Points in geometric point-free topology -- Abstract
Steve Vickers (School of Computer Science, University of Birmingham)
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This work is not new, but I have been asked to explain it again.

In both topos theory and predicative type theory it is recognized that point-set accounts of topology (in which a space is formulated as a set of points with extra structure) are unsatisfactory, and better replaced with point-free accounts. Their two apparently different approaches, one using frames and the other formal topologies, can be securely related when we understand that frame presentations are essentially equivalent to inductively generated formal topologies (and both to propositional geometric theories).

This looks bad news for practitioners, since points are such central components of a space. One cannot deny that topology is obscured when conducted in a purely point-free way, in terms solely of the frames or the formal topologies or the logical theories.

The good news is that topos theory provides techniques for reasoning with point-free spaces as though they had sufficient points. For example, there are two ways, depending on one's purpose, to define a map f: X -> Y.
(1) Thinking of it as a function, one can define the point f(x) of Y, given the point x of X.
(2) Thinking of it as a bundle, one can define (as point-free space) the fibre f*({y}), given the point y of Y.

What makes this work in topos theory is the fact that for each space X, the tops SX of sheaves over X provides a model of the logic being used.

But the techniques also rely on *geometricity*. The logic to use is not the whole of topos-valid logic, but the geometric fragment, preserved by inverse image functors of geometric morphisms. This includes finite limits, arbitrary colimits and free algebra constructions, but excludes powersets and function types.

The effect of this ability to transport along geometric morphisms is that geometric reasoning about points of X encompasses not only its global points (maps 1 -> X) but also its generalized points (maps W -> X for arbitrary W). Constructively it is rare for X to have enough global points (spatiality), but it always has enough generalized points.

I shall describe how the bundle idea allows us to define geometricity in a way that is more general than the above characterization, and includes constructions on internal spaces, and I shall relate it to examples in the current application of toposes to quantum theory.

I shall also discuss how much can be transferred to predicative mathematics. At present it seems that geometric reasoning does have predicative constent, but we lack a general metatheorem to encapsulate this.

Points in geometric point-free topology

Steve Vickers
School of Computer Science,
University of Birmingham

In topos theory: "Continuity is geometricity"
Geometric reasoning
$\Rightarrow$ point-free spaces have enough points

Why? - Gain access to generalized points
Same in predicative maths?

What does this define?

$\mathbb{R}$ = locale of "whose points are" Dedekind sections

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(L_1, R_1) + (L_2, R_2) = (\{q_1 + q_2 \mid q_i \in L_i\}, \{r_1 + r_2 \mid r_i \in R_i\})$

or: for any points x_1, x_2 of $\mathbb{R}$, $q, r \in \mathbb{Q}$

$$\begin{cases} q < x_1 + x_2 & \text{if } q = q_1 + q_2,\ q_i < x_i \\ x_1 + x_2 < r & \text{if } r = r_1 + r_2,\ x_i < r_i \end{cases}$$

Can these serve as definitions?

$\mathbb{R}$ = locale of Dedekind sections

If you define $\mathbb{R}$ = usual localic reals
then its points are (equivalent to) Dedekind sections

Describing the points
$\Rightarrow$ defining the locale ?

YES – if describe points as models of
a geometric theory.

Geometric theory of Dedekind sections

Propositional symbols $q<x$, $x<q$ $(q \in \mathbb{Q})$

Axioms

$$q<x \leftrightarrow \bigvee_{q<q'} q'<x$$
$$\top \to \bigvee_q q<x$$
$$x<r \leftrightarrow \bigvee_{r'<r} x<r'$$
$$\top \to \bigvee_r x<r$$
$$q<x \wedge x<q \to \bot$$
$$\top \to q<x \vee x<r \quad (\text{for } q<r)$$

Given model:
Dedekind section
(L, R) has
$L = \{q \mid q<x$
interpreted as true$\}$
$R = \{r \mid x<r$
interpreted as true$\}$

