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# Strongly Algebraic = SFP (Topically)

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Certain "Finite Structure Conditions" on a geometric theory are shown to be sufficient for its classifying topos to be a presheaf topos. The conditions assert that every homomorphism from a finite structure of the theory to a model factors via a finite model, and they hold in cases where the finitely presentable models are all finite.
The conditions are shown to hold for the theory of strongly algebraic (or SFP) information systems and some variants, as well as for some other theories already known to be classified by presheaf toposes.
The work adheres to geometric constructivism throughout and in consequence provides "topical" categories of domains (internal in the category of toposes and geometric morphisms) with an analogue of Plotkin's double characterization of strongly algebraic domains, by sets of minimal upper bounds and by sequences of finite posets.

## 1. Introduction

In (Vickers 1999), a constructive approach to domain theory is described in which the constructivist flavour is *geometric.* Based on the infinitary geometric logic (Makkai and Reyes 1977; Johnstone 1977), it relies on constructions that can not only be carried out internally in toposes (more specifically, Grothendieck toposes), but are also preserved by the inverse image functors of geometric morphisms. It is a stringent doctrine that in practice can avoid both the inherent choice of constructive type theory and the impredicativity possible in toposes. Nonetheless, it has the benefit that it fits well with topology: any class described geometrically has an intrinsic topology (possibly in Grothendieck's generalized sense of toposes), and any construction described geometrically is automatically continuous (again, possibly generalized as a geometric morphism). In (Vickers 1999), the ability to construct the limits characteristic of domain theory, including those used for solving recursive domain equations, was shown to arise directly and for very general reasons out of the geometric nature of the constructions. Moreover, the embedding-projection pairs, used in domain theory apparently as a trick, were rationally reconstructed as homomorphisms between models of a geometric theory.

The effects on domain theory are illustrated by the solution of domain equations. If $C$ is a category of domains and $F : C \to C$ is a domain construction then one seeks $D$ with $D \cong F(D)$ and it is calculated as a colimit of $\perp \to F(\perp) \to F^2(\perp) \to \cdots$ for a suitable trivial domain $\perp$. This will not work for arbitrary categories and functors, and so the general theory is complicated by explicit continuity structure and conditions such as the dcpo enrichment in the O-categories of (Wand 1979), as well as the apparently ad hoc use of embedding-projection pairs to get the right functoriality for $F$. By contrast, in a geometric mathematics the class $\mathrm{ob}(C)$ of objects of $C$ has to be the "class" (understood appropriately as a generalized space) of models of a geometric theory, and $F$ has to be constructed geometrically. Then the desired fixpoint exists in complete generality, provided only that $\mathrm{ob}(C)$ has an initial point – this is the algebraic completeness result of (Vickers 1999). The continuity structure is inherent in the geometric mathematics itself.

The geometric constructivism therefore paves the way for important simplifications of domain theory. In addition, an observational interpretation of it suggests its use as a specification language (Vickers 1995), and for both these reasons there is some intrinsic interest in testing out its scope in mathematical reasoning.

The present paper investigates a basic result of Plotkin's (Plotkin 1976) that shows the equivalence between two characterizations of his SFP domains. It requires some care even to formulate this geometrically, and so as well as continuing the geometric refoundation of domain theory the present work also explores new techniques for reasoning geometrically.

From a different perspective the work can also be interpreted as providing new answers to an old question in topos theory: When is a topos a presheaf topos? It is already known that classifiers for essentially algebraic theories are, and so are (toposes of sheaves over) algebraic dcpos as well as various other particular instances. We present here some new methods that give simple proofs not only for the domain-theoretic cases under consideration but also for other geometric theories where every model is a filtered colimit of finite models.

We call the approach "topical", because its categories of domains are internal in the category **Top** of Grothendieck toposes and geometric morphisms.

## 2. Technical background

### 2.1. *Geometric constructivism*

Geometric theories are described in various places such as (Makkai and Reyes 1977; Johnstone 1977; Mac Lane and Moerdijk 1992):

**Definition 2.1.** A many-sorted, first-order logical theory with a vocabulary of function and predicate symbols is *geometric* iff all its extralogical axioms are of the form $\forall xy\cdots.\ (\phi \to \psi)$, $\phi$ and $\psi$ being *geometric formulae* built using the connectives $\bigvee$ (disjunction, possibly infinitary), **false**, $\wedge$ (finitary conjunction), **true**, $=$ and $\exists$.

