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QUANTALES, OBSERVATIONAL LOGIC AND PROCESS SEMANTICS

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Abstract

Various notions of observing and testing processes are placed in a uniform algebraic framework in which observations are taken as constituting a quantale. General completeness criteria are stated, and proved in our applications.

1. Introduction

Our aim in this paper is to present a uniform algebraic framework for the study of various notions of observing and testing processes, and the equivalences they induce. Our approach differs from that of Hennessy (1988), in that we emphasize the logical and lattice-theoretic aspects, and consider a much wider range of equivalences. Also, our approach yields observational logics and denotational models for each of the computational situations we study, in a uniform fashion.

Our work is a continuation of Abramsky (1987 a and 1991) and Vickers (1989), but we expand on the remarks on p. 9 of Vickers (1989) to generalize and enrich the framework and encompass observations that may change the state of the process being observed. Algebraically, this corresponds to the passage from frames (also known as locales or complete Heyting algebras) to quantales (Mulvey 1986, Niefield and Rosenthal 1988, Rosenthal 1990); logically, to the passage from geometric logic to a geometric form of Girard's (1987) linear logic. Semantically, we are taking a further step towards the rapprochement between operational and denotational semantics that forms part of the programme of Abramsky (1987 a). Our detailed work on testing equivalences can also be seen as a continuation of the programme of Abramsky (1987 b).

The contents of the remainder of the paper are as follows, section by section.

2 *Observational logic and modules over quantales*

We discuss how the observational justification of geometric logic and frames generalizes to quantales when the observations may affect the object observed, and how the object observed must then take its value in a *module* that includes the different possible states.

3 *Background on quantales*

A short historical account of quantales and related ideas.

4 *The applications to processes*

4.1 Basic transition system semantics

4.2 The subbasic observations and relations involving them

5 *Some technicalities*

5.1 Testing and duality

5.2 Coverages

5.3 Coherence

2. Observational logic and modules over quantales

In Abramsky (1987 a and 91) and Vickers (1989) it is argued that propositional geometric logic (whose only connectives are finite conjunctions and arbitrary disjunctions) is the logic of finitely observable properties. This idea is conveniently presented in terms of the *topological systems* of Vickers (1989):

Definition 2.1 A *topological system* is a structure D = (pt D, $\models$, ΩD) where –

- pt D is a set of *points* (to be understood here as denotations of processes)
- ΩD is a frame of *opens* – a *frame* is a complete lattice satisfying the frame distributive law

  $$a \wedge \bigvee S = \bigvee \{a \wedge b : b \in S\}$$

  for arbitrary subsets S.
- $\models \subseteq \text{pt } D \times \Omega D$ is the *satisfaction relation,* subject to the conditions

  $$x \models \top \qquad (x \in \text{pt } D)$$
  $$x \models a \wedge b \text{ iff } x \models a \text{ and } x \models b \qquad (x \in \text{pt } D,\ a, b \in \Omega D)$$
  $$x \models \bigvee S \text{ iff } \exists a \in S.\ x \models a \qquad (x \in \text{pt } D,\ S \subseteq \Omega D)$$

If D and E are two topological systems, then a *continuous map* from D to E is a pair f = (pt f, Ωf) where pt f: pt D → pt E is a function, Ωf: ΩE → ΩD is a frame homomorphism (preserving finite meets and arbitrary joins), and pt f(x) $\models$ a iff x $\models$ Ωf(a).

Topological systems subsume both *topological spaces,* in which ΩD $\subseteq \wp$(pt D) with x $\models$ a iff x ∈ a, and *locales,* in which pt D is the set of frame homomorphisms from ΩD to $\mathbb{2}$, with x $\models$ a iff x(a) = **true**.

The idea is that finitely observable properties are closed under finite conjunctions and arbitrary disjunctions (to verify an infinite disjunction, we need only verify a single one of the disjuncts), the propositional connectives of *geometric logic,* but not under the other propositional connectives. These ideas form the basis for a computational interpretation of topology (since frames are the lattice-theoretic abstractions of topologies) and the development of "domain theory in logical form" using the ideas of Stone duality.

It is well known that frames are in fact complete Heyting algebras, but we consider the non-geometric operations such as infinite meets, implication and negation to be an accidental part of the structure. We therefore distinguish between the *first-class* geometric operators, and the other *second-class* operators of Heyting algebras. This distinction is reflected technically in the definition of frame homomorphisms, which are only required to preserve the first-class operations.

Let us re-examine the axioms for these geometric connectives more closely. The axioms for meets (other than distributivity over joins) can be presented as follows:

associativity:	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
unit law:	$a \wedge \top = a$
commutativity:	$a \wedge b = b \wedge a$
idempotence:	$a \wedge a = a$
top law:	$a \leq \top$ (i.e. $a \vee \top = \top$)

(Proof: the first four give the frame a semilattice structure under $\wedge$ and $\top$, and hence show that these are the binary and nullary meets for an ordering $\leq_\wedge$ defined by $a \leq_\wedge b$ iff $a \wedge b = a$. But using in addition the top law and distributivity one can show that $a \wedge b = a$ iff $b = a \vee b$, and so that $\leq_\wedge$ is the same as the original ordering $\leq$. Thus $\wedge$ and $\top$ are binary and nullary meet for the original complete lattice.)

Of these, the first two give a monoid structure under $\wedge$ and $\top$, and we shall leave these in place. The remaining three, however, incorporate certain assumptions about testing, and are worth closer scrutiny.

Commutativity says that the order in which we carry out the observations or tests is immaterial to the outcome.

Idempotence says that the number of times we perform a test is immaterial to the outcome.

The *top law* says that in the situation in which we have observed either a or nothing ($\top$, the trivial observation), but we don't know which, we have exactly the same knowledge as if we had observed nothing. This amounts to saying that all the possibilities that are consistent with the knowledge that a has been observed are also consistent with the knowledge that nothing has been observed.

These assumptions are sound provided that we stipulate that *carrying out an observation or test has no effect on the object being observed or tested*. If the object is conceived as a static, mathematical entity, this stipulation goes without saying; but if the object is a *process,* computational or physical, it is open to question. If we assume that processes have state, and change it as a result of our testing or observing them – and thereby interacting with them in some fashion – then these assumptions must be discarded.

Example 2.2 In a labelled transition system, as used extensively in work on the semantics of concurrency (see, e.g., Milner 1989), the transition $p \xrightarrow{\alpha} q$ combines the observation that p can perform the action α with p changing its state to q.

Thus we are led to the following definitions.

Definition 2.3 (Joyal and Tierney 1984) A *sup-lattice* is a complete join semilattice. We write 0 and $\top$ for its bottom and top elements.

A homomorphism between sup-lattices is a function that preserves all joins.

Of course, a sup-lattice is in fact a complete lattice, but the joins are the first-class operators, preserved by homomorphisms.

Definition 2.4 A *quantale* is a sup-lattice Q equipped with a monoid structure $(Q, \cdot, 1)$ and satisfying both complete distributive laws

$$a \cdot \bigvee S = \bigvee \{a \cdot b : b \in S\}$$
$$\bigvee S \cdot a = \bigvee \{b \cdot a : b \in S\}$$

A homomorphism between quantales is a function that preserves the first-class operations, namely 1, · and all joins.

The multiplication · can almost always be pronounced "then". Note that the identity 1 is to be a genuinely trivial observation under which nothing at all happens. It does not formalize silent actions such as Milner's τ.

The appropriate generalization of the notion of topological system must allow for states and their change. Thus instead of a satisfaction *predicate* $\models : X \times Q \to \mathbb{2}$ (where X contains the processes), we need a map

$\_\cdot\_ : X \times Q \to X$ $\qquad$ $x \cdot a$ = "what x changes to when the observation a is made"

However, we also need some structure on X.

Examples 2.5

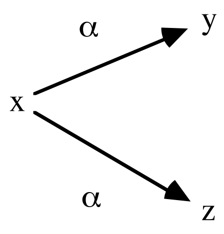


fig. 2.1

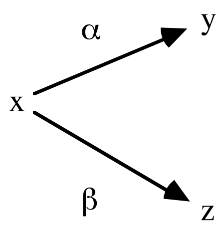


fig 2.2

(i) If a transition is non-deterministic, e.g. fig. 2.1, then we want to say $x \cdot \alpha = y \vee z$.

(ii) Note the special case where the action α is not possible from y: then $y \cdot \alpha = 0$.

(iii) Even if the system is deterministic, e.g. fig. 2.2, we still want to collect the results of disjunctive observations: $x \cdot (\alpha \vee \beta) = y \vee z$.

Thus we are led to:

Definition 2.6 Let Q be a quantale. A *(right) module* over Q is a sup-lattice M, together with a *module action* $\_\cdot\_ : M \times Q \to M$ satisfying

- $x \cdot 1 = x$
- $x \cdot (a \cdot b) = (x \cdot a) \cdot b$

- $x \cdot \bigvee S = \bigvee \{x{\cdot}a: a \in S\}$ $(S \subseteq Q)$
- $\bigvee S \cdot a = \bigvee \{x{\cdot}a: x \in S\}$ $(S \subseteq M)$

Let M and N be modules over Q, and let f: M → N be a sup-lattice homomorphism. f is a *module homomorphism* iff f(x·a) = f(x)·a for all x ∈ M, a ∈ Q.

Note that the term "module" is taken from ring theory (based on the analogy between quantales and rings exploited in Joyal and Tierney (1984)), and is nothing to do with the idea of modularization in computer programming.

As an immediate example, Q is a module over itself. The action is just multiplication. More generally, if f: Q_0 → Q is a quantale homomorphism, then Q is a module over Q_0 by x·a = x·f(a) (x ∈ Q, a ∈ Q_0).

A simple but important fact is that, as Q-module, Q is freely generated by 1: if M is any Q-module and x ∈ M, then there is a unique module homomorphism from Q to M that maps 1 to x (and a to x·a).

Example 2.7 Let M be a sup-lattice. The set E of all sup-lattice endomorphisms of M is a quantale, in which join is defined pointwise and product is composition. For any quantale Q, right Q-module structures on M are equivalent to quantale homomorphisms from Q to E.

Left modules are defined similarly, but with the action on the left. The only difference lies in the order in which multiplicands are applied, so if Q is commutative then right and left modules amount to the same thing.

We shall now give two families of examples. In the first, which arise from topology, observations are assumed not to affect the system being observed. All the same, the module structure is an important one in showing how our knowledge is changed by the observations we make. In the second family, which is essentially the theory of automata, we cannot observe anything without changing the system.

Examples 2.8

(i) *Topological systems*

Let D be a topological system.

- M = $\wp$(pt D), ordered by set inclusion, is a module over ΩD, with the action given by

  $$X{\cdot}a = \{x \in X: x \models a\} = X \cap \text{extent}(a)$$

  This is the *subspace module* for D.

  We do not expect anything here to change when we observe it. But the subspace X represents a whole range of possibilities for what is actually in front of us, and observing a enables us to cut it down to X·a.
- The open subsets form a submodule M_o.

- The closed subsets form a quotient module M_c of M under the closure mapping, so that the action in M_c is defined by

  $$X \cdot a = Cl(\{x \in X: x \models a\}) = Cl(X \cap extent(a))$$

  We leave it to the reader to verify that this defines an ΩD-module structure.

(ii) *Locales*
Let D be a locale. We can define ΩD-modules by reference to the points as in the previous examples, but there are some rather more localic constructions.

- We have already remarked that ΩD, the set of opens for D, is a module over itself. This is analogous to M_o above, and in fact is isomorphic to it if D is spatial.
- The opposite lattice to ΩD, which we write as (ΩD)^, is also a module over ΩD. Let us write $\hat{x} \in$ (ΩD)^ for the element corresponding to $x \in$ ΩD; then

  $$\hat{x} \cdot a = (a \rightarrow x)\hat{}$$

  where → is the (second-class) Heyting arrow operation in ΩD. Again, we leave it to the reader to verify that this makes a module. It is a special case of the duality theory in Section 5.1.

  We can interpret this action in terms of *guarantees*. If $\hat{x}$ is a guarantee that x won't be observed, then observing a converts it to a stronger guarantee, that nothing that meets a in x (i.e. nothing as strong as a→x) will be observed.
- We shall see a localic version of the subspace module in Example 5.1.4.

We now move on to the other family of examples.

Example 2.9 *Transition systems*
Let Act be a set, of *atomic actions*. A *transition system* (*labelled over* Act) is a set Proc equipped with a *transition relation* $\rightarrow \subseteq$ Proc × Act × Proc. We write –

$$p \xrightarrow{\alpha} q \quad \text{iff } (p, \alpha, q) \in \rightarrow \quad (p, q \in \text{Proc}, \alpha \in \text{Act})$$
$$p \xrightarrow{\alpha} \quad \text{iff } \exists q \in \text{Proc}.\ p \xrightarrow{\alpha} q$$

For $s \in$ Act*, we also define $p \xrightarrow{s} q$ by structural induction on s:

$$p \xrightarrow{1} q \quad \text{iff } q = p$$
$$p \xrightarrow{\alpha \cdot s} q \quad \text{iff } \exists r \in \text{Proc}.\ (p \xrightarrow{\alpha} r \wedge r \xrightarrow{s} q)$$

(Here we are writing 1 for the empty list and · for list concatenation, and we are identifying each element α of Act with the corresponding singleton list. We are really thinking of Act* in abstract terms as the free monoid generated by the elements of Act.)

Let Q = $\wp$(Act*), the set of sets of finite sequences of actions. Q is a quantale, with joins given by unions, and multiplication calculated pointwise:

$A \cdot B = \{s{\cdot}t: s \in A, t \in B\}$
$1 = \{1\}$

If Proc is a transition system over Act, then we can define a Q-module M = $\wp$(Proc), with the module action given by

$X{\cdot}A = \{q \in \text{Proc}: \exists p \in X, s \in A.\ p \xrightarrow{s} q\}$

It's unfortunate that the word "action" has established different meanings in the theories of processes and modules; we shall try our best to avoid confusion.

The significance of these examples is that they show that modules over quantales provide a common generalization of topological spaces – including the principal structures used in denotational semantics – and labelled transition systems – the basic structures of operational semantics.

We can say more about $\wp$(Act*).

Theorem 2.10

(i) Let S be a monoid. Then $\wp$(S), with multiplication defined elementwise, is the free quantale over the monoid S. (In other words, the construction $S \mapsto \wp(S)$ is a functor from monoids to quantales, left adjoint to the forgetful functor.)

(ii) Let Act be a set. Then $\wp$(Act*) is the free quantale over the set Act.

Proof (i) Let f: S → Q be a monoid homomorphism, with Q a quantale. If f is to factor via $\wp$(S), then it must extend to $f(X) = \bigvee\{f(x): x \in X\}$ for $X \subseteq S$; this proves uniqueness in the universal property. For existence, we must show that this does indeed define a quantale homomorphism, but this is readily checked.

(ii) This follows from (i), because Act* is the free monoid over Act. ⟧

The existence of free quantales justifies us in *presenting* quantales with generators and relations, a technique taken from universal algebra and used extensively for frames in Vickers (1989). We shall use the notation

$\text{Qu}\,\langle\, G \mid R \,\rangle$

for the quantale generated by a set G of *generators,* subject to a set R of *relations* on them (equations between expressions involving the generators).

We shall often work with presentations in which G (or some subset of generators) is not just a set, but has algebraic structure of its own which we want preserved in the quantale being presented. For instance, G may be a frame, hence itself a quantale. Then we use the notation

Qu ⟨ G (qua quantale) | R ⟩

for the presentation Qu ⟨ G | R' ⟩ in which R' contains the relations from R together with all relations $\bigvee_i a_{i1} \cdot a_{i2} \cdot \ldots \cdot a_{ik_i} = (\bigvee_i \bigwedge_j a_{ij})$ that express quantale relations already holding in G.

On the other hand, we may want only some of the structure of G preserved in the new quantale. Then we use notation such as Qu ⟨ G (qua poset) | R ⟩ for the presentation with implicit relations of the form $a \vee b = b$ ($a \leq b$ in G).

Modules can also be presented by generators and relations.

In presentations, we use the following abbreviations for algebraic theories:

Fr	frame
Qu	quantale
Qud	quantaloid
Q-Mod	left module over quantale Q
Mod-Q	right module over quantale Q
SupL	sup-lattice
SemiL	semilattice
Mon	monoid

Finally, we look at a particular case of Example 2.7, of quantales of sup-lattice endomorphisms.

Proposition 2.11 Let X be a set, and let M = ℘X be the free sup-lattice on X. The quantale E of sup-lattice endomorphisms of M is isomorphic to ℘(X×X), the set of relations from X to itself, where the unit 1 is equality, and multiplication is relational composition.

The set of subidentity endomorphisms is a sub*frame* of E.

Proof By freeness, the endomorphisms of M correspond to the functions from X to ℘X. Such a function is subidentity iff every x is mapped either to {x} or to Ø, and hence is determined by the set of elements x mapped to {x}. Union and composition correspond to union and intersection of these sets, so the subidentity endomorphisms form a subquantale of commuting idempotents.]]

Using this, we can deduce –

Proposition 2.12 Let Proc and Act be sets, let Q be the quantale ℘(Act*) and let M be the sup-lattice ℘(Proc). Then there is a bijection between –

- transition system structures on Proc labelled over Act, and
- right Q-module structures on the sup-lattice M.

Proof
A Q-module structure on M is a quantale homomorphism from Q to E, the set of sup-lattice endomorphisms of M, which is isomorphic to $\wp$(Proc×Proc). Since Q is free, such a quantale homomorphism is equivalent to a function from Act to $\wp$(Proc×Proc), i.e. a transition system structure on Proc.]

We have seen that quantales generalize both topological spaces and transition systems, and the combination of the two ideas will form the basis of our applications to processes in Section 4. Example 2.9 is used for the trace theory, and various refinements of it will yield a whole family of *testing equivalences* on processes that have been studied in the literature, in a uniform fashion, as instances of an algebraically formulated axiomatic framework.

3. Background on quantales

Although the term *quantale* was introduced very recently (Mulvey 1986), these structures have been studied for some time, in a surprising variety of contexts. The most topical connection is with Girard's linear logic (Girard 1987). We can enunciate the following slogan:

Quantales are to linear logic as frames are to intuitionistic logic.

To see this, recall that linear logic arises by dumping the structural rules for *contraction* and *thinning:*

$$\frac{\Gamma, A, A, \Delta \vdash B}{\Gamma, A, \Delta \vdash B} \qquad \frac{\Gamma, \Delta \vdash B}{\Gamma, A, \Delta \vdash B}$$

while we can also consider non-commutative linear logic (Girard 1989 a), which in addition dumps *exchange:*

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Algebraically, these three structural rules correspond precisely to the axioms for idempotence, top and commutativity, which we jettisoned to get quantales. Thus quantales give an algebraic semantics for non-commutative linear logic (and commutative quantales do the same for commutative linear logic). We interpret the additives ⊕ and & by join and meet, and ⊗ by the quantale multiplication.

We have the left and right linear implication interpreted as the *residuation* operations

$$a\backslash b = \bigvee \{c: a\cdot c \leq b\} \qquad b/a = \bigvee \{c: c\cdot a \leq b\}$$

so that $c \leq a\backslash b \Leftrightarrow a\cdot c \leq b$ and $c \leq b/a \Leftrightarrow c\cdot a \leq b$. (Categorically, since multiplying by a on the left preserves all joins, it has a right adjoint $b \mapsto a\backslash b$; similarly for multiplying on the right.) These coincide precisely when the quantale multiplication is commutative, yielding an interpretation of linear implication $\multimap$.

The *of course* operator, !(_), is interpreted in the commutative case by forcing idempotence and top; we can define !a as the greatest fixpoint of

$$!a = a \wedge 1 \wedge (!a \cdot !a)$$

Technically, we are taking the greatest localic conucleus of the quantale (Niefield and Rosenthal 1988). Lafont (1988) develops a similar interpretation of *of course* in more categorical terms as the cofree commutative comonoid.

Finally, given a commutative quantale Q and an element $\bot \in Q$ ("perphood"; this is not to be confused with the bottom element 0), we can define

$a^{\perp} = a \multimap \perp$

One then verifies that a $\mapsto a^{\perp\perp}$ is a quantic nucleus on Q (Niefield and Rosenthal 1988) and that the induced quotient quantale interprets full linear logic with involution. (Girard's phase semantics (e.g. Girard 1987) does this construction in the case where Q is $\wp$M, the free quantale over a commutative monoid M.)

Some precursors of quantales

The term *quantale* was introduced by Mulvey (1986) in connection with his work on non-commutative C*-algebras: he constructs quantales analogous to the spectra (which are locales) of commutative C*-algebras.

(Mulvey's definition does not assume the existence of a unit, 1 – this is reserved for *unital* quantales. Various slightly different definitions can be found in the literature, and the notion of quantale homomorphism may also vary.)

The earliest abstract definition we know of (non-unital) quantales is in Ward and Dilworth (1939). For any commutative ring, the set of its ideals is a quantale – this gives one of the most important families of examples –, and Ward and Dilworth used the abstract setting to discuss properties of ideals.

As we have seen in Theorem 2.10, another important class of quantales is that of the free quantales over monoids. For these, the residuations are defined by

$A/B = \{c: \forall b \in B.\ c{\cdot}b \in A\}$
$B\backslash A = \{c: \forall b \in B.\ b{\cdot}c \in A\}$

An obvious example is when the monoid is itself free, the monoid of strings over some alphabet. Some old work of Lambek (1958) on "categorial grammars" uses precisely these two notions of implication, / and \. As an example, consider the sentence "John never works." Let us start with the fact that "John" is a noun. We could follow by introducing notions of predicate, verb and adverb, but let us instead say that "works" is a kind of word that converts nouns into sentences when appended on the right (perhaps this is the definition of predicate):

works: noun \ sentence

Now we can use a higher order construction to express the idea that "never" qualifies predicates on the left:

never: (noun \ sentence) / (noun \ sentence)

Hence Lambek's work is related to non-commutative linear logic.

It is clear also that the * operator of regular algebra, defined on sets of words by

$A^* = \bigvee \{A^n: 0 \leq n\}$

is definable in the more general context of quantales. Conway (1971) makes this plain by defining quantales as "standard Kleene algebras". (However, his axiomatization on p.27 is incomplete – it's insufficient to justify his own theorems. With the extra axiom $\sum\{E\} = E$, his standard Kleene algebras are equivalent to quantales.) In particular, for free quantales, Conway's "derivatives" are linear implication: $\partial E/\partial a = \{a\}\backslash E$, and his "constant parts" compute the of course operator: $o[E] = !E$. He uses this notation to state an analogue of Taylor's Theorem.

Another example is the calculus of relations. Given a set X, the relations on X form a quantale $\wp(X\times X)$ in which join is union and the product is relational composition (Proposition 2.11).

$$x\ R{\cdot}S\ y \Leftrightarrow \exists z.\ x\ R\ z \text{ and } z\ S\ y$$
$$x\ R/S\ y \Leftrightarrow \forall z.\ (y\ S\ z \Rightarrow x\ R\ z)$$
$$x\ R\backslash S\ y \Leftrightarrow \forall z.\ (z\ R\ x \Rightarrow z\ S\ y)$$

These last two operations have been investigated recently by Hoare and He Jifeng (1987) as "weakest pre- and post-specifications". Their notation is different from Lambek's. If their ; is seen as the product, then they write Q\R for our R/Q and vice versa. Alternatively, their operators / and \ mean the same as Lambek's if the product is defined by $R{\cdot}S = S;R$.

A precursor of linear logic is relevance logic, which dumps the thinning rule. This has a semilattice semantics, a special case of Girard's phase semantics. Some relevant aspects are treated in Dunn (1986).

Joyal and Tierney (1984) have investigated commutative quantales and their modules, and from their work it is plain that the theory of quantales is in many respects very similar to that of rings. Readers familiar with ring theory will find this helpful.

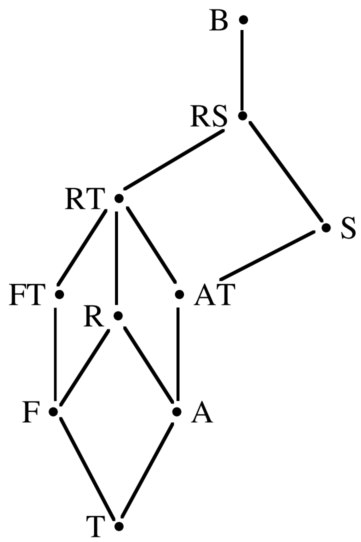
4. The applications to processes

4.1 Basic transition system semantics

We first map out the family of equivalences we shall consider. We are presently ignoring termination, divergence and internal – "τ" – actions, but, bearing that in mind, as far as we know the equivalences presented here include all those that have been proposed to date.

Our quantale methods deal with all these equivalences except bisimulation. After seeing the complexities of Section 6 (on the Ready Simulation), the reader will understand why we prefer to postpone a detailed treatment of bisimulation, but in any case it is by no means clear whether the same methods are applicable.

We can exhibit the following lattice diagram. It is partly inspired by R. van Glabbeek, whose terminology we follow.



The top of the lattice represents the *finest* equivalence (fewest processes identified), and the bottom, the coarsest.

B: bisimulation equivalence (Milner 1980, Park 1981)

RS: ready-simulation

= 2/3 bisimulation (Larsen and Skou 1989)

= denials equivalence (Bloom, Istrail and Meyer 1988)

S: simulation

RT: ready traces (Baeten, Bergstra and Klop 1985)

= barbed traces (Pnueli 1985)

FT: failure traces

= refusal equivalence (Phillips 1987)

AT: acceptance traces
R: ready sets (Olderog and Hoare 1986)
F: failures (Brookes, Hoare and Roscoe 1984)
= testing equivalence (Hennessy 1988)
A: acceptances
T: trace equivalence (Hoare 1985)

We recall some definitions.