Geometric theories

- first-order, many sorted
- formulae $\phi, \psi, \ldots$ use
  $\wedge \ \bigvee \ \top \ \bot \ = \ \exists$
  disjunctions can be infinitary (set indexed)
- axioms $\forall \vec{x}. (\phi \to \psi)$
  finite list of sorted variables
  formulae in context $\vec{x}$

Propositional symbols $q<x$, $x<q$ $(q \in \mathbb{Q})$
Axioms
$q<x \leftrightarrow \bigvee_{q<q'} q'<x$
$\top \to \bigvee_q q<x$
$x<r \leftrightarrow \bigvee_{r'<r} x<r'$
$\top \to \bigvee_r x<r$
$q<x \wedge x<q \to \bot$
$\top \to q<x \vee x<r$ (for $q<r$)

no sorts

Propositional geometric theory $\mathbb{T}$

frame presentation $\Omega[\mathbb{T}]$
point of $[\mathbb{T}]$ = model of theory $\mathbb{T}$
open $\in \Omega[\mathbb{T}]$ = formula

also:
inductively generated formal topology
basic open = finite set of prop^l symbols

Geometric description of points
defines the locale $[\mathbb{T}]$

Predicate geometric theories

Interpretations of sorts – sometimes constrained uniquely by geometric structure & axioms (up to isomorphism)

e.g. finite limits, arbitrary colimits, free algebras

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathcal{F}$ (finite powerset), List
- get finitely bounded $\forall$ in formulae

Convenient to treat these as part of "geometric mathematics"

e.g. Dedekind sections

Propositional symbols $q<x$, $x<q$ $(q \in \mathbb{Q})$

Axioms

$$q<x \leftrightarrow \bigvee_{q<q'} q'<x$$
$$\top \to \bigvee_q q<x$$
$$x<r \leftrightarrow \bigvee_{r'<r} x<r'$$
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$$q<x \wedge x<q \to \bot$$
$$\top \to q<x \vee x<r \quad (\text{for } q<r)$$

(essentially propositional)

Sorts none declared

Predicates $L, R \subseteq \mathbb{Q}$

Axioms
$$\forall q:\mathbb{Q}.\ (L(q) \leftrightarrow \exists q':\mathbb{Q}.\ (q<q' \wedge L(q')))$$
$$\top \to \exists q:\mathbb{Q}.\ L(q)$$
$$\forall r:\mathbb{Q}.\ (R(r) \leftrightarrow \exists r':\mathbb{Q}.\ (r'<r \wedge R(r')))$$
$$\top \to \exists r:\mathbb{Q}.\ R(r)$$
$$\forall q:\mathbb{Q}.\ (L(q) \wedge R(q) \to \bot)$$
$$\forall q,r:\mathbb{Q}.\ (q<r \to L(q) \vee R(r))$$

Maps defined pointwise

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(L_1, R_1) + (L_2, R_2) = (\{q_1+q_2 \mid q_i \in L_i\}, \{r_1+r_2 \mid r_i \in R_i\})$

or: for any points x_1, x_2 of $\mathbb{R}$, $q, r \in \mathbb{Q}$
$$\begin{cases} q < x_1+x_2 & \text{if } q = q_1+q_2,\ q_i < x_i \\ x_1+x_2 < r & \text{if } r = r_1+r_2,\ x_i < r_i \end{cases}$$

Locale map $X \xrightarrow{f} Y$

= frame homomorphism $\Omega X \xleftarrow{f^*} \Omega Y$

But point of X = map $1 \xrightarrow{x} X$

$\therefore f$ transforms points $x \mapsto f \circ x$

Example is true statement about how $+$ map acts on points

Is that all it is?

