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PREFRAME PRESENTATIONS PRESENT

Peter Johnstone,
Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge,
16 Mill Lane, Cambridge, CB2 1SB.

Steven Vickers,
Department of Computing, Imperial College,
180 Queen's Gate, London, SW7 2BZ.
email: sjv@doc.ic.ac.uk

Abstract

Preframes (directed complete posets with finite meets that distribute over the directed joins) are the algebras for an infinitary essentially algebraic theory, and can be presented by generators and relations. This result is combined with a general argument concerning categories of commutative monoids to give a very short proof of the localic Tychonoff Theorem.

It is also shown how frames can be presented as preframes, a result analogous to Johnstone's construction of frames from sites, and an application is given.

1. Introduction

It is well known (Joyal and Tierney [84]) that a coproduct of frames is a tensor product in a rather conventional sense, for it is universal for functions with a certain bilinear property:

> Let A and B be frames, let C be a sup-lattice, and let $\phi: A\times B \to C$ preserve all joins in each argument. Then there is a unique sup-lattice homomorphism $\phi': A\otimes B \to C$ such that $\phi'(a\otimes b) = \phi(a, b)$ for all $a \in A$, $b \in B$.

(A *sup-lattice* is a complete join semilattice, and a function between sup-lattices is a homomorphism if it preserves all joins.)

To put it another way, suppose for sup-lattices A and B the tensor product A⊗B is presented as

$$\begin{aligned}\text{SupLat}\langle\, & a\otimes b\ (a \in A,\ b \in B) \mid \\ & \textstyle\bigvee S \otimes b = \bigvee\{a\otimes b: a \in S\} \qquad (S \subseteq A) \\ & a \otimes \textstyle\bigvee T = \bigvee\{a\otimes b: b \in T\} \qquad (T \subseteq B) \quad \rangle\end{aligned}$$

If A and B are actually frames, then this tensor product serves as a frame coproduct. (This is directly analogous to the case of commutative rings, where coproducts are constructed as tensor products of Abelian groups.)

The uniqueness part of this universal property is obvious. Every element of A⊗B is a join of elements a⊗b, so a sup-lattice homomorphism is determined by its values on these elements.

There is another representation of elements of A⊗B. Let us write a⅋b (a "par" b) for the element a⊗**true** ∨ **true**⊗b. It is easily verified that

- ⅋ preserves finite meets and non-empty joins (in particular, directed joins) in each argument.
- ⅋ preserves all joins jointly in its two arguments.
- a⊗b = a⅋**false** ∧ **false**⅋b

Now a *finite* join of elements a⊗b is a finite join of finite meets of elements a⅋b, and hence, by finite distributivity, a finite meet of finite joins of elements a⅋b, hence a finite meet of elements a⅋b. Hence every element of A⊗B is a directed join of finite meets of elements a⅋b. We are going to investigate the possibility of defining functions from A⊗B that preserve finite meets and directed joins, by defining the values at elements a⅋b. A major purpose of this paper is to present the corresponding existence result, Theorem 4.3, and to use it to give what is probably the simplest proof yet discovered of the Localic Tychonoff Theorem.

What is happening here is that instead of the sup-lattice structure (joins) of frames, we are considering the *preframe* structure: following Banaschewski [88], we call a poset A a *preframe* if it has all finite meets and directed joins, with binary meets distributing over directed joins; and we call a function between preframes a *preframe homomorphism* if it preserves finite meets and directed joins. (Note – Gierz et al. [80] call preframes "meet continuous semilattices".)

The original result for the sup-lattice structure is quite cheap. The sup-lattice tensor product A⊗B can be presented as above, and it is then easy to prove that it is in fact a frame, the frame coproduct of A and B. The essential fact, that the presentation does indeed present a sup-lattice, comes from standard universal algebra, with only minor hitches from the infinitary joins in the theory of sup-lattices.

Our corresponding result for preframes follows a somewhat parallel route (put in a quite general setting in Section 4), but it is much harder to show that the same method of presentations works. The mathematical core of this paper lies in our result that for preframes also, a presentation by generators and relations does indeed present a preframe (Section 3). The underlying idea is that the theory of preframes is essentially algebraic. For *small* essentially algebraic theories, presentations by generators and relations do present, so for large theories (such as preframes) one can expect the same, provided that cardinality problems can be circumvented. For convenience, we summarize results about essentially algebraic theories in Section 2.

Finally, we prove a preframe analogue of the coverage theorem for frames (Johnstone [82]), which is best seen in a sup-lattice context, and use it in some applications.

2. Essentially algebraic theories

The infinitary analogue of finitary algebraic (or equational) theories has been fairly extensively studied by several people from Słominski [59] and Linton [66] onward; but much less attention has been paid to the infinitary analogue of what are variously called essentially algebraic theories (Freyd [72]), lim-theories (Coste [79]) or left exact theories (McLarty [86]): essentially the only works in this area are the paper of Isbell [72] and the monograph of Gabriel and Ulmer [71], neither of which takes an explicitly syntactic point of view (and the work of Gabriel and Ulmer applies only to small theories, which are insufficient for our purposes). However, in order to understand the theory of preframes, which is our main concern in this paper, it is necessary to regard it in this general context; so we devote this section to developing as much of the general theory as we shall need.

We shall follow the approach of Freyd [72] in presenting the syntax via partial operations whose domains are specified by equations, rather than that of Coste using primitive predicates and a "provably unique" existential quantifier. It is not hard to show that the expressive powers of the two approaches are identical, provided one allows many-sorted theories (partially ordered sets are a single-sorted theory in Coste's sense, but in our approach we must take two primitive sorts, one for the underlying set and one for the set of instances of the order relation).

An *essentially algebraic theory,* then, is specified by the following data:

(i) A set S of *sorts*.

(ii) A class O of *operation-symbols* ω, each of which is equipped with an *arity* f_ω which is a function from some index set I to S, and a *type* t_ω which is a single sort. (The intended interpretation is that, if the sorts have been interpreted by sets A_s, $s \in S$, then $\omega \in O$ will be interpreted as a partial function ω_A from $\prod_{i \in I} A_{f_\omega(i)}$ to A_{t_ω}.)

(iii) A well-founded partial ordering $<$ on O, with the property that the $<$-predecessors of any given operation-symbol form a set.

(iv) For each $\omega \in O$, a *domain declaration* which is a (possibly infinite) conjunction of equations ($t = t'$), where t and t' are terms of the same type, constructed in the usual way from variables and operation-symbols that preced ω in the ordering $<$. (The intended interpretation is that ω_A is defined exactly at those I-tuples where, for each equation ($t = t'$), the terms t_A and t'_A are defined and equal; if the domain declaration of ω is the empty conjunction, then the domain of ω_A is to be the whole of $\prod_{i\in I} A_{f_\omega(i)}$.)