Definition 4.1.1 Let Proc be a transition system over Act, $p \in \mathrm{Proc}$.

$$\begin{array}{ll}
\mathrm{traces}(p) = & \{s \in \mathrm{Act}^*: p \stackrel{s}{\Rightarrow}\} \\
R(p) = & \{\alpha \in \mathrm{Act}: p \stackrel{\alpha}{\Rightarrow}\} = \mathrm{traces}(p) \cap \mathrm{Act} \\
F(p) = & \mathrm{Act} - R(p) \\
\mathrm{accepts}(p) = & \{(s, X) \in \mathrm{Act}^* \times \wp_{\mathrm{fin}}(\mathrm{Act}): \exists q.\ (p \stackrel{s}{\Rightarrow} q \wedge X \subseteq R(q)\} \\
\mathrm{failures}(p) = & \{(s, X) \in \mathrm{Act}^* \times \wp_{\mathrm{fin}}(\mathrm{Act}): \exists q.\ (p \stackrel{s}{\Rightarrow} q \wedge X \subseteq F(q)\} \\
\mathrm{readies}(p) = & \{(s, X) \in \mathrm{Act}^* \times \wp(\mathrm{Act}): \exists q.\ (p \stackrel{s}{\Rightarrow} q \wedge X = R(q)\} \\
\mathrm{accept\text{-}traces}(p) = & \{(X_0, \alpha_1, \ldots, X_n): \exists q_0, \ldots, q_n. \\
& \quad (p = q_0 \stackrel{\alpha_1}{\to} q_1 \stackrel{\alpha_2}{\to} \ldots \stackrel{\alpha_n}{\to} q_n \\
& \qquad \wedge X_i \subseteq_{\mathrm{fin}} R(q_i)\ (0 \le i \le n))\} \\
\mathrm{failure\text{-}traces}(p) = & \{(X_0, \alpha_1, \ldots, X_n): \exists q_0, \ldots, q_n. \\
& \quad (p = q_0 \stackrel{\alpha_1}{\to} q_1 \stackrel{\alpha_2}{\to} \ldots \stackrel{\alpha_n}{\to} q_n \\
& \qquad \wedge X_i \subseteq_{\mathrm{fin}} F(q_i)\ (0 \le i \le n))\} \\
\mathrm{ready\text{-}traces}(p) = & \{(X_0, \alpha_1, \ldots, X_n): \exists q_0, \ldots, q_n. \\
& \quad (p = q_0 \stackrel{\alpha_1}{\to} q_1 \stackrel{\alpha_2}{\to} \ldots \stackrel{\alpha_n}{\to} q_n \\
& \qquad \wedge X_i = R(q_i)\ (0 \le i \le n))\} \\
p \precsim_T q \Leftrightarrow & \mathrm{traces}(p) \subseteq \mathrm{traces}(q) \\
p \precsim_A q \Leftrightarrow & \mathrm{accepts}(p) \subseteq \mathrm{accepts}(q) \\
p \precsim_F q \Leftrightarrow & \mathrm{failures}(p) \subseteq \mathrm{failures}(q) \\
p \precsim_R q \Leftrightarrow & \mathrm{readies}(p) \subseteq \mathrm{readies}(q) \\
p \precsim_{AT} q \Leftrightarrow & \mathrm{accept\text{-}traces}(p) \subseteq \mathrm{accept\text{-}traces}(q) \\
p \precsim_{FT} q \Leftrightarrow & \mathrm{failure\text{-}traces}(p) \subseteq \mathrm{failure\text{-}traces}(q) \\
p \precsim_{RT} q \Leftrightarrow & \mathrm{ready\text{-}traces}(p) \subseteq \mathrm{ready\text{-}traces}(q) \\
p \precsim_S q \Leftrightarrow & \forall p'.(p \stackrel{\alpha}{\to} p' \Rightarrow \exists q'.(q \stackrel{\alpha}{\to} q' \wedge p' \precsim_S q')) \\
p \precsim_{RS} q \Leftrightarrow & F(p) \subseteq F(q) \wedge \forall p'.(p \stackrel{\alpha}{\to} p' \Rightarrow \exists q'.(q \stackrel{\alpha}{\to} q' \wedge p' \precsim_{RS} q'))
\end{array}$$

Note that all these relations are *preorders* (reflexive and transitive); each preorder $\precsim_E$ has an associated equivalence $\sim_E$ defined by

$p \sim_E q \quad \Leftrightarrow \quad p \precsim_E q$ and $q \precsim_E p$

Also note that $\precsim_S$ and $\precsim_{RS}$ are defined recursively; the *greatest* fixpoints of the associated monotone inductive definitions are intended (cf. the definition of strong bisimulation in Milner (1989)).

(We haven't defined $\precsim_B$; this is *bisimulation*. The preorder is symmetric and so coincides with $\sim_B$. (See Milner 1989.))

An alternative description of these equivalences is furnished by the use of a modal logic, Hennessy-Milner logic (HML; Hennessy and Milner 1985). The idea is that two processes are equivalent iff they satisfy the same properties.

Given a set Act of atomic actions α, formulas ϕ of HML have the syntax

$\phi ::= \mathbf{true} \mid \mathbf{false} \mid \phi\wedge\psi \mid \phi\vee\psi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$

Given a transition system Proc over Act, a satisfaction relation $\models$ between processes $p \in$ Proc and formulas ϕ is defined by

$p \models \mathbf{true}$	always
$p \models \mathbf{false}$	never
$p \models \phi\wedge\psi$	$\Leftrightarrow p \models \phi$ and $p \models \psi$
$p \models \phi\vee\psi$	$\Leftrightarrow p \models \phi$ or $p \models \psi$
$p \models [\alpha]\phi$	$\Leftrightarrow \forall q.\ (p \xrightarrow{\alpha} q \Rightarrow q \models \phi)$
$p \models \langle\alpha\rangle\phi$	$\Leftrightarrow \exists q.\ (p \xrightarrow{\alpha} q$ and $q \models \phi)$

We define a number of sets of formulas of HML (for L_R and L_{RT}, we must assume that Act is finite):

L_B – all formulas

L_{RS} – formulas in which $[\alpha]$ occurs only in subformulas of the form $[\alpha]\mathbf{false}$

L_S – formulas with no occurrence of $[\alpha]$

L_{RT} – the smallest class of formulas containing **true** and such that if ϕ is in L_{RT} then so also, for each $S \subseteq$ Act (assumed finite), is

$$\langle\alpha_1\rangle\ldots\langle\alpha_m\rangle(\textstyle\bigwedge_{\beta\in S} [\beta]\mathbf{false} \wedge \bigwedge_{\beta\notin S} \langle\beta\rangle\mathbf{true} \wedge \phi)$$

L_{FT} – the smallest class of formulas containing **true** and such that if ϕ is in L_{FT} then so also are $[\alpha]\mathbf{false} \wedge \phi$ and $\langle\alpha\rangle\phi$

L_{AT} – the smallest class of formulas containing **true** and such that if ϕ is in L_{AT} then so also are $\langle\alpha\rangle\mathbf{true} \wedge \phi$ and $\langle\alpha\rangle\phi$

L_R – formulas of the form

$$\langle\alpha_1\rangle\ldots\langle\alpha_m\rangle(\textstyle\bigwedge_{\beta\in S} [\beta]\mathbf{false} \wedge \bigwedge_{\beta\notin S} \langle\beta\rangle\mathbf{true})$$

where $S \subseteq$ Act (assumed finite)

L_F – formulas of the form $\langle\alpha_1\rangle\ldots\langle\alpha_m\rangle(\bigwedge_j [\beta_j]\mathbf{false})$
L_A – formulas of the form $\langle\alpha_1\rangle\ldots\langle\alpha_m\rangle(\bigwedge_j \langle\beta_j\rangle\mathbf{true})$
L_T – formulas of the form $\langle\alpha_1\rangle\ldots\langle\alpha_m\rangle\mathbf{true}$

The following *Modal Characterization Theorem* summarizes some of the known results relating process equivalences to fragments of HML.

Theorem 4.1.2 Let Act be a finite set, and let Proc be an image-finite transition system over Act (i.e. $\{q: p \xrightarrow{\alpha} q\}$ is finite for every $p \in$ Proc, $\alpha \in$ Act). Let E be a process equivalence in {B, RS, S, RT, FT, AT, R, F, A, T}. Then for all p, q ∈ Proc,

$$p \subsetneq_E q \Leftrightarrow \forall\phi\in L_E.(p \vDash \phi \Rightarrow q \vDash \phi)$$

Proof
B,S: Hennessy and Milner (1985)
RS: Bloom, Istrail and Meyer (1988), Larsen and Skou (1989)
FT: Phillips (1987)
RT,AT: Minor modifications to the argument for FT
F,T: Brookes and Rounds (1983)
R,A: Minor modifications to the argument for F ⟧

4.2 The subbasic observations and relations involving them

It was suggested in Abramsky (1987 b) that different process equivalences represent equivalence of behaviour under different notions of how the processes can be tested or observed. Our algebraic treatment makes this very explicit by formalizing certain observations as the generators of quantales. On the analogy with topology, where a generating set of open sets is called a *subbasis,* we call our most fundamental observations *subbasic*. Fixing a set Act of process actions α, β, …, we define the following observations.

α is the observation that the action α has been performed, along with any associated change of state. The action may have been initiated by us or by the process autonomously; the observation is simply that, for whatever reason, it has happened.

$\alpha^\times$ is the *refusal* of α, the observation that the process has signalled its inability to perform α. The process does not change its state, although our state of knowledge about it changes (improves). This corresponds to the HML formula [α]**false**.

$\alpha^\surd$ is the *acceptance* of α, the observation that the process has signalled its ability to perform α, although it hasn't done it yet. For instance, we may have discovered α on a menu. Again, the process does not change its state. This corresponds to the HML formula ⟨α⟩**true**.

↲ is the observation that the process has *undone* its last action, returning to the state it was in before the action was done. Operationally, this represents the process's response to our pressing an "oops" button; in Section 6 we shall see it as the key to reducing a significant amount of our theory to topology.

We shall construct quantales out of various mixtures of these generators. We also want to make modules M out of transition systems Proc, so we must say how the subbasic observations act on them. The basic scheme takes M to be ℘(Proc). We extend Example 2.9 by defining

$$\{p\}\cdot\alpha = \{q \in \mathrm{Proc}: p \xrightarrow{\alpha} q\}$$

$$\{p\}\cdot\alpha^{\surd} = \begin{cases}\{p\} & \text{if } p \xrightarrow{\alpha} \\ \emptyset & \text{otherwise}\end{cases}$$

$$\{p\}\cdot\alpha^{\times} = \begin{cases}\{p\} & \text{if } \neg(p \xrightarrow{\alpha}) \\ \emptyset & \text{otherwise}\end{cases}$$

In all cases, this is extended to general subsets of Proc by distributivity over joins.

To implement ↲, we must take M = ℘(Proc×Proc*). In the pair (p, ps), p is the active process and ps, a list of processes, is a return stack. Then our actions are modified to

$$\{(p, ps)\}\cdot\alpha = \{(q, p::ps): p \xrightarrow{\alpha} q\}$$

$$\{(p, ps)\}\cdot\alpha^{\surd} = \begin{cases}\{(p, ps)\} & \text{if } p \xrightarrow{\alpha} \\ \emptyset & \text{otherwise}\end{cases}$$

$$\{(p, ps)\}\cdot\alpha^{\times} = \begin{cases}\{(p, ps)\} & \text{if } \neg(p \xrightarrow{\alpha}) \\ \emptyset & \text{otherwise}\end{cases}$$

$$\{(p, \mathbf{nil})\}\cdot\hookleftarrow = \emptyset$$
$$\{(p, q::qs)\}\cdot\hookleftarrow = \{(q, qs)\}$$

(We write "::" for the list "cons" operator.)

Without ↲, we can forget the return stack and regain the previous module.

In using these subbasics to generate a quantale, we must also specify some relations to hold between them.

The first relations stem from the fact that the refusals and acceptances are purely "propositional" in nature. This has two aspects. First, the process does not change its state when it affords such an observation, ϕ, say, and so we can say that $\phi \leq 1$. This means that any transition that could have taken place under ϕ could also have taken place under 1 – it is no transition at all. Second, ϕ's use to us lies only in what it tells us about the passive process. This means that order and multiplicity of such observations are immaterial. We now have relations

$\alpha^\times \leq 1$ (hence $\alpha^\times \cdot \alpha^\times \leq \alpha^\times$)
$\alpha^\times \leq \alpha^\times \cdot \alpha^\times$ (hence $\alpha^\times \cdot \alpha^\times = \alpha^\times$)
$\alpha^\times \cdot \beta^\times = \beta^\times \cdot \alpha^\times$

$\alpha^\surd \leq 1$
$\alpha^\surd \leq \alpha^\surd \cdot \alpha^\surd$
$\alpha^\surd \cdot \beta^\surd = \beta^\surd \cdot \alpha^\surd$

$\alpha^\times \cdot \beta^\surd = \beta^\surd \cdot \alpha^\times$

On the basis of these we extend our notation: if $X \subseteq_{\text{fin}}$ Act, then

$X^\times = \prod \{\alpha^\times : \alpha \in X\}$
$X^\surd = \prod \{\alpha^\surd : \alpha \in X\}$

The property that ↲ returns the process to a previous state tells us that if ϕ is a "propositional" observation then so is $\alpha \cdot \phi \cdot$↲. We postpone the detailed description of this to Section 6, but note for the present that we can define

$\alpha^\surd = \alpha \cdot$↲

The effect of this is that we can eliminate the generator $\alpha^\surd$ when α and ↲ are both present.

Next, we must say more specifically how these observations perform their intended roles. First, if a process refuses an action, it is not allowed then to do it.

$\alpha^\times \cdot \alpha = 0$
$\alpha^\times \cdot \alpha^\surd = 0$

Next, if a process has done α, then it must have been willing to do it.

$\alpha \leq \alpha^\surd \cdot \alpha$ (hence $\alpha = \alpha^\surd \cdot \alpha$)

Again, more complicated considerations apply when ↲ is present.

Finally, a process must always be prepared either to refuse or accept a process.

$1 \leq \alpha^\surd \vee \alpha^\times$ (hence $1 = \alpha^\surd \vee \alpha^\times$)

Proposition 4.2.1 All these relations are respected by the module actions we have defined.

Proof What this means is that when we interpret the subbasics as sup-lattice endomorphisms of M (= $\wp$(Proc) or $\wp$(Proc×Proc*)), they satisfy the relations. This is routine verification for the actions on singletons, as defined, and then it holds more generally by distributivity. For instance, take the relation $\alpha^\times \cdot \alpha = 0$.

$$\{p\}\cdot\alpha^{\times}\cdot\alpha = \begin{cases} \emptyset & \text{if } p \xrightarrow{\alpha} \\ \{p\}\cdot\alpha & \text{if } \neg(p \xrightarrow{\alpha}) \end{cases}$$

$$= \emptyset \qquad \text{in either case} \qquad]$$

This (with Example 2.7) assures us that when we present a quantale Q using a selection of these generators and relations, we can make a module over Q out of any transition system labelled over Act. Let us write *LTS* for the class of these modules over Q.

At this point, we can define two processes to be equivalent iff they "have the same capabilities in Q", i.e.

$$p \sim q \quad \text{iff} \quad \forall a \in Q.\ \{p\}\cdot a \neq \emptyset \Leftrightarrow \{q\}\cdot a \neq \emptyset$$

Our *first completeness criterion* is then that this equivalence should be some chosen equivalence E from the list in 4.1.1. This is a rather weak completeness (though there are cases where we need to assume finiteness of Act for it to be valid) that essentially says that we have chosen the right set of generators for Q.

Stronger is our *second completeness criterion,* which says that we have enough relations. Soundness has already appeared, the idea that if $a \leq b$ in Q then we should have $x\cdot a \leq x\cdot b$ for every x in any LTS module; completeness is the converse.

There remains a further step. Inherent in our definition of equivalence is the idea that the *meaning* of a process p is the set of its "capabilities", those a in Q for which $\{p\}\cdot a \neq \emptyset$. By pursuing this idea, we shall construct out of Q a semantic domain for processes. The crucial question is when two elements of Q are equivalent as process capabilities: in other words, given a and b in Q, when do we have that for every process p, $\{p\}\cdot a \neq \emptyset$ iff $\{p\}\cdot b \neq \emptyset$? We approach this through the preorder $\leq'$, $a \leq' b$ meaning "if a process can do a then it can also do b":

$$a \leq' b \quad \text{iff} \quad \forall x \in M \in \mathrm{LTS}.\ (x\cdot a \neq \emptyset \Rightarrow x\cdot b \neq \emptyset)$$

Here are some examples. (i)-(iv) are quite general, the rest are specific to our context with processes.

(i) if $a \leq b$, then $a \leq' b$
(ii) $a \leq' 1$
(iii) if $a_i \leq' b$ for all i in some indexing set, then $\bigvee_i a_i \leq' b$
(iv) if $a \leq' b$ then $c\cdot a \leq' c\cdot b$

(v) $b \leq' \alpha \vee \alpha^{\times}\cdot b$
(vi) $\alpha^{\surd} \leq' \alpha \leq' \alpha^{\surd}$

It's important to understand that the corresponding equivalence relation $='$, i.e. $\leq' \cap \geq'$, is not a quantale congruence, and we could not have included the primed relations in the presentation of Q. Intuitively, the difference between ≤ and ≤' is that ≤ "constrains the processes both before and after": if a ≤ b and $p \xrightarrow{a} q$ (by which we mean q ∈ {p}·a), then $p \xrightarrow{b} q$. On the other hand, if a ≤' b and $p \xrightarrow{a} q$, then all we know is that $p \xrightarrow{b} q'$ *for some* q'. In a sense, ≤ takes care to respect the *dynamics* of the transitions, whereas ≤' is more *static* – it is about properties of individual processes, what they can and cannot do.

Technically, the problem is seen as the one-sidedness of example (iv) above: a ≤' b does *not* necessarily imply a·c ≤' b·c. (For instance, α ≤' 1, but α·β ≰' β as one can see by considering a transition system $p \xrightarrow{\alpha} q \xrightarrow{\beta} r$.) However, it is clear from the rest that Q' = Q/=' is a *left module* over Q, and in fact a left module homomorphic image of Q. Our *third completeness criterion* is that the relations we give for ≤' (together with example (ii) above, which forces 1 to be top) are sufficient to present Q' as a left module homomorphic image of Q. We shall usually write j: Q → Q' for the natural homomorphism. Then Q' is generated by j(1) (with j(a) = a·j(1)), so we require

$$Q' \cong \text{Q-Mod}\left\langle j(1) \mid \begin{array}{ll} j(a) \le j(1) & (a \in Q) \\ \left.\begin{array}{l} j(a) \le j(✗) \vee j(\alpha \cdot a) \\ j(\alpha) \le j(\surd) \\ j(\alpha^{\surd}) \le j(✗) \\ \vdots \end{array}\right\} & \text{as required} \end{array}\right\rangle$$

From the definition of Q', it is clear that for each x ∈ M ∈ LTS we have a sup-lattice homomorphism from Q' to $\mathbb{2}$, mapping j(a) to ⊤ iff x·a ≠ 0. The set of these homomorphisms (which we write $\hat{Q}'$) is our semantic domain for processes, and in fact if we pick out the homomorphisms that arise from x's of the form {p}, then we can make them into a "master" transition system that is fully abstract in the sense that equivalence is just equality. Quite apart from any interest of its own, this is also a crucial step in our proofs of third completeness.

Summary

We now give an overview of our programme of using the algebraic framework of modules over quantales to analyse a process equivalence E. We set out the programme for the simpler cases when E is RT, AT, FT or T; for the others, certain modifications are made.

We assume that E includes a preorder $\precapprox_E$ on any transition system Proc over Act, and proceed in the following steps.

First we analyse E as arising from some combination of atomic observations. Let the quantale Q (or Q_E) be generated from these subbasic observations subject to the relevant relations, and let Q' (or Q'_E) be the left Q-module homomorphic image presented using the relevant primed relations. For each transition system Proc, we can define a Q-module structure on M = $\wp$(Proc).

We can now encapsulate what we need to show that Q, together with the homomorphism j: Q → Q', indeed fully captures E in three *completeness criteria:*

First completeness criterion: First, we show that if p and q are elements of a transition system Proc, then

$$p \precsim_E q \Leftrightarrow \forall a \in Q.(\{p\}\cdot a \neq \emptyset \Rightarrow \{q\}\cdot a \neq \emptyset)$$

This shows that the quantale characterizes the equivalence E.

Second completeness criterion: Next, we show that our "axiomatization" of the observational logic of E by the generators and relations presenting Q was indeed complete. This is the statement:

$$\forall a, b \in Q. (a \leq b \Leftarrow \forall x \in M \in LTS.\ x\cdot a \leq x\cdot b)$$

Algebraically, this is the statement that the modules of LTS are *jointly faithful* over Q.

Third completeness criterion: Finally, following the idea of the First Criterion, we note that each element a of Q can be thought of as an observable property of processes, satisfied by p iff {p}·a ≠ Ø. This suggests a preorder on Q:

$$a \leq' b \text{ iff } \forall x \in M \in LTS.\ (x\cdot a \neq \emptyset \Rightarrow x\cdot b \neq \emptyset)$$

The Third Completeness Criterion is that a ≤' b iff j(a) ≤ j(b). We can use this axiomatization of ≤' to build back a process model from the quantale itself, and this is *fully abstract* with respect to $\precsim_E$ in the sense that $\sim_E$ coincides with equality in the model.

What we shall see is that as we perform this construction in various specific cases, it yields various *denotational* process models, presented in the literature in fairly ad hoc fashion (e.g. failures, failure traces, etc), thus confirming our unification of the operational and denotational approaches.

We summarize in a table the different semantics we treat, and the generators and relations that they use. For brevity, we write the following for certain blocks of relations:

"$\alpha^\times$ propositional":

$$\alpha^\times \leq 1$$
$$\alpha^\times \leq \alpha^\times\cdot\alpha^\times$$
$$\alpha^\times\cdot\beta^\times = \beta^\times\cdot\alpha^\times$$

"$\alpha^\surd$ propositional" is similar.

Semantics	*Generating subbasics*	*Relations for* ≤	*Relations for* ≤' (include a ≤' 1)
T – Trace	α		
F – Failures (Testing equivalence)	$\alpha, \alpha^{\times}$, • (1)	$\alpha^{\times}$ propositional	$\bullet \cdot X^{\times} \le' \alpha \vee \bullet \cdot \alpha^{\times} \cdot X^{\times}$ (2) $1 \le' \bullet$
FT – Failure trace (Refusal equivalence)	$\alpha, \alpha^{\times}$	$\alpha^{\times}$ propositional $\alpha^{\times} \cdot \alpha = 0$	$s \le' \alpha \vee \alpha^{\times} \cdot s$ (2)
A – Acceptance	$\alpha, \alpha^{\surd}$, • (1)	$\alpha^{\surd}$ propositional	$\bullet \cdot \alpha^{\surd} =' \alpha$ $1 \le' \bullet$
AT – Acceptance trace	$\alpha, \alpha^{\surd}$	$\alpha^{\surd}$ propositional $\alpha \le \alpha^{\surd} \cdot \alpha$	$\alpha^{\surd} \le' \alpha$
R – Ready	$\alpha, \alpha^{\surd}, \alpha^{\times}$, • (1)	$\alpha^{\surd}$ propositional $\alpha^{\times}$ propositional $\alpha^{\times} \cdot \beta^{\surd} = \beta^{\surd} \cdot \alpha^{\times}$ $\alpha^{\times} \cdot \alpha^{\surd} = 0$ $1 \le \alpha^{\surd} \vee \square\alpha^{\times}$	$\bullet \cdot \alpha^{\surd} =' \alpha$ $1 \le' \bullet$
RT – Ready trace	$\alpha, \alpha^{\surd}, \alpha^{\times}$	$\alpha^{\surd}$ propositional $\alpha \le \alpha^{\surd} \cdot \alpha$ $\alpha^{\times}$ propositional $\alpha^{\times} \cdot \beta^{\surd} = \beta^{\surd} \cdot \alpha^{\times}$ $\alpha^{\times} \cdot \alpha^{\surd} = 0$ $1 \le \alpha^{\surd} \vee \square\alpha^{\times}$	$\alpha^{\surd} \le' \alpha$
S – Simulation	α, ↲	(3)	
RS – Ready simulation (2/3 Bisimulation, Denials equivalence)	$\alpha, \alpha^{\times}$, ↲	(3) $\alpha^{\times} \cdot \alpha = 0$ $1 \le \alpha \cdot$↲$\vee \square\alpha^{\times}$	

(1) The failure, acceptance and ready semantics have the property that propositional observations come right at the end, and cannot be followed by actions. Our treatment uses a categorical generalization, Sections 8 and 9, in which there are two *types* of process, live (and active) and dead (but subject to postmortem observation). • ("death", symbolized by a bullet) is the observation that a live process has changed into a dead one.

(2) $X^{\times}$ means a product of refusals $\beta^{\times}$, s means a product of refusals $\beta^{\times}$ and actions β.

(3) In the simulation and ready simulation semantics, which use the "undo" observation ↲, quite complicated propositional observations can be

constructed of the form $\alpha\cdot\ldots\cdot\lrcorner$. These are required to be commuting subidentity idempotents. Details of the relations required are given in Section 6.3.

5. Some technicalities

5.1 Testing and duality

We first summarize the duality theory for modules over quantales. This is well-known and is covered – at least for commutative quantales – in Joyal and Tierney (1984).

First, if M is any sup-lattice then we write $\hat{M}$, the *dual* of M, for the opposite lattice (still a sup-lattice); and if $x \in M$ we write $\hat{x}$ for x treated as an element of $\hat{M}$. Furthermore, if $f: M \to N$ is a sup-lattice homomorphism, then f has a right adjoint $g: N \to M$ that preserves meets and hence can be considered a sup-lattice homomorphism $\hat{f}: \hat{N} \to \hat{M}$:

$$\hat{f}(\hat{y}) = (\bigvee\{x: f(x) \le y\})^\wedge$$

In fact, ^ gives an order isomorphism between sup-lattice homomorphisms from M to N, and those from $\hat{N}$ to $\hat{M}$.

Note that elements of $\hat{M}$ are equivalent to sup-lattice homomorphisms from $\mathbb{2}$ to $\hat{M}$ (0 must map to 0, and $\top$ can map to anything) and hence to sup-lattice homomorphisms from M to $\mathbb{2}$, which we view as "tests" on elements of M.

Proposition 5.1.1 Let $f: M \to N$ and $g: N \to L$ be sup-lattice homomorphisms.

(i) $\hat{x}(y) = 0$ iff $y \le x$ ($x, y \in M$, so $\hat{x}: M \to \mathbb{2}$)

(ii) $\hat{f}(v)(x) = v(f(x))$ ($x \in M$, $v \in \hat{N}$, so $v: N \to \mathbb{2}$)

(iii) $\hat{\hat{M}} = M$ and $\hat{\hat{f}} = f$

(iv) $(f;g)^\wedge = \hat{g};\hat{f}$

(v) f is onto iff $\hat{f}$ is 1-1.

Proof (i)-(iv) are easily checked; we shall prove (v).

$\Rightarrow$: Suppose $\hat{f}(v) = \hat{f}(v')$. For any $y \in N$ we can find x such that $y = f(x)$, and then

$$v(y) = v(f(x)) = (\hat{f}(v))(x) = (\hat{f}(v'))(x) = v'(f(x)) = v'(y)$$

Therefore, $v = v'$.

$\Leftarrow$: We show that if f is 1-1 then $\hat{f}$ is onto, and then use (iii). If $x \in M$, then for all $z \in M$

$$\hat{x}(z) = 0 \Leftrightarrow z \le x \Leftrightarrow f(z) \le f(x)$$
$$\Leftrightarrow \hat{f}((f(x))^\wedge)(z) = (f(x))^\wedge(f(z)) = 0$$

Hence $\hat{x} = \hat{f}((f(x))^\wedge)$ and $\hat{f}$ is onto.]

Definition 5.1.2 Let Q be a quantale.

Let M be a *left* module over Q. Its dual $\hat{M}$ is a *right* Q-module by:

$(u\cdot a)(x) = u(a\cdot x) \qquad (u \in \hat{M}, a \in Q, x \in M)$

(Any a ∈ Q gives a sup-lattice endomorphism of M, and hence a dual sup-lattice endomorphism of $\hat{M}$. It acts on the opposite side because of the contravariance in 5.1.1 (iv).)

Note that if f is a homomorphism of (left) Q-modules, then $\hat{f}$ is a homomorphism of right Q-modules.

Part (v) of 5.1.1 tells us that quotient modules of M correspond to submodules of $\hat{M}$. This is a natural development of the ideas of *testing equivalence* presented in Hennessy and Plotkin (1987), where tests are understood as homomorphisms of finitary join semilattices. Each element of $\hat{M}$ can be thought of as a test on the elements of M for which x passes u iff $u(x) = \top$. A subset S of $\hat{M}$ gives rise naturally to a preorder on M,

$x \leqq y \text{ iff } \forall u \in S.\ (u(x) = \top \Rightarrow u(y) = \top)$

The equivalence corresponding to this preorder is a sup-lattice congruence, and so defines a quotient sup-lattice. We can replace S by the sub-sup-lattice it generates in $\hat{M}$, without changing the preorder. Finally, if S is a submodule of $\hat{M}$, then the quotient sup-lattice of M is a quotient module.

