A monad of valuation locales

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Abstract

If X is a locale then its valuation locale $\mathfrak{V}X$ has for its points the valuations on X. $\mathfrak{V}$ is the functor part of a strong monad on the category of locales, a localic analogue of the Giry monad. It is commutative, i.e. product valuations exist and a Fubini Theorem holds. An analogue of the Riesz Representation Theorem holds. Concrete representations are given for the tensor product of lattices and for the modular monoid. The work conforms with the constructive constraints of geometric logic.

1 Introduction

In point-free topology, a theory of integration can be got with measures replaced by *valuations*. These are like measures, but defined on the opens; and the condition of σ-additivity is replaced by Scott continuity. This approach avoids set-theoretic issues of measure theory, such as the question of whether there are unmeasurable sets, that would be extremely problematic in a point-free setting. It also avoids the need for a notion of measurable function, since in point-free topology all maps are continuous (although apparently discontinuous maps often arise because the reals are in many situations in effect given topologies of semicontinuity). However, it is restricted to those measures for which it is enough to define the measure on the opens.

Given a locale X, there is a *valuation* locale $\mathfrak{V}X$ whose points are the valuations on X. This was first described by Heckmann [Hec94], anticipated by work on the probabilistic powerdomain [JP89], and subsequently developed in [Vic08] and [CS09]. (There are minor variations in that some treatments restrict to probability valuations, where the mass of X is 1.) Our aim here is to develop the theory, showing that $\mathfrak{V}$ is the functor part of a commutative monad, and proving analogues of the Riesz Representation Theorem and the Fubini Theorem.

It is already known that some aspects of measure and integration can be conveniently summarized as a monad on a category of measurable spaces, based on a functor that, for each space, provides a space of measures on it. A well known example is the *Giry monad* [Gir81] on the category of measurable spaces. Another example on **Sets** is the distribution monad of [AB11] where, given a semiring R, the functor gives for each set X the set of functions $X \to R$ with finite support. In these examples the monad needs a strength in order that one may calculate double integrals, and be commutative to provide product measures and a Fubini Theorem.

A point-free analogue becomes particularly necessary for a topos-valid treatment, and it has been used in [HLS09] to introduce probability into quantum theory in a topos approach. Further developments [FRV11] have made it imperative to establish the basic properties of $\mathfrak{V}$.

On the way to showing that $\mathfrak{V}$ is a monad we also show a version of the Riesz Representation Theorem (33). Apart from its intrinsic interest, we also use it to justify our definition of the multiplication of the monad, defining it as a linear functional. The usual Riesz Theorem can be summarized roughly as follows. Suppose X is a measurable space, with σ-algebra $\mathcal{M}X$ of measurable subsets. A measure on X is then an assignment of non-negative reals to the measurable subsets and so lives in a subset $\mathrm{Meas}\,X$ of $\mathbb{R}^{\mathcal{M}X}$. More general than the measurable subsets, however, are the measurable functions $\mathcal{M}_{\mathbb{R}}X$ from X to $\mathbb{R}$ – in fact, the measurable sets A correspond bijectively with their characteristic functions χ_A, which are the measurable functions from X to $\{0, 1\}$.

If a measure m assigns real values $m(A)$ to measurable sets A, integration extends this assignment to one that assigns real values $\int f\ dm$ to measurable functions, in such a way that $\int \chi_A\ dm = m(A)$. Thus one may understand $\int -\ dm$ as a functional $\mathrm{Int}(m)$ from $\mathcal{M}_{\mathbb{R}}X$ to $\mathbb{R}$, so $\mathrm{Int} : \mathrm{Meas}\,X \to \mathbb{R}^{\mathcal{M}_{\mathbb{R}}X}$. A Riesz Theorem says that its image is precisely the linear functionals.

Coquand and Spitters [CS09] prove a version of this for the valuation locale. (Note that their valuation locale $\mathrm{VAL}(X)$ is a subspace of ours, being restricted to probability valuations with total mass 1. This has the important effect of enabling a theory of upper integrals and hence of Riemann integrals, since all their valuations are finite in the sense of [Vic08].) Their Riesz Theorem is restricted to the case where X is compact completely regular, and so, by a constructive version of a form of Gelfand-Naimark duality, corresponds to a Riesz space $R = C(X)$, the set of continuous maps $X \to \mathbb{R}$. They then show that their valuation locale $\mathrm{VAL}(X)$ is homeomorphic to a locale $\mathrm{INT}(R)$ of *integrals* on R, i.e. certain linear functions $R \to \mathbb{R}$.

In this paper we describe another localic Riesz Theorem, differing from the treatment in [CS09] in two main ways. First of all, it is completely general, working for arbitrary locales X. However, ours is not a simple generalization because we change the integrands (the integrable functions). In our constructive setting it is important to be clear which reals are in use. For instance, a valuation takes its values not in the extended, non-negative, *Dedekind* reals

$\mathbb{R}$, but in the *lower* reals $\overrightarrow{[0,\infty]}$. Constructively these are approximated from below, while classically they may be thought of as having the topology of lower semicontinuity. In our notation, the arrow shows the direction of the specialization order. For the lower reals it is the numerical order, for the upper reals it is the opposite. In [CS09] the integrands – the elements of their Riesz space – are maps from X to the Dedekind reals $\mathbb{R}$, possibly negative. They can integrate such functions (getting a Dedekind real), but the ability to do so depends on the compactness of X and the finiteness of their probability valuations. We instead integrate maps to the non-negative lower reals, using the lower integral of [Vic08].

We embed $\mathfrak{V}X$ in a locale whose points may be considered to be functionals defined on maps from X to the non-negative lower reals, and show that $\mathfrak{V}X$ then comprises the linear functionals. Our proof uses a detailed algebraic analysis of finitary lattices, including one whose elements play the role of simple functions.

2 Geometricity

The notion of geometricity will be important throughout this work, both in the development and in applications such as [FRV11] (which uses the valuation locale applied fibrewise in bundles). The technical foundation is set out in [Vic04], which relies heavily on geometricity of the double powerlocale, and we shall outline it here because similar considerations apply to the valuation locale.

Point-free topology in the form of locale theory works well in arbitrary toposes (even elementary toposes, though Grothendieck toposes are our main interest), and in fact works even better than point-set topology. There is a topos-valid notion of frame, in which, without going into details, the arbitrary joins in a frame A are expressed as a join morphism $\mathcal{P}A \to A$. To appreciate better the later discussion, we can divide the structure into a finitary part (A is a distributive lattice) and an infinitary part, the existence of directed joins, expressed as a morphism to A from the object of its directed subsets.

The well known technique of presenting a frame by generators and relations also works in toposes, so that given a presentation $\mathbb{T}$, it presents a frame $\Omega[\mathbb{T}]$. There is a range of possibilities for what $\mathbb{T}$ might be concretely. One very general one (Definition 1) is the *GRD-system* of [Vic04]. Another scheme that will be useful in the present paper is to structure $\mathbb{T}$ as *DL-site* (Definition 5), where the generators form a distributive lattice, with finitary meets and joins to be preserved, and the relations are of the form $a \leq \bigvee_i^{\uparrow} a_i$, the superscript $\uparrow$ denoting that the join is directed.

A core result is what we shall refer to as the *Localic Bundle Theorem* ([FS79], [JT84]). This says that, for any topos $\mathcal{E}$, there is an dual equivalence between internal frames in $\mathcal{E}$ and localic geometric morphisms with codomain $\mathcal{E}$ – we shall think of these as localic *bundles* over $\mathcal{E}$. Given an internal presentation $\mathbb{T}$ in $\mathcal{E}$ we thus also get an internal frame $\Omega[\mathbb{T}]$ (or locale $[\mathbb{T}]$) and a bundle (external) for which we shall write $\mathcal{E}[\mathbb{T}] \to \mathcal{E}$. The topos $\mathcal{E}[\mathbb{T}]$ is got from $\mathcal{E}$ by freely adjoining an internal model of $\mathbb{T}$.

The issue of geometricity arises when we consider how the relation between these three different accounts of internal locales – frames, presentations and bundles – is preserved under pullback along a geometric morphism $f : \mathcal{F} \to \mathcal{E}$. By definition part of f is its inverse image functor $f^* : \mathcal{E} \to \mathcal{F}$, and this can be applied to the ingredients of the presentation $\mathbb{T}$ to get a presentation $f^*\mathbb{T}$ in $\mathcal{F}$. It can also be applied to the frame $\Omega[\mathbb{T}]$ to get an object $f^*(\Omega[\mathbb{T}])$ but one should note that although this is a distributive lattice, it is not a frame in $\mathcal{F}$. The basic issue is with the existence of "all" joins, or, more carefully, all directed joins – the meaning of "all" is determined by the topos, and $f^*(\Omega[\mathbb{T}])$ has only the joins arising from $\mathcal{E}$. More technically, the join structure in $\mathcal{E}$ is given by a join map $\mathcal{P}\Omega[\mathbb{T}] \to \Omega[\mathbb{T}]$. We can apply f^* to get $f^*(\mathcal{P}\Omega[\mathbb{T}]) \to f^*(\Omega[\mathbb{T}])$, but since $\mathcal{P}$ is not preserved by inverse image functors (it is not geometric), we do not get all joins in $\mathcal{F}$.

On the other hand, pullback of bundles is just the categorical operation (strictly speaking: pseudopullback, so the squares commute up to isomorphism) in the category of toposes.

The inverse image functor f^* agrees with pullback of bundles in the following way. Each object X of $\mathcal{E}$ has a powerset $\mathcal{P}X$, and that is a frame; the bundles corresponding to frames of that kind are those for which the bundle map is a local homeomorphism. This gives an equivalence between objects of $\mathcal{E}$ and local homeomorphisms to $\mathcal{E}$, and the equivalence is preserved by the two forms of pullback. Thus geometricity as a property of constructions on topos objects, meaning preservation by inverse image functors f^*, can also be understood as preservation under pullback of the equivalent construction on local homeomorphisms.

Given a presentation $\mathbb{T}$ in $\mathcal{E}$, in $\mathcal{F}$ we have $f^*\mathbb{T}$, $f^*\Omega[\mathbb{T}]$ and $f^*p : f^*\mathcal{E}[\mathbb{T}] \to \mathcal{F}$. However, the connection between them has been lost because $f^*\Omega[\mathbb{T}]$ is not a frame, so is not $\Omega[f^*\mathbb{T}]$. Nonetheless, it is a fact [Vic04] that the frame $\Omega[f^*\mathbb{T}]$ corresponds to the bundle f^*p: in other words, $f^*\mathcal{E}[\mathbb{T}] \cong \mathcal{F}[f^*\mathbb{T}]$ over $\mathcal{F}$. Thus the connection between presentation and bundle is preserved by pullback.

For this reason, when we are interested in geometricity (as we are), there are good reasons for focusing on the presentation rather than the frame. Ideally we should like to ignore the frame altogether, and focus on the presentation $\mathbb{T}$ as presenting the bundle $\mathcal{E}[\mathbb{T}] \to \mathcal{E}$. In other words, we work as though the internal locale *is* the bundle, rather than that it *is* the frame. Experience shows that calculating the frame $\Omega[\mathbb{T}]$ as an object of $\mathcal{E}$ is often complicated and unhelpful and can be misleading, although in our current state of knowledge we still have to make use of the frame in some places. We discuss this below.[1]

[1] In predicative mathematics, where the powerclass is not a power*set,* the frame is also foundationally questionable. [MV10] has some first explorations of a setting where Grothendieck toposes are replaced by Joyal's arithmetic universes, pretoposes with parametrized list objects, which do not in general have powerobjects or exponentiation. Here the analogue of $\mathcal{E}[\mathbb{T}]$ is produced directly, by universal algebra. Interpreting maths in this setting corresponds to a very pure geometricity, in the fragment where the non-finite joins can be expressed internally in terms of algebraic objects such as the natural numbers object. The first results are encouraging but are still a long way from covering the geometricity needed in the present paper.

Geometricity as a property of topos-valid constructions on sets (objects of the toposes) is defined as preservation by inverse image functors f^*. In terms of bundles this readily extends to constructions on locales, or even toposes. Suppose $X \mapsto F(X)$ is a topos-valid construction on locales, defined in terms of their frames. By the localic bundle theorem it translates into a construction on localic bundles: if $X \to \mathcal{E}$ is a bundle, then we get $F_{\mathcal{E}}X \to \mathcal{E}$ by applying F to the internal frame of $X \to \mathcal{E}$ in $\mathcal{E}$. An especially good situation is where the bundle construction works fibrewise, since if we think of the bundle $X \to \mathcal{E}$ as a variable space parametrized by points of $\mathcal{E}$ then $F_{\mathcal{E}}X \to \mathcal{E}$ is got by applying F to this variable space. The fibres of a bundle are got by pulling back along the points of $\mathcal{E}$, and in order to deal with sensibly with the case where $\mathcal{E}$ has insufficient (global) points, we ask that F should be preserved by all pullbacks of bundles. Then we say that F is geometric.