(v) A class *A* of *axioms,* which are equations (t = t') between pairs of terms of the same type. (The intended interpretation is that a structure A for the language, as so far defined, will be a *model* for the theory provided, for each axiom (t = t'), the interpretations t_A and t'_A are equal at all tuples of elements of A where they are both defined.)

Of course, we say the theory is *single-sorted* if the set *S* is a singleton; in what follows we shall generally deal with the single-sorted theories in order to simplify notation. We say that the theory is *small* if the class *O* is a set (in which case *A* will also be a set, since we have only a set of possible terms).

The reader may have been surprised to see that our axioms all have the form of equations rather than Horn sequents; however, the presence of domain declarations allows us to achieve the effect of Horn sequents, by introducing extra operations that are restrictions of projections. For example, if we wish to assert that an equation (t = t') holds conditionally upon an equation (s = s'), we introduce a new operation α (whose arity includes the sorts of all the variables appearing in s or s', and whose type is that of one of the variables appearing in t or t'), with domain declaration (s = s'); we then add a new axiom $(\alpha(x_i)_{i\in I} = x_{i_0})$ and substitute $\alpha(x_i)_{i\in I}$ for x_{i_0} somewhere in t or t'.

The same device may also be used to turn all the axioms into directed equalities (t $\succ$ t') in the sense of Freyd and Scedrov [90] (intended interpretation: "if t is defined, then t' is also defined and equal to it") – though we cannot, in general, reduce them to assertions of the form "one side is defined iff the other is, and then they are equal".

Example 2.1 As previously mentioned, our primary interest in this paper is in the notion of *preframe* introduced by Banaschewski [88]: a preframe is a partially ordered set having finite meets (including a top element 1) and directed joins, such that binary meets distribute over directed joins. We give here a presentation of the theory of preframes as a (single-sorted) essentially algebraic theory. At the lowest level of the ordering on operations, we have a constant 1 and a binary operation $\wedge$, both with empty domain declarations (and we shall have axioms that say that ($\wedge$, 1) defines a semilattice structure on our underlying set A). To handle directed joins, we next introduce an operation $\bigvee^{\uparrow}_P$ of arity |P| (the underlying set of P) for each directed poset P; the domain declaration for $\bigvee^{\uparrow}_P$ will be the conjunction, over all pairs (p, q) in P with $p \leq q$, of the equations $(x_p \wedge x_q = x_p)$, so that $\bigvee^{\uparrow}_P (f)$ is defined, for a function f: P $\rightarrow$ A, iff f is order preserving. To ensure that $\bigvee^{\uparrow}_P (f)$ is the least upper bound of the image of f, it suffices to write down the axioms

$$\bigvee\nolimits^{\uparrow}_{\{p\}} (x_p) = x_p$$

for a singleton poset {p}, and

$$\bigvee\nolimits^{\uparrow}_P (x_p)_{p\in P} \wedge \bigvee\nolimits^{\uparrow}_Q (x_{h(q)})_{q\in Q} = \bigvee\nolimits^{\uparrow}_Q (x_{h(q)})_{q\in Q}$$

whenever $h: Q \to P$ is an order-preserving map between directed posets. Finally, we write down the distributive law as a scheme of axioms, one for each directed poset P:

$$x \wedge \bigvee\uparrow_P (y_p)_{p\in P} = \bigvee\uparrow_P (x \wedge y_p)_{p\in P}$$

For a small essentially algebraic theory $\mathbb{T}$, the forgetful functor from $\mathbb{T}$-models to **Set** (or to **Set**^{n} if $\mathbb{T}$ is many-sorted) has a left adjoint, just as in the algebraic case: the free $\mathbb{T}$-model on a set X is constructed in the usual way as the set of words (i.e. terms) in the elements of X, modulo $\mathbb{T}$-provable equality. The adjunction will not be monadic unless $\mathbb{T}$ is algebraic (i.e. has a presentation on which all the domain declarations are empty), but it will be possible to factor it as a tower of monadic adjunctions in the style of MacDonald and Stone [82], by expressing $\mathbb{T}$ as a union (indexed by some ordinal) of subtheories $\mathbb{T}_\alpha$, corresponding to the levels in the ordering on primitive operations, and successively constructing free functors from $\mathbb{T}_\alpha$-**Mod** to $\mathbb{T}_{\alpha+1}$-**Mod**. The tower may be of arbitrary height, as is shown by the following example:

Example 2.2 Let α be an ordinal, and let $\mathbb{T}_\alpha$ denote the theory having one unary operation ω_β for each $\beta < \alpha$ (ordered in the obvious way), the domain declaration of ω_β being the conjunction over all $\gamma < \beta$ of $(\omega_\gamma(x) = x)$, and no axioms. It is clear that, after β ordinal steps starting from **Set**, we cannot get any further than $\mathbb{T}_\beta$-**Mod**; in the free $\mathbb{T}_\alpha$-model on a $\mathbb{T}_\beta$-model, the operation $\omega_{\beta+1}$ has no fixed points, and so the later operations are never defined.

For large theories, even when the forgetful functor to **Set** has a left adjoint (and even when the arities of the generating operations, and the lengths of chains in the ordering on these operations, are bounded), it may not be possible to decompose the adjunction as a tower of monadic ones, as is shown by the following simple modification of the previous example: take a single unary operation ω_0 with empty domain declaration, and then a proper class of unary operations ω_α, all with domain declaration $(\omega_0(x) = x)$. The forgetful functor from $\mathbb{T}$-**Mod** to **Set** has a left adjoint, which sends a set X to the free $\mathbb{T}_1$-model on X; but the comparison functor from $\mathbb{T}$-**Mod** to $\mathbb{T}_1$-**Mod** has no left adjoint. Note, in particular, that the existence of free $\mathbb{T}$-models on sets is no guarantee of the existence of colimits (in particular, coequalizers) in $\mathbb{T}$-**Mod**.

Since the theory in which we are interested, that of preframes, is a large one, we shall have to investigate whether its category of models possesses this sort of structure. As far as colimits are concerned, Banaschewski [88] gave an explicit construction of coproducts; and we shall be able to infer the existence of coequalizers (though not to describe them all that explicitly) from our presentation theorem in the next section. For monadicity, the questions are more easily answered.

Lemma 2.3 The forgetful functor from **PreFrm** to **Set** has a left adjoint.

Moreover, the monadic length of the adjunction is 2.