NB $\mathbb{R}$ might not be spatial (have enough points)

Case-by-case reasoning

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(L_1, R_1) + (L_2, R_2) = (\{q_1+q_2 \mid q_i \in L_i\}, \{r_1+r_2 \mid r_i \in R_i\})$

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$$\begin{cases} q < x_1+x_2 & \text{if } q = q_1+q_2,\ q_i < x_i \\ x_1+x_2 < r & \text{if } r = r_1+r_2,\ x_i < r_i \end{cases}$$

- calculate inverse images of frame generators
$$+^*(q < x) = \bigvee\{(q_1 < x) \times (q_2 < x) \mid q_1 + q_2 = q\}$$
$$+^*(x < r) = \bigvee\{(x < r_1) \times (x < r_2) \mid r_1 + r_2 = r\}$$
- respects relations – extract from proof that pointwise defn constructs Dedekind section

Pointwise definition yields locale map

Can follow same process in formal topology

General metatheorem – in topos theory

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two propositional geometric theories
Consider a topos-valid result of the form –

> Let x be a model of $\mathbb{T}_1$.
> Then $f(x) \triangleq$ - - - - - - - - - - -
> is a model of $\mathbb{T}_2$

definition must be geometric

This defines a locale map $f : [\mathbb{T}_1] \longrightarrow [\mathbb{T}_2]$.
For every point $x : 1 \to [\mathbb{T}_1]$, $f \circ x$ is got by substituting x in def^n of $f(x)$.

How it works

① Every $\mathbb{T}$ has a topos of sheaves $\mathcal{S}[\mathbb{T}]$, a model of intuitionistic mathematics

② (Locale) maps $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$ $\approx$ geometric morphisms $\mathcal{S}[\mathbb{T}_1] \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{S}[\mathbb{T}_2]$ ($\perp$)

③ $\mathcal{S}[\mathbb{T}_2]$ classifies $\mathbb{T}_2$: $\mathcal{S}[\mathbb{T}_2]$ has generic model M of $\mathbb{T}_2$, and

maps $f : [\mathbb{T}_1] \longrightarrow [\mathbb{T}_2]$ $\approx$ models $f^* M$ of $\mathbb{T}_2$ in $\mathcal{S}[\mathbb{T}_1]$

Generalized points of $[\mathbb{T}_2]$ at stage $[\mathbb{T}_1]$

Point of $[\mathbb{T}_2]$ = model of $\mathbb{T}_2$

Metatheorem

- Let x_g be generic model of $\mathbb{T}_1$ in $\mathcal{S}[\mathbb{T}_1]$
- Construct $f(x_g)$, model of $\mathbb{T}_2$ in $\mathcal{S}[\mathbb{T}_1]$
- Get $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$

$\mathcal{S}[\mathbb{T}_2]$ classifies $\mathbb{T}_2$

General metatheorem – in topos theory
Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two propositional geometric theories
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Geometricity

Geometric constructions
- e.g. finite limits, arbitrary colimits, free algebras
  - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathcal{F}$ (finite powerset), List
  - get finitely bounded $\forall$ in formulae
- Convenient to treat these as part of geometric logic

are preserved by any f^*

NB not $\mathcal{P}$, function types

General notion (first attempt)

Construction is geometric if
- applicable in any (Grothendieck) topos
- preserved by inverse image functors f^* of geometric morphisms

Geometricity in pointwise definitions

Suppose $x : W \to [\mathbb{T}_1]$ a (generalized) point of $[\mathbb{T}_1]$

$x^*(x_g)$ model of $\mathbb{T}_1$ in $\mathcal{S}W$

f geometric $\Rightarrow$ x^* preserves f

$\therefore$ $f(x^*(x_g)) \cong x^*(f(x_g))$ models of $\mathbb{T}_2$ in $\mathcal{S}W$

corresponds to $f \circ x$

Deduce: $f \circ x$ always got by applying geometric definition of f

General metatheorem – in topos theory

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two propositional geometric theories

Consider a topos-valid result of the form –

> Let x be a model of $\mathbb{T}_1$.
> Then $f(x) \triangleq$ ....
> is a model of $\mathbb{T}_2$

definition must be geometric

This defines a locale map $f : [\mathbb{T}_1] \to [\mathbb{T}_2]$.
For every point $x : 1 \to [\mathbb{T}_1]$, $f \circ x$ is got by substituting x in def^n of $f(x)$.