There is an obvious issue of coherence here: if $f : \mathcal{F} \to \mathcal{E}$ then we want $f^*F_{\mathcal{E}}X \cong F_{\mathcal{F}}f^*X$, but we must take care over the question of how the isomorphisms fit together. Specifically, we want naturality with respect to bundle morphisms.

The major proof ingredient, as used in [Vic04] for geometricity of the double powerlocale $\mathbb{P}$, to to describe F by a geometric construction F' on presentations $\mathbb{T}$, so $F[\mathbb{T}] \cong [F'\mathbb{T}]$. See Lemma 6.

In order to address naturality for F' we must be able to work with morphisms between presentations. However, to describe a locale map in terms of presenting sites is complicated in general, for generators for one frame must be mapped to joins of finite meets of generators of the other, and to describe the way such an assignment respects the relations is not easy. Our work is much simplified if the assignment can map generators to generators and relations to relations, and this is most straightforward if we take the locales as presented by DL-sites derived fom the frame. Thus we shall need to use the frames in proving geometricity, in particular in proving functoriality of the construction and naturality of geometricty isomorphism.

2.1 Technicalities on presentations

We first look at GRD-systems, a very general way to present propositional geometric theories in a diagrammatic package.

Definition 1 *A* GRD-system *[Vic04] is a triple of sets* (G, R, D) *(of* generators, relations *and* disjuncts*) equipped with functions*

$$\begin{array}{ccc} & & D \\ & \rho \swarrow & \downarrow \pi \\ \mathcal{F}G & \underset{\lambda}{\longleftarrow} & R \end{array}$$

where $\mathcal{F}$ is the Kuratowski finite powerset. This presents a frame

$$\Omega[G, R, D] = \mathrm{Fr}\langle G \mid \bigwedge \lambda(r) \leq \bigvee_{\pi(d)=r} \bigwedge \rho(d) \ (r \in R)\rangle.$$

GRD-systems are themselves models of a predicate geometric theory, and so there is a natural notion of homomorphism for them. If $\mathbb{T}_i = (G_i, R_i, D_i)$ is a GRD-system ($i = 1, 2$), then a homomorphism $\theta : \mathbb{T}_1 \to \mathbb{T}_2$ is a triple of carrier functions $(\theta_G, \theta_R, \theta_D)$ preserving the structure maps λ, ρ, π. Unfortunately, these do not give rise to maps between the corresponding locales, either covariantly or contravariantly. Contravariance is the obvious one to try, seeking a frame homomorphism given by $G_1 \to G_2 \to \Omega[\mathbb{T}_2]$. When we check whether the relation for r is respected, we find it transforms to

$$\bigwedge \mathcal{F}\theta_G(\lambda(r)) \leq \bigvee_{\pi(d)=r} \bigwedge \mathcal{F}\theta_G(\rho(d)),$$

i.e.

$$\bigwedge \lambda(\theta_R(r)) \leq \bigvee_{\pi(d)=r} \bigwedge \rho(\theta_D(d)).$$

The relation $\theta_R(r)$ tells us that $\bigwedge \lambda(\theta_R(r)) \leq \bigvee_{\pi(d')=\theta_R(r)} \bigwedge \rho(d')$, but this is not strong enough unless θ_D maps the fibre of r onto the fibre of $\theta_R(r)$.

Definition 2 *A homomorphism $\theta : \mathbb{T}_1 \to \mathbb{T}_2$ of GRD-systems* preserves relations *iff, for each $r \in R_1$, θ_D maps the fibre of r in D_1 onto the fibre of $\theta_R(r)$ in D_2.*

Proposition 3 *Let a homomorphism $\theta : \mathbb{T}_1 \to \mathbb{T}_2$ of GRD-systems preserve relations. Then θ defines a map $[\theta] : [\mathbb{T}_2] \to [\mathbb{T}_1]$ by $\Omega[\theta](g) = \theta_G(g)$.*

Proof. The proof was outlined above. ■

Preserving relations is a geometric property of GRD-homomorphisms, and we now show that the geometricity of presentations is natural with respect to it.

Proposition 4 *Let $\theta : \mathbb{T}_1 \to \mathbb{T}_2$ be a relation preserving homomorphism of GRD-systems $\mathbb{T}_i = (G_i, R_i, D_i)$ in a topos $\mathcal{E}$, and let $f : \mathcal{E}' \to \mathcal{E}$ be a geometric morphism. Then the following square commutes for locales over $\mathcal{E}'$.*

$$\begin{array}{ccc} f^*[\mathbb{T}_2] & \cong & [f^*\mathbb{T}_2] \\ f^*[\theta] \downarrow & & \downarrow [f^*\theta] \\ f^*[\mathbb{T}_1] & \cong & [f^*\mathbb{T}_1] \end{array}$$

Proof. The proof in [Vic04] of the geometricity of presentations obtains the isomorphism $f^*[\mathbb{T}] \cong [f^*\mathbb{T}]$ by showing that a certain square

$$\begin{array}{lcl} \mathcal{E}'[f^*\mathbb{T}] & \longrightarrow & \mathcal{E}[\mathbb{T}] \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E} \end{array}$$

is a (pseudo)pullback of toposes. In the case of a homomorphism θ it therefore suffices to check commutativity of the square

$$\begin{array}{ccc} \mathcal{E}'[f^*\mathbb{T}_2] & \longrightarrow & \mathcal{E}[\mathbb{T}_2] \\ \mathcal{E}'[f^*\theta]\downarrow & & \downarrow\mathcal{E}[\theta] \\ \mathcal{E}'[f^*\mathbb{T}_1] & \longrightarrow & \mathcal{E}[\mathbb{T}_1] \end{array}$$

and this can be deduced from the discussion in [Vic04]. ■

We shall also be extensively using DL-sites, where the generators form a distributive lattice. Note that for us "distributive lattice" will always assume boundedness, i.e. both bottom and top. We shall use the phrase *topless distributive lattice* for an unbounded distributive lattice that has bottom but not necessarily a top.

Definition 5 *A* DL-site *is a pair $(L, \lhd)$ where L is a distributive lattice and the cover relation $\lhd$ is a set of pairs $(a, (a_i)_i)$ where $a \in L$ and $(a_i)_i$ is a directed family in L. We also require that $\lhd$ is join and meet stable: if $a \lhd (a_i)_i$, then $a \vee b \lhd (a_i \vee b)_i$ and $a \wedge b \lhd (a_i \wedge b)_i$ for every $b \in L$.*

The DL-site presents *the locale X given by*

$$\Omega X = \mathrm{Fr}\langle L \text{ (qua DL)} \mid a \leq \bigvee\nolimits_i^{\uparrow} a_i \text{ (for } a \lhd (a_i)_i)\rangle.$$

A topless DL-site *has exactly the same definition, except that L may be a topless distributive lattice. Its locale is given by*

$$\Omega X = \mathrm{Fr}\langle L \text{ (qua topless DL)} \mid \top \leq \bigvee\nolimits^{\uparrow} L, \\ a \leq \bigvee\nolimits_i^{\uparrow} a_i \text{ (for } a \lhd (a_i)_i)\rangle.$$

Note that if the topless DL L happens to have a top element, then the same locale is presented by both definitions.

If the cover relation is defined by sets R and D as in a GRD-system, then DL-sites are again the models of a geometric theory and have a natural notion of homomorphism. This time the carrier on the generators must be a lattice homomorphism. We say that a homomorphism *preserves relations* (or *preserves covers*) if it satisfies the condition as for GRD-systems. Note that the DL-site can be expressed as a GRD-system by turning "qua distributive lattice" (preservation of finitary meets and finitary joins) into relations; but preservation of relations will amount to the same according to either definition.

For any locale X there is a canonical DL-site on the frame ΩX, with relations $(\bigvee^{\uparrow} S) \leq \bigvee^{\uparrow}_{a \in S} a$ for each directed subset S of ΩX. Then a lattice homomorphism $\theta_G : \Omega X_1 \to \Omega X_2$ is a frame homomorphism iff it gives a cover preserving homomorphism of DL-sites by taking $\theta_R(S)$ to be the image of S under θ_R. The construction of canonical DL-sites is not geometric.

We can now put these results together into a pattern for proving geometricity of locale constructions. It an elaboration of [Vic04, Proposition 5.5], taking more care over the naturality.

Lemma 6 *Let $F : \mathbf{Loc} \to \mathbf{Loc}$ be an intuitionistically describable functorial construction of locales, giving an endofunctor $F_{\mathcal{E}}$ of each $\mathbf{Loc}/\mathcal{E}$. Let F' be a geometric construction from DL-sites to GRD-systems that is functorial with respect to relation preserving homomorphisms.*

Suppose we have an intuitionistic proof (valid over any topos) that if $\mathbb{T}$ is a DL-site then $[F'\mathbb{T}] \cong F[\mathbb{T}]$, the isomorphism being natural with respect to relation preserving homomorphisms. Then F is geometric: for any geometric morphism $f : \mathcal{E}' \to \mathcal{E}$ there is a natural isomorphism $f^* F_{\mathcal{E}} \cong F_{\mathcal{E}'} f^*$.

Proof. For a given locale X, let $\mathbb{T}$ be its canonical DL-site so $X = [\mathbb{T}]$. Then the isomorphism $f^* F_{\mathcal{E}} X \cong F_{\mathcal{E}'} f^* X$ is given by a sequence

$$F f^*[\mathbb{T}] \underset{(1)}{\cong} F[f^*\mathbb{T}] \underset{(2)}{\cong} [F' f^* \mathbb{T}] \underset{(3)}{\cong} [f^* F' \mathbb{T}] \underset{(4)}{\cong} f^*[F'\mathbb{T}] \underset{(5)}{\cong} f^* F[\mathbb{T}].$$

The isomorphisms here depend on (1) functoriality of F together with geometricity of presentations, (2) characterization of F', (3) geometricity of F', (4) geometricity of presentations and (5) characterization of F' again. Any locale map gives (non-geometrically) a relation preserving homomorphism between the canonical sites, and we can examine naturality on 5 squares corresponding to the division above. Commutativity for squares (1) and (4) follows from Proposition 4; for squares (2) and (5) by hypothesis; and for square (3) from naturality of geometric constructions. ■

3 Valuations

The standard definition of *valuation* on a distributive lattice L is that it is a function $m : L \to [0, \infty]$ such that $m(\bot) = 0$ and the *modular* law

$$mU + mV = m(U \cup V) + m(U \cap V)$$

holds. Here we shall allow m to take its values in the lower reals $\overrightarrow{[0, \infty]}$.

Note that the definition also makes sense when L is topless.

If L is a frame, then a valuation m is *continuous* if it preserves directed joins. We shall take "continuous" as understood in this situation.

For a locale X, we say that a valuation on X is a continuous valuation on its frame ΩX. [Vic08] defines a *valuation locale* $\mathfrak{V}X$ whose points are the valuations on X. A key result in that paper is a geometricity theorem for $\mathfrak{V}X$. We have generalized it slightly to allow for topless distributive lattices – in fact, the results can still be made to work for distributive $\vee$-semilattices.

Theorem 7 *Let X be a locale presented by a topless DL-site $(L, \lhd)$. Then valuations on X are equivalent to valuations m on L satisfying $m(a) \leq \bigvee_i^{\uparrow} m(a_i)$ if $a \lhd (a_i)_i$.*

Proof. The proof in [Vic08] still works in the topless case. But note that the conditions of join and meet stability should have been included in [Vic08] – they are clearly expected in the cited proof in [VT04]. ■

This description can be formulated as a presentation for $\mathfrak{V}X$, and also gives us a description of its generalized points. It will be important in our geometric treatment.

Theorem 8 $\mathfrak{V}$ *is geometric as a functor.*

Proof. We use Lemma 6. F is $\mathfrak{V}$, and the geometric construciton F' is implicit in Theorem 7 and more explicit in [Vic08]. It transforms a DL-site $\mathbb{T} = (L, \lhd)$ into a GRD-system with generators from $L \times \{q \in \mathbb{Q} \mid q > 0\}$. The generator (a, q) corresponds to the open comprising those valuations m for which $m(a) > q$. Now the isomorphism $[F'\mathbb{T}] \cong F[\mathbb{T}]$ says that the propositional theory of continuous valuations on $\Omega[\mathbb{T}]$ is equivalent to that of valuations on L respecting the covers, in other words the models of $F'\mathbb{T}$, and that is just the content of Theorem 7.

For naturality, suppose we have a relation preserving homomorphism $\theta : \mathbb{T}_1 \to \mathbb{T}_2$ between DL-sites. We want to show that the following square commutes.

$$\begin{array}{ccc} F[\mathbb{T}_2] & \cong & [F'\mathbb{T}_2] \\ F[\theta]\downarrow & & \downarrow[F'\theta] \\ F[\mathbb{T}_1] & \cong & [F'\mathbb{T}_1] \end{array}$$

Suppose m is a continuous valuation on $\Omega F[\mathbb{T}_2]$. Its images in $[F'\mathbb{T}_1]$ are the valuations on L_1 got by composing m with the corresponding maps

$$\begin{array}{ccc} \Omega[\mathbb{T}_2] & \longleftarrow & L_2 \\ [\theta]^*\uparrow & & \uparrow\theta_G \\ \Omega[\mathbb{T}_1] & \longleftarrow & L_1 \end{array}$$

and they are equal. ■

4 The frame of $R' \times X$

[2]Throughout this section we fix a locale X, presented by a DL-site $(L, \lhd)$.