Example 5.1.3 *Sublocales*
Let A be a frame. It is not hard to show, using commutativity, that the frame homomorphic images of A are the same as the A-module homomorphic images of A, and hence correspond to the A-submodules of $\hat{A}$. These are the subsets S of A that are closed under arbitrary meets, and for which if x ∈ S and a ∈ A, then a→x ∈ S; these are precisely the *sublocales* of A as described by Johnstone (1982) (or see Vickers 1989).

Example 5.1.4 *The sublocale module* (cf. Example 2.8)
Let D be a locale. N(ΩD) is the frame obtained by adjoining to ΩD complements a^c for every a∈ΩD, and the frame homomorphism to N(ΩD) from ΩD makes it an ΩD-module. Hence N(ΩD)^, which is isomorphic to the set of sublocales of D, is also an ΩD-module. If a ∈ ΩD, then, in N(ΩD)^, $\hat{a}$ and $\hat{a^c}$ are *respectively* the closed and open sublocales corresponding to a.

The dual homomorphism makes (ΩD)^ a homomorphic image of N(ΩD)^, and this is the analogue of what in spatial terms gives the closure of a subspace (see Example 2.8).

We can also show that the function $a \mapsto \hat{a^c}$ makes ΩD a submodule of N(ΩD)^, i.e. that $(a \wedge b)^{c\wedge} = \hat{a^c} \cdot b$. For $\hat{a^c} \cdot b = \hat{c}$, where $c = \bigvee\{e \wedge f^c : b \wedge e \wedge f^c \leq a^c\}$, but

$$b \wedge e \wedge f^c \leq a^c \Leftrightarrow b \wedge e \wedge a \leq f$$

so

$$c = \bigvee\{e \wedge (b \wedge e \wedge a)^c : e \in \Omega D\} = \bigvee\{e \wedge (b \wedge a)^c : e \in \Omega D\} = (a \wedge b)^c$$

Example 5.1.5 *Our main programme*
Suppose Q is a quantale, and M a right Q-module. We can define a *left* Q-module homomorphism from Q to $\hat{M}$ by mapping 1 to $\hat{0}$, i.e. $a \mapsto a \cdot \hat{0}$, i.e.

$$a \mapsto (x \mapsto 0 \text{ iff } x \cdot a = 0)$$

This dualizes to give a right Q-module homomorphism $Cap_M: M \to \hat{Q}$,

$$Cap_M(x)(a) = 0 \text{ iff } x \cdot a = 0$$

We think of Cap(x) (we shall usually omit the subscript M) as showing the *capabilities* of x, the a's that x "can do" ($x \cdot a \neq 0$).

Viewing $\hat{Q}$ as Q^{op}, we have that Cap(x) = (Ann(x))^, where Ann(x), the *annihilator* of x in Q, is $\bigvee\{a \in Q: x \cdot a = 0\}$.

From the Cap homomorphisms, we get a notion of equivalence on elements of Q-modules, for we can define $x \lesssim y$ iff $Cap(x) \leq Cap(y)$ – "anything x can do, so can y". In other words, for all a in Q, if $y \cdot a = 0$ then $x \cdot a = 0$. Note that every a in Q is a join of products of generators, so we get $Cap(x) \leq Cap(y)$ iff for every product s of generators, if $y \cdot s = 0$ then $x \cdot s = 0$.

In our contexts, M will usually be of the form ℘Proc, where Proc is a transition system. However, Q will vary, giving different orders on processes. Our first completeness for each Q will be that the induced order should be the same as some specified order on processes. Often, first completeness is trivial: for in many semantics, the capability (or meaning) of a process is defined to be a set of possible behaviours. We find that the behaviours correspond to products of generators for the quantale, and then a behaviour s is possible for a process iff $\{p\} \cdot s \neq \emptyset$.

Now, let $j: Q \to Q'$ be a surjective *left* Q-module homomorphism. (To present such a Q' as left Q-module, you need a single generator j(1), and some relations of the form $a \cdot j(1) \leq b \cdot j(1)$.) Depending on M, the homomorphism from Q to $\hat{M}$ that maps 1 to $\hat{0}$ may factor via j and Q' – what we need to show is that if $a \cdot j(1) \leq b \cdot j(1)$ is a presenting relation for Q', then $a \cdot \hat{0} \leq b \cdot \hat{0}$ in $\hat{M}$, i.e. for all $x \in M$, if $x \cdot a \neq 0$ then $x \cdot b \neq 0$ – "if x can do a, then it can do b".

(Note that the relations $a \cdot j(1) \leq j(1)$, or, more generally, $a \cdot b \cdot j(1) \leq \square a \cdot j(1)$, are automatically respected for arbitrary M. When we present our modules Q', we take these relations $a \cdot j(1) \le j(1)$ as understood, and possibly present some more, $a \cdot j(1) \le b \cdot j(1)$, in the form "$a \le' b$".)

If this homomorphism does factor via Q', $j(a) \mapsto a \cdot \hat{0}$, then we get a dual homomorphism from M to $\hat{Q}'$, $x \mapsto (j(a) \mapsto 0 \text{ iff } x \cdot a = 0)$. $\hat{Q}'$ is a right submodule of $\hat{Q}$, so this map is essentially Cap with restricted target.

We aim with each process semantics to capture the $\hat{Q}'$ that is large enough to contain each image Cap(M) for M = ℘Proc, but no larger – in other words, it's the submodule $N \le \hat{Q}$ generated by those images. Because each Cap(M) is a submodule, the elements of N are joins of elements of Cap(M)'s. Now let $j': Q \to \hat{N}$ be the dual of the inclusion $N \le \hat{Q}$. Then

$$\begin{aligned} j'(a) \le j'(b) &\Leftrightarrow \forall n \in N.\ (n(b) = 0 \Rightarrow n(a) = 0) \\ &\Leftrightarrow \forall \text{Proc}.\ \forall S \subseteq \text{Proc}.\ (\text{Cap}(S)(b) = 0 \Rightarrow \text{Cap}(S)(a) = 0) \\ &\Leftrightarrow \forall \text{Proc}.\ \forall S \subseteq \text{Proc}.\ (S \cdot b = \emptyset \Rightarrow S \cdot a = \emptyset) \\ &\Leftrightarrow \forall \text{Proc}.\ \forall p \in \text{Proc}.\ (\{p\} \cdot b = \emptyset \Rightarrow \{p\} \cdot a = \emptyset) \qquad (*) \end{aligned}$$

Hence our third completeness criterion will be that, for $j: Q \to Q'$, $j(a) \le j(b)$ iff (*) holds.

5.2 Coverages

When a frame is presented by generators and relations, it is well-known that every element can be written as a join of finite meets of generators. Hence, if the generators are closed under finite meets, every element is a join of generators and so can be represented by the set of generators less than it.

Now the relations can be written so that each takes the form $\bigvee X \le \bigvee Y$ for sets X and Y of generators, and this is equivalent to the set of relations $x \le \bigvee Y$ $(x \in X)$. Then, by replacing Y by $U = \{x \wedge y : y \in Y\}$, we can bring the relation to the form $x = \bigvee U$.

Johnstone (1982), in his *Coverage Theorem,* uses this idea to represent the elements of the frame precisely as sets of generators:

Consider a presentation

$$A = \text{Fr} \langle\ S \text{ (qua meet semilattice)} \mid \bigvee U = x\ (U \dashv x \text{ in } C)\ \rangle$$

where –

- S is a meet semilattice
- "qua meet semilattice" means that the finite meets in S are to be preserved in A
- $C \subseteq S \times ℘S$; we write $U \dashv x$ in C (U *covers* x) if $(x, U) \in C$
- If $U \dashv x$ then $u \le x$ for all $u \in U$

- C is a *coverage* in the following sense, that if $U \dashv x$ and $y \in S$ then $\{u \wedge y : u \in U\} \dashv x \wedge y$

Then A is isomorphic to the set C-Idl(S) of *C-ideals* of S, those lower closed subsets I of S such that if $U \dashv x$ and $U \subseteq I$, then $x \in I$.

There is something else that is not brought out in Johnstone's account. If S is any *poset,* and C is *any* set of cover relations, coverage or not, such that if $U \dashv x$ in C then x is an upper bound for U, then C-Idl(S) can be defined in the usual way; it is not hard to show that it satisfies

$$\text{C-Idl(S)} = \text{SupL} \langle \text{ S (qua poset)} \mid \bigvee U = x \ (U \dashv x \text{ in C}) \rangle$$

Thus the content of the coverage theorem is that certain presentations of frames can be converted directly into presentations of sup-lattices, and it enables us to define sup-lattice homomorphisms from abstractly presented frames into sup-lattices. This is a useful trick when we have quantales and modules around.

There is a good analogue from the algebra of rings. If R is an Abelian group and I is a subgroup, then the quotient group, constructed concretely as the set of cosets of I, is presentable as

$$R/I = \text{Group} \langle \text{ R (qua group)} \mid x = 0 \ (x \in I) \rangle$$

If it happens that R is a *ring,* and I an *ideal,* then the very same construction makes a quotient *ring,* presentable as

$$R/I = \text{Ring} \langle \text{ R (qua ring)} \mid x = 0 \ (x \in I) \rangle$$

The coverage theorem generalizes to quantales and modules, and we shall make extensive use of it.

Definition 5.2.1 Let S be a monoid.

A *cover relation* on S is a pair $U \dashv x$ ("U *covers* x") where $x \in S$, $U \subseteq S$.

A *coverage* on S is a set C of cover relations such that

$$(U \dashv x \text{ in C and } y, z \in S) \Rightarrow \{yuz : u \in U\} \dashv yxz \text{ in C}$$

A *C-ideal* in S is a subset $I \subseteq S$ such that if $U \dashv x$ is in C and $U \subseteq I$, then $x \in I$.

We write C-Idl(S) for the set of C-ideals in S.

If $U \subseteq S$, then we write C-$\langle U \rangle$ for the C-ideal generated by U, i.e. the intersection of all the C-ideals containing U.

Let Q be a quantale. A monoid homomorphism $f: S \to Q$ *transforms covers to joins* (with respect to C) iff

$$f(x) \leq \bigvee \{f(u) : u \in U\} \qquad \text{whenever } U \dashv x \text{ in C}$$

Note that although we still use Johnstone's language here, there are three related changes in meaning:

- For a cover relation $U \dashv x$, we do not require x to be an upper bound. (Of course, we couldn't anyway, because S is not ordered.)
- C-ideals are not required to be lower closed.
- In Theorem 5.2.2 (below), C-Idl(S) is generated by S qua set, not qua poset.

Theorem 5.2.2 Let S be a set, and C a set of cover relations on it (C need not be a coverage in any sense here). Then C-Idl(S) can be presented as

$$\text{SupL} \langle \text{ S (qua set) } | \bigvee U \geq x \ (U \dashv x \text{ in C}) \rangle$$

The injection of generators is $s \mapsto C\text{-}\langle\{s\}\rangle$.

Proof

Let M be a sup-lattice, and let f: S → M be a function respecting the relations: if $U \dashv x$ in C, then $\bigvee\{f(s): s \in U\} \geq f(x)$.

If f factors via a sup-lattice homomorphism f': C-Idl(S) → M, then it must be defined by

$$f'(I) = f'(\bigvee\{C\text{-}\langle\{s\}\rangle: s \in I\}) = \bigvee\{f(s): s \in I\}$$

This proves uniqueness.

For existence, we must show that f' defined thus preserves joins and makes $f'(C\text{-}\langle\{s\}\rangle) = f(s)$.

Lemma 5.2.2.1 If $T \subseteq S$, then $\bigvee\{f(s): s \in T\} = \bigvee\{f(s): s \in C\text{-}\langle T\rangle\}$.

Proof

≤ is clear. For ≥, let $a \in M$ be the LHS, and let $J = \{s \in S: f(s) \leq a\}$. J is a C-ideal containing T so $s \in C\text{-}\langle T\rangle \Rightarrow f(s) \leq a$.]]

Returning to the proof of Theorem 5.2.2, clearly $f'(C\text{-}\langle\{s\}\rangle) = f(s)$. If I_λ is a C-ideal for each $\lambda \in \Lambda$, then

$$f'(\bigvee_\lambda I_\lambda) = \bigvee\{f(s): s \in \bigcup_\lambda I_\lambda\} = \bigvee\{\bigvee\{f(s): s \in I_\lambda\}: \lambda \in \Lambda\}$$
$$= \bigvee\{f'(I_\lambda): \lambda \in \Lambda\}$$

In other words, f' preserves joins.]]]]

Theorem 5.2.3 Let S be a monoid and C a coverage on it. Then C-Idl(S) is a quantale, and it can be presented as

$$\text{Qu} \langle \text{ S (qua monoid) } | \bigvee U \geq x \ (U \dashv x \text{ in C}) \rangle$$

The injection of generators is again $s \mapsto C\text{-}\langle\{s\}\rangle$.

Proof

We now forget the concrete description of C-Idl(S), and treat it as being *defined* by the presentation in Theorem 5.2.2.

Fix $s \in S$. We map $S \to$ C-Idl(S) by $t \mapsto s\cdot t$. If $U \dashv x$ is in C, then so also is $\{s\cdot t: t \in U\} \dashv s\cdot x$, and so $\bigvee\{s\cdot t: t \in U\} \geq s\cdot x$: hence we get a sup-lattice homomorphism from C-Idl(S) to itself, $\bigvee T \mapsto \bigvee\{s\cdot t: t \in T\}$. We now have a function $\cdot$: S×C-Idl(S) → C-Idl(S) that distributes over joins on the right.

Now fix $a \in$ C-Idl(S). The function $s \mapsto s\cdot a$ similarly respects the relations, for suppose $U \dashv x$ in C and $a = \bigvee T$.

$$\bigvee\{s\cdot a: s \in U\} = \bigvee\{\bigvee\{s\cdot t: t \in T\}: s \in U\} = \bigvee\{\bigvee\{s\cdot t: s \in U\}: t \in T\}$$
$$\geq \bigvee\{x\cdot t: t \in T\} = x\cdot a$$

Hence we get a binary multiplication on C-Idl(S) that extends the multiplication on S and distributes over joins on both sides.

Now $1\cdot\bigvee T = \bigvee\{1\cdot t: t \in T\} = \bigvee T$ and similarly $(\bigvee T)\cdot 1 = \bigvee T$, so 1 (i.e. the unit of S considered as a generator of C-Idl(S), i.e. C-$\langle\{1\}\rangle$) is a unit.

Also, $\cdot$ is associative, for

$$(\bigvee T_1 \cdot \bigvee T_2)\cdot \bigvee T_3 = \bigvee\{t_1\cdot t_2: t_1 \in T_1, t_2 \in T_2\} \cdot \bigvee T_3$$
$$= \bigvee\{t_1\cdot t_2\cdot t_3: t_1 \in T_1, t_2 \in T_2, t_3 \in T_3\} = \ldots = \bigvee T_1 \cdot (\bigvee T_2 \cdot \bigvee T_3)$$

This proves that C-Idl(S) is a quantale.

As for the universal property, let Q be a quantale, and $f: S \to Q$ a monoid homomorphism that respects the relations. There is a unique sup-lattice homomorphism f': C-Idl(S) → Q extending this. Then $f'(1) = f(1) = 1$, and

$$f'(\bigvee T_1 \cdot \bigvee T_2) = f'(\bigvee\{t_1\cdot t_2: t_1 \in T_1, t_2 \in T_2\})$$
$$= \bigvee\{f(t_1\cdot t_2): t_1 \in T_1, t_2 \in T_2\} = \bigvee\{f(t_1)\cdot f(t_2): t_1 \in T_1, t_2 \in T_2\}$$
$$= \bigvee\{f(t_1): t_1 \in T_1\}\cdot\bigvee\{f(t_2): t_2 \in T_2\} = f'(\bigvee T_1)\cdot f'(\bigvee T_2)$$

Hence f' is a quantale homomorphism.]]

Let us repeat: under the hypotheses of Theorem 5.2.3,

$$\text{Qu}\,\langle\, S \text{ (qua monoid)} \mid \bigvee U \geq x \ (U \dashv x \text{ in C})\,\rangle$$
$$\cong \text{SupL}\,\langle\, S \text{ (qua set)} \mid \bigvee U \geq x \ (U \dashv x \text{ in C})\,\rangle$$

An arbitrary presentation Qu $\langle G \mid R \rangle$ can be constructed by a coverage on G*, the free monoid on G. Each relation in R can be expressed in the form

$$\bigvee_{\lambda\in\Lambda} s_\lambda = \bigvee_{\mu\in M} t_\mu \ (s_\lambda, t_\mu \in G^*)$$

i.e. $s_\lambda \leq \bigvee_\mu t_\mu$ and $t_\mu \leq \bigvee_\lambda s_\lambda$. The corresponding cover relations generate a coverage C, comprising

$$\{u\cdot t_\mu\cdot v: \mu \in M\} \dashv u\cdot s_\lambda\cdot v$$

$\{u \cdot s_\lambda \cdot v : \lambda \in \Lambda\} \dashv u \cdot t_\mu \cdot v$

for all u, v ∈ G*, and then Qu ⟨ G | R ⟩ is isomorphic to C-Idl(G*).

5.3 Coherence

We use in a number of places the properties of spectral locales, i.e. locales D for which ΩD is *coherent* in the following equivalent senses:

(i) ΩD can be presented without using infinite joins in the relations.

(ii) The compact opens KΩD are closed under finitary meets (note that compact elements are always closed under finitary joins; hence KΩD is a sublattice of ΩD), and every open is the directed join of the compact opens below it.

(iii) ΩD is the ideal completion of a distributive lattice.

Moreover, in a presentation as in (i), the generators and their finite meets are all compact. Further details may be found in, e.g., Johnstone (1982) or Vickers (1989). The most important property is that spectral locales are spatial.

Similar results hold, and with essentially the same proofs, for quantales. Let us define notions analogous to quantales and modules, but without infinitary joins:

- A *finitary quantale* is a set equipped with both a join semilattice structure and a monoid structure, multiplication distributing over finite joins.
- If Q is a finitary quantale, then a (left) *module* M over Q is a join semilattice equipped with an action: Q×M → M, with all the obvious laws holding.

Theorem 5.3.1 The following properties of a quantale Q are equivalent.

(i) Q can be presented without using infinite joins in the relations. (For such a presentation, the generators and their products are compact in Q.)

(ii) The compact elements KQ of Q are closed under products, and every element of Q is the directed join of the compact elements below it.

(iii) Q is the ideal completion of a finitary quantale.

Proof

The proof is essentially the same as for frames. The important steps are as follows. First, if Q_0 is a finitary quantale, then Idl(Q_0) is a quantale and in fact it is the free quantale over Q_0. Secondly, if Q ≅ Qu ⟨G | R⟩ is a quantale presentation that doesn't mention infinitary joins, then it can also be used to present a finitary quantale Q_0 ≅ FinQu ⟨G | R⟩. Then Q ≅ Idl(Q_0). (i) ⇒ (iii) is now immediate, and (iii) ⇒ (i) follows by taking a finitary quantale presentation of KQ and using it to present Q qua quantale. (ii) ⇔ (iii) is clear from the standard theory of ideal completions.]

Definition 5.3.2 Q is *coherent* if it satisfies the equivalent conditions of the Theorem.

Note – each quantale we present for the process semantics is coherent, and any product of generators is compact.

Theorem 5.3.3 Let Q be a coherent quantale, and let M be a left (or right) module over it. Then the following conditions on M are equivalent.

(i) M can be presented, qua Q-module, without using infinite joins or non-compact elements of Q in the relations. (For such a presentation, the generators are compact in M.)

(ii) The compact elements KM of M are closed under the actions of KQ, and every element of M is the directed join of the compact elements below it.

(iii) M is the ideal completion of a module over KQ.

Proof – Completely analogous.]

Definition 5.3.4 M is *coherent* if it satisfies the equivalent conditions of the Theorem.

Note – Let Q be one of the quantales we present for process semantics. For each one, we also present a left Q-module Q', a module homomorphic image of Q by a homomorphism j: Q → Q'. Q' is generated by a single element j(1), and can be presented using relations a·j(1) ≤ j(1) (where it suffices for a to be a product of generators of Q) and certain other relations of the form stated in 5.3.3 (i). It follows that Q' is coherent, and if a is a product of generators in Q then j(a) is compact. We shall use this fact extensively.

6 Ready simulation RS

The key idea is that the structure of Q_{RS} can be seen as analogous to that of polynomial rings. Any polynomial is a sum of monomials, each being a coefficient multiplied by a power (the *degree* of the monomial) of the indeterminate; any element of Q_{RS} is a join of "monomials", products of generators, which we shall now explain.

$\alpha^\times$ and $\alpha\cdot\lrcorner$ are "propositional" in nature in the sense that they observe properties of a process without (in the end) changing it. More generally, so are products of these (propositional conjunction) and those of the form $\alpha\cdot\phi\cdot\lrcorner$ (which observes $\langle\alpha\rangle\phi$ in Hennessy-Milner logic) where ϕ is propositional. In fact, we can always undo the actions of an observation to discover a property of the original process without changing it. These propositional observations constitute an important sub*frame* of Q_{RS}.

α and $\lrcorner$, on the other hand, are "transformational" in the sense that they almost certainly change the process.

Given any product x of generators α, $\lrcorner$ and $\alpha^\times$, we can eliminate its propositional parts by crossing out all $\alpha^\times$s and then, repeatedly, all occurrences of $\alpha\cdot\lrcorner$. What is left at the end is the "pure transformational part" of the original x, and it will be of the form $\lrcorner^e\cdot s$ where $s \in Act^*$. We call this the *degree* of x; x had propositional products (jointly playing the role of coefficient) interpolated between the symbols of its degree.

An important property of the degree is that it is representation invariant. If the relations of Q_{RS} allow us to rewrite x as y, then y is a join of products with the same degree as x.

Let us call a join of products of degree g a *monomial* of degree g, and write $Q_{RS}^{(g)}$ for the set of them, which is a sub-sup-lattice of Q_{RS}. Then every element of Q_{RS} can be written uniquely as a join of monomials of different degrees:

$$Q_{RS} \cong \prod_g Q_{RS}^{(g)}$$

Moreover, multiplication respects this decomposition in the sense that

$$Q_{RS}^{(g)}\cdot Q_{RS}^{(h)} \subseteq Q_{RS}^{(\text{degree } g\cdot h)}$$

This *graded* structure of Q_{RS} is crucial, because each $Q_{RS}^{(g)}$ is a coherent frame (hence a quantale, though not a subquantale of Q_{RS}), so that Q_{RS} is also a frame with spatial locale (though frame meet is not the quantale product). The spatiality helps in proving our completeness results. The graded structure is proved by presenting Q_{RS} by a coverage in which all the products of generators occurring in a given cover relation have the same degree. This allows us to conclude that as a sup-lattice, Q_{RS} is a coproduct of sup-lattices (and coproducts of sup-lattices are also products), one for each degree.

Q'_{RS} has a similar graded structure. As far as capabilities go, an observation x of degree $↲^e \cdot s$ is equivalent to $x \cdot ↲^{\text{length } s}$ (i.e. undo all the actions), which has degree $↲^e$. In fact, it turns out that

$$Q'_{RS} \cong \prod_e Q_{RS}(↲^e)$$

In our development, we find it convenient first to study $Q_{RS}(1)$, the frame of "propositional" observations. ("1" here is $↲^0 \cdot 1$, the identity element in the monoid of degrees.) This turns out to be isomorphic to a frame ΩD_{RS}, which we study in 6.1, which arises from the fragment L_{RS} of Hennessy-Milner logic (using $\alpha^\times$, $\langle\alpha\rangle$, $\wedge$ and $\bigvee$). (The corresponding locale D_{RS} is spatial, so it is perfectly in order to think of it as a topological space.) These observations are properties that can be applied to processes from arbitrary transition systems, so an arbitrary process p can be considered to denote a point $[\![p]\!]$ of the locale. But the points can be made into a transition system by

$$p \xrightarrow{\alpha} q \quad \text{iff } \forall\phi.\ (q \models \phi \Rightarrow p \models \langle\alpha\rangle\phi)$$

and this *master transition system* is in some sense characteristic for RS. It is thus possible to think of D_{RS} as being a semantic domain for processes, although (because we have not accounted for divergence) it has no bottom and the topology is not the Scott topology.

In 6.2 we study Q_{hf}, the part of Q_{RS} generated by monomials whose degrees do not contain ↲ – they are in Act*. This part is thought of as "history-free" (hf) in the sense that the observations do not assume anything to be on the return stack. The LTS modules can therefore be of the simple form $\wp$Proc, not $\wp$(Proc×Proc*). Q'_{hf} is the frame $Q_{RS}(1)$, i.e. ΩD_{RS}. The corresponding preorder on processes is $\subsetneqq_{RS}$, but we do not consider Q_{hf} a full observational analysis of $\subsetneqq_{RS}$ because it does not "explain how to observe $\langle\alpha\rangle$". (In Q_{RS} $\langle\alpha\rangle\phi$ is $\alpha \cdot \phi \cdot ↲$.)

Although for these reasons we see Q_{hf} as unsatisfactory from an observational point of view, we still feel it worthwhile to give a full treatment of it. First, it prepares the reader for the techniques (6.3) used for RS by presenting the same methods in a simpler setting; and, second, it will be mathematically useful when we come to the other history-free fragments (RT, FT, AT and T) in Section 7.

The first completeness result requires a restriction ("image closedness") to the transition systems, as would be expected from the theory of HML.

The second completeness result for Q_{hf} relies on a proof that it has the graded structure described above. As for the third completeness result, the quotient module Q'_{hf} is isomorphic to ΩD_{RS}: this is because an element of Q'_{hf} is supposed to represent abstractly an observation on processes p, "p·a ≠ 0" where a ∈ Q_{hf}, and these are in fact the same properties of processes as the RS propositions in ΩD_{RS}. It follows that $\hat{Q}'_{hf}$ is isomorphic to the set of closed subsets of D_{RS}, and in fact the capability homomorphism Cap: ℘Proc → $\hat{Q}'_{hf}$ maps X to the closure of {⟦p⟧:□p□∈ X}. Third completeness follows when one takes Proc to be the master transition system.

In 6.3 we construct Q_{RS} by including the "undo" observation ↲, and the results are similar but more complicated. Degrees now must include ↲'s, and Q'_{RS} is the frame of opens for a locale of stacks of points from D_{RS}.

6.1 The limited modal logic RS

Recall that Hennessy-Milner logic, HML, is a propositional logic of processes that uses finite conjunctions and disjunctions, and, for each action α ∈ Act, two modalities ⟨α⟩ ("possibly after α …") and [α] ("necessarily after α …"). When applied to a process p, ⟨α⟩ ϕ is interpreted as meaning that there is some q such that p $\xrightarrow{\alpha}$ q and q satisfies ϕ, while [α] ϕ means that for every q such that p $\xrightarrow{\alpha}$ q, q satisfies ϕ. We shall study a fragment *RS* that uses [α] **false** (written as $\alpha^{\times}$), and ⟨α⟩.

This has been studied previously by Larsen and Skou (1989) and Bloom, Istrail and Meyer (1988) under the name of "limited modal logic"; but since much of our work here can be seen as investigating limitations on the modal logic, we shall give a more precise name RS: it is the fragment of HML corresponding to ready simulation. In Section 7.5 we shall introduce the logic S corresponding to simulation, and implicit in the treatments of ready traces, failure traces, acceptance traces and traces, are the fragments of HML (no longer modal) corresponding to them.