Proof

The forgetful functor may be factored as

PreFrm → **SLat** → **Set**

where **SLat** is the category of meet-semilattices. Now **SLat** → **Set** has a left adjoint that sends a set X to the set $\wp_{\text{fin}}(X)$ of finite subsets of X, ordered by reverse inclusion, and **PreFrm** → **SLat** has a left adjoint sending a semilattice P to the set Idl(P) of ideals of P, ordered by inclusion (see Vickers [89], Theorem 9.1.5). Now **SLat** → **Set** is monadic, since semilattices are an algebraic theory; but in a free semilattice $\wp_{\text{fin}}(X)$ the upward closure of each element is finite, from which it follows easily that every ideal is principal, and so the monad on **Set** induced by the composite adjunction is (isomorphic to) the free semilattice monad. In other words, **SLat** is exactly the algebraic part of the essentially algebraic theory of preframes.

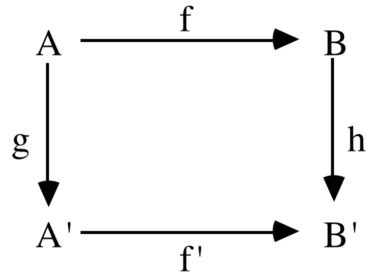
To complete the proof, we must show that the forgetful functor **PreFrm** → **SLat** is monadic; but we may do this directly, as follows. Let P be a semilattice, and suppose it has an algebra structure α: Idl(P) → P for the monad on **SLat** induced by the adjunction. Since α sends principal ideals to their generators and is order-preserving, we see that it must send each ideal of P to its join in P; so P has directed joins. Moreover, since α preserves binary meets, we see that binary meets distribute over directed joins in P, so it is a preframe; and its Idl-algebra structure is uniquely determined by its preframe structure.]

We note in passing that, once we have established the existence of coequalizers in **PreFrm**, Lemma 2.3 combined with Linton's theorem (Linton [69]) will enable us to "lift" more general colimits from **SLat** to **PreFrm**, without making use of Banaschewski's construction of coproducts.

In the case of a small theory $\mathbb{T}$, the decomposition of $\mathbb{T}$-**Mod** → **Set** into a tower of monadic functors, plus the fact that these functors have rank (i.e preserve α-filtered colimits for some cardinal α – just take a regular cardinal greater than the arities of all the generating operations of $\mathbb{T}$), ensure that the category $\mathbb{T}$-**Mod** is locally presentable (cf. Gabriel-Ulmer [71], Satz 10.3). Indeed, there is a converse: any locally α-presentable category is equivalent to the category of α-continuous set-valued functors on a small α-complete category C, and the theory of such functors may readily be presented as a (many-sorted) essentially algebraic theory (cf. Coste [79], Theorem 2.3.2, for the case $\alpha = \aleph_0$). Thus the categories of models of small essentially algebraic theories are, up to equivalence, exactly the locally presentable categories. In particular, all such categories are cocomplete.

Given a morphism f: A → B of models of an essentially algebraic theory $\mathbb{T}$, the set-theoretic image I of f is not in general a $\mathbb{T}$-model: the identifications made in passing from A to I may create "new" tuples of elements satisfying the domain declaration of some operation of $\mathbb{T}$, none of whose pre-images in A do so. However, $\mathbb{T}$-**Mod** does have image factorizations (even if $\mathbb{T}$ is large): that is, every morphism factors as a strong epimorphism followed by a monomorphism. To obtain the image of f: A → B in $\mathbb{T}$-**Mod**, we simply take the sub-$\mathbb{T}$-model I of B generated by the set-theoretic image I, i.e.

the intersection of all submodels that contain I. This factorization is not, in general, stable under pullback (and so the category $\mathbb{T}$-**Mod** is not in general regular), but it is at least functorial: that is, a commutative square



gives rise to a (unique) morphism from the $\mathbb{T}$-model image of f to that of f', since the inverse image of the latter under h is a sub-$\mathbb{T}$-model of B containing the set-theoretic image of f. (We shall need this observation, in the case of preframes, in our proof of the presentation theorem.)

Another point that should be noted is that strong epimorphisms in $\mathbb{T}$-**Mod** (that is, morphisms f: A → B that do not factor through any proper sub-$\mathbb{T}$-model of their codomain) need not be regular epimorphisms (that is, coequalizers): the coequalizer of the kernel-pair of f, if it exists, will be the $\mathbb{T}$-model freely generated by the set-theoretic image I, modulo the preservation of as much of the $\mathbb{T}$-model structure as exists in I, and this may not map injectively to B. This is another indication that, for large theories $\mathbb{T}$, the construction of coequalizers in $\mathbb{T}$-**Mod** can be expected to be a delicate matter.

Leaving this question aside for the moment, we conclude this section by briefly considering the notion of commutativity for essentially algebraic theories. For algebraic theories, the notion is well understood, but it is perhaps less widely appreciated that it makes perfectly good sense in our more general context. We say that a single-sorted essentially algebraic theory $\mathbb{T}$ is *commutative* if any two of its operations commute with each other, or, equivalently, if each operation is a homomorphism of $\mathbb{T}$-models. (To make sense of this second formulation, note that the commutativity of the subtheory $\mathbb{T}_1$ generated by the operations that precede a given operation ω ensures that the domain of ω_A is a sub-$\mathbb{T}_1$-model of the power of A corresponding to its arity.)

Proposition 2.4 Let $\mathbb{T}$ be a commutative essentially algebraic theory.

(i) The category $\mathbb{T}$-**Mod** has a symmetric closed structure, in which the internal hom [A, B] is the set of $\mathbb{T}$-model homomorphisms from A to B with operations defined pointwise.

(ii) If $\mathbb{T}$-**Mod** has a free functor and coequalizers (for instance, if $\mathbb{T}$ is small), then it also has a symmetric monoidal structure (⊗, I), where I is the free $\mathbb{T}$-model on one generator and (-)⊗A is left adjoint to [A, -].