The locale $R = \overrightarrow{[0,\infty]}$ has linear structure but is also a lattice, and is not far off being a Riesz algebra as described in [CS09]. The main difference, arising from our use of lower reals instead of Dedekind, is that it has no subtraction. Moreover, in order to be able to multiply continuously we must avoid the negative numbers. (Multiplying by negatives would be antitone with respect to the specialization order for lower reals, and hence discontinuous.) R is a localic semiring, and also a "localic frame" in that it is of the form $\mathbb{S}^{R'}$ for the locale $R' = \overleftarrow{[0,\infty)}$. It follows that maps $X \to R$ inherit this linear and lattice structure, and we shall exploit this in dealing with integration. In particular, the linear structure will be vital in formulating our Riesz Theorem, that valuations are linear functionals.

[2]where $R' = \overleftarrow{[0,\infty)}$

From $R \cong \mathbb{S}^{R'}$, we see that maps $X \to R$ are equivalent to maps $R' \times X \to \mathbb{S}$, i.e. opens of $R' \times X$, and so the linear and lattice structure on those maps must be apparent in the structure of $\Omega(R' \times X)$. In this section we examine this in some technical detail.

We first find a topless DL-site for $R' \times X$. Writing Q for the set of non-negative rationals, a topless distributive lattice, R' is presented by a topless DL-site on Q with covers $q \lhd \{q' \in Q \mid q' < q\}$ (if $q > 0$).

Lemma 9 *We can present $\Omega(R' \times X)$ as*

$$\begin{aligned}\mathrm{Fr}\langle Q \otimes L \text{ (qua topless DL)} \mid \top \leq \bigvee{}^{\uparrow}(Q \otimes L) \\ (q \times a) \vee u \leq \bigvee{}^{\uparrow}_{q'<q}(q' \times a) \vee u \text{ (if } q > 0) \\ (q \times a) \vee u \leq \bigvee{}^{\uparrow}_{i \in I}(q \times a_i) \vee u \text{ (if } a \lhd \{a_i \mid i \in I\})\rangle.\end{aligned}$$

Here $Q \otimes L$ is the $\vee$-semilattice tensor of Q and L, described in more detail in Section 4.1.

Proof. The first relation presents $\mathrm{Idl}(Q \otimes L) \cong \mathrm{Idl}\, Q \otimes \mathrm{Idl}\, L$. The remaining relations are the meet and join stabilized versions of the relations needed to give $\Omega R'$ and ΩX. ■

Corollary 10 *A valuation on $R' \times X$ is equivalent to a valuation $m : Q \otimes X \to \overrightarrow{[0, \infty]}$ satisfying the following conditions.*

1. $m((q \times a) \vee u) = \bigvee^{\uparrow}_{q'<q} m((q' \times a) \vee u)$ *($q > 0$).*
2. $m((q \times a) \vee u) \leq \bigvee^{\uparrow}_{i \in I} m((q \times a_i) \vee u)$ *if $a \lhd \{a_i \mid i \in I\}$.*

Proof. Combine Lemma 9 with Theorem 7. ■

In the rest of the Section we analyse $Q \otimes L$ in more detail. This is in effect an algebra of simple functions. It already appears in [CS09] in that role, though we shall anlyse its structure more precisely.

In Section 4.1 we describe $Q \otimes L$ concretely in a way that will relate to the definition of lower integral in [Vic08].

In Section 4.2 we relate it to the "modular monoid" $M(L)$, the free commutative monoid over L subject to $a + b = (a \vee b) + (a \wedge b)$ and $\bot = 0$. [CS09] derive a concrete representation of this from results in [HT48], and use its rationalization (with rational coefficients instead of integer) as an algebra of simple functions. We shall show that this rationalization is order isomorphic to our $Q \otimes L$.

In Section 5 this double characterization of $Q \otimes L$ will enable us to state and prove our Riesz theorem by working within $\mathfrak{V}(R' \times X)$.

4.1 Coproducts of distributive lattices

As is well known (see e.g. [Fra76]), if K and L are two distributive lattices then their coproduct can be constructed as a tensor product $K \otimes L$ with respect to their structure as $\wedge$- or $\vee$-semilattices. (This is directly analogous to the way a coproduct of commutative rings is their tensor product as Abelian groups.) In more detail, we say a function θ from $K \times L$ to a $\vee$-semilattice A is a $\vee$-*bimorphism* if θ preserves finite joins in each argument of θ when the other argument is fixed. Then the $\vee$-semilattice tensor $K \otimes L$ is characterized by being equipped with a universal $\vee$-bimorphism from $K \times L$, written $(a, b) \longmapsto a \times b$, such that any $\vee$-bimorphism θ factors uniquely via a $\vee$-semilattice homomorphism from $K \otimes L$, with $a \times b$ mapping to $\theta(a, b)$. This tensor exists for arbitrary $\vee$-semilattices K and L, but in the particular case where they are distributive lattices, so too is $K \otimes L$ and it is the distributive lattice coproduct. $a \times b$ then is the meet of the generators corresponding to a and b.

By a dual process, the coproduct can also be found as a $\wedge$-semilattice tensor. We shall still write $K \otimes L$ for that, since we know it is isomorphic to the $\vee$-semilattice tensor, but note that the $\wedge$-bimorphism is different. It maps (a, b) to the *join* of the generators, which we shall write as $a \odot b = a \times \top \vee \top \times b$. Then also $a \times b = a \odot \bot \wedge \bot \odot b$.

Regarding $K \otimes L$ as $\vee$-semilattice tensor, a typical element can be expressed – though not uniquely – as a finite join $\bigvee_i a_i \times b_i$. Hence $K \otimes L$ can be represented as a $\vee$-semilattice quotient of $\mathcal{F}(K \times L)$, since the Kuratowski finite powerset is a free semilattice, with $\cup$ as the semilattice operation. Our aim is to find its corresponding congruence preorder, a description of when $\bigvee_i a_i \times b_i \leq \bigvee_j a'_j \times b'_j$. Applying the distributive law,

$$\begin{aligned}\bigvee_{j=1}^{n} a'_j \times b'_j &= \bigvee_{j=1}^{n} (a'_j \odot \bot \wedge \bot \odot b'_j) \\ &= \bigwedge_{n=S \cup T} a'_S \odot b'_T\end{aligned}$$

where $a'_S = \bigvee_{j \in S} a'_j$ and the meet is over pairs (S, T) of finite sets such that $\{1, \ldots, n\} = S \cup T$. (We do not assume that S and T are disjoint; the non-disjoint pairs make no difference to the meet.) This redistribution reduces the problem to that of when $a \times b \leq c \odot d$. Geometrically, if a and c are subspaces of one space, and b and d subspaces of another, then $a \times b \leq c \odot d$ means that for every $x \in a$ and $y \in b$ we have either $x \in c$ or $y \in d$. Classically we could argue as follows: if $a \nleq c$ then there is some $x \in c - a$, so for all $y \in b$ we must have $y \in d$ and so $b \leq d$. Hence $a \times b \leq c \odot d$ iff $a \leq c$ or $b \leq d$, thus giving a solution to our problem. The classical argument has no direct application to current situation, but nonetheless if we take it as a guess we shall find it validated by our Proposition 11.

The statement of the proposition uses only the $\vee$-semilattice structure of K and L, and we find it convenient to prove it in the greater generality of *distributive $\vee$-semilattices* – those $\vee$-semilattices K for which the ideal completion, a complete join semilattice, is a frame. This can be characterized internally in two equivalent ways:

1. Whenever $x \leq \bigvee_{i=1}^{n} y_i$, then there are $x_i \leq y_i$ such that $x = \bigvee_{i=1}^{n} x_i$. (If K is a distributive lattice then we can take $x_i = x \wedge y_i$.)

2. If $z \leq x \vee y_i$ $(1 \leq i \leq n)$ then $z \leq x \vee y$ for some $y \leq$ all y_i.

Note that any topless distributive lattice is a distributive $\vee$-semilattice. That is the main situation here in which we shall need the extra generality.

Proposition 11 *Let K and L be distributive $\vee$-semilattices. We define a preorder $\leq$ on $\mathcal{F}(K \times L)$ by $A \leq B$ if for all $(a,b) \in A$ and decompositions $B = S \cup T$ we have $a \leq \bigvee\{c \mid (c,d) \in S\}$ or $b \leq \bigvee\{d \mid (c,d) \in T\}$. Then $\leq$ is a congruence preorder on the $\vee$-semilattice $(\mathcal{F}(K \times L), \cup)$, and there is a bimorphism $K \times L \to \mathcal{F}(K \times L)/\leq$, defined by $(a,b) \longmapsto \{(a,b)\}$, that makes $\mathcal{F}(K \times L)/\leq$ a tensor product $K \otimes L$.*

Proof. It is easy to see that $\leq$ is a congruence preorder and that the function defined is a bimorphism. It remains to show the universal property for a tensor product. Let $\phi : K \times L \to M$ be a bimorphism to a $\vee$-semilattice M. If it is to factor via $\mathcal{F}(K \times L)/\leq$, then it has to be as $\bar{\phi}(A) = \bigvee_{(a,b)\in A} \phi(a,b)$. We must show that this respects the preorder $\leq$.

For $A \leq B$, it suffices to consider A a singleton $\{(a,b)\}$ and B finitely enumerated $\{(c_i, d_i) \mid 1 \leq i \leq n\}$. We shall use induction on n. For the base case, $n = 0$, and with the only decomposition of $\emptyset$, we see that either $a \leq 0$ or $b \leq 0$. In either case $\phi(a,b) = 0 = \bar{\phi}(B)$.

Now suppose $B = \{(c_i, d_i) \mid 1 \leq i \leq n+1\}$. For every decomposition $n = S \cup T$, we get two decompositions of $n+1$ (putting the final index $n+1$ into either S or T), and we obtain

$$\begin{aligned} &a \leq c_S \text{ or } b \leq d_T \vee d_{n+1} \\ &a \leq c_S \vee c_{n+1} \text{ or } b \leq d_T \end{aligned}$$

where we write $c_S = \bigvee\{c \mid (c,d) \in S\}$ etc.[3] Hence we see either $a \leq c_S$ or $b \leq d_T$ or both $a \leq c_S \vee c_{n+1}$ and $b \leq d_T \vee d_{n+1}$.

It follows that the set of decompositions of n can be decomposed as $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ such that

$$\begin{aligned} &(S,T) \in \mathcal{D}_1 \Longrightarrow a \leq c_S \\ &(S,T) \in \mathcal{D}_2 \Longrightarrow b \leq d_T \\ &(S,T) \in \mathcal{D}_3 \Longrightarrow a \leq c_S \vee c_{n+1} \text{ and } b \leq d_T \vee d_{n+1}. \end{aligned}$$

For every $(S,T) \in \mathcal{D}_1 \cup \mathcal{D}_3$ we have $a \leq c_S \vee c_{n+1}$, and it follows by distributivity that there is some a' such that $a \leq a' \vee c_{n+1}$ and $a' \leq c_S$ for all $(S,T) \in \mathcal{D}_1 \cup \mathcal{D}_3$. Now we can find $a = a_1 \vee a_2$ where $a_1 \leq a'$ and $a_2 \leq c_{n+1}$.

[3]This notation is temporary. In Section 4.2 we shall use a similar notation but for meets.

Similarly, we can find $b = b_1 \vee b_2$ where $b_1 \leq d_T$ for all $(S,T) \in \mathcal{D}_2 \cup \mathcal{D}_3$, $b_2 \leq d_{n+1}$; also $b_2 \leq d_T$ for all $(S,T) \in \mathcal{D}_2$.

Thus $\phi(a,b) = \phi(a_1,b_1) \vee \phi(a_1,b_2) \vee \phi(a_2,b_1) \vee \phi(a_2,b_2)$. From the definitions, and using the fact that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ cover all decompositions of n, we see that $\{(a_1,b_1)\}$, $\{(a_1,b_2)\}$ and $\{(a_2,b_1)\}$ are all less than $\{(c_i,d_i) \mid 1 \leq i \leq n\}$ and so, by induction, $\phi(a_1,b_1)$, $\phi(a_1,b_2)$ and $\phi(a_2,b_1)$ are all less than $\bigvee_{i=1}^{n} \phi(c_i,d_i)$. Also $\phi(a_2,b_2) \leq \phi(c_{n+1},d_{n+1})$. This completes the proof. ■

Example 12 *Let Q be the $\vee$-semilattice of non-negative rationals, distributive because it is linearly ordered. Let L be a distributive lattice, and let A and B be in $\mathcal{F}(Q \times L)$. Then $A \leq B$ iff for every $(q,a) \in A$ we have $a \leq \bigvee\{b \mid \exists(r,b) \in B.\ q \leq r\}$. The condition of Proposition 11 says that for every decomposition $B = S \cup T$ we have $q \leq \max_{(r,b)\in S} r$ or $a \leq \bigvee_{(r,b)\in T} b$, so we get the result by taking $S = \{(r,b) \in B \mid r < q\}$ and $T = \{(r,b) \in B \mid q \leq r\}$. The converse is also clear.*

We can extend the order on $\mathcal{F}(Q \times L)$ to one on $(Q \times L)^*$ (the set of finite lists in $Q \times L$) in the obvious way. Now given $(q_i,a_i)_{i=1}^{n}$ we can assume up to equivalence that the q_is are in ascending order. Also, $(q_i,a_i)_i \equiv (q_i,a'_i)_i$ where $a'_i = \bigvee\{a_{i'} \mid q_i \leq q_{i'}\}$, so we can assume up to equivalence that in addition the a_is are in descending order. We say that $(q_i,a_i)_i$ is *sorted* if the q_is ascend and the a_is descend, so every element of $(Q \times L)^*$ can be put in sorted form. Furthermore, we can eliminate duplicates and copies of 0 amongst the q_is. We say that $(q_i,a_i)_i$ is *strictly sorted* if it is sorted and the q_is are non-zero and ascend strictly.