Note that we shall need the full generality of the infinitary disjunctions – or, at least, recursively infinite ones –, and not just the finitary *coherent* fragment (no infinitary disjunctions) described in (Mac Lane and Moerdijk 1992). We shall present theories using a notation influenced to some extent by the schemas of Z (Spivey 1989). Here is an example to present the theory of partially ordered sets.

$$
\begin{array}{l}
\text{Poset}[X] \\
\left[\ \leq \ \subseteq X \times X \right. \\
\left[\begin{array}{l}
\forall x : X.\ (\mathbf{true} \to x \leq x) \\
\forall x, y, z : X.\ (x \leq y \wedge y \leq z \to x \leq z) \\
\forall x, y : X.\ (x \leq y \wedge y \leq x \to x = y)
\end{array}\right.
\end{array}
$$

Here the top line names the theory, Poset, and lists the sorts (only one in this example) in square brackets. The next section declares the symbols, using $\subseteq$ for predicates (so $\leq$, as a subset of $X \times X$, is a binary predicate) and $\to$ for functions (none in this case). Finally come the axioms. For the first one we have written "$\forall x : X.\ (\mathbf{true} \to x \leq x)$" to emphasize its formal geometricity, but in practice one would write "$\forall x : X.\ x \leq x$".

As explained in (Vickers 1999), we shall extend the logical notation by allowing ourselves to use *sort constructors* for geometric constructions, that is to say for those constructions that can be characterized uniquely up to isomorphism using geometric structure and axioms. Equivalently, these are the constructions that can be carried out in any Grothendieck topos and are preserved by the inverse image functors of geometric morphisms, and it is then immediate that they include finite limits and arbitrary (small) colimits. Less obvious, but proved in (Johnstone 1977), is that free algebras for finitary algebraic theories are also geometric. In particular, this covers the finite powerset ($\mathcal{F}$), for it is a free semilattice, and we use this frequently in our theory presentations. A direct characterization in geometric logic is given in (Vickers 1999). In addition, universal quantification *bounded over finite sets* (i.e. $\forall x \in S. \cdots$ when $S$ is finite) can be expressed geometrically and so can be used as a connective in geometric formulae.

The geometric constructions do *not* include powerset or exponentials, and so the declarations of predicate and function symbols in schemas cannot be understood (*à la* Z) as declaring them as symbols of a particular sort. Rather, they specify *arities*, the sorts of arguments and results. Universal quantification bounded over arbitrary sets is also not geometric.

In slightly more detail, if a geometric theory is presented with base sorts $\sigma_1, \ldots, \sigma_k$ then a derived sort $\tau(\sigma_1, \ldots, \sigma_k)$ is a geometric construction of a new set out of $k$ old ones. (This can be made precise topos-theoretically as a geometric morphism $\tau : [\text{Set}]^k \to [\text{Set}]$ where [Set] is the object classifier, and this shows that $\tau$ is covariantly functorial in its arguments.) These can be used in the arities of functions $f : \tau_1 \to \tau_2$ and predicates $R \subseteq \tau$ (this covers $n$-ary functions and predicates, because we can tuple up the arguments). They can also appear in quantifications $\exists x : \tau$ and axioms $\forall x : \tau.\ (\phi(x) \to \psi(x))$, and there too we shall refer to such $\tau$s as arities.

The following definitions are similar to those of classical logic. However it is important to realise that the definition works not only in standard set theory, but, internally, in some other categories (specifically, in Grothendieck toposes).

**Definition 2.2.** A *structure* for a geometric theory is defined in the usual way: with a carrier set for each sort, and functions and subsets corresponding to function and predicate symbols. If in addition the axioms hold, then the structure is a *model* of the theory.

If $M$ and $N$ are two structures (or models), then a *homomorphism* $\phi : M \to N$ comprises a family of functions, one for each sort, between the corresponding carriers (once this is done for the base sorts, then it lifts to the derived sorts by their functoriality) such that $\phi$ also respects the structure in the following sense: (i) if $f$ is a function symbol then $f(\phi(x)) = \phi(f(x))$, and, (ii), if $P$ is a predicate symbol and $P(x)$ holds in $M$ then $P(\phi(x))$ holds in $N$.

### 2.2. *Strongly algebraic domains*

The SFP domains of (Plotkin 1976) have also been called "strongly algebraic" (e.g. (Smyth 1983)), and it is convenient for us to use the two terms to refer to two equivalent characterizations given by Plotkin.

**Definition 2.3.** (Classically) A *strongly algebraic information system* is a poset $X$ satisfying the following conditions.

1 $X$ is countable.
2 $X$ has a bottom element.
3 For each $S \in \mathcal{F}X$, its set MUB($S$) of minimal upper bounds in $X$ is finite, and also complete in the sense that every upper bound of $S$ is greater than a minimal one.
4 For each $S \in \mathcal{F}X$, there is some *finite* $T$ such that $S \subseteq T \subseteq X$ and $T$ is closed under MUB.