Proof

(i) Commutativity of $\mathbb{T}$ (plus an induction over the ordering on operations) implies that if ω is an operation of $\mathbb{T}$ (of arity J, say) and $(f_j)_{j\in J}$ is a family of homomorphisms from A to B that (pointwise) satisfy the domain declaration of ω, then the function

$$a \mapsto \omega_B(f_j(a))_{j\in J}$$

is again a homomorphism from A to B. So [A, B] has the structure required for a $\mathbb{T}$-model, and it satisfies the axioms since B does. Moreover, the assignment $(A, B) \mapsto [A, B]$ is easily seen to be a bifunctor, contravariant in the first argument and covariant in the second. For the symmetry, we observe that homomorphisms from A to [B, C] correspond to functions from A×B to C that are *bihomomorphisms,* i.e. are homomorphic in each variable provided the other is held constant, and these in turn correspond to homomorphisms from B to [A, C], yielding a natural isomorphism

$$[A, [B, C]] \cong [B, [A, C]]$$

(ii) To obtain the monoidal structure, we need to construct a universal bihomomorphism from A×B to A⊗B, i.e. one through which every bihomomorphism from A×B to C factors by a unique homomorphism from A⊗B to C. Under the extra hypotheses on $\mathbb{T}$, we may do this by first forming the free $\mathbb{T}$-model F on A×B, and then forming the coequalizer $F \twoheadrightarrow A\otimes B$ of $R \rightrightarrows F$, where R is the smallest congruence on F such that the composite $A\times B \rightarrowtail F \to A\otimes B$ is bihomomorphic. The remaining details are straightforward.]

Example 2.5 The theory of preframes, as defined earlier in this section, is commutative. It is well known that the theory of meet-semilattices is commutative, and the directed join operations commute with each other (and with themselves); so we need only verify that directed joins commute with finite meets. Now the distributive law tells us that, if $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are monotone nets indexed by the same directed set I, we have

$$\bigvee\nolimits^{\uparrow}_{I} (x_i)_{i\in I} \wedge \bigvee\nolimits^{\uparrow}_{I} (y_i)_{i\in I} = \bigvee\nolimits^{\uparrow}_{I\times I} (x_i \wedge y_j)_{(i,j)\in I\times I}$$

Then directedness of I tells us that the $(x_i \wedge y_i)$, $i \in I$, are cofinal among the $(x_i \wedge y_j)$ and so have the same join.

Although the commutativity of the theory of preframes does not seem to have been explicitly observed before, it is not exactly a new idea; in particular, it underlies the "Lawson duality" of continuous semilattices (see Johnstone [82], VII 2.11). A continuous semilattice is just a preframe which satisfies the additional requirement of continuity (expressed in the usual way in terms of the way-below relation); the *dual* of a continuous semilattice A is the poset $\hat{A}$ of Scott open filters of A (i.e. subsets that are upper closed, closed under finite meets and inaccessible by directed joins), ordered by inclusion. But a subset of A is a Scott open filter iff its characteristic function is a preframe homomorphism from A to $\mathbb{2} = \{0, 1\}$, so we may identify $\hat{A}$ with $[A, \mathbb{2}]$ as defined in

Proposition 2.4. Lawson duality is then the assertion that, for a continuous semilattice A, [A, $\mathbb{2}$] is also continuous, and the canonical mapping from A to [[A,□$\mathbb{2}$],□$\mathbb{2}$] is an isomorphism. (We note in passing that this duality cannot be extended to any larger class of preframes, since a preframe is continuous iff its Scott-open sets separate its points.)

The theory of frames, unlike that of preframes, is not commutative: although the distributive law implies that finite meets commute with directed joins, finite meets and finite joins can never commute in a nontrivial lattice. (If $a = d \leq b = c$, then $(a \wedge b) \vee (c \wedge d) = a$, but $(a \vee c) \wedge (b \vee d) = b$. Also, the presence of two distinct constants 0 and 1 is incompatible with commutativity.) In fact, the theory of preframes is a maximal commutative subtheory of frames. It is not unique with this property. Another such is the theory of complete join-semilattices (*sup-lattices*), obtained by retaining all the join operations (including 0) and discarding the finite meets. We shall see that in many ways the relationship between frames and preframes is similar to that between frames and sup-lattices. The latter was extensively investigated by Joyal and Tierney [84], and their work has served as a model for a large part of ours.

3. Preframe presentations present

First, recall Banaschewski's [88] theory of *prenuclei* on a frame: if A is a frame and ν_0 is a function from A to itself that is monotone and inflationary ($\nu_0(a) \geq a$), then ν_0 is a *prenucleus* if

$$\forall a, b \in A.\ \nu_0(a) \wedge b \leq \nu_0(a \wedge b)$$

If, in addition, ν_0 is idempotent, then it is a nucleus in the usual sense (see Johnstone [82]).

Banaschewski proves that for each prenucleus ν_0, there is a nucleus ν characterized by the property that each $\nu(a)$ is the least fixpoint of ν_0 greater than a. ν_0 and ν have the same fixpoints, which – by the standard theory for the nucleus ν – form a frame; it has the universal property of

$$\mathrm{Frm}\ \langle\ A\ (\text{qua Frm})\ |\ \nu_0(a) \leq a\ (a \in A)\ \rangle$$

Next, let us summarize some free constructions. Note for all of them that the concrete constructions show that the injections of generators are all 1-1.

Proposition 3.1

(i) The free preframe over a meet semilattice S is the ideal completion Idl(S).

(ii) The free frame over a meet semilattice S is the set Alex(S^{op}) of lower closed sets of S.

(iii) The free frame over a preframe A is the set of Scott closed subsets of A.

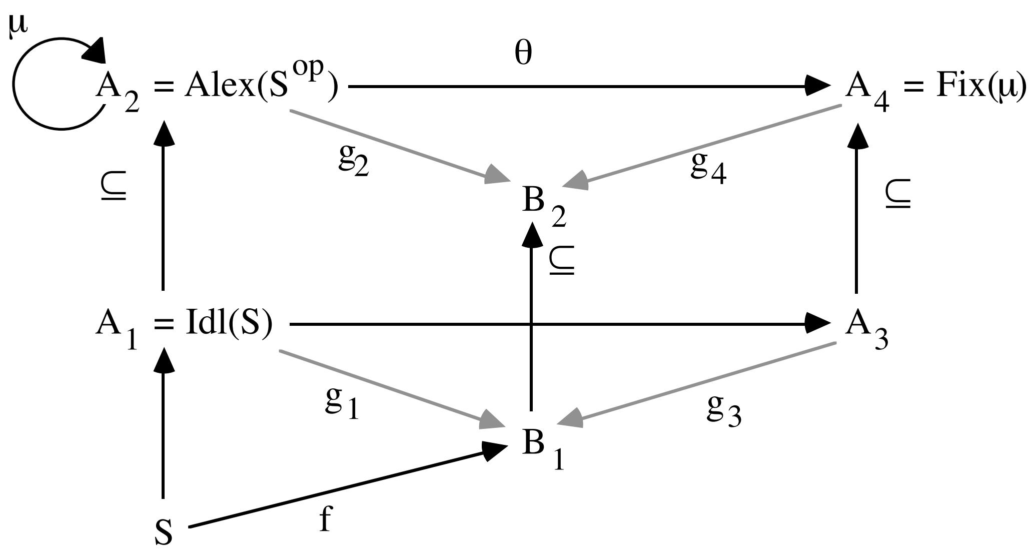
Proof (i) This has already been mentioned; it is Theorem 9.1.5 in Vickers [89].