Lemma 13 *Let $(q_i,a_i)_{i=1}^{n}$ and $(q_i,b_i)_{i=1}^{n}$ both be strictly sorted. Then $(q_i,a_i)_{i=1}^{n} \leq (q_i,b_i)_{i=1}^{n}$ iff $a_i \leq b_i$ for each i.*

Proof. This follows from the calculation of the example, using $\bigvee\{b_j \mid q_i \leq q_j\} = \bigvee\{b_j \mid i \leq j\} = b_i$. ■

Definition 14 *Let $(q_i,a_i)_{i=1}^{n}$ be arbitrary and let $(r_j)_{j=1}^{m}$ be a strictly ascending sequence of non-zeros in Q that includes all the elements q_i. Then the* refinement *of $(q_i,a_i)_{i=1}^{n}$ to $(r_j)_{j=1}^{m}$ is the sequence $(r_j,a_j^{(r)})_{j=1}^{m}$ where $a_j^{(r)} = \bigvee\{a_i \mid r_j \leq q_i\}$.*

It is obvious that $(q_i,a_i)_{i=1}^{n} \equiv (r_j,a_j^{(r)})_{j=1}^{m}$.

4.2 The modular monoid

Definition 15 *Let L be a distributive lattice (possibly topless). We write $M(L)$ for the* modular monoid *on L, presented as*

$$\mathbf{CMon}\langle L \mid a + b = (a \wedge b) + (a \vee b)\ (a,b \in L),\ 0 = \bot\rangle.$$

As [CS09] point out, this monoid, presented by universal algebra, can also be given a concrete representation described in [HT48].

Definition 16 *Let L be a distributive lattice. We define a preorder $\leq$ on L^* as follows. Suppose $x = (x_i)_{i=1}^m$ and $y = (y_j)_{j=1}^n$ are elements of L^*. If $I \subseteq m$ (understood as $\{1, \ldots, m\}$) we write x_I for $\bigwedge_{i \in I} x_i$, and similarly y_J if $J \subseteq n$. Then $x \leq y$ if, for every $I \subseteq m$, we have*

$$x_I \leq \bigvee \{y_J \mid J \in \mathcal{F}n, |J| = |I|\}.$$

Clearly, $x \leq y$ iff for all $k \in \mathbb{N}$ we have

$$\bigvee \{x_I \mid I \in \mathcal{F}m, |I| = k\} \leq \bigvee \{y_J \mid J \in \mathcal{F}n, |J| = k\}.$$

(Of course, it suffices to check the cases $k \leq m$. Note that the cardinalities of finite subsets of $\mathbb{N}$ are well defined, because $\mathbb{N}$ has decidable equality.) It follows that $\leq$ is a preorder; we shall write $\equiv$ for the corresponding equivalence relation.

Theorem 17 *$\equiv$ is a monoid congruence on L^* and $M(L) \cong L^* / \equiv$. It is a partially ordered, commutative, cancellation monoid.*

Proof. The result was stated in these terms in [CS09], using arguments largely present in [HT48]. Although the results in [HT48] are more explicitly stated for the case where L is a Boolean algebra, they have techniques for generalizing to arbitrary distributive lattices by embedding them in Boolean algebras. In any case, the proof techniques are general enough in themselves. ∎

One central lemma is so important that we shall state it separately.

Lemma 18 (Generalized Modularity Lemma) *Suppose $x_i \in L$ ($1 \leq i \leq m$). Then in $M(L)$ we have*

$$\sum_{i=1}^m x_i = \sum_{k=1}^m \bigvee \{x_I \mid I \in \mathcal{F}m, |I| = k\}.$$

(Note that it makes no difference if the upper bound m of the summation on the right is replaced any larger natural number.)

Proof. This is stated, with a proof sketch, in [HT48]. In more detail, one can use a double induction on m and j to show

$$\sum_{i=1}^{m+1} x_i = u_j + \bigvee_{I \in \mathcal{F}m, |I| = j} x_{I \cup \{m+1\}} + v_j$$

where

$$u_j = \sum_{k=1}^{j} \bigvee_{I \in \mathcal{F}(m+1), |I| = k} x_I$$
$$v_j = \sum_{k=j+1}^{m} \bigvee_{I \in \mathcal{F}m, |I| = k} x_I.$$

The case $j = 0$ is by using induction on m to see that $v_0 = \sum_{i=1}^{m} x_i$. For $1 \leq j \leq m$ we have, by induction on j,

$$\begin{aligned}
\sum_{i=1}^{m+1} x_i &= u_{j-1} + \bigvee_{I \in \mathcal{F}m, |I|=j-1} x_{I \cup \{m+1\}} + \bigvee_{I \in \mathcal{F}m, |I|=j} x_I + v_j \\
&= u_{j-1} + (\bigvee_{I \in \mathcal{F}m, |I|=j-1} x_{I \cup \{m+1\}} \vee \bigvee_{I \in \mathcal{F}m, |I|=j} x_I) \\
&\quad + (\bigvee_{I \in \mathcal{F}m, |I|=j-1} x_{I \cup \{m+1\}} \wedge \bigvee_{I \in \mathcal{F}m, |I|=j} x_I) + v_j \\
&= u_j + w + v_j
\end{aligned}$$

where

$$\begin{aligned}
w &= \bigvee_{I \in \mathcal{F}m, |I|=j-1} x_{I \cup \{m+1\}} \wedge \bigvee_{I \in \mathcal{F}m, |I|=j} x_I \\
&= \bigvee_{I \in \mathcal{F}m, |I|=j-1} \bigvee_{I' \in \mathcal{F}m, |I'|=j} x_{I \cup \{m+1\} \cup I'}.
\end{aligned}$$

It therefore remains only to show that

$$w = \bigvee_{I \in \mathcal{F}m, |I|=j} x_{I \cup \{m+1\}}.$$

Both directions are obvious.

The result now follows by taking $j = m$. ■

Proposition 19 *Let L be a distributive lattice. Then in $M(L)$ we have the following.*

1. *Every element can be written in* descending form *as $\sum_{k=1}^{m} y_k$ with $y_1 \geq \cdots \geq y_m$.*

2. *If $a = (a_i)_{i=1}^{m}$ is in descending form then $\bigvee \{a_I \mid I \in \mathcal{F}m, |I| = k\} = a_k$.*

3. *Suppose $a = (a_i)_{i=1}^{m}$ and $b = (b_j)_{j=1}^{n}$ are in descending form. Then $a \leq b$ iff $a_i \leq b_i$ for every i (after padding out the shorter sequence with zeros).*

4. *$M(L)$ is a topless distributive lattice, with binary meets and joins calculated termwise on descending forms.*

5. *$M(L)$ is* internally modular, *in the sense that $a + b = (a \vee b) + (a \wedge b)$ for all $a, b \in M(L)$.*

6. *Suppose $a = (a_i)_{i=1}^{m}$ and $b = (b_j)_{j=1}^{n}$ are in descending form. Then, in descending form,*

$$(a + b)_k = \bigvee_{k = k_1 + k_2} a_{k_1} \wedge b_{k_2}.$$

(We take $a_0 = b_0 = \top$.)

7. $M(L) \cong \mathbb{N} \otimes L$.

8. *$M(L)$ is a commutative semiring, with multiplication extended additively from $ab = a \wedge b$ $(a, b \in L)$ and with $\top_L$ as the unit element.*

Proof. (1) Given an arbitrary $\sum_{i=1}^m x_i$, take $y_k = \bigvee\{x_I \mid I \in \mathcal{F}m, |I| = k\}$ as in Lemma 18. If $I \in \mathcal{F}m, |I| = k+1$, then we can write $I = \{i\} \cup I'$ where $i = \min I$ and $I' = I - \{i\}$. Then $x_I = x_i \wedge x_{I'} \leq x_{I'} \leq y_k$, so $y_{k+1} \leq y_k$.

(2) If $I \in \mathcal{F}m, |I| = k$ then I has an element $i \geq k$. Hence $a_I \leq a_i \leq a_k$. On the other hand, a_k is amongst the a_Is, by taking $I = \{1, \ldots, k\}$.

(3) and (4) now follow immediately.

(5) Let $a = (a_i)_{i=1}^n$ and $b = (b_j)_{j=1}^n$ in descending form. Then

$$a + b = \sum_i (a_i + b_i) = \sum_i ((a_i \vee b_i) + (a_i \wedge b_i)) = (a \vee b) + (a \wedge b).$$

(6) Assume $m = n$, padding the shorter sequence with 0s. Then

$$\begin{aligned} a + b = \sum_{i=1}^n a_i + b_i &= \sum_{k=1}^{2n} \bigvee\{a_{\max I} \wedge b_{\max J} \mid I, J \in \mathcal{F}n, |I| + |J| = k\} \\ &= \sum_{k=1}^{2n} \bigvee_{k_1+k_2=k} \bigvee\{a_{\max I} \wedge b_{\max J} \mid I, J \in \mathcal{F}n, |I| = k_1, |J| = k_2\} \\ &= \sum_{k=1}^{2n} \bigvee_{k_1+k_2=k} a_{k_1} \wedge b_{k_2}. \end{aligned}$$

(7) Just as with Q, every element of $\mathbb{N} \otimes L$ can be put into strictly sorted form as $\bigvee_{i=1}^m q_i \times a_i$ with $q_i \in \mathbb{N}$. This can be refined to a form where the q_is are all the natural numbers from 1 to q_m. We map

$$\bigvee_{i=1}^{q_m} i \times a_i \leftrightarrow \sum_i (i - (i-1))a_i = \sum_i a_i.$$

This is monotone in both directions and so gives an order isomorphism.

(8) If $a \in L$ then the function $\alpha_a : L \to M(L)$ given by $\alpha_a(b) = a \wedge b$ is modular and so extends to a monoid homomorphism $\overline{\alpha}_a : M(L) \to M(L)$. Now if $a \in M(L)$, $a = \sum_i a_i$ for $a_i \in L$, define $\beta_a : L \to M(L)$ by $\beta_a(b) = \overline{\alpha}_b(a) = \sum_i a_i \wedge b$. This too is modular and so extends to a monoid homomorphism $\overline{\beta}_a : M(L) \to M(L)$. If $b = \sum_j b_j$ then we can define $ab = \overline{\beta}_a(b) = \sum_{ij} a_i \wedge b_j$. From this formula one calculates that this multiplication is commutative and associative and that $\top_L$ is the unit, and the derivation shows that it distributes over addition. ∎

We shall need to “rationalize” $M(L)$, i.e. extend it by rational coefficients.

Definition 20 *$M_Q(L)$ is the module of fractions of $M(L)$, pairs (a, r) $(a \in M(L)$, $1 \leq r \in \mathbb{N})$ modulo the preorder $(a, r) \leq (b, s)$ if $sa \leq rb$. We write $\frac{a}{r}$ for the equivalence class of (a, r).*

Proposition 21 *$Q \otimes L \cong M_Q(L)$.*

Proof. We map $\theta : M_Q(L) \to Q \otimes L$ by $\left(\sum_{i=1}^n a_i\right)/r \mapsto \bigvee_{i=1}^n (\frac{i}{r} \times a_i)$ if $\sum_i a_i$ is in descending form.

First, we show that the value of θ is unchanged if we replace $\left(\sum_{i=1}^n a_i\right)/r$ by $\left(s \sum_{i=1}^n a_i\right)/sr$.

$$\theta\left(s\left(\sum_{i=1}^n a_i\right)/sr\right) = \bigvee_{j=1}^{sn} (\frac{j}{sr} \times a'_j)$$

where $a'_j = a_i$, where i is the least such that $j \leq si$. This is because $s\sum_{i=1}^n a_i$, in descending form, has s copies of a_1, then s copies of a_2, and so on. On the other hand, we can also refine $\bigvee_{i=1}^n (\frac{i}{r} \times a_i)$ to the same sequence of rationals as $\bigvee_{j=1}^{sn} (\frac{j}{sr} \times a''_j)$ where $a''_j = \bigvee\{a_i \mid \frac{j}{sr} \leq \frac{i}{r}\}$. Hence $a''_j = a'_j$.