A *strongly algebraic domain* is a dcpo (directed complete poset) isomorphic to the ideal completion Idl($X$) of some strongly algebraic information system $X$. Its elements are the lower closed, directed subsets of $X$.

In (Vickers 1999) it is shown how to give a classically equivalent geometric theory of the information systems. It is not possible to characterize minimality of upper bounds geometrically unless the order $\sqsubseteq$ is decidable, and so MUB is replaced by a binary relation CUB ("complete set of upper bounds") on finite subsets. CUB is characterized geometrically, and condition (3) above is replaced by an axiom that every finite $S$ has a finite complete set of upper bounds. Similarly, condition (4) is replaced by an axiom that every finite $S$ is contained in a finite "CUB-closed" set.

A schema presentation follows. Notice the sort constructor $\mathcal{F}$ (finite powerset) and the finitely bounded universal quantification, extending the pure geometric logic. Because of these the theory is not necessarily coherent: even though no infinitary disjunctions appear explicitly, they are implicit in the geometric characterization of $\mathcal{F}$ and $\forall$.

**Definition 2.4.** The theory IS of *strongly algebraic information systems* is presented by the following schema.

$$
\begin{array}{l}
\mathrm{IS}[X] \\
\left[\begin{array}{l}
\sqsubseteq \subseteq X \times X \\
\mathrm{CUB} \subseteq \mathcal{F}X \times \mathcal{F}X
\end{array}\right. \\
\left[\begin{array}{ll}
\forall t : X.\ t \sqsubseteq t & (1) \\
\forall s, t, u : X.\ (s \sqsubseteq t \wedge t \sqsubseteq u \to s \sqsubseteq u) & (2) \\
\forall s, t : X.\ (s \sqsubseteq t \wedge t \sqsubseteq s \to s = t) & (3) \\
\forall S, T : \mathcal{F}X.\ (\mathrm{CUB}(S,T) \to \forall s \in S.\ \forall t \in T.\ s \sqsubseteq t) & (4) \\
\forall S, T : \mathcal{F}X, u : X.\ (\mathrm{CUB}(S,T) \wedge \forall s \in S.\ s \sqsubseteq u \to \exists t \in T.\ t \sqsubseteq u) & (5) \\
\forall S : \mathcal{F}X.\ \exists T : \mathcal{F}X.\ \mathrm{CUB}(S,T) & (6) \\
\forall S, T, T' : \mathcal{F}X.\ ((\forall s \in S.\ \forall t \in T.\ s \sqsubseteq t) \wedge \mathrm{CUB}(S,T') \wedge (\forall t' \in T'.\ \exists t \in T.\ t \sqsubseteq t') & (7) \\
\qquad\qquad \to \mathrm{CUB}(S,T)) & \\
\forall S : \mathcal{F}X.\ \exists T : \mathcal{F}X.\ (S \subseteq T \wedge (\forall U \in \mathcal{F}T.\ \exists V \in \mathcal{F}T.\ \mathrm{CUB}(U,V))) & (8)
\end{array}\right.
\end{array}
$$

Axioms (1)-(3) here say that $\sqsubseteq$ is a partial order. Axioms (4) and (5) say that if $\mathrm{CUB}(S,T)$ then $T$ is a (finite) complete set of upper bounds of $S$. Axiom (6) says that for every finite $S$ there is some $T$ with $\mathrm{CUB}(S,T)$ (so (4)-(6) correspond to condition (3) in Definition 2.3, sometimes known as the "$\frac{2}{3}$-SFP condition"), and (7) says that if $T$ is a complete set of upper bounds for $S$ (tested by reference to some $T'$ for which $\mathrm{CUB}(S,T')$ is already known), then $\mathrm{CUB}(S,T)$ holds. Axiom (8) says that every finite $S$ has a finite CUB closure, i.e. a finite superset $T$ for which $\forall U \in \mathcal{F}T. \exists V \in \mathcal{F}T. \mathrm{CUB}(U,V)$, corresponding to 2.3(4). Note that if $T$ is finite then so is $\mathcal{F}T$ (this is constructively non-trivial) and so the universal quantification $\forall U \in \mathcal{F}T$ is finitely bounded.

Although CUB is uniquely determined by $\sqsubseteq$, geometrically it is structure and not just a property. Having to preserve it therefore makes a difference to the homomorphisms of information systems: they correspond to adjunctions between the ideal completions (Vickers 1999). If we require the order to be decidable, then $\not\sqsubseteq$ is also part of the structure and the homomorphisms turn out to correspond to embedding-projection pairs (as used in Theorem 2.5).

We have also made two material changes to the notion in that we require neither a bottom element nor countability of the basis (conditions (1) and (2) in Definition 2.3). We shall return to the issue of the bottom in Theorem 4.8. In proving the main result for more general domains without bottom we have taken on a harder task; with bottoms there is a corresponding but easier result.