(ii) This comes immediately from the coverage theorem (Johnstone [82] Proposition II.2.11).
(iii) Banaschewski [88], Proposition 1. ⟧

Proposition 3.2 Let S be a meet semilattice, and let R be a set each of whose elements has the form (X, a) where $X = (x_i)_{i \in P}$ is a monotone net in S and a is an upper bound in S for $\{x_i: i \in P\}$.

Then PreFrm $\langle$ S (qua meet semilattice) $\mid \bigvee^{\uparrow} X = a$ ((X, a) $\in$ R) $\rangle$ exists.

Proof



Let A_1 and A_2 be the free preframe and frame over the meet semilattice S. We define $\mu:\square A_2 \to A_2$ by

$$\mu(U) = U \cup \{a \wedge b: b \in S, \exists X.\ (X, a) \in R, \forall i.\ x_i \wedge b \in U\}$$

Then μ is monotone and inflationary. It's also a prenucleus, for suppose $a \wedge b \in \mu(U) \cap V$ with (X,□a) in R and all $x_i \wedge b$ in U. Then each $x_i \wedge b \leq a \wedge b \in V$, so $a \wedge b \in \mu(U \cap V)$.

Let A_4 be the set of fixpoints of μ, the corresponding sublocale, let $\theta: A_2 \to A_4$ be the natural frame homomorphism, and let A_3 be the subpreframe of A_4 generated by the image of A_1 under θ (the preframe image of A_1 under θ, in the sense of Section 2). A_3 is the preframe we are trying to present. $A_1 \to A_3$ is a preframe epimorphism, and this will prove uniqueness in the universal property.

For existence, suppose B_1 is a preframe, and $f: S \to B_1$ is a meet semilattice homomorphism such that if (X, a) $\in$ R, then $f(a) \leq \bigvee^{\uparrow} (f \circ X)$. f factors via a preframe homomorphism $g_1: A_1 \to B_1$. Let B_2 be the free frame over the preframe B_1; note that the concrete construction of Proposition 3.1 tells us that the injection $B_1 \to B_2$ is 1-1. g_1 lifts to a frame homomorphism $g_2: A_2 \to B_2$, and $g_2 \circ \mu = g_2$, so that g_2 factors as $g_2 = g_4 \circ \theta$, where g_4 is a frame homomorphism. The inverse image of B_1 under g_4 is a subpreframe of B_4 containing the image of A_1, and hence containing A_3; so g_4 restricts to a preframe homomorphism $g_3: A_3 \to B_1$. ⟧

Theorem 3.3 PreFrm has coequalizers.

Proof Let f, g: A → B be two preframe homomorphisms. Let h: B → S be the meet semilattice coequalizer, and let R be generated by the set $\{(h(X), h(a)) \colon X \subseteq B$, X directed, $\bigvee^{\uparrow} X = a\}$. Then apply the previous Proposition.]]

It follows that one can present preframes by generators and relations.

4. Frames as monoids in **PreFrm**

We have seen that the theory of preframes is commutative; so, thanks to the presentation theorem of the last section, we can now assert that the category **PreFrm** has a symmetric monoidal structure, left adjoint to its closed structure.

To construct the preframe tensor product A⊗B of A and B, we first construct their meet-semilattice tensor product (P, say), and then equip it with the coverage R generated by all pairs (X⊗b, a⊗b) and (a⊗Y, a⊗b) where X and Y are monotone nets in A and B, with joins a and b, X⊗b denotes the monotone net (x⊗b | x ∈ X) and a⊗b denotes (for the moment) the image of (a, b) under the universal bihomomorphism from A×B to P. (Actually, as in the proof that the theory of preframes is commutative, we could simplify the presentation by considering only covers of the form $((x_i \otimes y_i \mid i \in I), a \otimes b)$ where $(x_i \mid i \in I)$ and $(y_i \mid i \in I)$ are monotone nets in A and B with the same index s I and with joins a and b. But we shall not have any use for this simplification.) Then the preframe A⊗B is that presented as ⟨P | R⟩. For the rest of this section, a⊗b will denote the image of (a, b) under the universal preframe bihomomorphism from A×B to A⊗B.

The unit of the monoidal structure is the free preframe on one generator; this is simply the two-element frame $\mathbb{2}$ = {0, 1}, the generator being the bottom element 0. The fact that it is a frame and not just a preframe is no accident, as we shall see in a moment.

First, we need a general result on symmetric monoidal categories. This is surely well-known, although we have not been able to find an explicit reference to it.

Lemma 4.1 Let *C* be a symmetric monoidal category. Then the category **CMon**(*C*) of commutative monoids (with respect to the tensor product ⊗) in *C* has finite coproducts, which are given by ⊗ and the unit I.

Proof

First, we observe that **CMon**(*C*) is closed under tensor products, in an obvious sense: if M and N are commutative monoids in *C* (with multiplications * and units e), then M⊗N has multiplication and unit

$$(M\otimes N)\otimes(M\otimes N) \xrightarrow{\cong} (M\otimes M)\otimes(M\otimes M) \xrightarrow{*\otimes *} M\otimes N$$
$$I \xrightarrow{\cong} I\otimes I \xrightarrow{e\otimes e} M\otimes N$$

Similarly, I carries a unique commutative monoid structure.

Moreover, there are canonical maps

$$M \xrightarrow{\cong} M\otimes I \xrightarrow{Id\otimes e} M\otimes N$$
$$N \xrightarrow{\cong} I\otimes N \xrightarrow{e\otimes Id} M\otimes N$$

and a codiagonal map from M⊗M to M given by * itself; it is easy to verify that these are monoid homomorphisms, and that they define the unit and counit of an adjuction between ⊗ (as a bifunctor from **CMon**(*C*) × **CMon**(*C*) to **CMon**(*C*)) and the diagonal functor. Hence M⊗N is the coproduct of M and N in **CMon**(*C*); and a similar argument establishes that I is initial in this category. ⟧

The best known application of this Lemma occurs when *C* is the category of Abelian groups. It then becomes the assertion that the coproduct of two commutative rings is (founded on) the tensor product of their additive groups. Interpreted when *C* is the category of sets (with its cartesian monoidal structure), it becomes the assertion that finite products and coproducts coincide in the category of commutative monoids (or in any full subcategory thereof that is closed under products, such as abelian groups or semilattices).