Summary

- Geometric reasoning transports along any f^*

$\therefore$ applies to generalized points as well as ordinary (global) points

- There are enough generalized points to define maps pointwise
- The generic point is already enough

SLOGAN: Continuity is geometricity

BUNDLES

Map
$$\begin{tikzcd} Y \arrow[d, "p"'] \\ X \end{tikzcd}$$
viewed as –

X-indexed family of spaces $p^{-1}(\{x\})$

fibres

Generalized fibre = pullback along generalized point

$$\begin{tikzcd} x^*Y \arrow[r] \arrow[d, "x^*p"'] & Y \arrow[d, "p"] \\ W \arrow[r, "x"'] & X \end{tikzcd}$$

Point-free bundles: two fundamental results

Joyal · Tierney · Fourman · Scott

① In any topos $\mathcal{E}$:
internal frames are dual to certain "localic" geometric morphisms with codomain $\mathcal{E}$
internal locales $\simeq$ external bundles

② $\mathbb{T}$ a geometric theory internal in $\mathcal{E}$
$p : \mathcal{F} \to \mathcal{E}$ bundle corr. to $\Omega[\mathbb{T}]$ in $\mathcal{E}$
$x : \mathcal{E}' \to \mathcal{E}$ a point of $\mathcal{E}$

Then fibre of p over x corresponds to theory $x^*(\mathbb{T})$

$$\begin{tikzcd} p^*\mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, "p"] \\ \mathcal{E}' \arrow[r, "x"'] & \mathcal{E} \end{tikzcd}$$

$x^*(\mathbb{T})$ theory in $\mathcal{E}'$

$\mathbb{T}$ theory in $\mathcal{E}$

Relativization

General idea:

internal property of frame in $\mathcal{E}$

corresponds to

external property of bundle $\begin{array}{c}\mathcal{F}\\ \downarrow p\\ \mathcal{E}\end{array}$

If external property preserved by pullback, then it holds fibrewise.

e.g. fibrewise discrete = local homeomorphism

Theorem (Joyal-Tierney): Object of topos (sheaf)

$\approx$ Internally discrete locale (frame = powerobject)

$\approx$ Internal locale X s.t. $X \xrightarrow{!} 1$, $X \xrightarrow{\Delta} X \times X$ both open

$\approx$ bundle a local homeomorphism

Local homeomorphisms preserved by pullback

– "fibrewise discrete"

Bundle pullback along f corresponds to applying inverse image functor f^*.

Geometricity (more generally)

Construction on bundles is geometric if preserved by pullback "works fibrewise"

- For local homeomorphisms: agrees with geometricity of topos-valid constructions
- For general bundle construction: geometricity $\approx$ geometric construction on geometric theories

Powerlocales: localic hyperspaces

		points = some subspaces of X
upper	$P_U X$	compact, fitted
lower	$P_L X$	overt, weakly closed
Vietoris	VX	compact, overt, weakly semifitted

Topos-valid definitions not geometric

e.g. $\Omega P_U X = \mathrm{Fr}\langle \Omega X \mid$ finite meets, directed joins preserved$\rangle$

BUT can reduce to geometric constructions on geometric theories

$\therefore$ All geometric.

Bundle for internal P_u in $\mathcal{S}X_2$

$$\begin{tikzcd}
 & (P_u/X_1)(f^*p) \cong f^*(P_u/X_2)(p) \arrow[rrr, bend left] \arrow[dd] \arrow[ddr] & & & (P_u/X_2)(p) \arrow[ddl] \\
 & & & Y \arrow[dd, "p"] & \\
f^*Y \arrow[urrr] \arrow[dr, "f^*p"'] & & & & \\
 & X_1 \arrow[rr, "f"'] & & X_2 &
\end{tikzcd}$$

Powersets etc.

X discrete $\Rightarrow$ pts of $P_L X$ = subsets of X

$\therefore$ $\wp X$ exists as a space $P_L X$, not a set

non-discrete (Scott topology)

Similarly: X, Y discrete

$\Rightarrow Y^X$ a space (point-open topology)

Ω exists as space $ (Sierpinski)

topos-valid construction of $\Omega, \wp$, exponentiation not geometric

Compactness ≈ geometric property of bundles

X compact

non-geometric

$\Leftrightarrow \wp X$ has finite subcover property

$\Leftrightarrow P_U X$ has strongly least point

left adjoint to $!$ $\quad 1 \underset{!}{\overset{\perp}{\rightleftarrows}} P_U X \quad \perp \dashv !$

geometric!