As for countability, to make this geometric would involve making the enumeration part of the geometric structure. This is conceivably possible but we have chosen not to attempt it. Dropping countability leads to changes in Plotkin's SFP ("Sequence of Finite Posets") result, to which we now turn.

**Theorem 2.5.** (Classically) A dcpo with bottom is strongly algebraic iff it is a colimit of a sequence of finite posets (with bottom) in the category of dcpos with bottom and embedding-projection pairs (an embedding-projection pair from $D$ to $E$ is a pair $(e,p)$ where $e : D \to E$ is left adjoint to $p$ and $e;p = \mathrm{Id}$).

It is this result that we make geometric in the present paper. The countability of basis for such an SFP domain is clear. However, existence of the sequence cannot be stated directly in a geometric form, and in effect we have to say instead that the domain is a *filtered* colimit of a canonically determined diagram of finite posets. In this generality it is quite natural to allow uncountable filtered diagrams and to forgo the countable basis property. A related issue is that in the traditional treatment there is some virtue in limiting the size of domains in order to get a small category of domains (or even a dcpo of domains). Fortunately, the geometric techniques work without that kind of size restriction and achieve the same purpose in a more constructive way.

### 2.3. *Toposes as generalized spaces*

We shall use geometric theories not as an exercise in formal logic, but to open up the hidden aspect of toposes, as generalized topological spaces. Although this has always been recognized as a fundamental part of their nature, it tends to get overshadowed by their more concrete manifestation as generalized universes of sets. Our contention is that the *class* of strongly algebraic information systems is captured better as a *generalized space* of information systems, which we write [IS], and this is the classifying topos of the geometric theory IS.

In an *un*generalized space, the topology can be described by stipulating the opens (the open subsets), but an equivalent approach is by stipulating the sheaves. The opens can then be recovered as the subsheaves of the terminal sheaf 1. This is, of course, an uncommon approach, for sheaves are much more complicated. However, Grothendieck uncovered a generalization of spaces in which the topological structure can only be described through the sheaves, for the opens are not enough.

*Example:* For the space [gp] whose points are all groups, one can define the sheaves to be the functors from **Gp** to **Set** that preserve filtered colimits. (The object part of such a functor is then the stalk function, defining the stalk of the sheaf at each point.) Because for any two groups $G$ and $H$ there is a homomorphism from $G$ to $H$ (for instance, the one that maps every element of $G$ to the identity element of $H$) it is easy to show that a subsheaf of 1, a sheaf for which every stalk is a subsingleton, must be constant, and it follows that the opens describe the indiscrete topology on [gp]. This is not enough to yield the sheaves just described, and so [gp] cannot be understood as an ungeneralized space.

Grothendieck also found examples where the sheaves were evident but the points were obscure or lacking, and in consequence the technical expression of generalized spaces – of *toposes* – has been austerely categorical and point-free. Nonetheless, in many situations it is possible to give the points more prominence, and this has been discussed in, for instance, (Moerdijk 1988). The geometric constructivism provides a systematic way of doing this.

We shall keep the point-free discussions separate by reserving the word "topos" for "generalized topological space" (as though topology were the study of toposes), and using the phrase *geometric universe* for the generalized universe of sets (the category of sheaves; so in the usual way of speaking the Grothendieck topos *is* the geometric universe). This is analogous to the distinction between locales and frames as made in (Johnstone 1982) (or between spaces and locales in (Joyal and Tierney 1984)). If $E$ is a topos, we write $\mathcal{S}E$ for its category of sheaves, and also $\Omega E$ for the frame of opens (subsheaves of 1). Then a map (geometric morphism) $f$ from $E$ to $F$ is an adjoint pair $(f^*, f_*)$ of functors between $\mathcal{S}E$ and $\mathcal{S}F$ such that the left adjoint (the inverse image functor) $f^* : \mathcal{S}F \to \mathcal{S}E$ preserves finite limits.

If $T$ is a geometric theory then its models (Definition 2.2) are the points of a generalized space $[T]$, the classifying topos of $T$. To understand this, consider its universal characterization: for any topos $E$, maps from $E$ to $[T]$ are equivalent to models of $T$ in $\mathcal{S}E$. In particular, by taking $E$ to be the one-point ungeneralized space 1, whose category of sheaves $\mathcal{S}1$ is **Set**, we see that the maps from 1 to $[T]$ (points of $[T]$) are just the models of $T$ in **Set**. If we generalize the notion of point to include arbitrary maps targeted at $[T]$, the source then being the *stage of definition* of the point, then the universal property of classifying topos says that the points of $[T]$ (at whatever stage) are equivalent to the models of $T$ (no matter where). $\mathcal{S}[T]$ is in effect the geometric universe freely generated by the generic model of $T$ (which corresponds to the identity map on $[T]$) and its construction is described in (Johnstone 1977; Mac Lane and Moerdijk 1992), but for the present paper we can ignore its internal details.