Now suppose $\left(\sum_{i=1}^n a_i\right)/r \leq \left(\sum_{i=1}^n b_i\right)/s$. By the previous argument we can assume that $r = s$ and so $ra \leq rb$, so $a \leq b$ and it follows that $\theta\left(\frac{a}{r}\right) \leq \theta\left(\frac{b}{r}\right)$.

We have now defined θ as a monotone map $M_Q(L) \to Q \otimes L$. It is surjective, since in any element of $Q \otimes L$ we can put the rational coefficients over a common denominator and then refine to the form $\bigvee_{i=1}^n (\frac{i}{r} \times a_i)$. Finally, it is injective, since if $\bigvee_{i=1}^n (\frac{i}{r} \times a_i) \leq \bigvee_{i=1}^n (\frac{i}{r} \times b_i)$ then $a \leq b$. ■

Proposition 22 *In $Q \otimes L$ we have, for all non-negative rationals r_i and s_j, and $a_i, b_j \in L$, that $\sum_{i=1}^m r_i a_i \leq \sum_{j=1}^n s_j b_j$ iff for all $I \in \mathcal{F}m$ we have $a_I \leq \sum\{b_J \mid J \in \mathcal{F}n, r_I \leq s_J\}$. (Here, e.g., $r_I = \sum_{i \in I} r_i$.)*

Proof. Stated in [CS09]. It suffices to prove in the case where the coefficients are all natural numbers. ■

We finish this section by proving the Principle of Inclusion and Exclusion in the context of $M(L)$.

Theorem 23 (Principle of inclusion and exclusion) *In $M(L)$ we have*

$$\bigvee_{i=1}^n a_i + \sum_{I \in \mathcal{F}^+ n, |I| \ even} a_I = \sum_{I \in \mathcal{F}^+ n, |I| \ odd} a_I.$$

($\mathcal{F}^+ n$ denotes the set of non-empty Kuratowski finite subsets of $\{1, \ldots, n\}$ – the non-emptiness is signified by the $+$ in $\mathcal{F}^+$.)

Proof. We use induction on n. The base case $n = 0$ is trivial, since both sides are 0. For $n+1$ we have

$$\begin{aligned}
\bigvee_{i=1}^{n+1} a_i &+ \sum_{I\in\mathcal{F}^+(n+1),|I|\text{ even}} a_I \\
&= \bigvee_{i=1}^{n} a_i \vee a_{n+1} + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I + \sum_{I\in\mathcal{F}^+n,|I|\text{ odd}} a_I \wedge a_{n+1} \\
&= \bigvee_{i=1}^{n} a_i \vee a_{n+1} + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I + \bigvee_{i=1}^{n}(a_i \wedge a_{n+1}) + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I \wedge a_{n+1} \\
&= \bigvee_{i=1}^{n} a_i \vee a_{n+1} + (\bigvee_{i=1}^{n} a_i) \wedge a_{n+1} + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I \wedge a_{n+1} \\
&= \bigvee_{i=1}^{n} a_i + a_{n+1} + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I \wedge a_{n+1} \\
&= a_{n+1} + \sum_{I\in\mathcal{F}^+n,|I|\text{ odd}} a_I + \sum_{I\in\mathcal{F}^+n,|I|\text{ even}} a_I \wedge a_{n+1} \\
&= \sum_{I\in\mathcal{F}^+(n+1),|I|\text{ odd}} a_I .
\end{aligned}$$

In step 2 we are using induction for the n elements $a_i \wedge a_{n+1}$, step 4 is the modular law, and step 5 is induction on the first n elements a_i. ■

5 The Riesz Theorem

We now turn to the Riesz Theorem. Again, throughout this section, X is a locale presented by a DL-site $(L, \triangleleft)$.

If m is a valuation on X then by universal algebra it gives a linear map $Q \otimes L \cong M_Q(L) \to R$. This then gives a valuation on $R' \times X$, so we get a map $\underline{\mathcal{I}} : \mathfrak{V}X \to \mathfrak{V}(R' \times X)$. From our structural analysis of $Q \otimes L$ we see that $\underline{\mathcal{I}}(m)$ agrees with the lower integral as defined in [Vic08].

But $\underline{\mathcal{I}}(m)$ is not an arbitrary valuation on $\mathfrak{V}(R' \times X)$, for it is defined by linearity. We elucidate the notion of linearity to define a sublocale of $\mathfrak{V}(R' \times X)$ comprising the linear valuations, and then our Riesz Theorem is that $\underline{\mathcal{I}}$ is a homeomorphism of $\mathfrak{V}X$ onto that sublocale.

5.1 Defining $\underline{\mathcal{I}}$

Definition 24 *Let L be a distributive lattice and $m : L \to R$ be a valuation. Then we define $\underline{m} : M_Q(L) \to R$ as the unique linear extension of m, i.e.*

$$\underline{m}\left(\frac{1}{r}\sum_i a_i\right) = \frac{1}{r}\sum_i m(a_i).$$

To see that this is well defined, first, by presentation of $M(L)$, m extends uniquely to a monoid homomorphism $M(L) \to R$, and then the extension to $M_Q(L)$ is clear.

Note from Proposition 19 (5) that any linear map $M_Q(L) \to R$ is also a valuation.

Proposition 25 *Let L be a distributive lattice and $m : L \to R$ be a valuation. If $(q_i, a_i)_i \in (Q \times L)^*$ is sorted, then*

$$\underline{m}\left(\bigvee_{i=1}^{n} q_i \times a_i\right) = \sum_{i=1}^{n} (q_i - q_{i-1}) m(a_i),$$

where we write $q_0 = 0$. Note that this lower real is unchanged if we remove duplicates and copies of 0 *from the q_is.*

Proof. Putting the rational coefficients q_i over a common denominator, it suffices to consider the case where they are all integers. Refining (see Definition 14) to the sequence of all natural numbers from 1 to q_n, and using Proposition 21, we have

$$\bigvee_{i=1}^{n} q_i \times a_i = \bigvee_{j=1}^{q_n} j \times b_j = \sum_{j=1}^{q_n} b_j$$

where $b_j = \bigvee\{a_i \mid j \leq q_i\}$, so $\underline{m}\left(\bigvee_{i=1}^{n} q_i \times a_i\right) = \sum_{j=1}^{q_n} m(b_j)$. Now

$$\begin{aligned} \sum_{i=1}^{n} (q_i - q_{i-1}) m(a_i) &= \sum_{i=1}^{n} \left(\sum_{q_{i-1} < j \leq q_i} 1 \right) m(a_i) \\ &= \sum_{i=1}^{n} \sum_{q_{i-1} < j \leq q_i} m(b_j) \\ &= \sum_{j \leq q_n} m(b_j). \end{aligned}$$

■

Theorem 26 *We can define a map $\underline{\mathcal{I}} : \mathfrak{V}X \to \mathfrak{V}(R' \times X)$ so that $\underline{\mathcal{I}}(m)$ extends $\underline{m}$. It is natural in X.*

Proof. By Corollary 10 it remains to show that the relations described there are respected.

First, we want $\underline{m}((q \times a) \vee u) = \bigvee^{\uparrow}_{q' < q} \underline{m}((q' \times a) \vee u)$. Let $(s_k)_{k=1}^{n}$ be a strictly sorted sequence that includes $q = s_i$ and all the coefficients from u, and also a value $q' = s_{i-1}$, immediately before q, that is not q or any of those coefficients. Let $(s_k, c_k)_k$ and $(s_k, c'_k)_k$ be the refinements of $(q \times a) \vee u$ and $(q' \times a) \vee u$ to $(s_k)_k$. Let $c = \bigvee\{r \mid (r, b) \text{ listed in } u, r \geq q\}$. Then $c_i = c_{i-1} = c'_{i-1} = c \vee a$, $c'_i = c$, and for all other indexes j we have $c_j = c'_j$.

The terms $(s_k - s_{k-1})m(c_k)$ and $(s_k - s_{k-1})m(c'_k)$ in the sums for $\underline{m}((q \times a) \vee u)$ and $\underline{m}((q' \times a) \vee u)$ agree for all k, and are independent of q', except for $k = i$ and $k = i - 1$. Hence it suffices to show for those two terms that

$$\begin{aligned}(q' - s_{i-2})m(c \vee a) + (q - q')m(c \vee a) &= (q - s_{i-2})m(c \vee a) \\ &= \bigvee\nolimits^{\uparrow}_{s_{i-2}<q'<q} ((q' - s_{i-2})m(c \vee a) + (q - q')m(c)).\end{aligned}$$

Suppose $(q - s_{i-2})m(c \vee a) > p$. We can suppose $p \geq 0$, since the right hand side is non-negative. Then we can find $m(c \vee a) > t > 0$ with $(q - s_{i-2})t > p$. Choose q' such that $qt > q't > s_{i-2}t + p$. Then $(q' - s_{i-2})m(c \vee a) > (q' - s_{i-2})t > p$.

Next, we want $\underline{m}((q \times a) \vee u) \leq \bigvee^{\uparrow}_i \underline{m}((q \times a_i) \vee u)$ if $a \lhd \{a_i\}_i$. Let $(s_k, c_k)_k$ be a strictly sorted form for $(q \times a) \vee u$. Then

$$c_k = c'_k \vee \begin{cases} a & \text{if } s_k \leq q \\ \bot & \text{if } s_k > q \end{cases}$$

where $c'_k = \bigvee\{b \mid (r, b) \text{ listed in } u, r \geq s_k\}$. If $s_k \leq q$ then by join stability $c_k \lhd \{c'_k \vee a_i\}_i$. Hence

$$\begin{aligned}\underline{m}((q \times a) \vee u) &= \bigvee\nolimits^{\uparrow}_i \left(\sum_{s_k \leq q} (s_k - s_{k-1})m(c'_k \vee a_i) + \sum_{s_k > q} (s_k - s_{k-1})m(c'_k) \right) \\ &= \bigvee\nolimits^{\uparrow}_i \underline{m}((q \times a_i) \vee u).\end{aligned}$$

Naturality can be checked by using the canonical DL-sites on the frames. ∎

We must check that $\underline{\mathcal{I}}$ agrees with the lower integral as already defined in [Vic08].

Proposition 27 *Let $f : X \to \overrightarrow{[0, \infty]} \cong \mathbb{S}^{\overleftarrow{[0,\infty)}}$ correspond to $U : \overleftarrow{[0, \infty)} \times X \to \mathbb{S}$, i.e. $U \in \Omega(\overleftarrow{[0, \infty)} \times X)$. Then for any valuation m on X, $\underline{\mathcal{I}}(m)(U) = \underline{\int} f \; dm$.*

Proof. $\underline{\mathcal{I}}(m)$ and the lower integral both preserve directed joins, so it suffices to check on opens U in the image of $Q \otimes L$. Let $(q_i, a_i)_{i=1}^n$ be strictly sorted, so $U = \bigvee_{i=1}^n (\overleftarrow{[0, q_i)} \times a_i)$ and $\underline{\mathcal{I}}(m)(U) = \sum_i (q_i - q_{i-1})m(a_i)$. The corresponding f is given by $f(x) = \sup\{q_i \mid x \vDash a_i\}$, and

$$\begin{aligned}\underline{\int} f \; dm &= \sup_{0<r_1<\cdots<r_m} \sum_j (r_j - r_{j-1})m(f^*\overrightarrow{(r_j, \infty]}) \\ &= \sup_{0<r_1<\cdots<r_m} \underline{m} \left(\bigvee_j r_j \times b_j \right)\end{aligned}$$

where $b_j = f^*\overrightarrow{(r_j, \infty]} = \bigvee\{a_i \mid r_j < q_i\}$. By Example 12 we see $\bigvee_j r_j \times b_j \leq \bigvee_i q_i \times a_i$, so $\underline{\int} f \; dm \leq \underline{\mathcal{I}}(m)(U)$. The reverse holds by taking $r_i = q_i - \varepsilon$, letting ε tend to 0, and using continuity as in the proof of Theorem 26. ∎

Proposition 28 *$\underline{\mathcal{I}}$ is geometric. That is to say, if $X \to \mathcal{E}$ is a localic bundle over a topos, giving $\underline{\mathcal{I}} : \mathfrak{V}_{\mathcal{E}} X \to \mathfrak{V}_{\mathcal{E}}(R' \times X)$ over $\mathcal{E}$, and $g : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism, then over $\mathcal{E}'$ the following diagram commutes.*

$$\begin{array}{rcl} g^*\mathfrak{V}_{\mathcal{E}} X & \cong & \mathfrak{V}_{\mathcal{E}'} g^* X \\ g^*\underline{\mathcal{I}} \downarrow & & \downarrow \underline{\mathcal{I}} \\ g^*\mathfrak{V}_{\mathcal{E}}(R' \times X) & \cong & \mathfrak{V}_{\mathcal{E}'}(R' \times g^* X) \end{array}$$

Proof. In fact this follows from the way $\underline{\mathcal{I}}$ was defined in terms of DL-sites. ■

Remark 29 *From this it follows that the lower integral is also geometric, because it can be defined from $\underline{\mathcal{I}}$. Hence structure defined using integration is geometric.*

5.2 Linear frames

For our Riesz Theorem we shall be interested in *integrals,* functionals that transform maps $X \to R$ into lower reals linearly. As we have seen, maps $X \to R$ are equivalent to opens of $R' \times X$, so we are interested in the linear structure – addition and scalar multiplication by non-negative rationals – on $\Omega(R' \times X)$. We shall also want to relate this to the same structure on $Q \otimes L$, which we can treat as an algebra of simple maps. We first look at this in an abstract setting.