Similarly:

X overt $\Leftrightarrow P_L X$ has strongly greatest point

$1 \underset{!}{\overset{\top}{\rightleftarrows}} P_L X \quad ! \dashv \top$

Fibrewise construction of bundle p over $[\mathbb{T}_1]$

> Let x be a model of $\mathbb{T}_1$.
> Then fibre $p^{-1}(\{x\}) \triangleq [\mathbb{T}_2(x)]$
> where $\mathbb{T}_2(x) \triangleq$ . . . .
> is a geometric theory

definition must be geometric

defines bundle $p: [\mathbb{T}_2] \longrightarrow [\mathbb{T}_1]$

(corresponds to $\mathbb{T}_2$ (generic model))

models of $\mathbb{T}_2$ = pairs (x, y) x a model of $\mathbb{T}_1$, y a model of $\mathbb{T}_2(x)$

Geometricity $\Rightarrow$ works fibrewise

Example: Affine scheme   A commutative ring

Geometric theory of prime coideals of A:

Sorts: none declared   A can be characterized geometrically

Predicate $C \subseteq A$

Axioms $C(0) \to \bot$ $\top \to C(1)$

$\forall a,b{:}A.\ (C(a+b) \to C(a) \vee C(b))$

$\forall a,b{:}A.\ (C(ab) \leftrightarrow C(a) \wedge C(b))$

Defines space - Zariski spectrum Spec(A)

For each prime coideal C: define local ring $A[C^{-1}]$ universally invert elements of C - geometric

Resulting local homeomorphism makes Spec(A) locally ringed space - affine scheme of A

Topos approach to quantum theory

Nijmegen version - Heunen - Landsman - Spitters

A C*-algebra

$\mathcal{C}(A)$ = poset of commutative subalgebras

Spectral bundle

Base space = $\mathrm{Idl}(\mathcal{C}(A))$ ideal completion

Fibre over C = Gelfand spectrum of C

commutative subalgebra ⇒ principal ideal

Uses Banaschewski · Mulvey topos-valid Gelfand duality

Not quite geometric! eg. completeness of C.

Fauser · Vickers: work in progress - make it geometric

Geometric reasoning: summary

- Generally working with spaces   cf. - e.g. - Taylor's ASD
- Geometric constructions include finite limits, coproducts, powerlocales
- Some spaces are discrete (sets)
- Some constructions preserve discreteness   finite limits, coproducts ✓
- Some lose it - eg. P_U, P_L
- Gain others - e.g. coequalizers, free algebras
- Can describe spaces in terms of sets using (set-presented) geometric theories

Predicative?

Bundle theorem ⇒

Topos internal spaces in SX $\simeq$ Slice of Loc Loc/X

? Use structure on Loc (& its slices) to justify geometricity principles

? Replicate structure on cat. FS of formal spaces.

e.g. - have powerlocales

- Works on slices? Preserved by pullback?

Bigger questions

?? How far can this be extended to generalized spaces (Grothendieck toposes)
= spaces of models of predicate geometric theories?
cf. "Cosheaves & connectedness in formal topology" Vickers

?? Working in pure geometric reasoning without function types?

cf. "An induction principle for consequence in arithmetic universes" Maietti Vickers

References ... (to my own papers)

"The double powerlocale & exponentiation" – detailed account of geometricity
"Locales & toposes as spaces" + connections with usual description of toposes
"Compactness in locales & formal topology"
"Some constructive roads to Tychonoff" – powerlocales in formal topology
"Topical categories of domains"
"Localic completion of generalized metric spaces" (I, II)
"The connected Vietoris powerlocale"
"A localic theory of lower & upper integrals" – case studies in domain theory & analysis