Furthermore, suppose we can give a *geometrically constructive* transformation $f$ of models of $T$ to models of $U$. By applying this to the generic model $M_g$ of $T$ in $\mathcal{S}[T]$ we get a model $f(M_g)$ of $U$ in $\mathcal{S}[T]$ and hence a map from $[T]$ to $[U]$. Without ambiguity we can also call this map $f$: for suppose we have a model $M$ of $T$ in $\mathcal{S}E$, corresponding to a map $x : E \to [T]$. Geometricity – preservation by inverse image functors – ensures that the model $f(M)$ of $U$ corresponds to the map $fx$, so the construction $f$ corresponds everywhere to postcomposition with the map $f$.

Taking this one stage further, if $f, g : [T] \to [U]$ are two maps, then a natural transformation $\alpha : f \to g$ can be described as a geometric construction of a homomorphism $\alpha(x)$ from $f(x)$ to $g(x)$, given a point $x$ of $[T]$. Note that the points and homomorphisms in effect put a category structure on $[T]$ (indexed by the stage of definition) that is quite different from $\mathcal{S}[T]$.

To summarize:

— By describing a class of models geometrically, we give it an intrinsic topology (possibly generalized).
— By constructing a transformation geometrically, we automatically ensure continuity (possibly as a geometric morphism).
— By constructing a natural transformation geometrically, we automatically ensure naturality.

Since the effect of these geometric techniques is to manipulate the points of the generalized spaces while apparently ignoring the topologies, we sometimes refer to them under the slogan of "topology-free spaces".

Note that the sheaves over a topos $E$, in other words the objects of $\mathcal{S}E$, are models in $\mathcal{S}E$ of the theory Set with one sort and no other symbols or axioms. Hence each sheaf corresponds to a map $f : E \to [\mathrm{Set}]$ where the classifying topos [Set] is also often called the *object classifier*. Spatially, this map takes each point $x$ of $E$ to a set $f(x)$, the *stalk* at $x$ for that sheaf.

### 2.4. *Finite sets*

We shall need to be careful about constructive variations of finiteness, so let us discuss here three different notions that arise. Fuller details can be found in (Johnstone 1977), and results in the geometric setting are summarized in (Vickers 1999).

**Definition 2.6.** We call a set $X$ *finite* iff it is *Kuratowski finite,* i.e., as an element of $\mathcal{P}X$, it is in the $\cup$-subsemilattice $\mathcal{F}X$ generated by the singletons.

This is our basic notion and is the one used in Section 2.1 for finite powersets and finitely bounded universal quantification. Since $\mathcal{F}X$ is a free semilattice over $X$, finiteness amounts to saying that the free semilattice has a top element. In other words, finite sets are the models of the following geometric theory:

$$\begin{array}{l} \mathrm{FinSet}[X] \\ \left[\ T : \mathcal{F}X \right. \\ \left[\ \forall x : X.\ x \in T \right. \end{array}$$

Note that, geometrically, finiteness is not a *property* of sets (i.e. extra axioms alone) but *structure* (an extra symbol), albeit uniquely determined. A consequence of this is that homomorphisms between finite sets, i.e. structure-preserving functions between models of the theory FinSet, must preserve the constant $T$ and hence must be surjective.

It is important to be aware that, by contrast with the classical situation, in a finite set the subsets need not be finite. However, complemented subsets are finite.

Stronger than finiteness is finiteness with decidable equality:

$$\begin{array}{l} \mathrm{FinDecSet}[X] \\ \left[\begin{array}{l} T : \mathcal{F}X \\ \neq\ \subseteq X \times X \end{array}\right. \\ \left[\begin{array}{l} \forall x : X.\ x \in T \\ \forall x : X.\ (x \neq x \to \mathbf{false}) \\ \forall x, y : X.\ (x = y \vee x \neq y) \end{array}\right. \end{array}$$

The following result will be important.