Another application of Lemma 4.1 was exploited by Joyal and Tierney [84]. There they considered frames as commutative monoids in the monoidal category **SupLat** of complete join semilattices (their *suplattices*), the monoid structure being given by finite meets. Of course, the general commutative monoid is a commutative quantale, not necessarily a frame; but, as Joyal and Tierney observed, frames may be characterized as those commutative quantales for which the multiplication is idempotent and the unit is the top element. These conditions cannot be expressed by commutative diagrams in **SupLat** – since the monoidal structure is not Cartesian, there is no diagonal map from A to A⊗A, which would be needed to express the idea of idempotency. However, they do suffice to recover the result (already known, of course, before the work of Joyal and Tierney – cf. Wigner [79]) that the coproduct of two frames coincides with their tensor product in **SupLat**.

Exactly the same arguments apply when we consider frames in relation to preframes. In fact, we have –

Lemma 4.2

(i) Let A be a frame. Then the binary join map $\vee: A\times A \to A$ is a preframe bihomomorphism, and the induced map from A⊗A to A gives A the structure of a commutative monoid in **PreFrm**.

(ii) A commutative monoid (A, *, e) in **PreFrm** is (derived in this way from) a frame iff $e(0) \leq a$ and $*(a\otimes a) = a$ for all $a \in A$.

(iii) The category **Frm**, regarded via (i) as a full subcategory of **CMon**(**PreFrm**), is closed under finite coproducts.

Proof

(i) That ∨ is a bihomomorphism for finite meets is just the (finite) distributive law; that it is a bihomomorphism for directed joins is a consequence of the fact that joins are

idempotent and commute with other joins. The rest follows from the fact that join is associative and commutative, and has the bottom element of A as a unit.
(ii) This is easy, and is exactly like Proposition 1 on p. 21 of Joyal and Tierney [84].
(iii) Let A and B be frames. To verify that $A\otimes B$, equipped with its commutative monoid structure, is a frame, we note first that its unit is $0_A\otimes 0_B$, which is clearly its least element (let us write it as 0). Thus we have $u*u \geq u*0 = u$ for all $u \in A\otimes B$, and it suffices to verify that $u*u \leq u$. But the set

$$\{u \in A\otimes B \mid u*u \leq u\}$$

is a sub-preframe of $A\otimes B$ that contains all the generators $a\otimes b$. $\rrbracket$

Theorem 4.3 The (underlying preframe of the) coproduct of two frames is the tensor product of their underlying preframes. $\rrbracket$

We note in particular that if A and B are frames, then their tensor product in **PreFrm** is order isomorphic to their tensor product in **SupLat**, because both are isomorphic to the frame coproduct. However, the generators of the two tensor products are different. If we think of A and B as the open-set lattices of locales X and Y, and of a and b as corresponding to the open sublocales U and V, then the **SupLat** generator $a\otimes b$ is the "open rectangle" $U\times V \subseteq X\times Y$. However, the **PreFrm** generators correspond to the complements of closed rectangles: if we (temporarily) write $a \parr b$ for the **PreFrm** generator corresponding to (a,b), to distinguish it from the **SupLat** generator $a\otimes b$, then we have

$$a \parr b = (a\otimes 1_B) \vee (1_A\otimes b)$$

In other words, $a \parr b$ corresponds to the complement of the closed rectangle $(X\backslash U)\times(Y\backslash V)$. Such complements of closed rectangles have been seen before, notably in the description of the Vietoris locale (Johnstone [85]). (They also make a brief appearance in the construction of the weak product of locales in Johnstone and Sun [88], although this seems to be largely coincidental.)

A further useful result, again analogous to one observed by Joyal and Tierney, is

Theorem 4.4 The forgetful functor from **Frm** to **PreFrm** creates filtered colimits.

Proof

Consider a diagram of frames (A_i), indexed by some filtered category I, with colimit A in **PreFrm**. Since the monoidal structure on **PreFrm** is closed, $A\otimes(-)$ preserves colimits, whence we deduce that $A\otimes A$ is the $I\times I$-indexed colimit of the $A_i\otimes A_j$, and hence (by filteredness of I) that it is the I-indexed colimit of the $A_i\otimes A_i$. Thus the $\otimes$-monoid structures on the A_i induce a $\otimes$-monoid structure on A, and arguments similar to those of Lemma 4.2 (ii) ensure that it is a frame structure. The fact that A, with this structure, is the colimit of the A_i in **CMon**(**PreFrm**) and hence in **Frm** is easily verified. $\rrbracket$

As an application of the last two results, we give what is probably the simplest proof yet discovered of the Tychonoff theorem for locales (cf. Ehresmann [57], Papert [67], Dowker and Strauss [76], Johnstone [81], Kříž [85], Banaschewski [88], Vermeulen [90], Coquand [90]). We note, following Banaschewski, that compactness is easily defined for preframes: a preframe A is *compact* iff its top element 1 is inaccessible by directed joins, or, equivalently, iff {1} is a Scott open filter in A. But a subset U ⊆ A is a Scott open filter iff its characteristic function is a preframe homomorphism from A to $\mathbb{2}$, from which we deduce –

Lemma 4.5

(i) A tensor product of two compact preframes is compact.
(ii) A colimit of a diagram of compact preframes and injective preframe homomorphisms is compact.

Proof

(i) Let A and B be compact preframes. The characteristic functions of $\{1_A\}$ and $\{1_B\}$ induce a preframe homomorphism from A⊗B to $\mathbb{2}\otimes\mathbb{2}$; but $\mathbb{2}\otimes\mathbb{2} \cong \mathbb{2}$, the isomorphisms being induced by the binary join map from $\mathbb{2}\times\mathbb{2}$ to $\mathbb{2}$. Thus we have a homomorphism h from A⊗B to $\mathbb{2}$ such that h(a⊗b) = 1 iff either a = 1 or b = 1, i.e. iff a⊗b is the top element of A⊗B. The fact that, for an arbitrary u ∈ A⊗B, we have h(u) = 1 iff u = 1, now follows from the fact that any such u may be reached from the generators a⊗b by taking finite meets and directed joins, and h preserves directed joins.
(ii) Let the vertices of the diagram be $(A_i \mid i \in I)$. Since the transition maps $A_i \to A_j$ in the diagram are all injective, the characteristic functions h_i: $A_i \to \mathbb{2}$ of the top elements of the A_is form a cone under the diagram, and so induce a homomorphism from its colimit to $\mathbb{2}$. the fact that this homomorphism is the characteristic function of the top element is proved as in (i).]

For the particular case of coproducts, the result of Lemma 4.5 (ii) is in Banaschewski's paper [88], although his proof is different – it involves an explicit construction of coproducts in **PreFrm**.

