mathbb{A}$

$\mathbb{N} = \mathrm{List}(1)$

① $\phi \hookrightarrow \mathbb{N}$ $\quad \left.\begin{array}{l}\phi(0)\\ \phi(n) \vdash_{n:\mathbb{N}} \phi(n+1)\end{array}\right\} \Longrightarrow \top \vdash_{n:\mathbb{N}} \phi(n)$

ϕ a subset of $\mathbb{N}$ closed under 0, suc

$\therefore$ = whole of $\mathbb{N}$

② $\phi, \psi \longrightarrow \mathbb{N}$ $\quad \left.\begin{array}{l}\phi(0) \vdash \psi(0)\\ \text{induction step?}\end{array}\right\} \overset{?}{\Longrightarrow} \phi(n) \vdash_{n:\mathbb{N}} \psi(n)$

Induction hypothesis:

fix n (generically), assume $\phi(n) \vdash \psi(n)$ — for that fixed n

Working in $\mathbb{A}[n:\mathbb{N}][\phi(n) \vdash \psi(n)] = \mathbb{A}'$ (say)

Induction step: In $\mathbb{A}'$ have $\phi(n+1) \vdash \psi(n+1)$

Can we deduce $\phi(n) \vdash_{n:\mathbb{N}} \psi(n)$ in $\mathbb{A}$? Yes!

Proof outline

- structure theorems $\mathbb{A}[u:U] \simeq \mathbb{A}/U$
  $\mathbb{A}[\phi \vdash \bot] \simeq$ "category of sheaves"
- "subspaces" open $\mathbb{A}[\top \vdash \psi]$, closed $\mathbb{A}[\phi \vdash \bot]$
  generate lattice $\cong \mathrm{BA}\langle \mathrm{Sub}_{\mathbb{A}}(1)\rangle$
  – classical logic of subspaces conservative over coherent logic of subobjects
- use Boolean manipulation of induction step to find properties in $\mathbb{A}$
- new induction lemma to deduce conclusion from those properties

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Structure theorems

→ structure theorems $\mathbb{A}[u:U] \simeq \mathbb{A}/U$
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$\mathbb{A}[u:U] \simeq \mathbb{A}/U$

- $\mathbb{A}/U$ an AU,
  $\mathbb{A} \xrightarrow{\xi} \mathbb{A}/U$ an AU functor
  $\xi(U) = \begin{array}{c} U\times U \\ \downarrow \\ U \end{array}$ has global element

$$\begin{tikzcd} U \arrow[r, "\Delta"] \arrow[dr] & U\times U \arrow[d] \\ & U \end{tikzcd}$$

$\therefore$ get $\mathbb{A}[u:U] \longrightarrow \mathbb{A}/U$

- Every object $\begin{array}{c} V \\ f\downarrow \\ U \end{array}$ of $\mathbb{A}/U$ is equalizer

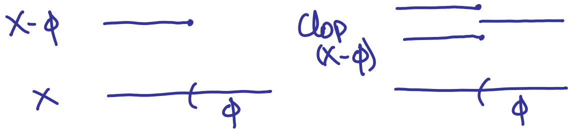
$$\begin{tikzcd} V \arrow[r, "\langle v{,}f\rangle"] \arrow[ddr] & V\times U \arrow[rr, "f\times U"] \arrow[dr] \arrow[dd] & & U\times U \arrow[ddll, bend left] \\ & & U \arrow[ur, "\Delta"'] \arrow[dl] & \\ & U & & \end{tikzcd}$$

  Map to corresponding equalizer in $\mathbb{A}[u:U]$
  Get AU functor $\mathbb{A}/U \longrightarrow \mathbb{A}[u:U]$
- Two functors give an equivalence

Structure theorems $\mathbb{A}[\phi \vdash \bot]$

$\phi \hookrightarrow 1$ in $\mathbb{A}$

Closed subspace is Stone over superspace



a Boolean algebra
$B_\phi = \mathrm{BA}\langle 1 \leq 0 \text{ (if } \phi)\rangle$

Sheaf = finitary sheaf F over B_ϕ

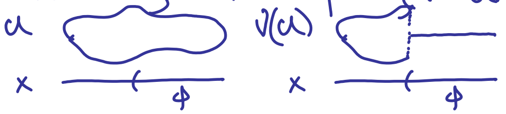
finitary sheaf: only finitary pasting

Presheaf $F(1) \to F(0)$, iso if ϕ

Sheaf $\Longleftrightarrow F(0) \cong 1$

$\therefore$ Sheaf = object U of $\mathbb{A}$ s.t. $U \to 1$ iso if ϕ

For any U: coequalizer $U\times\phi \rightrightarrows \begin{array}{c} U \\ \phi \end{array} \rightrightarrows U+\phi \twoheadrightarrow V(U)$



V a monad, μ iso

$\mathbb{A}[\phi \vdash \bot] \simeq \mathrm{Alg}(V)$

Subspaces

$\mathbb{A}[m^{-1}]$, m monic in $\mathbb{A}$

Preorder : $\mathbb{A}[m_1^{-1}] \leq \mathbb{A}[m_2^{-1}]$
if m_2 invertible in $\mathbb{A}[m_1^{-1}]$