**Proposition 2.7.** Let $X$ and $Y$ be sets, with $X$ finite decidable. Then exponentiation $Y^X$ is geometric.

*Proof.* The graph of any function from $X$ is finite, so the functions from $X$ to $Y$ can be identified with certain elements $S$ of $\mathcal{F}(X \times Y)$. Then the functionhood axioms of totality and single-valuedness can be expressed as geometric formulae, namely

$$\forall x \in X.\ \exists y \in Y.\ (x, y) \in S$$
$$\forall (x_1, y_1) \in S.\ \forall (x_2, y_2) \in S.\ (x_1 \neq x_2 \vee y_1 = y_2)$$

□

Of course, the notion of decidable equality is independent of finiteness, so there is a theory DecSet of sets with decidable equality (or, more briefly, *decidable sets*). Like finiteness, having decidable equality is not just a property, geometrically, but requires uniquely determined extra structure. A homomorphism of decidable sets must be monic, since it preserves inequality as well as equality, and a homomorphism of finite decidable sets must be an isomorphism.

Finally, stronger still is finiteness with a decidable total order, which we shall call *strong* finiteness:

$$\begin{array}{l} \mathrm{StrFin}[X] \\ \left[\begin{array}{l} T : \mathcal{F}X \\ < \subseteq X \times X \end{array}\right. \\ \left[\begin{array}{l} \forall x : X.\, x \in T \\ \forall x, y, z : X.\, (x < y \wedge y < z \to x < z) \\ \forall x : X.\, (x < x \to \mathbf{false}) \\ \forall x, y : X.\, (x = y \vee x < y \vee y < x) \end{array}\right. \end{array}$$

A homomorphism of such sets must be an order isomorphism.

**Definition 2.8.** If $n$ is a natural number, we write $\kappa_n$ for the set $\{i \in \mathbb{N} | 0 \leq i < n\}$ and call this the *finite ordinal* of $n$. This is identical to the "finite cardinal" $[n]$ defined elsewhere (Johnstone 1977), but we call it an ordinal to emphasize its ordering.

**Proposition 2.9.** StrFin is equivalent to the theory of a single natural number $n$.

*Proof.* Given $n$, then $\kappa_n$ is a finite set with decidable total order. Conversely, if $X$ is such then it can be put in unique order isomorphism with $\kappa_n$ where $n$ is its cardinality. (Note that cardinalities as natural numbers are well defined for finite decidable sets, though not for finite sets in general.) □

Thus although the class of strongly finite sets is large, it can for most purposes be replaced by the set $\mathbb{N}$ of natural numbers. This fact is rather crucial in our subsequent development, where we often need to find geometrically small categories.

Strong finiteness is preserved by some important geometric constructions, in particular $\times$, $+$ and $\mathcal{F}$. (More precisely, since the total order is not uniquely determined by the rest of the structure, there are canonical ways of supplying an order.) Proofs that finite decidability is preserved by $\times$ and $\mathcal{F}$ can be conveniently found in (Vickers 1999); $+$ is easy. $\kappa \times \lambda$ can be ordered by

$$(x, y) < (x', y') \text{ iff } x < x' \text{ or } (x = x' \text{ and } y < y'),$$

while $\kappa + \lambda$ can be ordered by letting all the elements of $\kappa$ come before all the elements of $\lambda$ – this is just ordinal arithmetic, of course.

As for $\mathcal{F}\kappa$, we can show that the obvious function from the list monoid $\kappa^*$ onto $\mathcal{F}\kappa$ is a retraction, split by a function $i : \mathcal{F}\kappa \to \kappa^*$ defined using the set recursion scheme described in (Vickers 1999): $i(\emptyset)$ is the empty list [] and

$$i(\{n\} \cup S) = \mathrm{ins}(n, i(S))$$

where ins is an easily implemented function to insert an element into an ordered list in the correct place, omitting duplicates. Thus for each finite set $S$ we get $i(S)$ an ordered list of the elements of $S$. $\kappa^*$ is not finite, but it has a decidable total order, the lexicographic order, and this can be restricted to $\mathcal{F}\kappa$.

Let us give a couple of sheaf-theoretic examples to show that these notions of finiteness are genuinely distinct.

— Let \$ be the Sierpinski locale, which can be formulated as the classifying topos for the theory of subsets $P$ of $\kappa_1 = \{0\}$. A sheaf over \$, an object of $\mathcal{S}\$$, comprises two sets, namely the stalks at $\emptyset$ and at $\{0\}$, and a function from the first to the second. In $\mathcal{S}\$$ we define an equivalence relation $\equiv$ on $\kappa_2 = \{0, 1\}$ by $x \equiv y$ iff $x = y \vee 0 \in P$. Then the quotient $\kappa_2/\equiv$, a sheaf, has stalks with two elements at $\emptyset$ and one at $\{0\}$. It is finite but not decidable, for if it were decidable then inequality of the two elements at $\emptyset$ would have to be preserved at $\{0\}$.

— Consider the twisted double cover of the circle, $O$. If $O$ is represented as the complex numbers of modulus 1, then the double cover can be represented as $z \mapsto z^2$ from $O$ to itself. This is a sheaf over $O$, and it is finite decidable (and has cardinality 2), but it cannot be globally ordered right round the circle.