Theorem 4.6 *(The Localic Tychonoff Theorem)*
A coproduct of compact frames is compact.
Proof
For finite coproducts, this is simply a special case of Lemma 4.5 (i), using Lemma 4.2□(iii). Just as in the case of rings (see, e.g., Bourbaki [70]), we may extend the result to infinite coproducts by regarding an infinite coproduct as a filtered colimit of finite coproducts (the transition maps being injective unless one of the factors in the product is degenerate – in which case the coproduct is degenerate, and so certainly compact), and using Theorem 4.4 and Lemma 4.5 (ii).]

Let us finish this section with an extension of Tychonoff that covers arbitrary Scott open filters of A and B, not just the case (compactness) when {1} is Scott open.

Proposition 4.7 (cf. Vickers [89], Lemma 6.4.3.)

Let A and B be frames. Then there is an order isomorphism between Scott open filters of A⊗B and subsets $U \subseteq A \times B$ satisfying –

- U is upper closed
- if $(a, b) \in U$ for all $a \in S$, where $S \subseteq_{fin} A$, then $(\bigwedge S, b) \in U$
- if $(\bigvee^{\uparrow} X, b) \in U$, where $X \subseteq A$ is directed, then $(a, b) \in U$ for some $a \in X$
- if $(a, b) \in U$ for all $b \in T$, where $T \subseteq_{fin} B$, then $(a, \bigwedge T) \in U$
- if $(a, \bigvee^{\uparrow} Y) \in U$, where $Y \subseteq B$ is directed, then $(a, b) \in U$ for some $b \in Y$

Proof Scott open filters of A⊗B are just preframe homomorphisms from A⊗B to the two-element frame $\mathbb{2}$. The subsets U described are just the preframe bihomomorphisms from A×B to $\mathbb{2}$.]]

Corollary 4.8 Let A and B be frames, and let F and G be Scott open filters in A and B. Then there is a Scott open filter H in A⊗B such that $a \invamp b \in H$ iff $a \in F$ or $b \in G$.
Proof Apply Proposition 4.7 to the set $U = \{(a, b): a \in F \text{ or } b \in G\}$.]]

This is actually another localic version of Tychonoff's Theorem, for the following reason. Let D and E be locales, and let F and G be Scott open filters in ΩD and ΩE. By the Hofmann-Mislove Theorem ([81] – the restriction to the spatial case is unnecessary; or see Vickers [89], Theorem 8.2.5), these correspond to compact saturated sets C_F and C_G of points of D and E, and H corresponds to a set C_H of points of D×E.

$(x, y) \in C_H \Leftrightarrow \forall u \in H.\ (x, y) \models u$
$\Leftrightarrow \forall a \in A, b \in B.\ (a \in F \text{ or } b \in G \Rightarrow (x, y) \models a \invamp b)$
$\Leftrightarrow \forall a \in A.\ (a \in F \Rightarrow (x, y) \models a \invamp \mathbf{false})$ and $\forall b \in B.\ (b \in G \Rightarrow (x, y) \models \mathbf{false} \invamp b)$
$\Leftrightarrow x \in C_F$ and $y \in C_G$

Hence $C_H = C_F \times C_G$. In other words, the product of compact saturated sets of points of D and E is still compact.

Although apparently more general than the finite case of Theorem 4.6, Corollary 4.8 could alternatively have been deduced from it using Lemma 3.4 of Johnstone [85] which enables one to reduce to compact sublocales of A and B.

5. The Preframe Version of the Coverage Theorem

In [82], Proposition II.2.11, Johnstone shows how to construct a frame as the set C-Idl(P) of "C-ideals" in a meet semilattice P, where C, a "coverage", is a set of relations of the form "X covers u" where $X \subseteq P$ and $u \in P$. C must also satisfy certain "meet stability" properties. It is also shown that C-Idl(P) has the universal properties of

$\mathbf{Frm}\langle$ P (qua meet semilattice) | $u \leq \bigvee X$ (whenever X covers u in C) $\rangle$

(“qua meet semilattice” means that all meet semilattice relations holding in P are to hold also in the frame being presented.)

Abramsky and Vickers [90] show how this construction has a specific technical meaning in the contex of sup-lattices. For regardless of whether P is a meet semilattice or C has the meet stability properties, the same definition of C-ideals leads to

$$\text{C-Idl}(P) \cong \text{SupLat}\,\langle\, P \text{ (qua poset)} \mid u \le \bigvee X \quad \text{(whenever } X \text{ covers } u \text{ in C)}\,\rangle$$

Hence the content of the result can be seen as being that provided P is a meet semilattice and C is meet stable, then

$$\text{Frm}\,\langle\, P \text{ (qua meet semilattice)} \mid u \le \bigvee X \quad \text{(whenever } X \text{ covers } u \text{ in C)}\,\rangle$$
$$\cong \text{SupLat}\,\langle\, P \text{ (qua poset)} \mid u \le \bigvee X \quad \text{(whenever } X \text{ covers } u \text{ in C)}\,\rangle$$

This facilitates the definition of *sup-lattice* homomorphisms out of frames, and is particularly useful in the work of Abramsky and Vickers, where functions are defined between frames and quantales. In practice, it is easy to work any frame presentation into the required form by putting in all finite meets of generators and any extra relations needed to give meet stability.

This understanding is related to a well-known result from the theory of rings. If R is a ring and I is a subgroup (of R as an additive group), then, *provided that I is an ideal,* we have

$$\text{Ring}\,\langle\, R \text{ (qua ring)} \mid r = 0 \text{ (whenever } r \in I)\,\rangle$$
$$\cong \text{Abelian Group}\,\langle\, R \text{ (qua Abelian Group)} \mid r = 0 \text{ (whenever } r \in I)\,\rangle$$

The purpose of this section is to give an analogous result enabling one to present frames as preframes, instead of sup-lattices.

Theorem 5.1 *The Preframe Version of the Coverage Theorem.*
Let P be a poset, and let C be a set of preframe relations of the form

$$\bigwedge S \le \bigvee^{\uparrow}_i \bigwedge S_i$$

where the sets S, S_i are all finite subsets of P. Let

$$A_1 = \text{PreFrm}\,\langle\, P \text{ (qua poset)} \mid C\,\rangle$$

(Every preframe presentation can be reduced to this form.)

Suppose in addition that

- P is a join semilattice,
- C is *join stable,* i.e. if $\bigwedge S \le \bigvee^{\uparrow}_i \bigwedge S_i$ is a relation in C, and $x \in P$, then the relation

  $$\bigwedge\{x \vee y: y \in S\} \le \bigvee^{\uparrow}_i \bigwedge\{x \vee y: y \in S_i\}$$

  is also in C.