Definition 30 *A frame ΩX is* linear *if it is equipped with a binary operation $+$ and scalar multiplication $Q \times \Omega X \to \Omega X$ satisfying the following conditions.*

1. *$+$ is Scott continuous, as is scalar multiplication by each $q \in Q$.*
2. *The operations make ΩX a module over the semiring Q, with $\perp$ as the zero element.*
3. *$U + V = (U \vee V) + (U \wedge V)$ for all $U, V \in \Omega X$.*

Similarly, we say a distributive lattice L is linear if it has the same constructions and conditions, but with Scott continuity weakened to monotonicity.

Proposition 31 *Let ΩX be a linear frame. Then any continuous linear map $m : \Omega X \to R$ is also modular.*

Proof. $m(U) + m(V) = m(U + V) = m((U \vee V) + (U \wedge V)) = m(U \vee V) + m(U \wedge V)$. ■

Clearly if ΩX is linear then the relations used to present $\mathfrak{V}X$ can be modified to express linearity instead of modularity, and by the proposition they present a sublocale $\mathfrak{L}X$ of $\mathfrak{V}X$.

Now suppose ΩX is presented by a DL-site $(L, \lhd)$ where L is linear. By [JMV08], if $\lhd$ is stable for addition and scalar multiplication then ΩX is a linear frame and the continuous linear maps $\Omega X \to R$ are equivalent to the monotone linear maps $L \to R$ that respect the covers. This enables us to present $\mathfrak{L}X$ geometrically, using generators derived from L instead of from the whole of ΩX.

Now consider an arbitrary locale X, presented by DL-site $(L, \lhd)$. Clearly $\Omega(R' \times X) \cong \mathbf{Loc}(X, R)$ inherits linear frame structure from R. By Lemma 9, $\Omega(R' \times X)$ is presented by a topless DL-site $(Q \otimes L, \lhd')$. As it stands, the cover relation described in the Lemma is not $+$-stable. The problem is that after adding elements, the directed joins have to be taken at several disjuncts $q \times a$ instead of just one. This could easily be repaired, but in fact we don't need to because we already know the linear frame structure on $\Omega(R' \times X)$. The following proposition shows that if $m : Q \otimes L \to R$ is monotone and linear, then the unique continuous lifting to $\Omega(R' \times X)$ is also linear.

Proposition 32 *With X presented by $(L, \lhd)$ as above, the injection of generators $Q \otimes L \to \Omega(R' \times X)$ is a Q-algebra homomorphism (it is linear* and *preserves multiplication).*

Proof. $\bigvee_i q_i \times a_i$ maps to $f : X \to R$ defined by

$$f(x) = \{q \mid (\exists i)\ q < q_i \text{ and } x \vDash a_i\} = \sup\{q_i \mid x \vDash a_i\}.$$

Suppose $a^\alpha \in Q \otimes L$ $(\alpha = 1, 2)$, $a^\alpha = \frac{1}{n} \sum_i a_i^\alpha$ in descending form. Then $f^\alpha(x) = \sup\{\frac{i}{n} \mid x \vDash a_i^\alpha\}$. We see

$$\begin{aligned}\left(f^1 + f^2\right)(x) &= \sup\{\frac{i+j}{n} \mid x \vDash a_i^1 \wedge a_j^2\} \\ &= \sup\{\frac{k}{n} \mid x \vDash \bigvee_{i+j=k} a_i^1 \wedge a_j^2\}\end{aligned}$$

and by Proposition 19 this is the f corresponding to $a^1 + a^2$.

Clearly the injection of generators preserves scalar multiples. It remains to check multiplication. For $a \in L$, the corresponding map is the *characteristic* map $\chi_a(x) = \sup(\{0\} \cup \{1 \mid x \vDash a\})$, which also has the property $\chi_a \chi_b = \chi_{a \wedge b}$. (Note that χ_a is continuous, because R has the topology of lower semicontinuity.) From linearity, we have $f^\alpha = \frac{1}{n} \sum_i \chi_{a_i^\alpha}$ and so

$$f^1 f^2 = \frac{1}{n^2} \sum_{ij} \chi_{a_i^1} \chi_{a_j^2} = \frac{1}{n^2} \sum_{ij} \chi_{a_i^1 \wedge a_j^2}$$

and this is the f got from $a^1 a^2 = \frac{1}{n^2} \sum_{ij} a_i^1 \wedge a_j^2$. ■

We can now put the results together into our Riesz Theorem. It asserts that valuations on X are equivalent to linear functionals on maps $X \to R$.

Theorem 33 (Riesz Theorem) *Let X be a locale, presented by a DL-site $(L, \lhd)$. Then $\underline{\mathcal{I}} : \mathfrak{V}X \to \mathfrak{V}(R' \times X)$ factors as $\mathfrak{V}X \cong \mathfrak{L}(R' \times X) \hookrightarrow \mathfrak{V}(R' \times X)$.*

Proof. Suppose m is in $\mathfrak{V}X$. Since $\underline{m}$ is linear on $Q \otimes L$ by definition, it follows by continuity and Proposition 32 that $\underline{\mathcal{I}}(m)$ is linear on $\Omega(R' \times X)$. Hence $\underline{\mathcal{I}}$ factors via $\mathfrak{L}(R' \times X)$. Note that $m(a) = \underline{m}(1 \times a)$, so m can be recovered from $\underline{m}$.

In the reverse direction, if m' is in $\mathfrak{L}(R' \times X)$ then by Proposition 32 m' is linear on $Q \otimes L$ and so is determined by a unique valuation m on L, $m(a) = m'(1 \times a)$. We see that m preserves covers because m' does, so m extends to a valuation on X. This gives a map $\mathfrak{L}(R' \times X) \to \mathfrak{V}X$. We see $m' = \underline{\mathcal{I}}(m)$, and it follows that the two maps between $\mathfrak{V}X$ and $\mathfrak{L}(R' \times X)$ are mutually inverse. ■

6 The valuation monad

We now show that $\mathfrak{V}$ is the functor part of a monad on **Loc**. As before, throughout this section, R will denote $\overrightarrow{[0, \infty]}$.

The unit is straightforward; it is the Dirac measure.

Definition 34 *Let X be a locale. Then $\delta : X \to \mathfrak{V}X$ is defined by*

$$\delta(x)(U) = \sup\{1 \mid x \vDash U\}.$$

Although on the face of it this is non-geometric, it can be made geometric by considering a presentation of ΩX.

Proposition 35 *Let X be a locale. If x is a point of X, then $\delta(x)$ is characterized as an integral by $\underline{\int} f \ d\delta(x) = f(x)$.*

Proof. $\sum_i (q_i - q_{i-1})\delta(x)(f^*\overrightarrow{(q_i, \infty]}) = \sum_i \sup\{q_i - q_{i-1} \mid q_i < f(x)\} = \sup\{q_i \mid q_i < f(x)\}$, so the lower integral is $f(x)$. Uniqueness follows from the equivalence between valuations and integrals. ■

Proposition 36 *δ is natural in X.*

Proof. Suppose $g : X \to Y$ and $f : Y \to R$. Then

$$\underline{\int} f \ d\delta_Y(g(x)) = f \circ g(x) = \underline{\int} f \circ g \ d\delta(x) = \underline{\int} f \ d\mathfrak{V}g(\delta(x))$$

and so $\delta_Y(g(x)) = \mathfrak{V}g(\delta(x))$. ■

We now turn to the multiplication $\mu : \mathfrak{V}^2 \to \mathfrak{V}$. We shall reduce it to the case $X = 1$, noting that $\mathfrak{V}1 \cong R$.

Definition 37 *The map $\mu_1 : \mathfrak{V}R \to R$ is defined by $\mu_1(m) = \underline{\int} \mathrm{Id} \ dm$.*

Proposition 38 *Let X be a locale and M a point of $\mathfrak{V}^2 X$. Then there is a unique valuation $\mu(M)$ on X such that*

$$\underline{\int} f \ d\mu(M) = \underline{\int} \mu_1 \circ \mathfrak{V}f \ dM$$

for all $f : X \to R$.

Proof. By the Riesz Theorem, it suffices to show that $\mu_1 \circ \mathfrak{V}f$ is linear in f, and this holds because $\mu_1 \circ \mathfrak{V}f(m) = \underline{\int} \mathrm{Id}\ \ d\mathfrak{V}f(m) = \underline{\int} f\ dm$. ∎

Proposition 39 *μ gives a natural transformation $\mathfrak{V}^2 \to \mathfrak{V}$.*

Proof. Suppose $g : X \to Y$, $f : Y \to R$ and M a point of $\mathfrak{V}^2 X$. Then

$$\begin{aligned}\underline{\int} f\ d\mu(\mathfrak{V}^2 g(M)) &= \underline{\int} \mu_1 \circ \mathfrak{V}f\ \ d\mathfrak{V}^2 g(M) = \underline{\int} \mu_1 \circ \mathfrak{V}f\ \circ \mathfrak{V}g\ dM \\ &= \underline{\int} \mu_1 \circ \mathfrak{V}(f\ \circ g)\ dM = \underline{\int} f\ \circ g\ d\mu(M) \\ &= \underline{\int} f\ \ d\mathfrak{V}g(\mu(M))\end{aligned}$$

and so $\mu(\mathfrak{V}^2 g(M)) = \mathfrak{V}g(\mu(M))$.

Geometricity follows because in Proposition 38 it suffices to have the equation for *simple* f, i.e. (given a DL-site (L, C) for X) those arising from elements of $Q \otimes L$. ∎

Lemma 40 *We note the following properties.*

1. *Suppose $a : 1 \to R$ and b is a point of $\mathfrak{V}1$. They may both be considered points of R. Then $\underline{\int} a\ db = ab$.*
2. *μ_1 is the component μ_X for $X = 1$.*
3. *$\mu_1 \circ \mathfrak{V}\mu_1 = \mu_1 \circ \mu_R$*

Proof. (1) $b = b\delta(*)$, where $*$ is the unique point of 1, and by linearity $\underline{\int} a\ db = b \underline{\int} a\ d\delta(*) = ba(*) = ab$.

(2) Let m be a point of $\mathfrak{V}^2 1 \cong \mathfrak{V}R$. If $a : 1 \to R$ then $\underline{\int} a\ d\mu_1(m) = a\mu_1(m) = a \underline{\int} \mathrm{Id}\ \ dm$. On the other hand, taking μ as defined for general X,

$$\underline{\int} a\ d\mu(m) = a \underline{\int} 1\ d\mu(m) = a \underline{\int} \mu_1 \circ \mathfrak{V}1\ dm.$$

It thus suffices to show that $\mu_1 \circ \mathfrak{V}1 = \mathrm{Id}$. This holds because

$$\mu_1(\mathfrak{V}1(b)) = \underline{\int} \mathrm{Id}\ \ d\mathfrak{V}1(b) = \underline{\int} 1\ db = b$$

by part (1).

(3) Let M be a point of $\mathfrak{V}^3 1 \cong \mathfrak{V}^2 R$. If $a : 1 \to R$ then

$$\begin{aligned}\underline{\int} a\ d\mu_1(\mathfrak{V}\mu_1(M)) = a\mu_1(\mathfrak{V}\mu_1(M)) &= a \underline{\int} \mathrm{Id}\ \ d\mathfrak{V}\mu_1(M) \\ &= a \underline{\int} \mu_1\ dM \\ &= a \underline{\int} \mathrm{Id}\ \ d\mu_R(M) = \underline{\int} a\ d\mu_1(\mu_R(M)).\end{aligned}$$

∎

Theorem 41 $(\mathfrak{V}, \delta, \mu)$ *form a monad on* **Loc***.*

Proof. We have three equations to prove. In each case we suppose $f : X \to R$.