These counterexamples notwithstanding, it is still possible to use the convenience of strong finiteness when dealing with arbitrary finite sets, for every finite set is a surjective image of a finite ordinal. The reason is that for any set $X$ the natural homomorphism $\theta : X^* \to \mathcal{F}X$ ($X^*$ the free monoid over $X$, $\mathcal{F}X$ the free semilattice) is surjective, its image being a subsemilattice of $\mathcal{F}X$ that contains all the generators. Hence if $X$ is finite then it is $\theta(l)$ for some $l : X^*$. The list $l$ contains all the elements of $X$ (possibly with repetitions – we can't do anything about that), and so is a surjection from $\kappa_n$ ($n$ the length of $l$) to $X$. Note that the existential proof is satisfied with lists locally, but they cannot necessarily be put together as a global element of $X^*$. This is most evident in the second example, where $X$ is the double cover of $O$. $X$ has no global elements, so the only global list is the empty list []. But we can find an open cover of $O$ by two large arcs, on each of which we can consistently order the two partial sections of $X$.

### 2.5. *Ind-completion and presheaf categories*

Plotkin's SFP characterization uses colimits of $\omega$-chains of finite posets, and we shall in effect need to generalize this to filtered colimits. This suggests the use of an ind-completion, a completion by filtered colimits, and in topical terms this amounts to using presheaf toposes: for, as remarked in (Johnstone and Joyal 1982), the ind-completion of $C$ is equivalent to the category of points of the presheaf topos $\mathbf{Set}^C$. We now explain this in more detail.

A good description of the ind-completion Ind-$C$ of a category $C$ can be found in (Johnstone 1982). Its objects are filtered diagrams in $C$, but it also embeds fully and faithfully in the presheaf category $\mathbf{Set}^{C^{op}}$, extending the Yoneda embedding $\mathcal{Y}$. In a fuller account in (Grothendieck and Verdier 1972) one sees that the presheaves in the image of Ind-$C$ are exactly the flat presheaves $F$, i.e. those for which the category of elements

$\int_C F$ is filtered. (This is the notation of (Mac Lane and Moerdijk 1992). Grothendieck and Verdier write $C_{/F}$.) These include the representable presheaves $\mathcal{Y}(U)$ ($U$ an object of $C$), defined by $\mathcal{Y}(U)(V) = C(V, U)$. The objects of $\int_C F$ are pairs $(U, x)$ where $U$ is an object of $C$ and $x \in F(U)$, and a morphism from $(U, x)$ to $(V, y)$ is a morphism $f : U \to V$ in $C$ such that $x = F(f)(y)$.

$C$ can almost be recovered from Ind-$C$ through the notion of finite presentability: an object $X$ of a category $D$ is *finitely presentable* iff $D(X, -) : D \to \mathbf{Set}$ preserves filtered colimits. In Ind-$C$, an object is finitely presentable iff it is a retract of a representable, and it can be deduced that if in $C$ all idempotents split then $C$ is equivalent to the category of all finitely presentable objects of Ind-$C$.

**Proposition 2.10.** If $C$ is a small category, then the geometric theory $\mathrm{FlatPreSh}_C$ of flat presheaves over $C$ is presented by –

$$
\begin{array}{l}
\mathrm{FlatPreSh}_C[F] \\
\left[\begin{array}{l}
p : F \to \mathrm{ob}(C) \\
\mu : \mathrm{ar}(C) \times_{\mathrm{ob}(C)} F \to F \text{ (we write } f \cdot x \text{ for } \mu(f,x)\text{, defined iff } \mathrm{tar}(f) = p(x))
\end{array}\right. \\
\left[\begin{array}{l}
\forall x : F, f : \mathrm{ar}(C).\ (\mathrm{tar}(f) = p(x) \to p(f \cdot x) = \mathrm{src}(f)) \\
\forall x : F.\ \mathrm{Id} \cdot x = x \\
\forall x : F, f, g : \mathrm{ar}(C).\ (\mathrm{tar}(f) = \mathrm{src}(g) \wedge \mathrm{tar}(g) = p(x) \to (f; g) \cdot x = f \cdot (g \cdot x)) \\
\exists x : F.\ \mathbf{true} \\
\forall x, y : F.\ \exists z : F.\ \exists f, g : \mathrm{ar}(C).\ (x = f \cdot z \wedge y = g \cdot z) \\
\forall y : F, f, g : \mathrm{ar}(C).\ (f \cdot y = g \cdot y \to \exists z : F.\ \exists h : \mathrm{ar}(C).\ (y = h \cdot z \wedge f; h = g; h))
\end{array}\right.
\end{array}
$$

*Proof.* See (Johnstone 1977). The set $F$ as presented here is the internal way of describing a presheaf. To get the contravariant functor one defines $F(U) = \{x \in F | p(x) = U\}$ for each object $U$ of $C$. □

The last three axioms are the flatness. All the rest just presents the theory of presheaves. The homomorphisms for the theory are the presheaf morphisms (natural transformations).