Definition 6.1.1 An *RS-frame* (over some tacitly understood set Act) is a frame A equipped with constants $\alpha^{\times}$ and unary operations ⟨α⟩ (α ∈ Act) such that –

- ⟨α⟩ preserves all joins
- $\alpha^{\times}$ is the complement of ⟨α⟩**true**.

A homomorphism of RS-frames is a frame homomorphism that also preserves $\alpha^{\times}$ and ⟨α⟩.

This describes an infinitary algebraic theory, RS-Fr: its operators are the frame operators (joins of arbitrary arity and finite meets), $\alpha^{\times}$ and ⟨α⟩.

Definition 6.1.2 An *RS-locale* is a locale whose frame of opens is equipped with an RS-frame structure.

An *RS-map* between RS-locales is a continuous map whose inverse image part is an RS-frame homomorphism.

Example 6.1.3 Let Proc be a transition system over Act. Then considered as a discrete space (i.e. with frame $\wp$Proc), it is an RS-locale. For we can define

$\alpha^\times \quad = \{p \in \text{Proc}: \neg(p \xrightarrow{\alpha})\}$

$\langle\alpha\rangle\, S \quad = \{p \in \text{Proc}: \exists q \in S.\ p \xrightarrow{\alpha} q\} \quad (S \in \wp\text{Proc})$

Note that, rather trivially, the transition structure can be recovered from the RS-locale structure: for

$$p \xrightarrow{\alpha} q \Leftrightarrow \forall S \in \wp\text{Proc}.\ (q \in S \Rightarrow p \in \langle\alpha\rangle\, S)$$

(For $\Leftarrow$, take S = {q}.) We extend this to arbitrary RS-locales.

Definition 6.1.4 Let D be an RS-locale. Then pt D is a transition system over Act, with

$$p \xrightarrow{\alpha} q \text{ iff } \forall\phi \in \Omega D.\ (q \vDash \phi \Rightarrow p \vDash \langle\alpha\rangle\, \phi)$$

Proposition 6.1.5 Let D be a spatial RS-locale.

(i) For each point p, the set $p{\cdot}\alpha = \{q: p \xrightarrow{\alpha} q\}$ is closed.

(ii) $p \vDash \langle\alpha\rangle\, \phi \Leftrightarrow \exists q \in \text{pt } D.\ (q \vDash \phi \wedge p \xrightarrow{\alpha} q)$

$p \vDash \alpha^\times \Leftrightarrow \neg(p \xrightarrow{\alpha})$

In other words, the *extent* homomorphism from ΩD to $\wp$(pt D), extent(ϕ) being $\{p \in \text{pt } D: p \vDash \phi\}$, is an RS-frame homomorphism.

Proof

(i) Suppose $q \in \text{Cl}(p{\cdot}\alpha)$. If $q \vDash \phi$, then ϕ meets p·α: so we can find $q' \vDash \phi$ with $p \xrightarrow{\alpha} q'$, and so $p \vDash \langle\alpha\rangle\, \phi$. Hence $p \xrightarrow{\alpha} q$ and $q \in p{\cdot}\alpha$.

(ii) Consider the first equivalence. $\Leftarrow$ is easy. For $\Rightarrow$, suppose the RHS does not hold: then for all $q \vDash \phi$ we have $\neg(p \xrightarrow{\alpha} q)$, so there is some ψ such that $q \vDash \psi$ but $p \nvDash \langle\alpha\rangle\, \psi$. Hence $\phi \le \bigvee \{\psi: p \nvDash \langle\alpha\rangle\, \psi\}$; then $\langle\alpha\rangle\phi \le \bigvee \{\langle\alpha\rangle\psi: p \nvDash \langle\alpha\rangle\, \psi\}$, so $p \nvDash \langle\alpha\rangle\, \phi$.

The second equivalence follows because $p \vDash \alpha^\times \Leftrightarrow p \nvDash \langle\alpha\rangle$ **true**.]]

These results provide the link between transition systems and RS-locales; and the RS-frames give us an algebraic (or logical) handle on them.

Theorem 6.1.6 There is a final RS-locale D_{RS}. It is spectral, and each operation ⟨α⟩ preserves compactness.

Proof We construct an initial RS-frame ΩD_{RS}.

First, consider the theory RSDL of “RS-distributive lattices”, defined by forgetting the possibility of infinite joins in RS-frames. An RSDL is a distributive lattice, equipped with $\alpha^\times$ and ⟨α⟩, such $\alpha^\times$ is the complement of ⟨α⟩**true** and ⟨α⟩

preserves all *finite* joins. We show that the forgetful functor from **RS-Fr** to **RSDL** has a left adjoint, namely the ideal completion functor (see, for example, Vickers 1989, chapter 9).

Let K be an RSDL. The unary operation $\langle\alpha\rangle$ on K, preserving finite joins, extends uniquely to a unary operation on Idl(K) preserving all joins, so Idl(K) is an RS-frame. Now let A be another, with $f': K \to A$ an RSDL homomorphism. As a distributive lattice homomorphism this extends uniquely to a frame homomorphism $f: \mathrm{Idl}(K) \to A$, which, it is not hard to show, is an RS-frame homomorphism.

RSDL is a finitary algebraic theory, so ordinary universal algebra shows that there is an initial RSDL. Let us call it $K\Omega D_{RS}$, in other words we are defining D_{RS} to be its spectrum. By construction, D_{RS} is spectral and $\langle\alpha\rangle$ preserves compactness. Then $\Omega D_{RS} = \mathrm{Idl}(K\Omega D_{RS})$ is the initial RS-frame and D_{RS} is the final RS-locale. ⟧

Corollary 6.1.7 Let Proc be a transition system. Then Proc can be made into a topological system D = (Proc, ΩD_{RS}) satisfying conditions (ii) of Proposition 6.1.5 ($p \in$ Proc, $\phi \in \Omega D_{RS}$).

In other words, the opens of D_{RS} can be treated as observations on the processes of *any* transition system.

Or – in other words, any process p in any transition system denotes a point $[\![p]\!]$ of D_{RS}: we think of the points of D_{RS} as being abstract processes.

Proof The discrete space Proc is an RS-locale (Example 6.1.3), so there is a unique RS-frame homomorphism from ΩD_{RS} to $\wp$ Proc. This makes D a topological system. The rest comes from the definition of $\alpha^{\times}$ and $\langle\alpha\rangle$ in $\wp$ Proc. ⟧

Now consider 6.1.7 in the case where Proc = pt D_{RS}. By 6.1.5 (ii), the unique RS-frame homomorphism from ΩD_{RS} to $\wp$ Proc is the extent homomorphism, and it follows that the topological system constructed in 6.1.7 is the original locale D_{RS}. Note also that if $p \in$ pt D_{RS}, then $[\![p]\!] = p$.

We call pt D_{RS} the *(RS-) master transition system* for Act.

A set X of points affords the observation ϕ iff $\{x \in X: x \models \phi\} \neq \emptyset$, i.e. iff $X \models \Diamond\phi$. This implies that we cannot distinguish between X and its closure, so the more general elements of modules $\wp$(Proc) denote closed subsets of D_{RS}, i.e. points of P_L D_{RS}, where P_L is the lower(or "Hoare") power locale construction, defined by

$$\Omega P_L\, D = \mathrm{Fr}\,\langle\, \Diamond a\ (a \in \Omega D) \mid \Diamond \text{ preserves all joins}\,\rangle \cong \mathrm{Fr}\,\langle\, \Omega D \text{ (qua SupL)}\,\rangle$$

For any transition system Proc, the topological system (Proc, ΩD_{RS}) induces a topology on Proc, and an associated specialization preorder $\sqsubseteq$.

Definition 6.1.8 Let Proc be a transition system over Act. Proc is *image closed* (with respect to RS) iff for all $p \in$ Proc and $\alpha \in$ Act, the set

$p \cdot \alpha = \{q \in \mathrm{Proc}.\ p \xrightarrow{\alpha} q\}$

satisfies $\downarrow(p \cdot \alpha) = \mathrm{Cl}(p \cdot \alpha)$: its downward closure in the specialization preorder $\sqsubseteq$ is equal to its topological closure. (Equivalently, $\downarrow(p \cdot \alpha)$ is topologically closed.)

The following proposition is the core of first completeness for RS, which will be proved in Theorem 6.3.11.

Proposition 6.1.9 Let Proc be an image closed transition system over Act.

Then for all p, q ∈ Proc,

$p \sqsubseteq q \Leftrightarrow p \subsetapprox_{RS} q$

Proof

We use the usual analysis of the definition (4.1.1) of $\subsetapprox_{RS}$, defining an operation G on relations on Proc by

$p\ G(R)\ q \qquad \text{iff } F(p) \subseteq F(q) \wedge \forall p'.(p \xrightarrow{\alpha} p' \Rightarrow \exists q'.(q \xrightarrow{\alpha} q' \wedge p'\ R\ q'))$

Then $\subsetapprox_{RS}$ is the largest relation R satisfying $R \subseteq G(R)$.

Even without image closedness, we can show that $\subsetapprox_{RS}$ is contained in $\sqsubseteq$. For let U be the set

$\{\phi \in \Omega D_{RS}: \forall p, q \in \mathrm{Proc}.\ (p \subsetapprox_{RS} q \wedge p \models \phi \Rightarrow q \models \phi)\}$

U contains every refusal $\alpha^{\times}$, and is closed under finite meets and arbitrary joins. Also if $\phi \in U$ then $\langle\alpha\rangle\phi \in U$, for suppose $p \subsetapprox_{RS} q$ and $p \models \langle\alpha\rangle\phi$. Then for some p', $p \xrightarrow{\alpha} p'$ and $p' \models \phi$, so for some q', $q \xrightarrow{\alpha} q'$ and $p' \subsetapprox_{RS} q'$, so $q' \models \phi$, and $q \models \langle\alpha\rangle\phi$. It follows that U is the whole of ΩD_{RS}, so for all ϕ in ΩD_{RS}, if $p \subsetapprox_{RS} q$ and $p \models \phi$ then $q \models \phi$.

We next show that $\sqsubseteq\ \subseteq G(\sqsubseteq)$. Suppose $p \sqsubseteq q$; certainly $F(p) \subseteq F(q)$. Suppose $p \xrightarrow{\alpha} p'$. If $p' \models \phi \in \Omega D_{RS}$, then $p \models \langle\alpha\rangle\phi$ and so also $q \models \langle\alpha\rangle\phi$. Therefore there is some $q' \in q \cdot \alpha$ such that $q' \models \phi$. It follows that p' is in the topological closure of $q \cdot \alpha$, but by assumption this coincides with the lower closure. Hence there is some $q' \in q \cdot \alpha$ such that $p' \sqsubseteq q'$, as required. ⟧

Note two particularly important cases. In any topological space, the lower closure of a finite set of points is closed, and so the theorem can be applied to any *image-finite* transition system, one in which $p \cdot \alpha$ is always finite. Secondly, it can be applied when $p \cdot \alpha$ is always closed, because a closed set is its own lower closure; and this covers by definition the case of the master transition system pt D_{RS}.

6.2 The history-free fragment hf

We saw in the previous paragraph how to view the opens of D_{RS} as observations on an arbitrary transition system Proc; hence the frame ΩD_{RS}, *qua* quantale, acts on a module M = ℘ Proc:

$$X\cdot\phi = \{p \in X: p \vDash \phi\}$$

(It is routine to check that this makes M a right ΩD_{RS} module.) We now investigate the quantale that incorporates both these observations and the elements of Act.

Definition 6.2.1 The *history-free quantale* (over Act) is

$$Q_{hf} = \mathrm{Qu}\ \langle \Omega D_{RS} \text{ (qua quantale), Act} \mid \alpha\cdot\phi = \langle\alpha\rangle\phi\cdot\alpha\cdot\phi \rangle$$

We also define the left quotient module Q'_{hf} of $Q_{hf}\cdot\top$ by

$$\langle\alpha\rangle\phi \le' \alpha\cdot\phi$$

Recall how this means a presentation of Q'_{hf} as left Q_{hf} module:

$$Q'_{hf} = Q_{hf}\text{-Mod} \langle\ Q_{hf} \text{ (qua } Q_{hf}\text{-Mod)} \mid a \le 1 \quad (a \in Q_{hf}), \quad \langle\alpha\rangle\phi \le \alpha\cdot\phi \quad (\phi \in \Omega D_{RS}, \alpha \in \text{Act}) \ \rangle$$

Note that in Q'_{hf} we actually have equality between $\langle\alpha\rangle\phi$ and $\alpha\cdot\phi$, for

$$\alpha\cdot\phi = \langle\alpha\rangle\phi\cdot\alpha\cdot\phi \le' \langle\alpha\rangle\phi$$

Theorem 6.2.2

(i) ΩD_{RS} is a left Q_{hf} module isomorphic to Q'_{hf}.

(ii) The corresponding quotient map t: $Q_{hf} \to \Omega D_{RS}$ is then given by

$$\phi_0\cdot\alpha_1\cdot\phi_1\cdot\ldots\cdot\alpha_n\cdot\phi_n \mapsto \phi_0 \wedge \langle\alpha_1\rangle(\phi_1 \wedge \langle\alpha_2\rangle(\ldots \langle\alpha_n\rangle\phi_n)\ldots))$$

(Note: as a left module over itself, Q_{hf} is freely generated by 1. Hence as a left Q_{hf}-module homomorphism, t is uniquely determined by its mapping 1 to **true**.)

(iii) If p is a process in any transition system Proc labelled over Act, and $a \in Q_{hf}$, then $\{p\}\cdot a \neq \emptyset \Leftrightarrow p \vDash t(a)$.

Proof

(i) The action of Q_{hf} on ΩD_{RS} is defined by

$$\phi\cdot\psi = \phi \wedge \psi$$
$$\alpha\cdot\psi = \langle\alpha\rangle\psi$$

Some checking needs to be done here. What we are doing is using the universal property of Q_{hf} (deriving from its presentation by generators and relations) to define a quantale homomorphism from Q_{hf} to the sup-lattice endomorphism quantale of ΩD_{RS}. The checks, all routine, are –

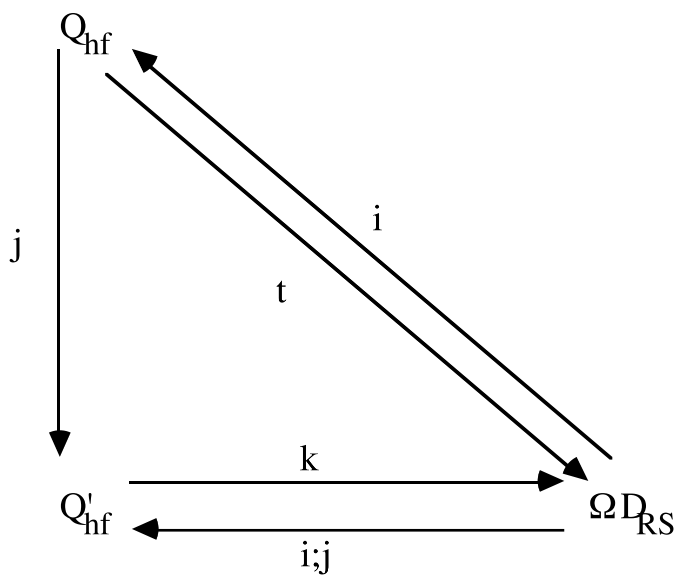
- $\phi\cdot-$ and $\alpha\cdot-$ are both sup-lattice endomorphisms (they preserve all joins)
- that "qua quantale" is respected for the generators of Q_{hf} that come from ΩD_{RS}, i.e.

$$\mathbf{true}\cdot\psi = \psi$$
$$(\phi_1 \wedge \phi_2)\cdot\psi = \phi_1\cdot(\phi_2\cdot\psi)$$
$$(\bigvee_i \phi_i)\cdot\psi = \bigvee_i (\phi_i\cdot\psi)$$

- that the other relations are respected, i.e.

$$\alpha\cdot(\phi\cdot\psi) = \langle\alpha\rangle\phi\cdot(\alpha\cdot(\phi\cdot\psi))$$

We now define maps in this diagram:



Here (and by "module" we mean "left Q_{hf} module"),

- i is the injection of generators. It is a *quantale* homomorphism.
- t is the module homomorphism determined by $1 \mapsto$ **true** (using the fact that Q_{hf} is freely generated by 1).
- j is the module quotient homomorphism.
- k is the module homomorphism determined by t = j;k. This exists because t respects the relations given in presenting Q'_{hf}.

Now although i is only a quantale homomorphism, it is nonetheless the case that i;j is a module homomorphism: for (we need check only the actions of the generators of Q_{hf}, and we shall also be more explicit than usual about i)

$$\alpha\cdot(i;j)(\psi) = j(\alpha\cdot i(\psi)) = j(i(\langle\alpha\rangle\psi)) = (i;j)(\alpha\cdot\psi)$$

$i(\phi)\cdot(i;j)(\psi) = j(i(\phi)\cdot i(\psi)) = j(i(\phi\wedge\psi)) = (i;j)(i(\phi)\cdot\psi)$

We show that the module homomorphisms k and i;j are mutually inverse.

Since j is onto, to show that k;i;j = Id it suffices to show that j;k;i;j = j; and for that, because Q_{hf} is freely generated as a module by 1, it suffices to show that 1 has the same image under the two homomorphisms.

$(j;k;i;j)(1) = (t;i;j)(1) = (i;j)(\mathbf{true}) = j(1)$

To show that i;j;k = Id, we have

$(i;j;k)(\psi) = (i;t)(\psi) = t(i(\psi)\cdot 1) = i(\psi)\cdot\mathbf{true} = \psi\wedge\mathbf{true} = \psi$

(ii) is clear from the module action on ΩD_{RS}.
(iii) By definition of the action of ΩD_{RS} on M = ℘Proc, p ⊨ t(a) iff $\{p\}\cdot i(t(a)) \neq Ø$; and because j(i(t(a))) = (j;k;i;j)(a) = j(a), and using the arguments of Example 5.1.5, this occurs iff $\{p\}\cdot a \neq Ø$. ⟧

Immediately, we have –

Theorem 6.2.3 hf satisfies the third completeness criterion.
Proof
Let Proc be the master transition system pt D_{RS}, and let M = ℘Proc.

Suppose we have a, b ∈ Q_{hf} with t(a) ≰ t(b). By spatiality of D_{RS} we can find a point x ∈ Proc that satisfies t(a) but not t(b), so by (iii) of the above theorem we have $\{x\}\cdot a \neq Ø$, but $\{x\}\cdot b = Ø$. ⟧

Another consequence of Theorem 6.2.2 is (using spatiality of D_{RS}) that $\hat{Q}'_{hf}$ is isomorphic to the sup-lattice of closed subsets of D_{RS}, i.e. elements of $(\Omega D_{RS})^{op}$. We can therefore consider the capability maps Cap_{RS}: M → $\hat{Q}'_{hf}$ as giving each element of M a meaning as a closed set of points of D_{RS}.

Proposition 6.2.4 Let Proc be a transition system labelled over Act, and let M = ℘Proc. Then Cap_{RS}: M → $(\Omega D_{RS})^{op}$ can be defined by

$Cap(X) = Cl(\{[\![p]\!]: p \in X\})$

Proof First note that Cap preserves joins, so it suffices to show that Cap({p}) is $Cl(\{[\![p]\!]\})$ for each p in Proc.

Considering $Cl(\{[\![p]\!]\})$ as an element of $(\Omega D_{RS})^{\wedge} \subseteq \hat{Q}_{hf}$, and writing t as in 6.2.2 (ii), we have

$Cl(\{[\![p]\!]\})(a) = 0 \Leftrightarrow t(a) \leq Cl(\{[\![p]\!]\})^c = \bigvee\{\phi\in\Omega D_{RS}: p \not\models \phi\}$
$\Leftrightarrow p \not\models t(a)$
$\Leftrightarrow \{p\}\cdot a = Ø$ by 6.2.2 (iii)

$\Leftrightarrow$ Cap({p})(a) = 0 by definition of Cap ⟧

Theorem 6.2.5 (1st completeness for hf)
Let Proc be an image closed transition system over Act.
Then for all p, q ∈ Proc,

$$p \sqsubseteq q \Leftrightarrow \mathrm{Cap}_{hf}(\{p\}) \leq \mathrm{Cap}_{hf}(\{q\}) \Leftrightarrow p \precsim_{RS} q$$

Proof This just extends Proposition 6.1.9 in the light of Proposition 6.2.4:

$$\mathrm{Cap}_{hf}(\{p\}) \leq \mathrm{Cap}_{hf}(\{q\}) \Leftrightarrow [\![p]\!] \sqsubseteq [\![q]\!] \Leftrightarrow p \sqsubseteq q$$ ⟧

Second completeness for hf
We now concentrate on the graded structure of Q_{hf} and second completeness. For each $s = \alpha_1 \cdot \ldots \cdot \alpha_n \in \mathrm{Act}^*$ we first define a locale $D^{(s)}$, a sublocale of the product locale $(D_{RS})^{n+1}$, whose points are the sequences $(p_0, \ldots, p_n)$ of points of D_{RS} such that for each $0 \leq i < n$ we have $p_i \xrightarrow{\alpha_{i+1}} p_{i+1}$. The generating opens are symbols $\phi^{(i)}$ ($\phi \in \Omega D_{RS}$, $0 \leq i \leq n$) with

$$(p_0, \ldots, p_n) \models \phi^{(i)} \text{ iff } p_i \models \phi$$

Definition 6.2.6 Let $s = \alpha_1 \cdot \ldots \cdot \alpha_n \in \mathrm{Act}^*$. Define the locale $D^{(s)}$ by

$\Omega D^{(s)} = \mathrm{Fr}\,\langle \phi^{(i)}\ (\phi \in \Omega D_{RS}, 0 \leq i \leq n) \mid$
for each i, the frame relations in ΩD_{RS} are preserved in the $\phi^{(i)}$'s,
$\phi^{(i+1)} \leq (\langle \alpha_{i+1} \rangle \phi)^{(i)}$ $\rangle$

The locale $D^{(s)}$ is spectral, for, using the fact that $\langle\alpha\rangle$ preserves compactness, it can also be presented as

$\Omega D^{(s)} = \mathrm{Fr}\,\langle \phi^{(i)}\ (\phi \in K\Omega D_{RS}, 0 \leq i \leq n) \mid$
for each i, the lattice relations in $K\Omega D_{RS}$ are preserved in the $\phi^{(i)}$'s,
$\phi^{(i+1)} \leq (\langle \alpha_{i+1} \rangle \phi)^{(i)}$ $\rangle$

Anticipating the next proposition, given $\phi_i \in \Omega D_{RS}$ ($0 \leq i \leq n$), we shall write $\phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n$ for $\bigwedge \{(\phi_i)^{(i)}\}$ in $\Omega D^{(s)}$. This is the image of the open $\phi_0 \otimes \ldots \otimes \phi_n$ of $(D_{RS})^{n+1}$ and so such opens form a basis for $D^{(s)}$. (The symbol "⊗" is used here, as in Vickers (1989), to represent what in spatial terms would be the Cartesian product of open sets.)

Proposition 6.2.7 $Q_{hf} \cong \prod_{s \in \mathrm{Act}^*} \Omega D^{(s)}$ as sup-lattices.

Proof First, note that the category of sup-lattices has *biproducts:* a Cartesian product A×B is also a coproduct, with injections

$a \mapsto (a, 0)$ $\qquad$ $b \mapsto (0, b)$

(See MacLane (1971). For sup-lattices, this works for infinitary products too. See Joyal and Tierney (1984).) Therefore, to get the homomorphism from right to left, we define its restrictions to the $\Omega D^{(s)}$'s.

If $s = \alpha_1 \cdot \ldots \cdot \alpha_n$, then from the coverage theorem for frames we have

$$\begin{aligned} \Omega D^{(s)} \cong \mathrm{Fr}\,\langle\, & (\Omega D_{RS})^{n+1} \text{ (qua meet semilattice) } \mid \\ & (\phi_0, \ldots, \bigvee S, \ldots, \phi_n) \leq \bigvee \{(\phi_0, \ldots, \phi, \ldots, \phi_n) : \phi \in S\} \\ & (\phi_0, \ldots, \phi_i, \phi_{i+1}, \ldots, \phi_n) \leq (\phi_0, \ldots, \phi_i \wedge \langle \alpha_{i+1} \rangle \phi_{i+1}, \phi_{i+1}, \ldots, \phi_n) \,\rangle \end{aligned}$$

$$\cong \mathrm{SupL}\,\langle\, (\Omega D_{RS})^{n+1} \text{ (qua poset) } \mid \text{ the same relations } \rangle$$

This maps into Q_{hf} by $(\phi_0, \ldots, \phi_n) \mapsto \phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n$ (as suggested by our notation introduced above).

For the morphism from left to right, note that the RHS is actually a quantale. Given s and $t = \beta_1 \cdot \ldots \cdot \beta_m$ in Act*, we have a pairing from $\Omega D^{(s)} \otimes \Omega D^{(t)}$ to $\Omega D^{(s \cdot t)}$,

$$\begin{aligned} & (\phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n, \psi_0 \cdot \beta_1 \cdot \ldots \cdot \beta_m \cdot \psi_m) \\ & \quad \mapsto \phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot (\phi_n \wedge \psi_0) \cdot \beta_1 \cdot \ldots \cdot \beta_m \cdot \psi_m \end{aligned}$$

and this extends by distributivity to a quantale multiplication on the whole biproduct. Now we can map from left to right by

$$\phi \mapsto \phi \in \Omega D^{(1)}$$
$$\alpha \mapsto \mathbf{true} \cdot \alpha \cdot \mathbf{true} \in \Omega D^{(\alpha)}$$

It is not hard to show that these two maps are mutually inverse.]]

We call the elements of $\Omega D^{(s)}$ *monomials,* of *degree* s and say that Q_{hf} is *graded* over Act*.

Corollary 6.2.8 As a sup-lattice, Q_{hf} is a frame. We write $D^{(*)}$ for the corresponding locale; it is the disjoint sum (union) $\sum_{s \in Act^*} D^{(s)}$ and is spatial.]]

Theorem 6.2.9 hf satisfies the second completeness criterion.
Proof Consider the transition system Proc, the disjoint union of pt D_{RS} and pt $D^{(*)}$, with

$((p_0, p_1, \ldots, p_n), \alpha \cdot s) \xrightarrow{\alpha} ((p_1, \ldots, p_n), s)$	
$((p_0, \ldots, p_n), s) \xrightarrow{\alpha} q$	provided $p_0 \xrightarrow{\alpha} q$ in D_{RS}
$p \xrightarrow{\alpha} q$	provided $p \xrightarrow{\alpha} q$ in D_{RS}

We think of the points of $D^{(*)}$ as being pairs

$$(\mathbf{p}, s) = ((p_0, p_1, \ldots, p_n), \alpha_1 \cdot \ldots \cdot \alpha_n)$$

where $s \in$ Act* and $\mathbf{p} \in$ pt $D^{(s)}$ (see Definition 6.2.6 for a concrete description). The idea is that among these points, $(\mathbf{p}, s)$ should be thought of as p_0 with its transitions closely constrained to follow the α_i's to p_n but satisfying all the observations in ΩD_{RS} that p_0 does. It is to make this second part work that we need the separate copy of pt D_{RS}. There are in effect two copies of each point p of D_{RS}: once "bald", and once as a pair ((p),□1).