Then A_1 is isomorphic to $A_2 = \mathrm{Frm}\langle P \text{ (qua } \vee\text{-semilattice)} \mid C \rangle$, the generators corresponding under the isomorphism in the obvious way.

Proof

First, we show that A_1 is a frame.

$0 \in P$ is bottom in A_1, for $\{a \in A_1: a \geq 0\}$ is a subpreframe containing the generators. (Hence there is a preframe homomorphism from A_1 to this set which, when composed with the inclusion, gives the identity map on A_1: so the inclusion is onto.)

Next, if $x \in P$, we can define a preframe endomorphism θ_x of A_1 by $y \mapsto x \vee y$. $x \mapsto \theta_x$ is monotone and respects the relations in C, and so defines a preframe homomorphism ϕ from A_1 to $[A_1, A_1]$. Writing $a \oplus b$ for $\phi(a)(b)$, $\oplus$ is a preframe bihomomorphism and $x \oplus y = (x \vee y)$ for $x, y \in P$. Now,

- $b \leq a \oplus b$, i.e. $\mathrm{Id} \leq \phi(a)$, i.e. $y \leq \phi(a)(y)$ for all $y \in P$: for the set of such a is a subpreframe containing the generators.
- $a \oplus b = b \oplus a$. For first, if $x \in P$ then $\{b: x \oplus b = b \oplus x\}$ is a subpreframe containing the generators; then, fixing b, $\{a: a \oplus b = b \oplus a\}$ is a subpreframe containing the generators.
- $a \oplus a = a$, $\{a: a \oplus a = a\}$ being a subpreframe containing the generators. For $1 \oplus 1 \geq 1$; if $a \oplus a = a$ and $b \oplus b = b$, then

  $$(a \wedge b) \oplus (a \wedge b) = a \oplus a \wedge b \oplus b \wedge a \oplus b \wedge b \oplus a = a \wedge b \wedge a \oplus b = a \wedge b$$

  and if S is a directed set of such elements, then

  $$\bigvee^{\uparrow} S \oplus \bigvee^{\uparrow} S = \bigvee^{\uparrow} \{a \oplus b: a, b \in S\} = \bigvee^{\uparrow} \{c \oplus c: c \in S\} = \bigvee^{\uparrow} S$$

We have now shown that $\oplus$ is a binary join in A_1. We might as well write it as $\vee$.

$\vee$ in A_1 distributes over $\wedge$, so A_1 is a distributive lattice, and hence a frame; also, the injection of generators preserves finite joins.

We now know that we can define a frame homomorphism $\alpha: A_2 \to A_1$ and a preframe homomorphism $\beta: A_1 \to A_2$, both mapping generators to generators in the obvious way. α is a preframe homomorphism, so $\beta;\alpha$ is the identity on A_1. It remains to show that β is a frame homomorphism, after which we know that $\alpha;\beta$ is also the identity. Fixing $y \in P$, the set $\{a \in A_1: \beta(a \vee y) = \beta(a) \vee y\}$ is a subpreframe containing the generators; and then fixing a, the set $\{b \in A_1: \beta(a \vee b) = \beta(a) \vee \beta(b)\}$ is a subpreframe containing the generators.]]

This result could in fact have been used in the proof of Theorems 4.3 and 4.4, though we preferred to put those results in a more general context. We give here another sample application concerning the upper and lower power locales. These, decomposing the Vietoris construction into two parts, were first studied as topologies in Michael [51]. They are also well-known in computer science following the work of Smyth [78, 83]; see, for instance, Vickers [89].

Suppose D is a locale. Following Vickers [89], we write ΩD for the corresponding frame "of opens", and if f: $D \to E$ is a continuous map between locales, we write Ωf:□ΩE□$\to \Omega D$ for its inverse image map.

The *upper power locale* $P_U D$ is defined by $\Omega P_U D = \mathrm{Frm} \langle \Omega D \text{ (qua preframe)} \rangle$.

The *lower power locale* $P_L D$ is defined by $\Omega P_L D = \mathrm{Frm} \langle \Omega D \text{ (qua sup-lattice)} \rangle$.

If $a \in \Omega D$, then we write $\Box a$ and $\Diamond a$ for the corresponding generators of $\Omega P_U D$ and $\Omega P_L D$; so $\Box$ preserves finite meets and directed joins, while $\Diamond$ preserves all joins.

Proposition 5.2 *The upper and lower power locale functors commute.*

Let D be a locale. Then $P_U P_L D \cong P_L P_U D$.

Proof We define mutually inverse frame homomorphisms

$$\Omega\theta: \Omega P_L P_U D \to \Omega P_U P_L D \qquad \Omega\phi: \Omega P_U P_L D \to \Omega P_L P_U D$$

for which $\Omega\theta(\Diamond \Box a) = \Box \Diamond a$ and $\Omega\phi(\Box \Diamond a) = \Diamond \Box a$.

$\Omega\theta$: This must be equivalent to a sup-lattice homomorphism from $\Omega P_U D$ to $\Omega P_U P_L D$ taking $\Box a$ to $\Box \Diamond a$. Johnstone's coverage theorem for frames, i.e. the "sup-lattice version", tells us that

$$\begin{aligned}\Omega P_U D &\cong \mathrm{Frm} \langle \Box a \ (a \in \Omega D \text{ qua } \wedge\text{-semilattice}) \mid \Box \text{ preserves } \textstyle\bigvee^{\uparrow} \rangle \\ &\cong \mathrm{SupLat} \langle \Box a \ (a \in \Omega D \text{ qua poset}) \mid \Box \text{ preserves } \textstyle\bigvee^{\uparrow} \rangle\end{aligned}$$

and $\Box a \mapsto \Box \Diamond a$ does indeed preserve directed joins.

$\Omega\phi$: This must be equivalent to a preframe homomorphism from $\Omega P_L D$ to $\Omega P_L P_U D$. The preframe version of the coverage theorem tells us that

$$\begin{aligned}\Omega P_L D &\cong \mathrm{Frm} \langle \Diamond a \ (a \in \Omega D \text{ qua } \vee\text{-semilattice}) \mid \Diamond \text{ preserves } \textstyle\bigvee^{\uparrow} \rangle \\ &\cong \mathrm{PreFrm} \langle \Diamond a \ (a \in \Omega D \text{ qua poset}) \mid \Diamond \text{ preserves } \textstyle\bigvee^{\uparrow} \rangle\end{aligned}$$

and then, as before, $\Diamond a \mapsto \Diamond \Box a$ preserves directed joins. ⟧

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