Semilattice :

$$\mathbb{A}[m_1^{-1}] \wedge \mathbb{A}[m_2^{-1}] = \mathbb{A}[(m_1+m_2)^{-1}] \simeq \mathbb{A}[m_1^{-1}][m_2^{-1}]$$

If $\phi, \psi \hookrightarrow 1$:
open ψ = $\mathbb{A}[(\psi \hookrightarrow 1)^{-1}]$ $\mathbb{A}[\top \vdash \psi]$
closed $\neg\phi$ = $\mathbb{A}[(0 \hookrightarrow \phi)^{-1}]$ $\mathbb{A}[\phi \vdash \bot]$
crescent $\neg\phi \wedge \psi$
cocrescent $\mathbb{A}[(\phi \wedge \psi \hookrightarrow \phi)^{-1}]$ $\mathbb{A}[\phi \vdash \psi]$

- structure theorems $\mathbb{A}[u:U] \simeq \mathbb{A}/U$
  $\mathbb{A}[\phi \vdash \bot] \simeq$ "category of sheaves"
- → "subspaces" open $\mathbb{A}[\top \vdash \psi]$, closed $\mathbb{A}[\phi \vdash \bot]$
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Boolean algebra of subspaces generated by opens & closeds

If $a = \bigwedge_{i=1}^{n} (\neg\phi_i \vee \psi_i) \in BA\langle Sub_{\mathbb{A}}(1)\rangle$ write

$$\sigma(a) = \bigwedge_{i=1}^{n} \mathbb{A}[\phi_i \vdash \psi_i]$$ – meet of cocrescents

- σ preserves joins
- σ is order embedding

conservativity

- closed subspace $\neg\phi$
  is Boolean complement of ϕ
- $\mathbb{A}[\phi \vdash \psi] = \neg\phi \vee \psi$

use structure theorems!
Note $\mathbb{A}[\top \vdash \psi] = \mathbb{A}[*:\psi]$
$\simeq \mathbb{A}/\psi$

Induction hypothesis

$\mathbb{A}[n:\mathbb{N}][\phi(n) \vdash \psi(n)]$

– as subspace of $\mathbb{A}[n:\mathbb{N}]$ is
$\neg\phi(n) \vee \psi(n)$

Induction step $\neg\phi(n) \vee \psi(n) \leq \neg\phi(n+1) \vee \psi(n+1)$

$\neg\phi(n) \leq \neg\phi(n+1) \vee \psi(n+1)$ $\psi(n) \leq \neg\phi(n+1) \vee \psi(n+1)$

$\phi(n+1) \leq \phi(n) \vee \psi(n+1)$ $\phi(n+1) \wedge \psi(n) \leq \psi(n+1)$

$\therefore \phi(n+1) \vdash \phi(n) \vee \psi(n+1)$ $\phi(n+1) \wedge \psi(n) \vdash \psi(n+1)$
in $\mathbb{A}[n:\mathbb{N}]$ conservativity

$\therefore \phi(n+1) \vdash_{n:\mathbb{N}} \phi(n) \vee \psi(n+1)$ $\phi(n+1) \wedge \psi(n) \vdash_{n:\mathbb{N}} \psi(n+1)$
in $\mathbb{A}$ $\mathbb{A}[n:\mathbb{N}] \simeq \mathbb{A}/\mathbb{N}$

- structure theorems $\mathbb{A}[u:U] \simeq \mathbb{A}/U$
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Induction lemma

If $\phi(0) \vdash \psi(0)$
$\phi(n+1) \vdash_{n:\mathbb{N}} \phi(n) \vee \psi(n+1)$
$\phi(n+1) \wedge \psi(n) \vdash_{n:\mathbb{N}} \psi(n+1)$ then $\phi(n) \vdash_{n:\mathbb{N}} \psi(n)$

For $k:\mathbb{N}$ can define $A(k) = \{j \in \mathbb{N} \mid j \leq k, \phi(j), \dots, \phi(k)\}$
$f_k : A(k) \to \psi(k)$ by recursion on $j+k$

Cases for $f_k(j)$: 0 if $j = k = 0$ $\phi(0) \vdash \psi(0)$

If $j = k > 0$: $\phi(j) \vdash \phi(j-1) \vee \psi(j)$
$f_k(j-1)$ done

If $j < k$: $f_{k-1}(j) \Rightarrow \psi(k-1)$, $\phi(k) \wedge \psi(k-1) \vdash \psi(k)$
Given ϕ_n, $f_n(n) \Rightarrow \psi(n)$

- structure theorems $\mathbb{A}[u:U] \simeq \mathbb{A}/U$
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Conclusions

- Can prove implications by induction even though not cartesian closed
- More general induction principles too
- Some results analogous to those for lattice of sublocales
- Some structure theorems for some classifying AUs
- More plausibility to general idea: use AUs to provide finitary fragment of geometric logic, strictly stronger than coherent