Next, before continuing the proof of the Theorem, we prove two lemmas.

Lemma 6.2.9.1

(i) $[\![p]\!] = p$

(ii) $[\![(\mathbf{p}, s)]\!] = p_0$

Proof (i) is clear from the remarks after Corollary 6.1.7.
(ii): Let U be the subframe of ΩD_{RS} comprising those elements φ such that for all (**p**,□s),**p**, s) ⊨ φ iff p_0 ⊨ φ. We show that U is an RS-subframe of ΩD_{RS}, and hence is the whole of ΩD_{RS}.

Suppose φ ∈ U; we show ⟨α⟩φ ∈ U. If $p_0 \xrightarrow{\alpha} q \models \phi$ then $(\mathbf{p}, s) \xrightarrow{\alpha} q \models \phi$ and conversely. If $(\mathbf{p}, s) \xrightarrow{\alpha} (\mathbf{p}', s') \models \phi$, then $p_0 \xrightarrow{\alpha} p_1 \models \phi$.

The refusal $\alpha^\times$ is in U because it is the complement of ⟨α⟩**true**.]]

Lemma 6.2.9.2 If a □∈$\Omega D^{(s)}$, then

$(p_n, 1) \in \{(\mathbf{p}, s)\}\cdot a \Leftrightarrow \mathbf{p} \models a$

Proof – by induction on n; the base case follows from Lemma 6.2.9.1 (ii).

Without loss of generality, we can assume a is a product of generators, not just a join of such:

$a = \phi_0\cdot\alpha_1\cdot\ldots\cdot\alpha_n\cdot\phi_n$

Let **p**', s' and a' be defined by **p**' = (p_1, …, p_n) and so on. Then

$(p_n, 1) \in \{(\mathbf{p}, s)\}\cdot a \Leftrightarrow \{(\mathbf{p}, s)\}\cdot\phi_0 \neq \emptyset \;\&\; (p_n, 1) \in \{(\mathbf{p}', s')\}\cdot a'$
$\Leftrightarrow p_0 \models \phi_0 \;\&\; \mathbf{p}' \models a'$ (by 6.2.9.1 (ii) and by induction)
$\Leftrightarrow \mathbf{p} \models a$]]

We continue the proof of the Theorem 6.2.9 by showing that M = ℘Proc is faithful over Q_{hf}. Suppose a ≰ b in Q_{hf}. Then for some $s = \alpha_1\cdot\ldots\cdot\alpha_n$, we have $a_s \not\leq b_s$ where a_s and b_s are the components in $\Omega D^{(s)}$, so by spatiality there is some **p** = (p_0,□p_1, …, p_n) ∈pt $D^{(s)}$ satisfying a_s but not b_s. Then $(p_n, 1)$ is in $\{(\mathbf{p}, s)\}\cdot a_s$ but not $\{(\mathbf{p}, s)\}\cdot b_s$, so $\{(\mathbf{p}, s)\}\cdot a_s \not\leq \{(\mathbf{p}, s)\}\cdot b_s$.]]]]

We conjecture that the module ℘(pt D_{RS}) is also faithful for Q_{hf}.

6.3 The Ready Simulation quantale Q_{RS}

The structure and completeness results for Q_{RS} are handled in a similar way to those for Q_{hf}, so we shall be less detailed. The main process-theoretic issues seem to be covered by hf; what RS does is to explain the generating observations in Q_{hf}, which include the whole system of opens for ΩD_{RS}, in terms of the much simpler generators using $\hookleftarrow$.

Definition 6.3.1 Let Act be a set, and let E be the alphabet $\{\alpha, \alpha^{\times}: \alpha \in \text{Act}\} \cup \{\hookleftarrow\}$. Certain elements of E* are defined to be *balanced,* inductively as follows:

- 1, the empty string, is balanced.
- $\alpha^{\times}$ is balanced for each α.
- if s and t are balanced, then so is s·t
- if s is balanced, then so is $\alpha\cdot s\cdot\hookleftarrow$ for each α.

Definition 6.3.2 Let Act be a set. The quantale Q = Q_{RS}(Act) is presented by

$$
\begin{array}{lll}
Q = \mathrm{Qu}\,\langle\, \alpha, \alpha^{\times}, \hookleftarrow (\alpha \in \mathrm{Act}) \mid & & \\
 & s\cdot s = s \le 1 & \text{if s is balanced} \\
 & s\cdot t = t\cdot s & \text{if s and t are balanced} \\
 & \alpha^{\times}\cdot\alpha = 0 & \\
 & \alpha\cdot s \le \alpha\cdot s\cdot\hookleftarrow\cdot\alpha & \text{if s is balanced (hence } \alpha\cdot s = \alpha\cdot s\cdot\hookleftarrow\cdot\alpha\cdot s) \\
 & 1 \le \alpha\cdot\hookleftarrow \vee \square\alpha^{\times} & \text{(hence equality here)} \quad \rangle
\end{array}
$$

No extra relations are needed for the testing preorder: Q' is just Q·⊤.

Proposition 6.3.3

$Q_{RS} \cong \mathrm{Qu}\,\langle\, Q_{hf}$ (qua quantale), $\hookleftarrow \mid \alpha\cdot\phi\cdot\hookleftarrow = \langle\alpha\rangle\phi$ $(\phi \in \Omega D_{RS})\rangle$

Proof

The relations of Q_{RS} imply that the joins of balanced products of generators form a subframe of Q_{RS}, and moreover it is an RS-frame with $\langle\alpha\rangle\phi = \alpha\cdot\phi\cdot\hookleftarrow$. There is therefore a unique RS-frame homomorphism to it from ΩD_{RS}. Together with the assignments $\alpha \mapsto \alpha$ and $\hookleftarrow \mapsto \hookleftarrow$, this defines quantale homorphisms first from Q_{hf} to Q_{RS}, and next from the RHS to Q_{RS}.

The inverse homomorphism is more obvious.]

Given a transition system Proc, we have already seen in Section 4 how to define the right Q_{RS}-module structure on M = $\wp$(Proc × Proc*).

We next show how, like Q_{hf}, Q_{RS} can be graded. This time the degrees are elements not of Act*, but of the monoid

$$G_{RS} = \mathrm{Mon}\,\langle\, \mathrm{Act}, \hookleftarrow \mid \alpha\cdot\hookleftarrow = 1\ (\alpha \in \mathrm{Act})\,\rangle$$

One can show that the elements of this can be reduced uniquely to the form $\checkmark^e \cdot s$ where $s \in Act^*$, and (writing $|s|$ for the length of s)

$$\checkmark^e \cdot s \cdot \checkmark^f \cdot t = \begin{cases} \checkmark^{e+f-|s|} \cdot t & \text{if } f \geq |s| \\ \checkmark^e \cdot s' \cdot t & \text{if } s = s' \cdot s'' \text{ where } |s''| = f \end{cases}$$

Proposition 6.3.4 $Q_{RS} \cong \prod_{e\geq 0, s\in Act^*} \Omega(D_{RS}{}^e \times D^{(s)})$ as sup-lattices.

If $g = \checkmark^e \cdot s \in G_{RS}$, then we shall write $D_{RS}{}^{(g)}$ for $D_{RS}{}^e \times D^{(s)}$.

Proof

From the coverage theorem for frames, we have

$$\begin{aligned} \Omega D_{RS}{}^{(g)} \cong \text{SupL} \langle\, & (\Omega D_{RS})^{e+n+1} \text{ (qua poset) } | \\ & (\phi_0, \ldots, \bigvee S, \ldots, \phi_n) \leq \bigvee \{(\phi_0, \ldots, \phi, \ldots, \phi_n) : \phi \in S\} \\ & (\phi_0, \ldots, \phi_i, \phi_{i+1}, \ldots, \phi_n) \leq (\phi_0, \ldots, \phi_i \wedge \langle \alpha_{i+1-e} \rangle \phi_{i+1}, \phi_{i+1}, \ldots, \phi_n) \\ & \qquad (e \leq i < e+n) \,\rangle \end{aligned}$$

This maps into Q_{RS} by

$$(\phi_0, \ldots, \phi_{e+n}) \mapsto \phi_0 \cdot \checkmark \cdot \ldots \cdot \checkmark \cdot \phi_e \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_{e+n}$$

For the morphism from left to right, again the RHS can be made a quantale. Given $g = \checkmark^e \cdot \alpha_1 \cdot \ldots \cdot \alpha_m$ and $h = \checkmark^f \cdot \beta_1 \cdot \ldots \cdot \beta_n$ in G_{RS}, we have an associative pairing $\mu_{g,h} : \Omega D_{RS}{}^{(g)} \otimes \Omega D_{RS}{}^{(h)} \to \Omega D_{RS}{}^{(g \cdot h)}$, defined by

$$\mu_{g,h}((\phi_0, \ldots, \phi_{e+m}), (\psi_0, \ldots, \psi_{f+n}))$$
$$= \begin{cases} (\phi_0, \ldots, \phi_{e+m} \wedge \psi_0, \ldots, \psi_{f+n}) & \text{if } f = 0 \text{ or } m = 0 \\ \mu_{g',h'}((\phi_0, \ldots, \phi_{e+m-1} \wedge \langle \alpha_m \rangle (\phi_{e+m} \wedge \psi_0)), (\psi_1, \ldots, \psi_{f+n})) & \\ \qquad \text{otherwise, where } g' = \checkmark^e \cdot \alpha_1 \cdot \ldots \cdot \alpha_{m-1} & \\ \qquad\qquad\qquad\qquad h' = \checkmark^{f-1} \cdot \beta_1 \cdot \ldots \cdot \beta_n & \end{cases}$$

and this extends by distributivity to a quantale multiplication on the whole biproduct. Now we can map from left to right by

$$\alpha^\times \mapsto (\alpha^\times) \in \Omega D_{RS}{}^{(1)}$$
$$\alpha \mapsto (\mathbf{true}, \mathbf{true}) \in \Omega D_{RS}{}^{(\alpha)}$$
$$\checkmark \mapsto (\mathbf{true}, \mathbf{true}) \in \Omega D_{RS}{}^{(\checkmark)}$$

It is not hard to show that these two maps are mutually inverse. ⟧

We call the elements of $\Omega D_{RS}{}^{(g)}$ *monomials,* of *degree* g.

Theorem 6.3.5 RS satisfies the second completeness criterion.

Proof

Suppose $a \not\leq b$ in Q_{RS}; without loss of generality we can assume they are monomials, of degree $g = \checkmark^e \cdot s$. Then there is a point of $D_{RS}{}^{(g)}$ satisfying a but not b; it is of the form $(p_0, \ldots, p_e, \ldots, p_{e+|s|})$ where $p_{e+i} \xrightarrow{a_{i+1}} p_{e+i+1}$ $(0 \leq i < |s|)$. Let Proc be the transition system constructed in Theorem 6.2.9 (second completeness for hf). The corresponding module for RS is $\wp$(Proc×Proc*), i.e. the set of sets of non-empty lists of processes from Proc. Let P be the list $(p_0, \ldots, p_{e-1}, (\mathbf{p}', s))$, where p_i $(0 \leq i < e)$ is the "bald" process p_i in Proc, and $\mathbf{p}'$ is the sequence $(p_e, \ldots, p_{e+|s|})$. Then for all $c \in \Omega D_{RS}^{(g)}$ we have $P \models c$ iff $(p_{e+|s|}) \in P{\cdot}c$, so $P{\cdot}a \not\leq P{\cdot}b$. ⟧

To deal with third completeness, we look more closely at the structure of Q'_{RS}. Just as with Q'_{hf}, it is the set of opens for a locale, this time that of non-empty finite and infinite lists of points of D_{RS}.

Definition 6.3.6 The locale $D_{RS}^{+\omega}$ is defined by

$$\Omega D_{RS}^{+\omega} = \mathrm{Fr}\left\langle \phi^{(k)}\ (\phi \in \Omega(\text{lift } D_{RS})\ (\text{qua frame});\ k \geq 0) \mid \begin{array}{l} (\text{lift } \mathbf{true})^{(0)} = \mathbf{true} \\ (\text{lift } \mathbf{true})^{(k+1)} \leq (\text{lift } \mathbf{true})^{(k)} \end{array} \right\rangle$$

(Recall that if D is a locale, then lift D, or $D_\perp$, has the points of D together with a new bottom $\perp$, and the opens of D together with a new **true**. Strictly, if ϕ is an open in D, then the corresponding open in lift D is written *lift* ϕ; but we shall normally only bother with this in the uniquely necessary case when ϕ = **true**. lift **true** is satisfied by all points except the new $\perp$.)

Proposition 6.3.7 $D_{RS}^{+\omega}$ is spectral, and its points are the sequences $(p_k)_{k\geq 0}$ where either $p_k = \perp$ or p_k is a point of D_{RS}, $p_0 \neq \perp$, and if $p_k = \perp$ then $p_{k+1} = \perp$. (Hence the points are the finite and infinite non-empty lists of points of D_{RS}.) ⟧

Proposition 6.3.8 $Q'_{RS} \cong \Omega D_{RS}^{+\omega}$ as sup-lattices.

Proof

We first give a coverage presentation for $\Omega D_{RS}^{+\omega}$, which will enable us to show that it is a left Q_{RS}-module.

Let S be the set

$$S = \{(\phi_k)_{k\geq 0} \in (\Omega\text{lift } D_{RS})^\omega: \begin{array}{l} \phi_0 \leq \text{lift } \mathbf{true} \\ \forall k.\ (\phi_{k+1} \leq \text{lift } \mathbf{true} \Rightarrow \phi_k \leq \text{lift } \mathbf{true}) \\ \exists k.\ \phi_k = \mathbf{true} \end{array} \}$$

This is a meet semilattice. It inherits its binary meets from $(\Omega\text{lift } D_{RS})^\omega$, and its top element is (lift **true**, **true**, …). We can now use the coverage theorem for frames to show that

$$\Omega D_{RS}^{+\omega} \cong \mathrm{SupL}\left\langle S\ (\text{qua poset}) \mid (\ldots, \bigvee X, \ldots) \leq \bigvee \{(\ldots, \phi, \ldots): \phi \in X\} \right\rangle$$

This presentation as a *sup-lattice* enables us to define a left Q_{RS}-action on $\Omega D_{RS}{}^{+\omega}$ as follows.

$\psi\cdot(\phi_0, \phi_1, \ldots) = (\psi\wedge\phi_0, \phi_1, \ldots)$ $\qquad (\psi \in \Omega D_{RS})$
$\alpha\cdot(\phi_0, \phi_1, \phi_2, \ldots) = (\langle\alpha\rangle\phi_0\wedge\phi_1, \phi_2, \ldots)$
$\lrcorner\cdot(\phi_0, \phi_1, \phi_2, \ldots) = (\text{lift } \mathbf{true}, \phi_0, \phi_1, \phi_2, \ldots)$

We can now define a left Q_{RS} module homomorphism t from Q_{RS} itself to $\Omega D_{RS}{}^{+\omega}$ by mapping 1 to **true**, and this obviously factors via Q'_{RS} as t = j;t' where j: $Q_{RS} \to Q'_{RS}$ is the quotient map. t' is an isomorphism. For its inverse f, consider an element (ϕ_i) of S, where $\phi_i \leq$ lift **true** if $i \leq k$, and ϕ_i = **true** otherwise. Then $f((\phi_i)) = j(\phi_0\cdot\lrcorner\cdot\ldots\cdot\lrcorner\cdot\phi_k)$. The sup-lattice presentation of $\Omega D_{RS}{}^{+\omega}$ enables us to see that this defines a sup-lattice morphism, and then it is readily checked that it is a left Q_{RS} module homomorphism and that it is inverse to t'. ⟧

Proposition 6.3.9 The elements of $\hat{Q}'_{RS}$ can be identified with the closed sets of points of $D_{RS}{}^{+\omega}$. ⟧

Theorem 6.3.10 RS satisfies the third completeness criterion.

Proof

Let M = $\wp$(Proc×Proc*) be the Q_{RS}-module derived from the master transition system Proc = pt D_{RS}. We can then identify Proc×Proc* with a subset of pt□$D_{RS}{}^{+\omega}$, and $\hat{Q}'_{RS}$ with the set of closed sets in $D_{RS}{}^{+\omega}$, and just as in Theorem 6.2.4 we show that the homomorphism Cap coincides with topological closure.

If F is a closed subset of pt $D_{RS}{}^{+\omega}$, then it is the topological closure of F∩(Proc×Proc*), i.e. the set of finite lists in F. Hence if a, b ∈ Q_{RS} with $j(a) \not\leq j(b)$ then, considering j(a)^ and j(b)^ as closed sets, there is a finite list x ∈ Proc×Proc* in j(b)^–j(a)^. {x}·b = Ø but {x}·a ≠ Ø. ⟧

Theorem 6.3.11 Let Proc be an image closed transition system, and p, q ∈ Proc. Then

$$\mathrm{Cap}_{RS}(p) \leq \mathrm{Cap}_{RS}(q) \Leftrightarrow p \subsetneqq_{RS} q$$

Proof ⟦p⟧ and ⟦q⟧ can be considered singleton sequences of points of D_{RS}, i.e. points of $D_{RS}{}^{+\omega}$ in the image of the continuous map (-): $D_{RS} \to D_{RS}{}^{+\omega}$; but this has a postinverse, Head: $D_{RS}{}^{+\omega} \to D_{RS}$, and it follows that

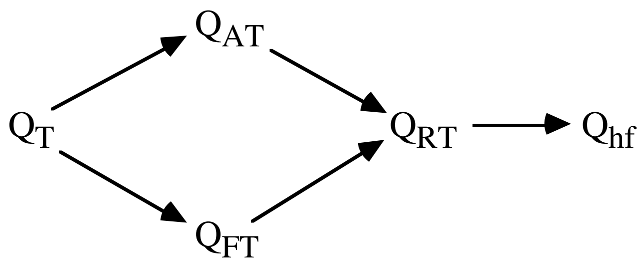
$$\mathrm{Cap}_{RS}(p) \leq \mathrm{Cap}_{RS}(q) \Leftrightarrow \mathrm{Cap}_{hf}(p) \leq \mathrm{Cap}_{hf}(q)$$

We can now use Theorem 6.2.5. ⟧

7. Some coarser semantics

We discuss next the following semantics: Ready Trace (RT), Failure Trace (FT), Acceptance Trace (AT), Trace (T) and Simulation (S). S is somewhat different from the others, in that it uses the "undo" mechanism ↵ and hence needs a return stack in the same way as RS did. Its treatment (Section 7.5) follows the same lines as RS, but more simply. The other four (RT, FT, AT and T) bear many resemblances to each other, so let us first sketch their treatment.

Second completeness is handled by comparison with hf (which was one reason for treating hf fully). We have a system of quantale homomorphisms:



and the main burden of the proof is to show that they are all 1-1. Given (Theorem 6.2.9) that hf has second completeness, we can then use the following simple lemma:

Lemma 7.0.1 Let Q be a quantale, and let *M* be a jointly faithful class of right Q-modules (so when Q is one of our quantales for process semantics and *M* is the class of modules arising from transition systems, this is second completeness).

Let R be a subquantale of Q. All Q-modules are then also R-modules; let M_R be the class of right R-modules corresponding to the Q-modules in *M.*

Then M_R is jointly faithful for R.

Proof

Suppose $a \not\leq b$ in R: then $a \not\leq b$ in Q, so there are some $M \in M$ and some $x \in M$ such that $x \cdot a \not\leq x \cdot b$. Then $M \in M_R$.]]

To prove that the quantale homomorphisms are 1-1, we show that the quantales are actually graded subquantales of Q_{hf}. Recall that the elements of Q_{hf} of degree s (a sequence of elements of Act) formed a coherent frame $\Omega D^{(s)}$. We use localic methods to show that the elements of Q (Q_{RT}, Q_{FT}, etc.) of degree s form a sub*frame* of $\Omega D^{(s)}$.

For *third completeness,* the proofs run by constructing transitions systems out of $\hat{Q}'$. What we should like to say is that $\hat{Q}'$ is a transition system with $x \xrightarrow{\alpha} y$ iff $y \leq x \cdot \alpha$, and then if $j(a) \not\leq j(b)$ (writing j for the natural homomorphism from Q to Q'), then x

$= j(b)^\wedge$ is a process that can do a but not b. This is nonsense as it stands; for instance, we always have $x \xrightarrow{\alpha} 0$, so every x can do every α. But by restricting the x's (omitting 0, for a start) we can construct a suitable *master transition system* whose processes are certain elements of $\hat{Q}'$.

Recall that for each semantics there is a homomorphism Cap: $\wp$Proc $\to \hat{Q}'$ for any transition system Proc, which gives a meaning to each process. However, not every element of $\hat{Q}'$ can arise as Cap({p}) for p a single process in Proc. For the "propositional observations" ϕ, such as refusals $\alpha^\times$ and acceptances $\alpha^\surd$, we must have that if $\{p\}\cdot\phi \neq \emptyset$ then $\{p\}\cdot\phi = \{p\}$, and this property must also hold for Cap({p}). For each semantics (the precise definition will vary according to the semantics), we therefore define an element $p \in \hat{Q}'$ to be *pointlike* iff

$$p \neq 0 \text{ and for every propositional observation } \phi.\ (p\cdot\phi \neq 0 \Rightarrow p\cdot\phi = p)$$

Lemma 7.0.2 Suppose $p \in \hat{Q}'_{RT}$ is pointlike and $\alpha \in$ Act. Then –

(i) If $p\cdot\alpha = 0$, then $p\cdot\alpha^\surd = 0$ and $p\cdot\alpha^\times = p$.
(ii) If $p\cdot\alpha \neq 0$, then $p\cdot\alpha^\surd = p$ and $p\cdot\alpha^\times = 0$.

Proof We give a proof that also works (as appropriate) in FT and AT.

Recall that ^ converts elements of Q' into the corresponding elements of $\hat{Q}' \cong (Q')^{op}$. If $x \in \hat{Q}'$ and $a \in Q$, then

$$x\cdot a = 0 \Leftrightarrow x(j(a)) = x(a\cdot j(1)) = (x\cdot a)(j(1)) = 0 \Leftrightarrow j(a) \leq \hat{x} \Leftrightarrow x \leq j(a)^\wedge$$

where we write $j: Q \to Q'$ for the natural homomorphism.

(i) $p\cdot\alpha = 0$ iff $j(\alpha) \leq \hat{p}$; but $\alpha^\surd \leq' \alpha$, i.e. $j(\alpha^\surd) \leq j(\alpha)$, so $p\cdot\alpha^\surd = 0$.
$1 \leq' \alpha \vee \alpha^\times$, so if $p\cdot\alpha^\times = 0$ we have $j(\alpha)$ and $j(\alpha^\times)$ both $\leq \hat{p}$, so $j(1) \leq \hat{p}$, i.e. $p = 0$ – contradiction.

(ii) $\alpha = \alpha^\surd\cdot\alpha$, so if $p\cdot\alpha^\surd = 0$ then $p\cdot\alpha = 0$.
$\alpha^\times\cdot\alpha = 0$, so if $p\cdot\alpha^\times = p$ then $p\cdot\alpha = 0$.]

The pointlike elements of $\hat{Q}'$ can be made into a transition system by

$$p \xrightarrow{\alpha} q \quad \text{iff } q \leq p\cdot\alpha$$

The crucial property of the pointlikes is the following, proved separately for the different semantics:

every element of $\hat{Q}'$ *is a join of pointlikes*

This implies that $\neg(p \xrightarrow{\alpha})$ iff $p\cdot\alpha = 0$; also, it allows us to prove third completeness once we have the following result, which we shall prove for RT, although the proofs are similar in the other semantics. It shows that a pointlike p, considered as an element of the master transition system, is its own capability.

Lemma 7.0.3 Suppose $p \in \hat{Q}'_{RT}$ is pointlike and $a \in Q_{RT}$. Then $p \cdot a = 0$ iff $\{p\} \cdot a = \emptyset$.
Proof Without loss of generality we can assume a is a product of generators, so it suffices to prove the result first when a = 1 (this is obvious), and then for a of the form $\alpha \cdot a'$, $\alpha^{\times} \cdot a'$ or $\alpha^{\surd} \cdot a'$ where the result is assumed for a'.

$$p \cdot \alpha \cdot a' = 0 \Leftrightarrow \forall \text{ pointlike } q \leq p \cdot \alpha.\ q \cdot a' = 0$$
$$\Leftrightarrow \forall q \in \{p\} \cdot \alpha.\ \{q\} \cdot a' = \emptyset \Leftrightarrow \{p\} \cdot \alpha \cdot a' = \emptyset$$

When $p \cdot \alpha = 0$, we have (using Lemma 7.0.2) both $p \cdot \alpha^{\surd} \cdot a' = 0$ and $\{p\} \cdot \alpha^{\surd} \cdot a' = \emptyset$, while

$$p \cdot \alpha^{\times} \cdot a' = 0 \Leftrightarrow p \cdot a' = 0 \Leftrightarrow \{p\} \cdot a' = \emptyset \Leftrightarrow \{p\} \cdot \alpha^{\times} \cdot a' = \emptyset$$

The case $p \cdot \alpha \neq 0$ is similar.]]

Theorem 7.0.4 *Third Completeness*
Proof
Suppose $a \not\leq' b$ in Q, i.e. $j(b)^\wedge \not\leq j(a)^\wedge$. Then there is a pointlike p such that $p \leq j(b)^\wedge$, i.e. $p \cdot b = 0$, but $p \not\leq j(a)^\wedge$, i.e. $p \cdot a \neq 0$. In terms of the master transition system, this can be restated as $\{p\} \cdot b = \emptyset$, but $\{p\} \cdot a \neq \emptyset$.]]

Let us also prove the following:

Lemma 7.0.5 If Proc is the master transition system, then Cap: $\wp\,\text{Proc} \to \hat{Q}'$ is the join map, $\text{Cap}(S) = \bigvee S$.
Proof Since Cap preserves joins, it suffices to prove that for a singleton {p} we have Cap({p}) = p. This follows because, using Lemma 7.0.3, if $a \in Q$ then

$$\text{Cap}(\{p\})(a) = 0 \Leftrightarrow \{p\} \cdot a = \emptyset \Leftrightarrow p \cdot a = 0 \quad]]$$

Using this lemma, we can now characterize the pointlike elements as precisely those than can arise as Cap({p}) for p a process in some transition system.

This definition of the “pointlike elements” of $\hat{Q}'$ is quite general once one is given the quantale Q, the left module homomorphism j: $Q \to Q'$, and a subframe A of Q (comprising the propositional observations in our cases; for S and RS, A is isomorphic to Q'): $x \in \hat{Q}'$ is pointlike iff it is non-zero and for every $\phi \in A$, $x \cdot \phi$ is either x or 0. This generalizes the definition of points of a locale, for suppose that Q, Q' and A are all the same, namely a frame A. $\hat{a} \cdot b = (b \to a)^\wedge$, and one can then show that $\hat{a} \in \hat{A}$ is pointlike (i.e. non-zero and $\hat{a} \cdot b$ is always either 0 or $\hat{a}$) iff a is a prime element of A. Our central lemma, that every element of $\hat{Q}'$ is a join of pointlikes, becomes here just a statement of spatiality of the locale of A. (Then A is the frame of opens for a sober topological space, and $\hat{A}$ is the family of closed sets. The pointlike elements of $\hat{A}$ correspond to the irreducible closed sets, which – by sobriety – are in bijection with the points.)