$\mu \circ \delta\mathfrak{V} = \mathrm{Id}$: If m is a point of $\mathfrak{V}X$ then

$$\underline{\int} f \ d\mu(\delta(m)) = \underline{\int} \mu_1 \circ \mathfrak{V}f \ d\delta(m) = \mu_1(\mathfrak{V}f(m)) = \underline{\int} \mathrm{Id} \ d\mathfrak{V}f(m) = \underline{\int} f \ dm.$$

$\mu \circ \mathfrak{V}\delta = \mathrm{Id}$: If m is a point of $\mathfrak{V}X$ then

$$\begin{aligned} \underline{\int} f \ d\mu(\mathfrak{V}\delta(m)) &= \underline{\int} \mu_1 \circ \mathfrak{V}f \ d\mathfrak{V}\delta(m) = \underline{\int} \mu_1 \circ \mathfrak{V}f \circ \delta_X \ dm \\ &= \underline{\int} \mu_1 \circ \delta_R \circ f \ dm \\ &= \underline{\int} f \ dm \text{ by the first equation at } X = 1. \end{aligned}$$

$\mu \circ \mu\mathfrak{V} = \mu \circ \mathfrak{V}\mu$: If $\mathcal{M}$ is a point of $\mathfrak{V}^3 X$ then

$$\begin{aligned} \underline{\int} f \ d\mu(\mu(\mathcal{M})) &= \underline{\int} \mu_1 \circ \mathfrak{V}f \ d\mu(\mathcal{M}) = \underline{\int} \mu_1 \circ \mathfrak{V}(\mu_1) \circ \mathfrak{V}^2 f \ d\mathcal{M} \\ &= \underline{\int} \mu_1 \circ \mu_R \circ \mathfrak{V}^2 f \ d\mathcal{M} \text{ by the Lemma} \\ &= \underline{\int} \mu_1 \circ \mathfrak{V}f \circ \mu_X \ d\mathcal{M} = \underline{\int} \mu_1 \circ \mathfrak{V}f \ d\mathfrak{V}\mu(\mathcal{M}) \\ &= \underline{\int} f \ d\mu(\mathfrak{V}\mu(\mathcal{M})). \end{aligned}$$

■

Remark 42 *From Proposition 28 and Remark 29 we see that the monad structure is also geometric, since it was defined in terms of integration.*

7 Product valuations

If m and n are valuations on X and Y, then the product valuation $m \times n$ on $X \times Y$ is the unique one such that $(m \times n)(U \times V) = m(U)n(V)$. However, both existence and uniqueness here are non-trivial. We first look at existence in a monad-theoretic way, using a strength of the monad $\mathfrak{V}$. This will give us two maps $\mathfrak{V}X \times \mathfrak{V}Y \to \mathfrak{V}(X \times Y)$ that both satisfy the required condition for making product valuations. Their equality is commutativity of the strong monad, and for this we shall need a uniqueness result.

Definition 43 *The strength* $t_{XY} : X \times \mathfrak{V}Y \to \mathfrak{V}(X \times Y)$ *is defined by* $t_{XY}(x, m) = \mathfrak{V}\langle x, Y\rangle(m)$*.*

Note that the definition is not special to $\mathfrak{V}$, but is a consequence of its geometricity. Also, x here is a generalized point, so the definition is relying on the geometricity of $\mathfrak{V}$ in order to make sense.

Proposition 44 *t is a strength for the monad $\mathfrak{V}$.*

Proof. First we show that t interacts correctly with the product structure. For the standard isomorphism $1 \times X \cong X$ we show

$$\mathfrak{V}(\cong)(t_{1X}(*, m)) = \mathfrak{V}(\cong \circ \langle *, X\rangle)(m) = \mathfrak{V}(X)(m) = m.$$

Next, modulo the associativity isomorphism $(X \times Y) \times Z \cong X \times (Y \times Z)$ we have

$$\begin{aligned} t_{X,Y\times Z}((X \times t_{YZ})(x, y, m)) &= t_{X,Y\times Z}(x, t_{YZ}(y, m)) \\ &= \mathfrak{V}\langle x, Y \times Z\rangle \circ \mathfrak{V}\langle y, Z\rangle(m) \\ &= \mathfrak{V}\langle x, y, Z\rangle(m) = t_{X\times Y,Z}(x, y, m). \end{aligned}$$

Now we show that t interacts correctly with the monad structure. For the unit, we have

$$t_{XY}(x, \delta(y)) = \mathfrak{V}\langle x, Y\rangle(\delta(y)) = \delta(\langle x, Y\rangle(y)) = \delta(x, y).$$

Finally, for the multiplication, first note that $t_{XY} \circ \langle x, \mathfrak{V}Y\rangle = \mathfrak{V}\langle x, Y\rangle$, because

$$t_{XY} \circ \langle x, \mathfrak{V}Y\rangle(m) = t_{XY}(x, m) = \mathfrak{V}\langle x, Y\rangle(m).$$

Hence

$$\begin{aligned} \mu(\mathfrak{V}t_{XY}(t_{X,\mathfrak{V}Y}(x, M))) &= \mu(\mathfrak{V}t_{XY} \circ \mathfrak{V}\langle x, \mathfrak{V}Y\rangle(M)) \\ &= \mu(\mathfrak{V}^2\langle x, Y\rangle(M)) = \mathfrak{V}\langle x, Y\rangle(\mu(M)) \\ &= t_{XY}(x, \mu(M)). \end{aligned}$$

■

The product valuation then would arise from commutativity of the monad, i.e. commutativity of the diagram

$$\begin{array}{ccccc} \mathfrak{V}X \times \mathfrak{V}Y & \xrightarrow{t'_{X,\mathfrak{V}Y}} & \mathfrak{V}(X \times \mathfrak{V}Y) & \xrightarrow{\mathfrak{V}t_{X,Y}} & \mathfrak{V}^2(X \times Y) \\ t_{\mathfrak{V}X,Y} \downarrow & & & & \downarrow \mu \\ \mathfrak{V}(\mathfrak{V}X \times Y) & \xrightarrow[\mathfrak{V}t'_{X,Y}]{} & \mathfrak{V}^2(X \times Y) & \xrightarrow{\mu} & \mathfrak{V}(X \times Y) \end{array}$$

where the costrength $t'_{XY} : \mathfrak{V}X \times Y \to \mathfrak{V}(X \times Y)$ is defined from the strength with the obvious swap morphisms.

Remark 45 *Once commutativity is established, it will follow from abstract monad theory that the strength can be defined as a product, $t_{XY}(x, m) = \delta(x) \times m$ and similarly for t'_{XY}.*

Let us temporarily write, for valuations m and n on X and Y,

$$m \triangleright n = \mu \circ \mathfrak{V}t_{X,Y} \circ t'_{X,\mathfrak{V}Y}(m,n),$$
$$m \triangleleft n = \mu \circ \mathfrak{V}t'_{X,Y} \circ t_{\mathfrak{V}X,Y}(m,n).$$

Proposition 46 *If $f : X \times Y \to R$ then*

$$\underline{\int} f \ d(m \triangleright n) = \underline{\int\int} f \ dn \ dm,$$
$$\underline{\int} f \ d(m \triangleleft n) = \underline{\int\int} f \ dm \ dn.$$

Proof.

$$\begin{aligned}\underline{\int} f \ d(m \triangleright n) &= \underline{\int} f \ d\mu(\mathfrak{V}t_{X,Y}(t'_{X,\mathfrak{V}Y}(m,n))) \\ &= \underline{\int} \mu_0 \circ \mathfrak{V}f \circ t_{X,Y} \ dt'_{X,\mathfrak{V}Y}(m,n) \\ &= \underline{\int} \mu_0 \circ \mathfrak{V}f \circ t_{X,Y} \circ \langle X, n\rangle \ dm.\end{aligned}$$

Now

$$\begin{aligned}\mu_0 \circ \mathfrak{V}f \circ t_{X,Y} \circ \langle X, n\rangle(x) = \mu_0 \circ \mathfrak{V}f \circ t_{X,Y}(x,n) &= \underline{\int} \mathrm{Id} \ d\mathfrak{V}f \circ t_{X,Y}(x,n) \\ &= \underline{\int} f \ dt_{X,Y}(x,n) = \underline{\int} f \circ \langle x, Y\rangle \ dn.\end{aligned}$$

It follows that $\underline{\int} f \ d(m \triangleright n)$ is got as a double integral $\underline{\int\int} f \ dn \ dm$. A symmetric calculation shows that $m \triangleleft n$ gives the double integral $\underline{\int\int} f \ dm \ dn$. ∎

Proposition 47 *Let m and n be valuations on X and Y. Then both $m \triangleright n$ and $m \triangleleft n$ have the property required for a product valuation $m \times n$.*

Proof. Let U and V be opens of X and Y. Then

$$(m \triangleright n)(U \times V) = \underline{\int} \chi_{U \times V} \ d(m \triangleright n) = \underline{\int\int} \chi_{U \times V} \ dn \ dm$$

by Proposition 46. But the inner integral $\underline{\int} \chi_{U \times V} \ dn$, evaluated at x, is $n(V)\chi_U(x)$. Hence by linearity the double integral is $m(U)n(V)$. The proof for $m \triangleleft n$ is by symmetry. ∎

This has proved existence for the product valuation. For uniqueness we first prove some lemmas.

Lemma 48 *Let u_i $(1 \le i \le n)$, x and y be lower reals such that $\sum_i u_i + x \le \sum_i u_i + y$ and $u_i \le y$ for all i. Then $x \le y$.*

Proof. Note that the lower reals are not a cancellation monoid and some kind of condition such as our $u_i \leq y$ is needed. For if $\sum_i u_i = \infty$ then $\sum_i u_i + x \leq \sum_i u_i + y$ for all x and y. In this example, our extra condition implies that $\infty = \frac{1}{n}\sum_i u_i \leq y$ and so $x \leq y$. In the proof, the condition enables us to take a bound on the rationals in the u_is.

The case $n = 0$ is obvious, so we assume $n > 0$. Suppose $p < x$. Then we have $p < p + \varepsilon < x$ for some $\varepsilon > 0$. We show by induction on k that if $u_i > q_i \in \mathbb{Q}$ $(1 \leq i \leq n)$ and $np - \sum_i q_i \leq k\varepsilon$, then $p < y$. For the base case $k = 0$, we can find $p \leq q_i$ for some i, and by our condition $u_i \leq y$ we have $p < y$. Now suppose $k \geq 1$. We can find $q'_i < u_i$ and $p' < y$ such that $\sum_i q_i + p + \varepsilon = \sum_i q'_i + p'$. If $p \leq p'$ then $p < y$. Otherwise $p' < p$ and so $\sum_i q_i + \varepsilon < \sum_i q'_i$ and

$$np - \sum_i q'_i < np - \sum_i q_i - \varepsilon \leq (k-1)\varepsilon,$$

so by induction $p < y$. ■

Corollary 49 *Let u_i $(1 \leq i \leq n)$ and v be lower reals. Then the equation $\sum_i u_i + x = v$ has at most one solution x such that $u_i \leq x$ for all i.*

Proposition 50 *Let X and Y be locales presented with DL-sites on lattices L and M. Then any valuation m on $X \times Y$ is uniquely determined by its values on opens $a \times b$ $(a \in L,\ b \in M)$.*

Proof. By continuity of valuations, it suffices to consider opens in $L \otimes M$, i.e. of the form $\bigvee_{i=1}^n a_i \times b_i$. Now consider the Principle of Inclusion and Exclusion, Theorem 23, in $M(L \otimes M)$: it says that

$$\bigvee_{i=1}^n a_i \times b_i + \sum_j c_j \times d_j = \sum_k c'_k \times d'_k$$

where each $c_j \times d_j$ or $c'_k \times d'_k$ is in that form because it is a meet of elements $a_i \times b_i$, and so is also less than $\bigvee_{i=1}^n a_i \times b_i$. By linearity it follows that

$$m\left(\bigvee_{i=1}^n a_i \times b_i\right) + \sum_j m(c_j \times d_j) = \sum_k m(c'_k \times d'_k)$$

with each $m(c_j \times d_j) \leq m\left(\bigvee_{i=1}^n a_i \times b_i\right)$. Hence by Corollary 49, $m\left(\bigvee_{i=1}^n a_i \times b_i\right)$ is uniquely determined by the values $m(c_j \times d_j)$ and $m(c'_k \times d'_k)$. ■

Hence product measures are unique, and in particular $m \lhd n = m \rhd n$. We shall henceforth write it as $m \times n$.

Theorem 51 *$\mathfrak{V}$ is commutative.*

Theorem 52 (Fubini Theorem) *Let m and n be valuations on locales X and Y, and let $f : X \times Y \to R$. Then*

$$\underline{\int\int} f\ dm\ dn = \underline{\int\int} f\ dn\ dm = \underline{\int} f\ d(m \times n).$$

Proof. Combine Proposition 46 with the uniqueness of product measures. ■

8 Probability and subprobability valuations

[Hec94] and [CS09] work with a locale of *probability* valuations m, for which $m(X) = 1$. These are the points of a semifitted sublocale (i.e. a meets of opens and closeds) $\mathfrak{V}^{(1)}X$ of $\mathfrak{V}X$, defined by relations $\top \vdash m(X) > q$ (for all $q < 1$) and $m(X) > 1 \vdash \bot$. Similarly there is a locale $\mathfrak{V}^{(\leq 1)}X$ of subprobability valuations m with $m(X) \leq 1$. It is a closed sublocale of $\mathfrak{V}X$. Our aim in this section is to show that they too are monads.

Theorem 53 *$\mathfrak{V}^{(1)}$ and $\mathfrak{V}^{(\leq 1)}$ are both commutative strong monads on* **Loc**.

Proof. We prove the case of $\mathfrak{V}^{(1)}$ in detail; $\mathfrak{V}^{(\leq 1)}$ is similar.

First, $\mathfrak{V}^{(1)}$ is a functor, since if $f : X \to Y$ then $\mathfrak{V}f(m)(Y) = m(f^*Y) = m(X)$: so $\mathfrak{V}f$ restricts to a map $\mathfrak{V}^{(1)}f : \mathfrak{V}^{(1)}X \to \mathfrak{V}^{(1)}Y$.