We can relate some of these techniques to existing results on the failures semantics F. Brookes *et al.* (1984) describe a complete semilattice of failure sets, i.e. subsets of Act* × $\wp_{\mathrm{fin}}$Act satisfying certain conditions. We shall see in Proposition 9.1.10 that this complete semilattice is isomorphic to what we might as well for the moment call $\hat{Q}'_F$ (though for technical reasons it appears in Section 9.1 as $(\hat{Q}'_F)_*$). Then Brookes *et al.* explicitly identify our ∨ (their ⊓) with non-deterministic choice, and our ≥ (their $\xrightarrow{\langle\rangle}$) with the resolution of non-deterministic choice by internal, invisible changes. In these terms, a pointlike element represents a process that is *initially* deterministic, in other words it has no internal choices than can be resolved before an action takes place. The actions are still non-deterministic, so a transition p $\xrightarrow{\alpha}$ q between pointlikes (q ≤□pα) represents an action followed by a resolution of internal choice.

This view is also reflected in DeNicola and Hennessy (1987).

Brookes *et al.* consider a stronger notion of determinism, that the process will never display any internal choice (though it may display non-determinism governed by external choice). This is formalized in the following way. The trace semantics T cannot distinguish between (in CCS notation) α·p + α·q and α·(p+q), and so does not require us to consider actions to be non-deterministic. The master transition system for the trace semantics comprises *all* non-zero elements of $\hat{Q}'_T$), and so gives a deterministic transition system by p $\xrightarrow{\alpha}$ q iff q = p·α. The deterministic elements of $\hat{Q}'_F$ are defined to be those of the form $\mathrm{Cap}_F(\{p\})$ where $0 \neq p \in \hat{Q}'_T$.

For the failure semantics, our result (Theorem 9.1.8) that every element of $\hat{Q}'_F$ is a join of pointlikes can be strengthened: every element is a join of deterministic elements in the strong sense (Blamey 1991). This corresponds to the Deistic idea, expressed in Hoare (1985) as well as in Blamey (1991), that non-determinism can (often, at least) be viewed as an implementor’s choice between deterministic processes. Since the formalization of determinism makes sense also for the other semantics we deal with, it would be interesting to know whether it is still true for them that every element of $\hat{Q}'$ is a join of deterministic elements.

7.1 Ready trace semantics RT

Definition 7.1.1 Let Act be a set. We present the quantale

$$Q = Q_{RT}(\mathrm{Act}) = \mathrm{Qu}\,\langle\, \alpha, \alpha^{\times}, \alpha^{\surd}\ (\alpha \in \mathrm{Act}) \mid$$
$$\alpha^{\times}\cdot\alpha^{\times} = \alpha^{\times} \leq 1$$

$$\alpha^{\surd}\cdot\alpha^{\surd} = \alpha^{\surd} \leq 1$$
$$\alpha^{\times}\cdot\beta^{\times} = \beta^{\times}\cdot\alpha^{\times}$$
$$\alpha^{\surd}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\surd}$$
$$\alpha^{\times}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\times}$$
$$\alpha^{\times}\cdot\alpha^{\surd} = 0$$
$$\alpha \leq \alpha^{\surd}\cdot\alpha$$
$$1 \leq \alpha^{\surd} \vee \Box\alpha^{\times} \qquad \rangle$$

The testing preorder is presented by

$$\alpha^{\surd} \leq' \alpha \qquad\qquad (\alpha \in \mathrm{Act})$$

In other words,

$$Q' = Q'_{RT}(\mathrm{Act}) = \text{Q-Mod}\ \langle\ Q\ (\text{qua Q-Mod})\ |$$
$$\top \leq 1$$
$$\alpha^{\surd} \leq \alpha \qquad (\alpha \in \mathrm{Act}) \quad \rangle$$

If Act is finite, then an entire menu X can be observed as a finite product of acceptances and refusals, $X^{\surd}\cdot(\mathrm{Act}-X)^{\times}$.

Proposition 7.1.2 The above system is presented in terms of coverages by –

- $S = S_{RT}$ is the monoid generated by symbols α, $\alpha^{\surd}$ and $\alpha^{\times}$ ($\alpha \in \mathrm{Act}$), subject to relations making the $\alpha^{\surd}$s and $\beta^{\times}$s commuting idempotents. Its elements can be written uniquely as $\phi_0\cdot\alpha_1\cdot\phi_1\cdot \ldots\cdot\alpha_n\cdot\phi_n$ where each ϕ_i is of the form $X_i^{\surd}\cdot Y_i^{\times}$ for some X_i, $Y_i \subseteq_{fin}$ Act.
- The coverage $C = C_{RT}$ comprises the cover relations (with $s, t \in S_{RT}$)

$$\{s\cdot t\} \dashv s\cdot\alpha^{\surd}\cdot t$$
$$\{s\cdot t\} \dashv s\cdot\alpha^{\times}\cdot t$$
$$\{s\cdot\alpha^{\surd}\cdot\alpha\cdot t\} \dashv s\cdot\alpha\cdot t$$
$$\emptyset \dashv s\cdot\alpha^{\surd}\cdot\alpha^{\times}\cdot t$$
$$\{s\cdot\beta^{\surd}\cdot t,\ s\cdot\beta^{\times}\cdot t\} \dashv s\cdot t$$

Proof
Q is mapped to C-Idl(S) in the obvious way. For the inverse isomorphism, the obvious monoid homomorphism from S_{RT} to Q clearly transforms covers to joins. These homomorphisms are mutually inverse. ⟧

Proposition 7.1.3 The obvious homomorphism from Q_{RT} to Q_{hf} is 1-1.
Proof
We show that Q_{RT} has a graded structure, each component of which is contained in a component of Q_{hf}.

First, define the Stone locale D_{RT} by

$$\Omega D_{RT} = \mathrm{Fr}\,\langle\, \alpha^{\times}, \alpha^{\surd}\ (\alpha \in \mathrm{Act}) \mid$$
$$\alpha^{\surd} \wedge \alpha^{\times} = \textbf{false}$$
$$\alpha^{\surd} \vee \alpha^{\times} = \textbf{true} \quad \rangle$$

Its points are the subsets Z of Act, with $Z \vDash \alpha^{\times}$ iff $\alpha \notin Z$, $Z \vDash \alpha^{\surd}$ iff $\alpha \in Z$.

If $s = \alpha_1 \cdot \ldots \cdot \alpha_n \in \mathrm{Act}^*$, then the elements of S_{RT} of the form $\phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n$ constitute a $\wedge$-semilattice, with

$$\phi_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n \le \phi_0' \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \phi_n' \qquad \text{iff each } X_i \supseteq X_i',\ Y_i \supseteq Y_i'$$
$$(\phi_i = X_i^{\surd} \cdot Y_i^{\times},\ \phi_i' = X_i'^{\surd} \cdot Y_i'^{\times})$$

C restricts to a coverage in Johnstone's sense on this semilattice, presenting a frame $\Omega D_{RT}^{(s)}$. The corresponding locale $D_{RT}^{(s)}$ is spectral, a sublocale of $(D_{RT})^{n+1}$, and its points are the sequences (Z_i) $(0 \le i \le n;\ Z_i \subseteq \mathrm{Act})$ such that $\alpha_{i+1} \in Z_i$ $(0 \le i < n)$.

Because the cover relations in C do not mix elements from different components (different values of s), it follows that $Q_{RT} \cong \prod_{s \in \mathrm{Act}^*} \Omega D_{RT}^{(s)}$ as sup-lattices.

The homomorphism from Q_{RT} to Q_{hf} restricts to a frame homomorphism from $\Omega D_{RT}^{(s)}$ to $\Omega D_{hf}^{(s)}$, corresponding to the continuous map from $D_{hf}^{(s)}$ to $D_{RT}^{(s)}$ that takes $(p_0, \ldots, p_n)$ to $(R(p_0), \ldots, R(p_n))$. By spatiality, to show that this is 1-1 on opens we show that it is surjective on points.

Let $(Z_i)_0^n$ be a point of $D_{RT}^{(s)}$, and define a transition system

$$\mathrm{Proc} = \{p_i : 0 \le i \le n\} \cup \{q\}$$
$$p_i \xrightarrow{\alpha_{i+1}} p_{i+1} \qquad (0 \le i < n)$$
$$p_i \stackrel{\beta}{\not\rightarrow} q \qquad (\beta \in Z_i)$$

Then the sequence $([\![p_i]\!])_0^n$ is a point of $D_{hf}^{(s)}$ mapping to $(Z_i)_0^n$.]]

This proves second completeness; we now turn to third completeness.

Definition 7.1.4 An element $p \in \hat{Q}'_{RT}$ is *pointlike* iff $p \ne 0$ and for every $\phi \in \Omega D_{RT}$, $p \cdot \phi$ is either 0 or p.

Note that for any x and α,

$$x \cdot \alpha^{\times} = 0 \Rightarrow x = x \cdot (\alpha^{\surd} \vee \alpha^{\times}) = x \cdot \alpha^{\surd}$$
$$x \cdot \alpha^{\times} = x \Rightarrow 0 = x \cdot \alpha^{\times} \cdot \alpha^{\surd} = x \cdot \alpha^{\surd}$$

Hence in the definition it suffices to consider ϕ of the form $\alpha^{\times}$, or, alternatively, ϕ of the form $\alpha^{\surd}$.

Lemma 7.1.5 Let $x \in \hat{Q}'_{RT}$, and let $U \subseteq \mathrm{Act}$. Define –

$$p \quad = x \wedge \textstyle\bigwedge_{\alpha \in U} j(\alpha^{\times})^{\wedge} \wedge \bigwedge_{\alpha \notin U} j(\alpha^{\surd})^{\wedge}$$

(i) Provided p is non-zero, then it is pointlike and $U = \{\alpha: p\cdot\alpha^{\surd} = p\}$.

(ii) Let $q \le x$ be a pointlike such that $U = \{\alpha: q\cdot\alpha^{\surd} = q\}$. Then $q \le p$.

Proof

(i) If $\alpha \in U$, then $p\cdot\alpha^{\times} \le j(\alpha^{\times})^{\wedge}\cdot\alpha^{\times} = 0$, so $p = p\cdot(\alpha^{\surd} \vee \alpha^{\times}) = p\cdot\alpha^{\surd}$.

If $\alpha \notin U$, then $p\cdot\alpha^{\surd} \le j(\alpha^{\surd})^{\wedge}\cdot\alpha^{\surd} = 0$.

(ii) $q \le x$. If $\alpha \in U$ then $q\cdot\beta^{\times} = 0$, so $q \le j(\beta^{\times})^{\wedge}$. Similarly, if $\alpha \notin U$, then $q \le j(\alpha^{\surd})^{\wedge}$. Hence $q \le p$. ⟧

Lemma 7.1.6 Let $u, v \in Q'_{RT}$ and $\alpha \in Act$.

If both $v \le u \vee j(\alpha^{\surd})$ and $v \le u \vee j(\alpha^{\times})$, then $v \le u$.

Proof

$v = (\alpha^{\surd} \vee \alpha^{\times})\cdot v \le \alpha^{\surd}\cdot(u \vee j(\alpha^{\times})) \vee \alpha^{\times}\cdot(u \vee j(\alpha^{\surd})) = (\alpha^{\surd} \vee \alpha^{\times})\cdot u = u$ ⟧

Theorem 7.1.7 Every element of $\hat{Q}'_{RT}$ is a join of pointlikes.

Proof Let Proc be the set of pointlike elements of $\hat{Q}'_{RT}$, and let $x \in \hat{Q}'_{RT}$. We must show that $x = \bigvee\{p \in Proc: p \le x\}$, i.e. $\hat{x} \ge \bigwedge\{\hat{p}: p \in Proc, \hat{x} \le \hat{p}\}$. In other words, if $v \in Q'_{RT}$ and $v \not\le \hat{x}$, then there is a pointlike $p \le x$ such that $v \not\le \hat{p}$. Using Lemma 7.1.5, it suffices to find $U \subseteq Act$ such that

$$v \not\le \hat{x} \vee \bigvee_{\alpha\notin U} j(\alpha^{\surd}) \vee \bigvee_{\alpha\in U} j(\alpha^{\times})$$

By coherence of Q and Q', we can assume without loss of generality that v is compact.

Now call a pair (U, V) of subsets of Act *good* iff

$$v \not\le \hat{x} \vee \bigvee_{\alpha\in V} j(\alpha^{\surd}) \vee \bigvee_{\alpha\in U} j(\alpha^{\times})$$

$(\emptyset, \emptyset)$ is good, and if (U, V) is good then U and V are disjoint – for $j(\alpha^{\surd}) \vee j(\alpha^{\times}) = j(1)$. Also, using compactness of v, if we have a chain of good pairs then the (componentwise) union is also good. We can therefore apply Zorn's lemma to find a maximal good pair (U, V). Write $u = \hat{x} \vee \bigvee_{\alpha\in V} j(\alpha^{\surd}) \vee \bigvee_{\alpha\in U} j(\alpha^{\times})$. If γ is in Act – $(U\cup V)$, then by maximality we have $v \le u \vee j(\alpha^{\surd})$ and $v \le u \vee j(\alpha^{\times})$. Hence by Lemma 7.1.6 $v \le u$, a contradiction. Therefore $Act = U\cup V$ and we are done. ⟧

Now using the discussion at the start of Section 7, we have –

Theorem 7.1.8 The second and third completeness criteria hold for RT. ⟧

Theorem 7.1.9 Let Proc be a transition system over a *finite* set Act, and let p, q be elements of Proc. Then

$$Cap_{RT}(p) \le Cap_{RT}(q) \Leftrightarrow p \sqsubseteq_{RT} q$$

Proof ⇐ (the argument here works even when Act is infinite): If each ϕ_i is $U_i^{\surd}\cdot V_i^{\times}$, then $Cap_{RT}(p)(\phi_0\cdot\alpha_1\cdot\phi_1\cdot\ldots\cdot\alpha_n\cdot\phi_n) \neq 0$ iff there is some $(X_0, \alpha_1, \ldots, X_n)$ in ready-traces(p) such that $U_i \subseteq X_i$ and $V_i \subseteq X_i^c$, so $Cap_{RT}(p)$ is determined by ready-traces(p).

⇒: $(X_0, \alpha_1, \ldots, X_n) \in$ ready-traces(p) iff $Cap_{RT}(p)(\phi_0\cdot\alpha_1\cdot\phi_1\cdot\ldots\cdot\alpha_n\cdot\phi_n) \neq 0$ where $\phi_i = X_i^{\surd}\cdot(Act–X_i)^{\times}$. ⟧

Example 7.1.10 Some restriction is necessary in 7.1.9, for consider –

$$\begin{array}{ll} \text{Act} & = \{\alpha\} \cup \{\beta_i: i \in \omega\} \\ \text{Proc} & = \{p\} \cup \{p'\} \cup \{q_S: S \subseteq \omega\} \cup \{r\} \\ p \xrightarrow{\alpha} q_S & \text{if } S \text{ is finite} \\ p' \xrightarrow{\alpha} q_S & \text{for all } S \\ q_S \xrightarrow{\beta_i} r & \text{iff } i \in S \end{array}$$

Then Cap(p) = Cap(p'), ready-traces(p) ≠ ready-traces(p').

7.2 Failure trace semantics FT

Definition 7.2.1 If Act is a set, then we define the quantale

$$\begin{aligned} Q = Q_{FT}(Act) = Qu \langle \alpha, \alpha^{\times}\ (\alpha \in Act) \mid \\ \alpha^{\times} \leq 1 \\ \alpha^{\times} \leq \alpha^{\times}\cdot\alpha^{\times} \\ \alpha^{\times}\cdot\beta^{\times} = \beta^{\times}\cdot\alpha^{\times} \\ \alpha^{\times}\cdot\alpha = 0 \quad \rangle \end{aligned}$$

The testing preorder is presented by

$$s \leq' \alpha \vee \alpha^{\times}\cdot s \qquad \text{(s a product of actions and refusals)}$$

There is an obvious homomorphism from Q_{FT} to Q_{RT}, and in Q'_{RT} we have

$$s = 1\cdot s \leq \alpha^{\surd}\cdot s \vee \alpha^{\times}\cdot s \leq \alpha^{\surd} \vee \alpha^{\times}\cdot s \leq \alpha \vee \alpha^{\times}\cdot s$$

so that the composite homomorphism from Q_{FT} to Q'_{RT} factors via Q'_{FT}.

Proposition 7.2.2 The above system is presented in terms of coverages by –

- The monoid $S = S_{FT}$ is generated by symbols α and $\alpha^{\times}$ ($\alpha \in Act$) subject to relations requiring the $\alpha^{\times}$'s to be commuting idempotents. Every element can be expressed uniquely as $X_0^{\times}\cdot\alpha_1\cdot X_1^{\times}\cdot \ldots\cdot\alpha_n\cdot X_n^{\times}$ with $X_i \subseteq_{fin} Act$.
- The coverage $C = C_{FT}$ comprises the cover relations

$$\{s\cdot t\} \dashv s\cdot\alpha^{\times}\cdot t$$

$\emptyset \dashv s\cdot\alpha^\times\cdot\alpha\cdot t$]]

Theorem 7.2.3 FT satisfies the second completeness criterion.
Proof We use the coverage presentation to describe the graded structure of Q_{FT} and hence show that the obvious homomorphism from Q_{FT} to Q_{RT} is 1-1.

First, a spectral locale D_{FT} is defined by

$\Omega D_{FT} = \mathrm{Fr}\langle \alpha^\times\ (\alpha \in \mathrm{Act})\rangle$

Its points are the subsets Z of Act, the same as for D_{RT}, with $Z \vDash \alpha^\times$ iff $\alpha \notin Z$. In fact D_{RT} is patch D_{FT}.

Next, iff $s = \alpha_1\cdot\ldots\cdot\alpha_n \in \mathrm{Act}^*$, then the elements of S of the form $X_0^\times\cdot\alpha_1\cdot X_1^\times\cdot\ldots\cdot\alpha_n\cdot X_n^\times$ constitute a $\wedge$-semilattice $S^{(s)}$; and the joins of such elements in Q_{FT} constitute a frame $\Omega D_{FT}^{(s)}$. The corresponding locale $D_{FT}^{(s)}$ is spectral, and its points are the same as those of $D_{RT}^{(s)}$. Hence $\Omega D_{FT}^{(s)}$ can be considered a subframe of $\Omega D_{RT}^{(s)}$.

Finally, $Q_{FT} \cong \prod_{s\in \mathrm{Act}^*} \Omega D_{FT}^{(s)}$ as sup-lattices, so Q_{FT} is a subquantale of Q_{RT}.
]]

Definition 7.2.4 An element p of $\hat{Q}'_{FT}$ is *pointlike* iff $p \neq 0$ and for all $\alpha \in \mathrm{Act}$, $p\cdot\alpha^\times$ is either 0 or p.

We shall prove, as usual (Theorem 7.2.8), that every element of $\hat{Q}'_{FT}$ is a join of pointlikes; but first let us prove some analogues of Lemmas 7.1.5 and 7.1.6.

Lemma 7.2.5 Let $x \in \hat{Q}'_{FT}$, and let $U \subseteq \mathrm{Act}$. Define –

$p \quad = x \wedge \bigwedge_{\alpha\in U} j(\alpha^\times)^\wedge \wedge \bigwedge_{\alpha\notin U} j(\alpha)^\wedge$

(i) Provided p is non-zero, then it is pointlike and $U = \{\alpha: p\cdot\alpha^\times = 0\}$.

(ii) Let $q \leq x$ be a pointlike such that $U = \{\alpha: q\cdot\alpha^\times = 0\}$. Then $q \leq p$.

Proof
(i) If $\alpha \in U$, then $p\cdot\alpha^\times = 0$. Now suppose $\alpha \notin U$, so $j(\alpha) \leq \hat{p}$: we must show $p \leq p\cdot\alpha^\times$, i.e. for all products of generators $s \in Q_{FT}$, if $j(s) \leq (p\cdot\alpha^\times)^\wedge$, i.e. $p\cdot\alpha^\times\cdot s = 0$, i.e. $j(\alpha^\times\cdot s) \leq \hat{p}$, then $j(s) \leq \hat{p}$, i.e. $p\cdot s = 0$. This is clear, because $j(s) \leq j(\alpha \vee \alpha^\times\cdot s)$.
(ii) As in Lemma 7.1.5, with – if $\alpha \notin U$, then $q\cdot\alpha = q\cdot\alpha^\times\cdot\alpha = 0$, so $q \leq j(\alpha)^\wedge$.]]

For an analogue of 7.1.6, we must do more work. In fact, we show how Q'_{FT} may be constructed concretely using Theorem 5.2.2.

Lemma 7.2.6 Let S_{FT} be as described in Proposition 7.2.2, and let C' be the set of cover relations (with $s, t \in S_{FT}$)

$\{s\} \dashv s\cdot t$

$\{s\cdot t\} \dashv s\cdot\alpha^{\times}\cdot t$
$\varnothing \dashv s\cdot\alpha^{\times}\cdot\alpha\cdot t$
$\{s\cdot\alpha, s\cdot\alpha^{\times}\cdot t\} \dashv s\cdot t$

Then C'-Idl(S_{FT}) is isomorphic to Q'_{FT}, with s (as generator in C'-Idl(S_{FT})) corresponding to j(s).

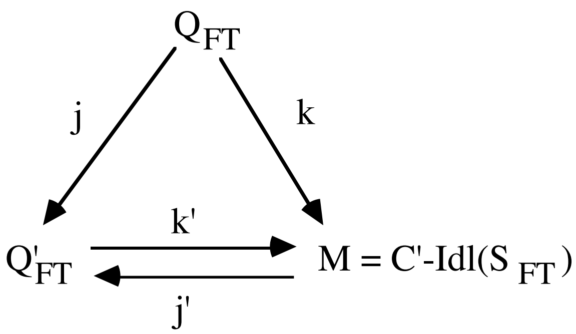
Proof

Let us write M for C'-Idl(S_{FT}), which we may think of (by Theorem 5.2.2) as the sup-lattice generated by S_{FT} subject to relations $s\cdot t \leq s$, etc. Let us write k(s) for $s \in S_{FT}$ considered as a generator of M; concretely, $k(s) = C'\text{-}\langle\{s\}\rangle$.

Each s defines a sup-lattice homomorphism of M by $k(t) \mapsto k(s\cdot t)$. Considering the particular cases of $s = \alpha$ and $s = \alpha^{\times}$, we see that this makes M a left Q_{FT}-module.

Consider the module homomorphism from Q_{FT} to M defined by mapping 1 to $k(1) = S_{FT}$, i.e. $a \mapsto a\cdot k(1)$. This maps s to $s\cdot k(1) = k(s)$, so we shall write $k: Q_{FT} \to M$ for this homomorphism. k respects the defining relations for Q'_{FT}, so factors as $k = j;k'$. k' maps j(s) to k(s).

Now define the sup-lattice homomorphism $j': M \to Q'_{FT}$ by $k(s) \mapsto j(s)$. This preserves the actions of α and $\alpha^{\times}$, and so is a Q_{FT}-module homomorphism.



$j;k';j' = j$ (they both map 1 to j(1)), so $k';j' = id$. Also, $k'(j'(k(s))) = k(s)$ for every $s \in S_{FT}$, so $j';k' = id$. Hence $M \cong Q'_{FT}$ with $k(s) \leftrightarrow j(s)$. ⟧

Lemma 7.2.7 Let $v, u \in Q'_{FT}$, and $\alpha \in$ Act.

If both $v \leq u \vee j(\alpha)$ and $v \leq u \vee j(\alpha^{\times})$, then $v \leq u$.

Proof

It suffices to take v of the form j(s), $s \in S_{FT}$. It follows that in C'-Idl(S_{FT}) (as defined in Lemma 7.2.6), $s \in C'\text{-}\langle I \cup \{\alpha^{\times}\}\rangle$ where I is $\{s \in S_{FT}: j(s) \leq u\}$, the C'-ideal corresponding to u. We show that for any C'-ideal I,

$$C'\text{-}\langle I \cup \{\alpha^{\times}\}\rangle = I \cup \{X^{\times}\cdot r: X^{\times}\cdot\alpha \in I, r \in S_{FT}\} \qquad (*)$$

Let us write J for the RHS of this equation.

$\supseteq$: Suppose $X^{\times}\cdot\alpha \in I$. Then $X^{\times}\cdot\alpha^{\times}\cdot r \in C'\text{-}\langle\{\alpha^{\times}\}\rangle$, so $X^{\times}\cdot r \in C'\text{-}\langle I \cup \{\alpha^{\times}\}\rangle$.

$\subseteq$: $\alpha^\times \cdot \alpha \in I$, so $\alpha^\times \in J$. It remains to show that J is a C'-ideal. The only part of difficulty is to show that if $s \cdot \beta$ and $s \cdot \beta^\times \cdot t$ are both in J, then so is $s \cdot t$.

If $s \cdot \beta = X^\times \cdot r$, where $X^\times \cdot \alpha \in I$, then s begins with $X^\times$ so $s \cdot t$ does too.

Otherwise, $s \cdot \beta \in I$. If $s \cdot \beta^\times \cdot t \in I$, then $s \cdot t \in I$ because I is a C'-ideal. Otherwise, $s \cdot \beta^\times \cdot t = X^\times \cdot r$, where $X^\times \cdot \alpha \in I$. Again, if s starts with $X^\times$ then we are done. The only other possibility is that $s = Y^\times \cdot Y'^\times$, $t = Z^\times \cdot t'$ and $X = Y \cup \{\beta\} \cup Z$. $s \cdot \beta$ and $s \cdot \beta^\times \cdot Z^\times \cdot \alpha$ are both in I, so $s \cdot Z^\times \cdot \alpha$ is too, and it follows that $s \cdot t = s \cdot Z^\times \cdot t' \in J$.

This completes the proof of the claim (*). Let us return to the main result. Because $j(s) \le u \vee j(\alpha^\times)$, (*) tells us that either $j(s) \le u$ or $s = X^\times \cdot s'$ where $j(X^\times \cdot \alpha) \le u$; and in the latter case, we have $j(s) = j(X^\times \cdot s) \le X^\times \cdot u \vee X^\times \cdot j(\alpha) \le u$. ⟧

Theorem 7.2.8 Every element x of $\hat{Q}'_{FT}$ is a join of pointlikes.

Proof

The proof is very similar to that of Theorem 7.1.7. After Lemma 7.2.5, it suffices to show that if $v \in Q'_{FT}$ is compact and $v \not\le \hat{x}$, then there is some $U \subseteq$ Act such that $v \not\le \hat{x} \vee \bigvee_{\alpha \in U} j(\alpha^\times) \vee \bigvee_{\alpha \notin U} j(\alpha)$. Just as in 7.1.7, we can find a pair (U, V) of subsets of Act, maximal with respect to

$$v \not\le \hat{x} \vee \bigvee_{\alpha \in U} j(\alpha^\times) \vee \bigvee_{\alpha \in V} j(\alpha),$$

and then use Lemma 7.2.7 to show that Act is the disjoint union of U and V. ⟧

Theorem 7.2.9 The second and third completeness criteria hold for FT. ⟧

Note – Lemma 7.2.6 can be put in a more general setting – as indeed we did in a previous draft of this paper (Abramsky and Vickers 1990). What makes C'-Idl(S_{FT}) a left module over Q_{FT} is really a one-sided coverage condition:

if $X \dashv u$ in C', then $\{s \cdot x: x \in X\} \dashv s \cdot u$ for all $s \in S_{FT}$

The same method can be used for other semantics that we treat, giving more concrete representations of Q'.

One interesting byproduct is a more concrete representation of $\hat{Q}'$, and hence of "abstract processes". An element of Q'_{FT} can be represented as a C'-ideal of S_{FT}. An element of $\hat{Q}'_{FT}$ is the same, but to make the representation reflect the reversed ordering, let us represent it by a C'-*co*ideal, i.e. the complement in S_{FT} of a C'-ideal. Elements of $\hat{Q}'_{FT}$ are then represented by sets F of sequences

$$s = X_0^\times \cdot \alpha_1 \cdot X_1^\times \cdot \ldots \cdot \alpha_n \cdot X_n^\times$$

such that if $s \in F$ then –

- if $k \le n$ and $Y_i \subseteq X_i$ $(0 \le i \le k)$, then $Y_0^\times \cdot \alpha_1 \cdot Y_1^\times \cdot \ldots \cdot \alpha_k \cdot Y_k^\times \in F$

- $\alpha_{i+1} \notin X_i$
- for every $k \le n$ and every $\beta \in \mathrm{Act}$, F contains at least one of

$$X_0^{\times}\cdot\alpha_1\cdot X_1^{\times}\cdot\ldots\cdot\alpha_k\cdot X_k^{\times}\cdot\beta$$
$$X_0^{\times}\cdot\alpha_1\cdot X_1^{\times}\cdot\ldots\cdot\alpha_k\cdot X_k^{\times}\cdot\beta^{\times}\cdot\alpha_{k+1}\cdot\ldots\cdot\alpha_n\cdot X_n^{\times}$$

This is exactly the sort of treatment that is traditionally given for the failures semantics F (see Section 9.1). Since C'-ideals are closed under arbitrary intersections, join in $\hat{Q}'_{FT}$ is represented by *union* of C'-coideals. The right action of Q_{FT} on Q'_{FT} is represented by

$$F\cdot s = \{t \in S_{FT}: s\cdot t \in F\}$$

If Proc is a transition system and $X \subseteq \mathrm{Proc}$, then $\mathrm{Cap}_{FT}(X)$ is represented by the C'-coideal $\{s \in S_{FT}: X\cdot s \neq \emptyset\}$; in particular, $\mathrm{Cap}_{FT}(\{p\})$ = failure-traces(p) for $p \in \mathrm{Proc}$. We deduce –

Theorem 7.2.10 Let Proc be a transition system and $p, q \in \mathrm{Proc}$. Then

$$\mathrm{Cap}_{FT}(p) \le \mathrm{Cap}_{FT}(q) \Leftrightarrow p \precsim_{FT} q \qquad]\!]$$

7.3 Acceptance trace semantics AT

Definition 7.3.1 If Act is a set, then we define the quantale

$$Q = Q_{AT}(\mathrm{Act}) = \mathrm{Qu}\langle\, \alpha, \alpha^{\surd}: \alpha \in \mathrm{Act} \mid$$
$$\alpha^{\surd} \le 1$$
$$\alpha^{\surd} \le \alpha^{\surd}\cdot\alpha^{\surd}$$
$$\alpha^{\surd}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\surd}$$
$$\alpha \le \alpha^{\surd}\cdot\alpha \quad \rangle$$

The testing preorder is presented by

$$\alpha^{\surd} \le' \alpha$$

Theorem 7.3.2 The above system is presented in terms of coverages by –

- The monoid $S = S_{AT}$ is generated by symbols α and $\alpha^{\surd}$ ($\alpha \in \mathrm{Act}$), subject to relations requiring the $\alpha^{\surd}$'s to be commuting idempotents. Every element can be expressed uniquely as $X_0^{\surd}\cdot\alpha_1\cdot X_1^{\surd}\cdot\ldots\cdot\alpha_n\cdot X_n^{\surd}$ with $X_i \subseteq_{fin} \mathrm{Act}$.
- The coverage $C = C_{AT}$ comprises the cover relations (with $s, t \in S_{AT}$)

$$\{s\cdot t\} \dashv s\cdot\alpha^{\surd}\cdot t$$
$$\{s\cdot\alpha^{\surd}\cdot\alpha\cdot t\} \dashv s\cdot\alpha\cdot t \qquad]\!]$$

Theorem 7.3.3 AT satisfies the second completeness criterion.
Proof Just like Theorem 7.2.3 for FT.

First, a spectral locale D_{AT} is defined by

$\Omega D_{AT} = Fr \langle \alpha^{\surd} (\alpha \in Act) \rangle$

Again the points are the subsets Z of Act; this time $Z \vDash \alpha^{\surd}$ iff $\alpha \in Z$. D_{AT} is the spectral dual of D_{FT}. ($K\Omega D_{AT}$ and $K\Omega D_{FT}$ are opposite lattices – actually, here they're isomorphic as well as anti-isomorphic.)

For $s = \alpha_1 \cdot \ldots \cdot \alpha_n \in Act^*$, the elements of S_{AT} of the form $X_0{}^{\surd} \cdot \alpha_1 \cdot X_1{}^{\surd} \cdot \ldots \cdot \alpha_n \cdot X_n{}^{\surd}$ constitute a $\wedge$-semilattice $S_{AT}{}^{(s)}$; and the joins of such elements in Q_{AT} constitute a frame $\Omega D_{AT}{}^{(s)}$. The corresponding locale $D_{AT}{}^{(s)}$ is spectral, and its points are the same as those of $D_{RT}{}^{(s)}$. The obvious homomorphism from Q_{AT} to Q_{RT} restricts to a frame homomorphism from $\Omega D_{AT}{}^{(s)}$ to $\Omega D_{RT}{}^{(s)}$, corresponding to the continuous map from $D_{RT}{}^{(s)}$ to $D_{AT}{}^{(s)}$ whose points part is the identity. By spatiality, it is therefore 1-1 on opens.

Finally, $Q_{AT} \cong \prod_{s \in Act^*} \Omega D_{AT}{}^{(s)}$ as sup-lattices, so Q_{AT} is a subquantale of Q_{RT}, and second completeness follows using 7.0.1. ⟧

Definition 7.3.4 An element $p \in \hat{Q}'_{AT}$ is *pointlike* iff $p \neq 0$ and for every $\alpha \in Act$, $p \cdot \alpha^{\surd}$ is either 0 or p.

Before we prove the main step (Theorem 7.3.6), that every element of $\hat{Q}'_{AT}$ is a join of pointlikes, let us prove two lemmas about the structure of Q'_{AT}.

Lemma 7.3.5 Let s be a product of generators in Q_{AT}. Then –

(i) j(s) is completely coprime in Q'_{AT}.

(ii) For all $\alpha \in Act$, if $j(s) \leq j(\alpha^{\surd})$ then $s = \alpha^{\surd} \cdot s$.

Proof

(i) The proof is by a kind of glueing, freely adjoining a new join structure. It relies on the fact that there are no joins in the relations for Q_{AT} and Q'_{AT}, so there is no means by which j(s) can interact with joins in a non-trivial way.

Let R be the set of lower closed subsets of Q'_{AT}. R is a sup-lattice, join just being union; in fact, it is the free sup-lattice over Q'_{AT} qua poset. Also, R can be made a left Q_{AT}-module, with

$$\alpha \cdot S = \downarrow\{\alpha \cdot u : u \in S\}$$
$$\alpha^{\surd} \cdot S = \downarrow\{\alpha^{\surd} \cdot u : u \in S\}$$

(To prove this, observe that the actions $\alpha \cdot –$ and $\alpha^{\surd} \cdot –$ as defined preserve all joins, so we have mapped the generators of Q_{AT} into the sup-lattice endomorphism quantale of R; and the relations of Q_{AT} are respected, so we have a quantale homomorphism from Q_{AT} to the endomorphism quantale, i.e. a Q_{AT}-module structure on R. Note how the fact that the relations of Q_{AT} are respected relies on their containing no joins.)

Now we can define a Q_{AT}-module homomorphism f from Q_{AT} to R by mapping 1 to Q'_{AT}, i.e. the top element of R. By induction on the length of t, we can show that if t is a product of generators in Q_{AT}, then $f(t) = \downarrow j(t)$. For t = 1 this holds by definition. If it holds for t, then

$$f(\alpha \cdot t) = \alpha \cdot f(t) = \alpha \cdot \downarrow j(t) = \downarrow \{\alpha \cdot u: u \le j(t)\} = \downarrow \alpha \cdot j(t) = \downarrow j(\alpha \cdot t)$$

and similarly for $f(\alpha^{\surd} \cdot t)$.

f respects the relations used in presenting Q'_{AT}, so f factors as j;g where g: $Q'_{AT} \to R$ is a Q_{AT}-module homomorphism.

Next write $\bigvee$ for the join map from R to Q'_{AT}. This respects the actions $\alpha \cdot -$ and $\alpha^{\surd} \cdot -$, and hence is also a Q'_{AT}-module homomorphism. $j;g;\bigvee$ maps 1 to j(1), so $j;g;\bigvee = j$ and hence $g;\bigvee$ is the identity on Q'_{AT}.

Now suppose $j(s) \le \bigvee X$, where $X \subseteq Q'_{AT}$. Applying g, we get

$$\downarrow j(s) = f(s) \subseteq \bigcup \{g(u): u \in X\}$$

Hence $\downarrow j(s) \subseteq$ some g(u) with $u \in X$. Applying $\bigvee$, we get $j(s) \le u$.

(ii) Let $s = X_0^{\surd} \cdot \alpha_1 \cdot X_1^{\surd} \cdot \ldots \cdot \alpha_n \cdot X_n^{\surd}$ where $X_i \subseteq$ Act and $X_i^{\surd}$ means the product of the $\beta^{\surd}$s, $\beta \in X_i$. Define a transition system Proc = $\{p_i: 0 \le i \le n\} \cup \{q\}$ with

$$p_i \xrightarrow{\alpha_{i+1}} p_{i+1}$$
$$p_i \xrightarrow{\beta} q \qquad \text{iff } \beta \in X_i$$

Then $\{p_0\} \cdot s$ is non-empty (it contains p_n), so because $j(s) \le j(\alpha^{\surd})$ we have $\{p_0\} \cdot \alpha^{\surd}$ also non-empty. Hence either $\alpha \in X_0$ or $\alpha = \alpha_0$; in each case, $j(\alpha^{\surd} \cdot s) = j(s)$.]]

Theorem 7.3.6 Every element x of $\hat{Q}'_{AT}$ is a join of pointlikes.

Proof

It suffices (cf. Theorem 7.1.7) to show for every product s of generators of Q_{AT} that if $j(s) \not\le \hat{x}$ (i.e. $x \cdot s \ne 0$), then there is some pointlike $p \le x$ such that $p \cdot s \ne 0$. Let $U = \{\alpha: s = \alpha^{\surd} \cdot s\}$, i.e. X_0 if $s = X_0^{\surd} \cdot \alpha_1 \cdot X_1^{\surd} \cdot \ldots \cdot \alpha_n \cdot X_n^{\surd}$, and define

$$p \quad = (x \wedge \textstyle\bigwedge_{\alpha \notin U} j(\alpha^{\surd})^{\wedge}) \cdot U^{\surd}$$

$p \le x$ and p is either pointlike or 0 ($p \cdot \alpha^{\surd}$ is 0 if $\alpha \notin U$, p if $\alpha \in U$). It remains to show that $p \cdot s \ne 0$, i.e. (because $U^{\surd} \cdot s = s$) $(x \wedge \bigwedge_{\alpha \notin U} j(\alpha^{\surd})^{\wedge}) \cdot s \ne 0$, i.e.

$$j(s) \not\le \hat{x} \vee \textstyle\bigvee_{\alpha \notin U} j(\alpha^{\surd})$$

But this is so, for otherwise (using Lemma 7.3.5 (i)) we'd have either $j(s) \le \hat{x}$, contradicting $x \cdot s \ne 0$, or $j(s) \le j(\alpha^{\surd})$ for some $\alpha \notin U$, and by Lemma 7.3.5 (ii) this contradicts the definition of U.]]

Theorem 7.3.7 Let Proc be a transition system and p, q ∈ Proc. Then

$Cap_{AT}(p) \leq Cap_{AT}(q) \Leftrightarrow p \subsetneqq_{AT} q$

Proof If $s \in S_{AT}$, then

$Cap_{AT}(p)(s) \neq 0 \Leftrightarrow \{p\} \cdot s \neq \emptyset \Leftrightarrow s \in$ accept-traces(p)

Hence,

$Cap(p) \leq Cap(q) \Leftrightarrow$ accept-traces(p) ⊆ accept-traces(q)]]

7.4 Trace semantics T

Definition 7.4.1 Let Act be a set. Then the quantale $Q = Q_T(Act)$ is the free quantale on the set Act.

The testing preorder is presented with no relations.

This being the simplest case of all, the reader should have no trouble proving –

Theorem 7.4.2

(i) T satisfies the second and third completeness criteria.

(ii) For any transition system Proc, and processes p and q ∈ Proc,

$Cap_T(p) \leq Cap_T(q) \Leftrightarrow p \subsetneqq_T q$]]

For concreteness, let us note that –

- Q_T is $\wp(Act^*)$.
- Q'_T is the set of subsets X of Act* such that if s ∈ X then s·t ∈ X; in other words, the upper-closed subsets of Act* under the prefix ordering; in other words the Alexandrov opens of Act*; in other words the Scott opens of the Kahn domain $Act^{*\omega}$ of finite and infinite lists from Act.
- $\hat{Q}'_T$ is the set of prefix-closed subsets of Act*; in other words the set of Scott closed subsets of $Act^{*\omega}$.
- The only "propositional" elements of Q_T are 0 and 1, so all elements of Q'_T can be considered pointlike.

7.5 Simulation semantics S

This semantics uses just the actions α and ↲: not the refusals. Compared with RS, there is therefore a restriction on the topological observations in ΩD_{RS} that can arise.

Definition 7.5.1 Let Act be a set. We present the quantale

$Q = Q_S(Act) = Qu \langle \alpha, ↲ (\alpha \in Act) \mid$

$s \cdot s = s \leq 1$	if s is balanced
$s \cdot t = t \cdot s$	if s and t are balanced

$$\alpha\cdot s \le \alpha\cdot s\cdot\hookleftarrow\cdot\alpha \qquad \text{if s is balanced (hence } \alpha\cdot s = \alpha\cdot s\cdot\hookleftarrow\cdot\alpha\cdot s) \qquad \rangle$$

No extra relations are needed for the testing preorder.

S is essentially a junior version of RS that lacks refusals. We sketch a theory of "S-locales" exactly analogous to RS-locales, but without the refusals.

Definition 7.5.2 An *S-frame* (over Act) is a frame A equipped with operations $\langle\alpha\rangle$ ($\alpha \in$ Act) that preserve all joins.

A homomorphism of S-frames is a frame homomorphism that preserves $\langle\alpha\rangle$.

We can now develop a theory just as in Section 6.1:

- Define *S-locales* and *S-maps*.
- Every transition system, considered as a discrete space, is an S-locale.
- The points of any S-locale form a transition system:

$$p \xrightarrow{\alpha} q \text{ iff } \forall\phi \in \Omega D.\ (q \models \phi \Rightarrow p \models \langle\alpha\rangle\,\phi)$$

- If D is a spatial S-locale, then

$$p \models \langle\alpha\rangle\,\phi \Leftrightarrow \exists q \in \mathrm{pt}\ D.\ (q \models \phi \wedge p \xrightarrow{\alpha} q)$$

- There is a final S-locale D_S and it is spectral.
- Any transition system Proc can be made into a topological system D = (Proc,□ΩD_S).
- Hence an element p of a transition system denotes a point $[\![p]\!]_S$ of D_S.

Proposition 7.5.3 $K\Omega D_S$ can be identified with the least sublattice of $K\Omega D_{RS}$ that is closed under $\langle\alpha\rangle$ for all α.

Proof

The proof of Theorem 6.1.6 used the theory of "RS-distributive lattices", and in an analogous way $K\Omega D_S$ is the initial "S-distributive lattice". $K\Omega D_{RS}$ is also an S-distributive lattice, so we get the unique homomorphism between them corresponding to the unique S-map from D_{RS} to D_S. This is surjective on points: if p $\in$ pt D_S, then $[\![p]\!] \in$ pt D_{RS} maps to it. (Note that the function $p \mapsto [\![p]\!]$ is not continuous.) Hence it is injective on opens.]]

We can now grade Q_S over G_{RS} just as in Section 6.3, but replacing D_{RS} by D_S throughout. The obvious maps from $D_{RS}{}^{(x)}$ to $D_S{}^{(x)}$ are surjective on points, and hence by spatiality (these locales are spectral) injective on opens; hence Q_S can be considered a subquantale of Q_{RS}.

Similarly, Q'_S can be identified with the opens of a spectral locale $D_S{}^{+\omega}$ of non-empty finite and infinite lists from D_S, and the natural map from $D_{RS}{}^{+\omega}$ to $D_S{}^{+\omega}$ is surjective on points.

Hence,

Theorem 7.5.4 S satisfies the second and third completeness criteria.]]

Finally, we deal with the preorder on processes. As in RS, we must make concessions to infinities in the transition system.

Definition 7.5.5 Let Proc be a transition system over Act. Proc is *image closed* (with respect to S) iff for all $p \in \text{Proc}$ and $\alpha \in \text{Act}$, the set $p{\cdot}\alpha = \{q \in \text{Proc}.\ p \xrightarrow{\alpha} q\}$ satisfies $\downarrow_S(p{\cdot}\alpha) = \text{Cl}_S(p{\cdot}\alpha)$: its downward closure in the specialization preorder $\sqsubseteq_S$ is equal to its topological closure (equivalently, $\downarrow_S X$ is topologically closed). The subscripts S mean that the topology is understood to be that imposed by ΩD_S.

Theorem 7.5.6 Let Proc be a transition system over Act that is image closed with respect to S. Then for all $p, q \in \text{Proc}$,

$$p \sqsubseteq_S q \Leftrightarrow \text{Caps}_S(\{(p)\}) \leq \text{Caps}_S(\{(q)\}) \Leftrightarrow p \subsetsim_S q$$

Proof The proof is similar to that of Theorems 6.2.5 and 6.3.11.]]

8. Quantaloids

In this part, we show how to generalize the foregoing theory to include a notion of typing on the processes. An observation has a source type and a target type, so that if the observed object starts off with the source type, it afterwards has the target type. This is obviously category theory. Our principal application is to the failure (F) and similar semantics, where there are two types: "live" and "dead". In this simple context, the general theory is far more than is needed, but we sketch it for its independent interest and in the expectation that it will find a use.

It is important to realise that although the new generalizations look forbiddingly complex, the methods used are essentially the same.

The corresponding generalization in ring theory, from rings to *ringoids,* has already been studied in, e.g., Mitchell (1972).

Definition 8.1 A *quantaloid* is a small sup-lattice enriched category, in other words, a small category such that

- each hom set is a sup-lattice
- morphism composition (multiplication) distributes over all joins on both sides.

A homomorphism of quantaloids is a functor that preserves all joins.

If Q is a quantaloid (or indeed any category) and i, j are objects in it, then we write Q_{ij} for the hom set Q(i,□j).

A *right Q-module* is a functor from Q to the category **SupL** of sup-lattices that preserves all joins (one can think of **SupL** as being a large quantaloid, so a right Q-module is simply a quantaloid homomorphism from Q to **SupL**).

In more more concrete terms, a module M involves –

- for each object i of Q, a sup-lattice M_i
- for each pair of objects i, j, an action (_·_): $M_i \times Q_{ij} \to M_j$

and these satisfy –

- $x\cdot 1 = x$ $\quad$ ($x \in M_i$, $1 = 1_i$)
- $x\cdot(a\cdot b) = (x\cdot a)\cdot b$ $\quad$ ($x \in M_i$, $a \in Q_{ij}$, $b \in Q_{jk}$)
- $(\bigvee X)\cdot a = \bigvee\{x\cdot a: x \in X\}$ $\quad$ ($X \subseteq M_i$, $a \in Q_{ij}$)
- $x\cdot(\bigvee Y) = \bigvee\{x\cdot a: a \in Y\}$ $\quad$ ($x \in M_i$, $Y \subseteq Q_{ij}$)

A homomorphism from one Q-module, M, to another, N, is a natural tranformation whose components preserve all joins. In other words,

- for each object i, there is a function f_i (the i is often omitted here) from M_i to N_i

- $f(\bigvee X) = \bigvee\{f(x): x \in X\}$ $(X \subseteq M_i)$
- $f(x \cdot a) = f(x) \cdot a$ $(x \in M_i, a \in Q_{ij})$

A *left Q-module* is a similar functor from Q^{op} to **SupL**.

The duality theory for modules over quantaloids is similar to that for modules over quantales. If M is a right module over a quantaloid Q, then the left module $\hat{M}$ is defined by $(\hat{M})_i = (M_i)^\wedge$, and if $a \in Q_{ij}$ then its left action from $(\hat{M})_j$ to $(\hat{M})_i$ is the dual of its right action from M_i to M_j.

Generators and Relations for Quantaloids

We state without proof that quantaloids can be presented by generators and relations. What in the quantale case is a *set* of generators must here be a directed graph: the nodes and edges of the graph are to become objects and morphisms of the quantaloid. In deriving new expressions from these, we must take care to multiply and join only when the products and joins are defined. Thus the general expression has a source i and a target j, and is a join of paths from i to j through the graph. Each relation takes the form $e_1 \leq e_2$. Usually, e_1 and e_2 will have the same source and target as each other; the only point in not doing this is to identify objects, and this might as well be done in the original graph of generators.

The coverage theorem for quantales can be generalized to quantaloids (and in fact this was done in Abramsky and Vickers (1990)). However, there is no great gain in doing this in the simple cases we consider.

Yoneda's Lemma

In Section 2 it was mentioned that a quantale Q, qua left (or right) module over itself, is freely generated by 1. We prove here a corresponding result for quantaloids that can also be seen as an analogue of Yoneda's Lemma.

Let Q be a quantaloid, and let j be an object of Q. We define the left Q-module Q_{-j} by $(Q_{-j})_i = Q_{ij}$. The action is defined by multiplication in Q: if $a \in Q_{ki}$ and $b \in (Q_{-j})_i$, then $a \cdot b \in (Q_{-j})_k$. (Thinking of Q just as a category, Q_{-j} is the representable functor for object j.)

Theorem 8.2 *(Quantaloid version of Yoneda's Lemma)*

Q_{-j}, qua left Q-module, is freely generated by 1_j: in other words, if M is a left Q-module and $x \in M_j$, then there is a unique left Q-module homomorphism from Q_{-j} to M that maps 1_j to x.

Proof If f is such a homomorphism and $a \in Q_{ij}$, then $f(a) = f(a \cdot 1_j) = a \cdot f(1_j) = a \cdot x$. This proves uniqueness. For existence, show that f thus defined is indeed a homomorphism.]

The quantaloid analogue of the Yoneda embedding is the quantaloid homomorphism from Q to Q-**Mod** given on objects by $j \mapsto Q_{-j}$. Then Theorem 8.2 tells us that the elements of Q_{jk} are in bijection with the Q-module homomorphisms from Q_{-j} to Q_{-k}. It is easy to see that this defines a faithful quantaloid homomorphism.

Of course, we can also define right Q-modules Q_{i-} and get a right-handed version of Theorem 8.2.

Capabilities

Let Q be a quantaloid and M a right Q-module. For each object j, we have a left Q-module homomorphism from Q_{-j} to $\hat{M}$ defined by $1_j \mapsto \hat{0}_j$, i.e. (for $a \in Q_{ij}$)

$$a \mapsto a \cdot \hat{0}_j, \text{ i.e.} \qquad a \mapsto (x \mapsto 0 \text{ iff } x \cdot a = 0\ (x \in M_i))$$

Hence we have a homomorphism from $\prod_j Q_{-j}$ to $\hat{M}$ (recall that for modules over a quantale, products and coproducts coincide; the same goes for modules over quantaloids). Its dual is Cap_M: $M \to (\prod_j Q_{-j})^\wedge$, $x \mapsto (a \mapsto 0 \text{ iff } x \cdot a = 0)$. Just as described in Example 5.1.5, we wish to describe a left Q-module quotient Q' of $\prod_j Q_{-j}$ such that $\hat{Q}'$ is the submodule of $(\prod_j Q_{-j})^\wedge$ generated by the images of Cap_M as M ranges over the modules derived from transition systems. We shall again write j for the natural homomorphism from $\prod_j Q_{-j}$ to Q'.

If $a \in Q_{ij}$, then $j(a) \in Q'_i$ should be thought of as the observation a viewed as a "static property". If $x \in M_i$, then the property $x \cdot a \neq 0$ ("x can do a") is a property of x, static in the sense that it is not interested in the change to $x \cdot a$. If $b \in Q_{ik}$ (notice that b must have the same source i as a, but not necessarily the same target), then j(a) should equal j(b) iff a and b represent the same static property, i.e. for all x, $x \cdot a = 0$ iff $x \cdot b = 0$.

We next generalize Lemma 7.0.1.

Lemma 8.3 Let f: $R \to Q$ be a quantaloid homomorphism.

(i) Every Q-module M gives rise to an R-module $M^{(f)}$, with

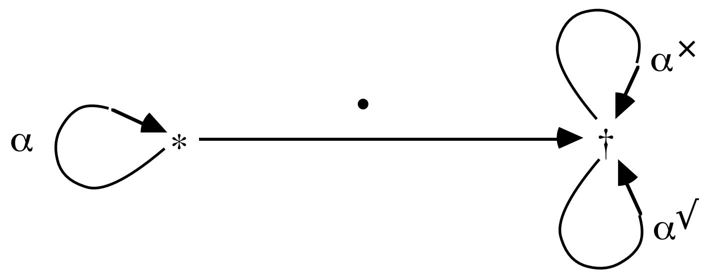
$$M_i^{(f)} = M_{f(i)}, \qquad x \cdot a = x \cdot f(a)$$

(ii) If f is faithful, and *M* is a jointly faithful class of right modules for Q (i.e. if $a \not\leq b$ in Q_{ij}, then there are $M \in$ *M* and $x \in M_i$ such that $x \cdot a \not\leq x \cdot b$), then $\{M^{(f)}: M \in$ *M*$\}$ is jointly faithful for R.]]

9. Typed semantics

We deal here with three semantics: Ready (R), Failure (F) and Acceptance (A). These are closely related to RT, FT and AT, but they have the property that after a refusal or acceptance, no more pure actions are possible: the process has deadlocked in some sense. This may be seen as somewhat user-unfriendly, but these semantics, particularly the Failures equivalence F –□which we consider first, in most detail –, have proved very popular (cf. Hoare 1985).

We discuss them as typed semantics, with two types (objects of the quantaloid) *live* (*) and *dead* (†). Actions are observed of a live process, and refusals and acceptances are postmortem observations. We also introduce an observation • ("bullet") to mark the transition from life to death. These ingredients can be illustrated in a directed graph, which generates the Ready quantaloid Q_R:



9.1 Failures semantics F

Definition 9.1.1 Let Act be a set. We present the quantaloid

$$
\begin{aligned}
Q = Q_F(\mathrm{Act}) = \\
\mathrm{Qud}\,\langle\ & \alpha: * \to * \quad (\alpha \in \mathrm{Act}), \\
& \bullet: * \to \dagger, \\
& \alpha^\times: \dagger \to \dagger\ (\alpha \in \mathrm{Act})\ | \\
& \alpha^\times \cdot \alpha^\times = \alpha^\times \le 1_\dagger \\
& \alpha^\times \cdot \beta^\times = \beta^\times \cdot \alpha^\times \quad \rangle
\end{aligned}
$$

The testing preorder is presented by

$$1_* \le' \bullet$$
$$\bullet \cdot X^\times \le' \alpha \vee \bullet \cdot (X \cup \{\alpha\})^\times \qquad (\alpha \in \mathrm{Act})$$

We shall describe the technical meaning of this testing preorder in more detail later, but for the moment consider the intended statements about processes. The first inequality ($1_* \le' \bullet$) says that if a process is live, then it can die. For the second, suppose that p is a live process, and that after its death, a postmortem examination reveals that it would refuse the actions in X: $p\cdot\bullet\cdot X^\times \neq 0$. Consider whether p could have done α. If so, then $p\cdot\alpha \neq 0$; if not, then a more careful postmortem examination would reveal that p would refuse the actions in $X\cup\{\alpha\}$: $p\cdot\bullet\cdot(X\cup\{\alpha\})^\times \neq 0$. Notice how the quantaloid elements $\bullet\cdot X^\times$, α and $\bullet\cdot(X\cup\{\alpha\})^\times$ all have the same source (*), but their targets vary. This is allowed in these "≤'" relations, which are really comparing elements of $\prod_j Q_{ij}$ for some i. (See Section 8.)