Next, for any x in X we have $\delta(x)(X) = 1$ and so $\delta(x)$ in $\mathfrak{V}^{(1)}X$. Hence δ is a natural transformation from $\mathrm{Id}_{\mathbf{Loc}}$ to $\mathfrak{V}^{(1)}$.

Now we show that μ gives a multiplication. Suppose M is in $\mathfrak{V}^{(1)}\mathfrak{V}^{(1)}X$. Let $i : \mathfrak{V}^{(1)}X \hookrightarrow \mathfrak{V}X$ be the sublocale inclusion. Then we can make M a point of $\mathfrak{V}^2X$ by applying $\mathfrak{V}i \circ i : \mathfrak{V}^{(1)}\mathfrak{V}^{(1)}X \to \mathfrak{V}\mathfrak{V}^{(1)}X \to \mathfrak{V}\mathfrak{V}X$. Then

$$\mu(\mathfrak{V}i(i(M)))(X) = \underline{\int} 1 \; d\mu(\mathfrak{V}i(i(M))) = \underline{\int} \mu_1 \circ \mathfrak{V}1 \circ i \; di(M).$$

where $1 = \chi_X : X \to \overrightarrow{[0,\infty]}$ is the constant map 1. Now if m is in $\mathfrak{V}^{(1)}X$ then

$$\mu_1 \circ \mathfrak{V}1 \circ i(m) = \underline{\int} \mathrm{Id} \;\; d\mathfrak{V}1 \circ i(m) = \underline{\int} \mathrm{Id} \circ 1 \; di(m) = m(X) = 1$$

so $\mu_1 \circ \mathfrak{V}1 \circ i = 1$ and

$$\underline{\int} \mu_1 \circ \mathfrak{V}1 \circ i \; di(M) = \underline{\int} 1 \; di(M) = M(\mathfrak{V}^{(1)}X) = 1.$$

Now for product valuations we have that if m and n are in $\mathfrak{V}^{(1)}X$ and $\mathfrak{V}^{(1)}Y$ then $(m \times n)(X \times Y) = m(X)n(Y) = 1$. Then from Remark 45 it follows that the strength t_{XY} factors as $X \times \mathfrak{V}^{(1)}Y \to \mathfrak{V}^{(1)}(X \times Y)$. ■

9 Covaluations and upper integrals

As well as valuations, [Vic08] also discusses *covaluations.* In effect, a covaluation on a locale X is a valuation on the *closed* sublocales of X, defined as a continuous map from ΩX to the upper reals $\overleftarrow{[0,\infty]}$. The covaluations are the points of a locale $\mathfrak{C}X$. There is also a notion of upper integral, in which the integrand is a map from X to $\overleftarrow{[0,\infty]}$ and the measure is a covaluation. The upper integral is more complicated in that it has to be provided with a bound, and because of this our results for valuations and the lower integral do not appear to transfer in full. Nonetheless, the algebraic results of Section 4 can be applied in a dual way to develop some analogous results for the covaluations and upper integrals, and we outline some calculations here.

On a distributive lattice L, we define a *covaluation* to be a function $m : L \to \overleftarrow{[0,\infty]}$ that is monotone with respect to the specialization order on the upper reals (and hence numerically antitone), has $m(\top) = 0$ and satisfies the modular law.

We shall also need to deal with sites $(L, \lhd)$ where L is a bottomless distributive lattice. This is because our construction analogous to that of $Q \otimes L$ will use Q^{op}, which is bottomless. For topless DL-sites, it was natural to require $\top$ in the presented frame to be $\bigvee^{\uparrow} L$, and this meant that for a nominally topless L that happened to have a top, the two sites – one as topless DL, one as DL – present the same frame. In the bottomless case we do not have an analogous candidate for $\bot$, and so the $\bot$ in the frame will not agree with a $\bot$ that happens to exist already in L. When it comes to dcpo homomorphisms from the frame, this means we shall have to specifiy explicitly the image of the newly adjoined $\bot$. To avoid confusion we shall not use sites based on bottomless distributive lattices, but instead put in the bottoms explicitly.

Theorem 54 *Let X be a locale.*

Suppose X is presented by a DL-site $(L, \lhd)$. Then covaluations on X are equivalent to covaluations m on L satisfying $m(a) \geq \inf_i m(a_i)$ if $a \lhd (a_i)_i$.

Proof. As stated in [Vic08], the proof is analogous to that of Theorem 7. ■

In the rest of the section, we shall change our previous notation and reverse the upper and lower roles of R and R'. R' will be $\overrightarrow{(0,\infty]}$ and R will be $\overleftarrow{[0,\infty]} \cong \mathbb{S}^{R'}$. As before, Q is the set of non-negative rationals, and we shall write Q_∞ for $Q \cup \{\infty\}$. Then R' is presented by a DL-site on Q_∞^{op} with covers $q \lhd \{r \in Q \mid q < r\}$ for each $q \in Q$.

Lemma 55 *We can present $R' \times X$ as a DL site on $Q_\infty^{op} \otimes L$, with covers $(q \odot a) \wedge u \lhd \{(q' \odot a) \wedge u \mid q < q'\}$ ($q \in Q$) and $(q \odot a) \wedge u \lhd \{(q \odot a_i) \wedge u \mid i \in I\}$ if $q \in Q_\infty$ and $a \lhd \{a_i \mid i \in I\}$.*

Proof. The proof is analogous to that of Lemma 9, but this time we analyse $Q_\infty^{op} \otimes L$ as a tensor of $\wedge$-semilattices when finding the meet and join stabilized versions of the relations. ■

Corollary 56 *A covaluation on $R' \times X$ is equivalent to a covaluation $m : Q_\infty^{op} \otimes X \to \overleftarrow{[0,\infty]}$ satisfying the following conditions.*

1. *$m((q \odot a) \wedge u) = \bigvee^{\uparrow}_{q<q'} m((q' \odot a) \wedge u)$ for each $q \in Q$,*

2. $m((q \odot a) \wedge u) \leq \bigvee^{\uparrow}_{i \in I} m((q \odot a_i) \wedge u)$ *for each* $q \in Q_\infty$*, where* $a \lhd \{a_i \mid i \in I\}$.

Proof. Combine Lemma 55 with Theorem 54. ∎

We now wish to define an upper analogue of the integration map $\underline{\mathcal{I}} : \mathfrak{V}X \to \mathfrak{V}(\overleftarrow{[0,\infty)} \times X)$, of the form $\overline{\mathcal{I}} : \mathfrak{C}X \to \mathfrak{C}(\overrightarrow{(0,\infty]} \times X)$. Suppose m is a covaluation on X. Then as a map from L^{op} to R it is monotone (with respect to the numerical order on R), modular, and maps bottom to 0, and hence factors via a linear map $\overline{m} : M_Q(L^{op}) \to R$. By Proposition 21, $M_Q(L^{op}) \cong Q \otimes L^{op}$, where the tensor is for $\vee$-semilattices; but this is $(Q^{op} \otimes L)^{op}$ using the $\wedge$-semilattice tensor, so we see that $\overline{m} : Q^{op} \otimes L \to R$ is a covaluation with $\overline{m}(q \odot a) = qm(a)$. An element $\bigwedge_{i=1}^n q_i \odot a_i$ is now in sorted form if both the the q_is and the a_is ascend with i, and

$$\overline{m}\left(\bigwedge_{i=1}^n q_i \odot a_i\right) = \sum_{i=1}^n (q_i - q_{i-1}) m(a_i)$$

We extend this linearly to $Q_\infty^{op} \otimes L$ by $\overline{m}(\infty \odot a) = \infty m(a) = \infty$. (Note that even $\infty \cdot 0 = \infty \cdot \inf_{q>0} q = \infty$ by continuity.) Thus $\overline{m}\left(\bigwedge_{i=1}^n q_i \odot a_i \wedge \infty \odot a\right) = \infty$. Then using Corollary 56 we get a covaluation $\overline{\mathcal{I}}(m)$ on $R' \times X$, so we have defined $\overline{\mathcal{I}} : \mathfrak{C}X \to \mathfrak{C}(\overrightarrow{(0,\infty]} \times X)$.

Relating $\overline{\mathcal{I}}$ to the upper integral as already defined in [Vic08] is slightly complicated. The identity map $\mathrm{Id} : \overleftarrow{[0,\infty]} \to \overleftarrow{[0,\infty]}$ uncurries to the evaluation map $\mathrm{ev} : \overrightarrow{(0,\infty]} \times \overleftarrow{[0,\infty]} \to \mathbb{S}$ (with $\mathrm{ev}(q,r) = \top$ iff $r < q$), so ev is an open of $\overrightarrow{(0,\infty]} \times \overleftarrow{[0,\infty]}$. If m is a covaluation on $\overleftarrow{[0,\infty]}$ then we calculate $\overline{\mathcal{I}}(m)(\mathrm{ev})$ as the inf over values $\overline{m}(u)$ with $u \in Q_\infty^{op} \otimes L$ and $u \leq \mathrm{ev}$. Suppose $u = \bigwedge_{i=1}^n q_i \odot r_i$ in strictly sorted form. Here each q_i represents the open $\overrightarrow{(q_i,\infty]}$ of $\overrightarrow{(0,\infty]}$, and r_i is the open $\overleftarrow{[0,r_i)}$ of $\overleftarrow{[0,\infty]}$. Redistributing, we find

$$u = \bigvee_{n = S \cup T} q_{\max S} \times r_{\min T} = 0 \times r_1 \vee \bigvee_{i=1}^{n-1} q_i \times r_{i+1} \vee q_n \times \top.$$

From this we deduce that $u \leq \mathrm{ev}$ iff (i) $r_1 = 0$, (ii) $r_{i+1} \leq q_i$ for each i, and (iii) $q_n = \infty$. Unfortunately, this implies that $\overline{m}(u) = \infty$, so $\overline{\mathcal{I}}(m)(\mathrm{ev}) = \infty$ for all m – it is a consequence of linearity of $\overline{\mathcal{I}}(m)$, in fact.

From the argument above, we may as well restrict our attention to those u in which $r_{i+1} = q_i$. Reindexing, we see that

$$\mathrm{ev} = \bigvee{}^{\uparrow}\{\bigwedge_{i=1}^n r_i \odot r_{i-1} \wedge \infty \odot r \mid 0 = r_0 < r_1 < \cdots < r_n \text{ and } r \leq r_n\}.$$

In [Vic08] the upper integral $\overline{\int}_K \mathrm{Id}\ dm$ is parametrized by a compact fitted sublocale K of $\overleftarrow{[0,\infty]}$ and is defined to be the infimum of the values $\overline{m}\left(\bigwedge_{i=1}^n r_i \odot r_{i-1}\right)$ such that $0 = r_0 < r_1 < \cdots < r_n$ as above and $\overleftarrow{[0,r_n)}$ is a neighbourhood of K.

It follows that

$$\overline{\int}_K \mathrm{Id}\ dm = \overline{\mathcal{I}}(m)(\nu_K(\mathrm{ev}))$$

where ν_K is the nucleus corresponding to the sublocale K: $\nu_K(U)$ is the biggest V such that $V \leq U$ modulo K. Note that in this situation ν_K is Scott continuous [Esc01].

10 Conclusions

As a development of [Vic08] and [CS09] we have proved a comprehensive set of new, constructive results for the valuation locale and the lower integral that are analogous to those known in the standard theory of measures and integrals, but with measurable spaces and measures replaced by locales and valuations, and the Dedekind reals replaced by lower reals (general taken to be non-negative). Our locales are completely unrestricted. The new results include the following.

1. A detailed algebraic analysis of the modular monoid and the algebra of simple maps (which were defined in [CS09] from results in [HT48]), showing their double nature as monoids and as semilattice tensors.
2. Our version of the Riesz Theorem, showing that valuations are equivalent to integrals, i.e. linear functionals.
3. The valuation locale is the functor part $\mathfrak{V}$ of a monad on the category of locales. In proving this we made use of the Riesz Theorem in defining the multiplication of the monad, with $\mu(M)$ defined as an integral.
4. $\mathfrak{V}$ is a commutative strong monad, thus defining product valuations with a Fubini Theorem holdiing.
5. (3) and (4) also hold for the probability valuation locale $\mathfrak{V}^{(1)}$, as used in [CS09], and the subprobability valuation locale $\mathfrak{V}^{(\leq 1)}$.

Because of the "lower" nature of the theory, using lower reals and lower integrals, this is not a true localic form of the standard theory, nor of the Riesz Theorem in [CS09]. (In the [CS09] version the integrands take their values in the Dedekind reals and the locales are restricted to be compact, completely regular.) Nonetheless, we hope that the work here, together with parallel work on the covaluations and upper integrals of [Vic08], will combine to give the results of [CS09]. Preliminary calculations of Section 9 suggest that the "upper" theory is not so clean as the lower in general.

Geometricity has played a pervasive role in the development here, allowing us to deal with point-free locales in a natural manner as though they had enough points. It is also essential in applications such as [**?**], where it allows the valuation locale construction to be applied fibrewise to bundles.

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