Let us now investigate the completeness criteria. First completeness is obvious, for $(s, X) \in \text{failures}(p)$ iff $\{p\}\cdot s\cdot\bullet\cdot X^\times \neq \emptyset$ (and also $\{p\}\cdot s \neq \emptyset$ iff $\{p\}\cdot s\cdot\bullet\cdot\emptyset^\times \neq \emptyset$). For second completeness, we examine the structure of Q_F in more detail.

Proposition 9.1.2

$(Q_F)_{**} \cong \wp(\text{Act}^*)$

$(Q_F)_{\dagger\dagger} \cong \Omega D_{FT}$ (defined as in Theorem 7.2.3)

$(Q_F)_{\dagger *} \cong 0$

$(Q_F)_{*\dagger} \cong \wp(\text{Act}^*) \otimes \Omega D_{FT}$

Here, ⊗ represents the tensor product of sup-lattices (see Joyal and Tierney 1984).

Proof

We can use the four given sup-lattices to construct a quantaloid on objects * and †, and then it is easy to define mutually inverse quantaloid homomorphisms between it and Q_F.]]

Theorem 9.1.3 The second completeness criterion holds for F.

Proof

We look at the four hom-sets of Q_F in turn. In each, we suppose we have $a \not\leq b$, and then using 9.1.2 we construct a process p for which $\{p\}\cdot a \not\subseteq \{p\}\cdot b$.

**: $a, b \in \wp(\text{Act}^*)$. Find $s = \alpha_1\cdot\ldots\cdot\alpha_n \in a-b$, and define Proc = $\{p_i: 0 \leq i \leq n\}$ with $p_i \xrightarrow{\alpha_{i+1}} p_{i+1}$. Then p_n is in $\{p_0\}\cdot a$ but not in $\{p_0\}\cdot b$.

††: $a, b \in \Omega D_{FT}$, and this is spectral, and hence spatial. Therefore there is a point $X \subseteq \text{Act}$ of D_{FT} that satisfies a but not b ($X \vDash \alpha^\times$ iff $\alpha \notin X$). Define Proc = {p, q} with $p \xrightarrow{\alpha} q$ iff $\alpha \in X$. Then for all $\phi \in \Omega D_{FT}$ we have that $X \vDash \phi$ iff $p \in \{p\}\cdot\phi$, and so p is in $\{p\}\cdot a$ but not $\{p\}\cdot b$.

*†: Because $\wp(\text{Act}^*)$ and ΩD_{FT} are both frames, their sup-lattice tensor product is the same as their frame coproduct, i.e. the frame of opens for the product locale $\text{Act}^*\times D_{FT}$. Because the two locales are locally compact, the product is spatial. Hence we can find $s = \alpha_1\cdot\ldots\cdot\alpha_n \in \text{Act}^*$ and $X \in \text{pt } D_{FT}$ such that (s, X) satisfies a but not b. Let Proc = $\{p_i: 0 \leq i \leq n\} \cup \{q\}$, with $p_i \xrightarrow{\alpha_{i+1}} p_{i+1}$, $p_n \xrightarrow{\alpha} q$ iff $\alpha \in X$. Every

element c of $\wp$(Act*) ⊗ ΩDFT can be expressed in the form $\bigvee_\lambda \{t_\lambda\} \otimes U_\lambda{}^\times$ (where $t_\lambda \in$ Act*, $U_\lambda \subseteq_{fin}$ Act). We have

$$(s, X) \models c \Leftrightarrow \exists\lambda.\ s = t_\lambda \text{ and } X \cap U_\lambda = \emptyset \Leftrightarrow p_n \in \{p_0\} \cdot c$$

Hence p_n is in $\{p_0\}\cdot a$ but not $\{p_0\}\cdot b$.

†*: This is trivial. ⟧

We now turn to third completeness. Recall from Section 8 that Q'_F is to be a left module over Q_F, so that it has two sup-lattices $(Q'_F)_*$ and $(Q'_F)_\dagger$. Moreover, Q'_F is a quotient of $Q_{-*} \times Q_{-\dagger}$, so we have –

$j_*: Q_{**} \times Q_{*\dagger} \to (Q'_F)_*$
$j_\dagger: Q_{\dagger *} \times Q_{\dagger\dagger} \to (Q'_F)_\dagger$

The map $j_\dagger$ is not interesting. $Q_{\dagger *}$ is the 0 sup-lattice, and there are no ≤' relations at † except for those requiring $j(1_\dagger)$ to be top, which it is anyway ($Q_{\dagger\dagger}$ is a frame). Hence $(Q'_F)_\dagger \cong Q_{\dagger\dagger} \cong \Omega D_{FT}$. Third completeness at † requires that if $a \not\leq b$ in $Q_{\dagger\dagger}$, then there is a process p with $\{p\}\cdot a \neq 0$ but $\{p\}\cdot b \neq 0$; the process constructed in 9.1.3 for second completeness (case ††) also suffices here.

For j_* we prove an analogue of Lemma 7.2.6. Note that $j(1_*) = j(\bullet)$, so that $(Q'_F)_*$ is generated by the image of $Q_{*\dagger}$ under j_*.

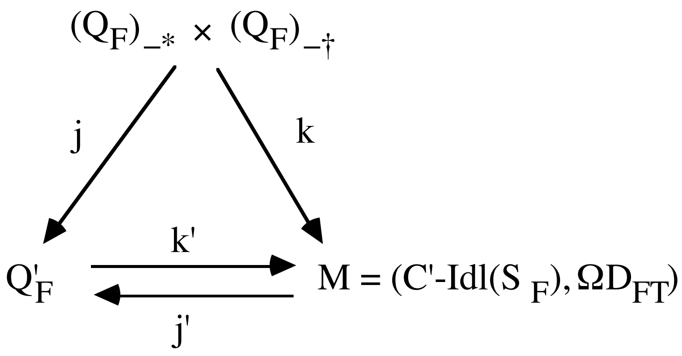
Lemma 9.1.4 Let $S_F = \text{Act}^* \times \wp_{fin}(\text{Act})$; we write $(s, X) \in S_F$ as $s\cdot\bullet\cdot X^\times$ as though it were a product of generators of Q_F, and feel free to omit s if it is 1, or $X^\times$ if X is Ø. Let C' be the set of cover relations (with s, t ∈ Act*)

$$\{s\cdot\bullet\} \dashv s\cdot t\cdot\bullet\cdot X^\times$$
$$\{s\cdot\bullet\cdot X^\times\} \dashv s\cdot\bullet\cdot\alpha^\times\cdot X^\times$$
$$\{s\cdot\alpha\cdot\bullet,\ s\cdot\bullet\cdot\alpha^\times\cdot X^\times\} \dashv s\cdot\bullet\cdot X^\times$$

Then C'-Idl(S_F) is isomorphic to $(Q'_F)_*$, with $s\cdot\bullet\cdot X^\times$ (as generator in C'-Idl(S_F)) corresponding to $j(s\cdot\bullet\cdot X^\times)$.

Proof

Let us write M_* for C'-Idl(S_F) and $M_\dagger$ for ΩD_{FT}. One can then show that M (with these two components) is a left Q_F-module. Much as in Lemma 7.2.6, one can then define left Q_F-module homomorphisms



and show that j' and k' are mutually inverse. ⟧

Lemma 9.1.5 Let $v, u \in (Q'_F)_*$, and $\alpha \in \mathrm{Act}$.

If both $v \le u \vee j(\alpha)$ and $v \le u \vee j(\bullet\cdot\alpha^\times)$, then $v \le u$.

Proof The obvious quantaloid homomorphism from Q_F to Q_{FT} makes Q'_{FT} a left Q_F-module, and we get a Q_F-module homomorphism $f: Q'_F \to Q'_{FT}$. On C'-ideals (Lemmas 7.2.6 and 9.1.4), f_* maps a C'_F ideal I to $\{s\cdot X^\times\cdot t: s\cdot\bullet\cdot X^\times \in I, t \in S_{FT}\}$. This is readily checked to be 1-1. Hence if we have v, u and α as stated, then in Q'_{FT} we have

$$f_*(v) \le f_*(u) \vee j(\alpha) \quad \text{and} \quad f_*(v) \le f_*(u) \vee j(\alpha^\times)$$

It follows from Lemma 7.2.7 that $f_*(v) \le f_*(u)$, and so $v \le u$. ⟧

Definition 9.1.6 Let $p \in (\hat{Q}'_F)_*$. Then p is *pointlike* iff $p \ne 0$, and for all $\alpha \in \mathrm{Act}$, if $p\cdot\bullet\cdot\alpha^\times \ne 0$ then $p\cdot\alpha = 0$.

Lemma 9.1.7 Let $x \in (\hat{Q}'_F)_*$. If $x\cdot\alpha = 0$, then $x\cdot\bullet\cdot\alpha^\times = x\cdot\bullet$.

Proof

We must show $(x\cdot\bullet\cdot\alpha^\times)^\wedge \le (x\cdot\bullet)^\wedge$. Suppose $j(X^\times) \le (x\cdot\bullet\cdot\alpha^\times)^\wedge$, i.e. $x\cdot\bullet\cdot\alpha^\times\cdot X^\times = 0$. Then $j(\bullet\cdot X^\times) \le j(\alpha) \vee j(\bullet\cdot\alpha^\times\cdot X^\times) \le \hat{x}$ and $j(X^\times) \le (x\cdot\bullet)^\wedge$. ⟧

In FT we could also prove the converse of the analogue of this result by straightforward algebraic means (using $\alpha^\times\cdot\alpha = 0$), but that does not seem to be possible here – though the converse is still true, and follows from the next theorem. Hence we need a formally stronger definition of pointlikeness.

Theorem 9.1.8 Every element x of $(\hat{Q}'_F)_*$ is a join of pointlikes.

Proof

Let $v \in (Q'_F)_*$ be compact, with $v \not\le \hat{x}$ (the usual coherence arguments still work). We must show that $v \not\le \hat{p}$ for some pointlike $p \le x$. Just as with FT, it suffices to find a set $U \subseteq \mathrm{Act}$ such that $v \not\le \hat{p}$ where $p = x \wedge \bigwedge_{\alpha\in U} j(\bullet\cdot\alpha^\times)^\wedge \wedge \bigwedge_{\alpha\notin U} j(\alpha)^\wedge$. By Zorn’s lemma we find a pair (U, V) of subsets of Act maximal with respect to

$v \nleq \hat{\chi} \vee \bigvee_{\alpha \in U} j(\bullet \cdot \alpha^{\times}) \vee \bigvee_{\alpha \in V} j(\alpha)$

and then Lemma 9.1.5 shows that Act is the disjoint union of U and V.]]

Theorem 9.1.9 F satisfies the third completeness criterion.
Proof
Just as in Section 7. Let Proc be the set of pointlike elements of $(\hat{Q}'_F)_*$, made into a transition system by $p \xrightarrow{\alpha} q$ iff $q \leq p \cdot \alpha$. Then for every a in Q with source *, we have $\{p\} \cdot a = \emptyset$ iff $p \cdot a = 0$. (As before, reduce to the case where a is a product of generators, and then use induction on the length.) Now suppose a and b have source *, and $j(a) \nleq j(b)$. Then there is some pointlike p with $p \leq j(b)^\wedge$, $p \nleq j(a)^\wedge$, so $\{p\} \cdot a \neq \emptyset$, $\{p\} \cdot b = \emptyset$.]]

Let us conclude this section by observing that Lemma 9.1.4 provides us with a direct link to existing work.

Proposition 9.1.10 The elements of $(\hat{Q}'_F)_*$ are in 1-1 order-preserving correspondence with the sets $F \subseteq Act^* \times \wp_{fin}(Act)$ satisfying:

(F1) $(s \cdot t, X) \in F \Rightarrow (s, \emptyset) \in F$
(F2) if $(s, Y) \in F$ and $X \subseteq Y$ then $(s, X) \in F$
(F3) if $(s, X) \in F$ then $(s, X \cup \{\alpha\}) \in F$ or $(s \cdot \alpha, \emptyset) \in F)$ $(\alpha \in Act)$

Proof
$Act^* \times \wp_{fin}(Act)$ is the S_F of Lemma 9.1.4, and then the sets F described above are the complements of the C'-ideals (let us call them C'-*co*ideals). The C'-ideals themselves are in 1-1 order-preserving correspondence with the elements of $(Q'_F)_*$.

Note that the set of coideals is closed under unions, and hence that joins in $(\hat{Q}'_F)_*$ are represented by unions of the corresponding coideals.]]

The conditions in this proposition are taken directly from Brookes, Hoare and Roscoe (1984) (or from Hoare 1985, p. 130). They also include the condition:

(F0) $(1, \emptyset) \in F$

which is to say that F is non-empty. Of course, the empty coideal plays an essential part in our mathematics. As to its process theoretic meaning, it can perhaps be thought of as a “nullary internal choice” that cannot even exist, let alone perform any actions or refusals.

The extra work in our account has related the formalism based on failure sets (satisfying F1-3 above) both to the algebra with generators and relations (using the coverage theorem) and to the transitions systems (results analogous to Blamey’s (1991)).

9.2 Ready semantics R

The development here is very similar to that for F, so we shall do no more than sketch the sequence of results.

Definition 9.2.1 Let Act be a set. We present the quantaloid

$$
\begin{aligned}
& Q = Q_R(\mathrm{Act}) = \\
& \quad \mathrm{Qud}\ \langle\ \alpha: * \to * \quad (\alpha \in \mathrm{Act}), \\
& \qquad \bullet: * \to \dagger, \\
& \qquad \alpha^{\times}, \alpha^{\surd}: \dagger \to \dagger\ (\alpha \in \mathrm{Act})\ | \\
& \qquad \alpha^{\times}\cdot\alpha^{\times} = \alpha^{\times} \le 1_{\dagger} \\
& \qquad \alpha^{\surd}\cdot\alpha^{\surd} = \alpha^{\surd} \le 1_{\dagger} \\
& \qquad \alpha^{\times}\cdot\beta^{\times} = \beta^{\times}\cdot\alpha^{\times} \\
& \qquad \alpha^{\surd}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\surd} \\
& \qquad \alpha^{\times}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\times} \\
& \qquad \alpha^{\times}\cdot\alpha^{\surd} = 0_{\dagger\dagger} \\
& \qquad 1_{\dagger} \le \alpha^{\surd} \vee \square\alpha^{\times} \quad \rangle
\end{aligned}
$$

The testing preorder is presented by

$$
\begin{aligned}
& \bullet\cdot\alpha^{\surd} =' \alpha \qquad (\alpha \in \mathrm{Act}) \\
& 1_{*} \le' \bullet
\end{aligned}
$$

Theorem 9.2.2 *(First completeness for R)*
If Act is finite, then the R-order on processes is $\subsetneq_R$.

Proof (s, X) is in readies(p) iff $\{p\}\cdot s\cdot X^{\surd}\cdot(\mathrm{Act}-X)^{\times} \neq \emptyset$, and $\{p\}\cdot s\cdot Y^{\surd}\cdot Z^{\times} \neq \emptyset$ iff there is some $X \subseteq \mathrm{Act}$ such that $(s, X) \in \mathrm{readies}(p)$, $Y \subseteq X$ and $Z \cap X = \emptyset$. ⟧

Proposition 9.2.3

$$
\begin{aligned}
& (Q_R)_{**} \cong \wp(\mathrm{Act}^*) \\
& (Q_R)_{\dagger\dagger} \cong \Omega D_{RT} \qquad \text{(defined as in Proposition 7.1.3)} \\
& (Q_R)_{\dagger *} \cong 0 \\
& (Q_R)_{*\dagger} \cong \wp(\mathrm{Act}^*) \otimes \Omega D_{RT}
\end{aligned}
$$

⟧

Theorem 9.2.4 The second completeness criterion criterion holds for R. ⟧

Lemma 9.2.5 Let $S_R = \mathrm{Act}^* \times \wp_{\mathrm{fin}}(\mathrm{Act}) \times \wp_{\mathrm{fin}}(\mathrm{Act})$; we write $(s, X, Y) \in S_R$ as $s\cdot\bullet\cdot X^{\surd}\cdot Y^{\times}$ as though it were a product of generators of Q_R. Let C' be the set of cover relations (with $s, t \in \mathrm{Act}^*$, $\phi = X^{\surd}\cdot Y^{\times}$ for some X, Y)

$$
\begin{aligned}
& \emptyset && \dashv s\cdot\bullet\cdot\alpha^{\surd}\cdot\alpha^{\times}\cdot\phi \\
& \{s\cdot\bullet\cdot\alpha^{\surd}\cdot\phi,\ s\cdot\bullet\cdot\alpha^{\times}\cdot\phi\} && \dashv s\cdot\bullet\cdot\phi \\
& \{s\cdot\bullet\} && \dashv s\cdot t\cdot\bullet\cdot\phi
\end{aligned}
$$

$\{s\cdot\bullet\cdot\phi\}$ ⊣ $s\cdot\bullet\cdot\alpha^{\surd}\cdot\phi$
$\{s\cdot\bullet\cdot\phi\}$ ⊣ $s\cdot\bullet\cdot\alpha^{\times}\cdot\phi$
$\{s\cdot\alpha\cdot\bullet\}$ ⊣ $s\cdot\bullet\cdot\alpha^{\surd}$
$\{s\cdot\bullet\cdot\alpha^{\surd}\}$ ⊣ $s\cdot\alpha\cdot\bullet$

Then C'-Idl(S_R) is isomorphic to $(Q'_R)_*$, with $s\cdot\bullet\cdot X^{\surd}\cdot Y^{\times}$ (as generator in C'-Idl(S_R)) corresponding to $j(s\cdot\bullet\cdot X^{\surd}\cdot Y^{\times})$. ⟧

Lemma 9.2.6 Let v, u ∈ $(Q'_R)_*$, and α ∈ Act.

If both $v \leq u \vee j(\bullet\cdot\alpha^{\surd})$ and $v \leq u \vee j(\bullet\cdot\alpha^{\times})$, then $v \leq u$.

Proof The crucial calculations are that if I is a C'-ideal in S_R, then –

$$\text{(i)}\qquad C'\text{-}\langle I \cup \{\bullet\cdot\alpha^{\surd}\}\rangle = \begin{cases} S_R & \text{if } \bullet\cdot\alpha^{\times} \in I \\ I \cup \bigcup\{\beta s\cdot\bullet\cdot\phi : s\cdot\phi \in S_R, \bullet\cdot\alpha^{\times}\cdot\beta^{\surd} \in I\} & \\ \qquad \cup \bigcup\{\bullet\cdot\phi : \bullet\cdot\alpha^{\times}\cdot\phi \in I\} & \text{otherwise} \end{cases}$$

$$\text{(ii)}\qquad C'\text{-}\langle I \cup \{\bullet\cdot\alpha^{\times}\}\rangle = \begin{cases} S_R & \text{if } \bullet\cdot\alpha^{\surd} \in I \\ I \cup \bigcup\{\beta s\cdot\bullet\cdot\phi : s\cdot\phi \in S_R, \bullet\cdot\alpha^{\surd}\cdot\beta^{\surd} \in I\} & \\ \qquad \cup \bigcup\{\bullet\cdot\phi : \bullet\cdot\alpha^{\surd}\cdot\phi \in I\} & \text{otherwise} \end{cases}$$

It is then not hard to show that the intersection of these two is I. ⟧

Definition 9.2.7 Let p ∈ $(\hat{Q}'_R)_*$. Then p is *pointlike* iff p ≠ 0, and for all α ∈ Act, one of $p\cdot\bullet\cdot\alpha^{\surd}$ and $p\cdot\bullet\cdot\alpha^{\times}$ is zero (so the other is $p\cdot\bullet$).

Theorem 9.2.8 Every element x of $(\hat{Q}'_R)_*$ is a join of pointlikes. ⟧

Theorem 9.2.9 R satisfies the third completeness criterion. ⟧

9.3 Acceptance semantics A

Definition 9.3.1 Let Act be a set. We present the quantaloid

$Q = Q_A(\text{Act}) =$
Qud ⟨ $\alpha: * \to *$ (α ∈ Act),
$\bullet: * \to \dagger$,
$\alpha^{\surd}: \dagger \to \dagger$ (α ∈ Act) |
$\alpha^{\surd}\cdot\alpha^{\surd} = \alpha^{\surd} \leq 1_{\dagger}$
$\alpha^{\surd}\cdot\beta^{\surd} = \beta^{\surd}\cdot\alpha^{\surd}$ ⟩

The testing preorder is presented by

$\bullet\cdot\alpha^{\surd} =' \alpha$ (α ∈ Act)
$1_* \leq' \bullet$

First completeness is obvious.

Proposition 9.3.2

$(Q_A)_{**} \cong \wp(\mathrm{Act}^*)$
$(Q_A)_{\dagger\dagger} \cong \Omega D_{AT}$ (defined as in Proposition 7.3.3)
$(Q_A)_{\dagger *} \cong 0$
$(Q_A)_{*\dagger} \cong \wp(\mathrm{Act}^*) \otimes \Omega D_{AT}$]]

Theorem 9.3.3 The second completeness criterion criterion holds for A.]]

Lemma 9.3.4 Let $S_A = \mathrm{Act}^* \times \wp_{fin}(\mathrm{Act})$; we write $(s, X) \in S_A$ as $s{\cdot}\bullet{\cdot}X^{\surd}$ as though it were a product of generators of Q_A. Let C' be the set of cover relations (with s, t ∈ Act*)

$\{s{\cdot}\bullet\} \dashv s{\cdot}t{\cdot}\bullet{\cdot}\phi$
$\{s{\cdot}\bullet{\cdot}\phi\} \dashv s{\cdot}\bullet{\cdot}\alpha^{\surd}{\cdot}\phi$
$\{s{\cdot}\bullet{\cdot}\phi\} \dashv s{\cdot}\bullet{\cdot}\alpha^{\times}{\cdot}\phi$
$\{s{\cdot}\alpha{\cdot}\bullet\} \dashv s{\cdot}\bullet{\cdot}\alpha^{\surd}$
$\{s{\cdot}\bullet{\cdot}\alpha^{\surd}\} \dashv s{\cdot}\alpha{\cdot}\bullet$

Then C'-Idl(S_A) is isomorphic to $(Q'_A)_*$, with $s{\cdot}\bullet{\cdot}X^{\surd}$ (as generator in C'-Idl(S_A)) corresponding to $j(s{\cdot}\bullet{\cdot}X^{\surd})$.]]

Lemma 9.3.5

(i) $j(s{\cdot}\bullet{\cdot}X^{\surd})$ is completely coprime in $(Q'_A)_*$.
(ii) If $j(s{\cdot}\bullet{\cdot}X^{\surd}) \le j(\bullet{\cdot}\alpha^{\surd})$ in $(Q'_A)_*$, then either $s = \alpha{\cdot}s'$ for some s', or s = 1 and $\alpha \in X$.

Proof
(i) C'-ideals in S_A are closed under arbitrary union.
(ii) $C'\text{-}\langle\{\bullet{\cdot}\alpha^{\surd}\}\rangle = \{\alpha{\cdot}t{\cdot}\bullet{\cdot}Y^{\surd}: t{\cdot}\bullet{\cdot}Y^{\surd} \in S_A\} \cup \{\bullet{\cdot}Y^{\surd}: \alpha \in Y \subseteq_{fin} \mathrm{Act}\}$.]]

Definition 9.3.6 Let $p \in (\hat{Q}'_A)_*$. Then p is *pointlike* iff p ≠ 0, and for all α ∈ Act, $p{\cdot}\bullet{\cdot}\alpha^{\surd}$ is either 0 or $p{\cdot}\bullet$.

Theorem 9.3.7 Every element x of $(\hat{Q}'_A)_*$ is a join of pointlikes.
Proof
We must show that if $j(s{\cdot}\bullet{\cdot}X^{\surd}) \not\le \hat{x}$, then there is a pointlike p ≤ x such that $j(s{\cdot}\bullet{\cdot}X^{\surd}) \not\le \hat{p}$. Define $U \subseteq_{fin}$ Act to be {α} if $s = \alpha{\cdot}s'$, X if s = 1: so $j(s{\cdot}\bullet{\cdot}X^{\surd}) \le j(\bullet{\cdot}U^{\surd})$. Define

$$p = x \wedge \bigwedge_{\beta \notin U} j(\bullet{\cdot}\beta^{\surd})^{\wedge}$$

If β ∉ U then $p{\cdot}\bullet{\cdot}\beta^{\surd} = 0$; we show that if γ ∈ U then $p{\cdot}\bullet{\cdot}\gamma^{\surd} = p{\cdot}\bullet$, i.e. if $Y \subseteq_{fin}$ Act and $p{\cdot}\bullet{\cdot}Y^{\surd} \ne 0$, then $p{\cdot}\bullet{\cdot}\gamma^{\surd}{\cdot}Y^{\surd} \ne 0$. But if $p{\cdot}\bullet{\cdot}Y^{\surd} \ne 0$ then $Y \subseteq U$, so it suffices to show that $p{\cdot}\bullet{\cdot}\gamma^{\surd}{\cdot}U^{\surd} \ne 0$, i.e. (because γ ∈ U) $p{\cdot}\bullet{\cdot}U^{\surd} \ne 0$. If $p{\cdot}\bullet{\cdot}U^{\surd} = 0$, then

$$j(\bullet\cdot U^{\surd}) \leq \hat{X} \vee \bigvee_{\beta \notin U} j(\bullet\cdot\beta^{\surd})$$

By Lemma 9.3.5 (i), we have either $j(\bullet\cdot U^{\surd}) \leq \hat{X}$, which contradicts $j(s\cdot\bullet\cdot X^{\surd}) \not\leq \hat{X}$, or $j(\bullet\cdot U^{\surd}) \leq j(\bullet\cdot\beta^{\surd})$ for some $\beta \notin U$, which gives a contradiction by 9.3.5 (ii).]]

Theorem 9.3.8 A satisfies the third completeness criterion.]]

10. Concluding remarks

There are some obvious directions in which our results need to be extended to give a fully adequate treatment of process semantics.

First, we have not considered unobservable actions (τ) and divergence in this paper. Furthermore, we have taken a "syntax-free" approach to transition systems, ignoring the algebraic structure of process expressions. Finally, we have not considered *causal* semantics, as we have focused purely on sequential observations.

One tantalizing question left open is whether bisimulation yields to the methods of analysis and description available in our framework. We hope that an answer to this question, in either direction, will shed some light on the current debate about which process equivalences are based on reasonable notions of observation (Abramsky 1987 b, Bloom, Istrail and Meyer 1988, Larsen and Skou 1989).

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