Note that in $\mathrm{ar}(C) \times_{\mathrm{ob}(C)} F$ we have used pullback as a geometric sort constructor. This can be justified categorically on the grounds that all finite limits are geometric. Alternatively, one can use logic directly, defining $\mu$ as a ternary relation on $\mathrm{ar}(C) \times F \times F$ and using geometric axioms to specify that it is the graph of a partial function on $\mathrm{ar}(C) \times F$, defined at $(f, x)$ iff $\mathrm{tar}(f) = p(x)$.

**Definition 2.11.** We write $\hat{}C$ for the classifying topos of $\mathrm{FlatPreSh}_C$. It is the topical analogue of Ind-$C$ in that the points and homomorphisms of $\hat{}C$ correspond to the objects and morphisms of Ind-$C$, so we call it the *topical ind completion* of $C$.

Diaconescu's theorem, for which see (Johnstone 1977), asserts (in the traditional way of speaking) that $\mathrm{FlatPreSh}_C$ is classified by $\mathbf{Set}^C$: for us that means that $\mathcal{S}\hat{}C$ is equivalent to $\mathbf{Set}^C$, hence our notation. (A referee remarked that the Grothendieck school have used a similar notation, $\widehat{C}$ for $\mathbf{Set}^{C^{op}}$.)

In the next Theorem we summarize some of the main known properties of $\hat{}C$.

**Theorem 2.12.** Let $C$ be a small category.

1 For each object $U$ of $C$, the representable presheaf $\mathcal{Y}(U)$, defined by $\mathcal{Y}(U)(V) = C(V,U)$, is flat.
2 Every model of $\mathrm{FlatPreSh}_C$ is a filtered colimit of representables.
3 For any object $U$ of $C$ and model $F$ of $\mathrm{FlatPreSh}_C$ there is a bijection between homomorphisms from $F_U$ to $F$ and elements of $F(U)$.
4 The sheaves of $\hat{\ }C$ (the maps – or geometric morphisms – from $\hat{\ }C$ to [Set]) are equivalent to covariant functors from $C$ to **Set**.
5 Each object $U$ of $C$ gives rise to a sheaf of $\hat{\ }C$, $F \mapsto F(U)$ – that is to say, for each point $F$ of $\hat{\ }C$, the stalk at $F$ is the set $F(U)$.
6 For any topos $E$, the category of maps from $\hat{\ }C$ to $E$ is equivalent to the category of functors from $C$ to the category of global points of $E$ (i.e. maps from 1 to $E$).

*Proof.* We sketch the constructions.

1 This was referred to at the start of the subsection.
2 Given a model $F$, we can construct the filtered category $\int_C F$ (see above). To each object $(U, x)$ we attach the representable model $\mathcal{Y}(U)$ and thereby get a filtered diagram of representables; $F$ is its colimit.
3 This is Yoneda's Lemma.
4 This is Diaconescu's Theorem. Note that by restricting a sheaf to the representable models we obviously get a functor from $C$ to **Set**; the deep part is to show the equivalence.
5 Note that the construction $F \mapsto F(U)$ is geometric. As a functor from $C$ to **Set**, the sheaf takes $V$ to $\mathcal{Y}(V)(U) = C(U,V)$.
6 See (Johnstone 1977). Once a map or natural transformation is defined on the representables, its action of the rest of the flat presheaves is determined by (2).

□

An interesting question asks which toposes are topical ind-completions – in more conventional terms, which toposes are presheaf toposes. In showing that a topos is equivalent to some $\hat{\ }C$, in practice we try to identify the finitely presentable models. Since finite presentability in itself is not a geometric property, it is used as a guide rather than a formal part of the geometric reasoning. Our main result, Theorem 4.7, answers the particular case of the classifier for strongly algebraic information systems, and the techniques used are also applied to some other examples. To prepare for it, and to illustrate the "topology-free space" methods of (Vickers 1999), let us first review some well-known examples from (Johnstone and Wraith 1978; Mac Lane and Moerdijk 1992).

**Theorem 2.13.** Let $T$ be a finitary algebraic theory. Then its classifying topos $[T]$ is equivalent to $\hat{\ }\mathrm{fp}_T$ where $\mathrm{fp}_T$ is the category of finitely presented $T$-algebras. ("Finitely" here is in a strong sense – the set of generators and the set of relations must both be finite ordinals.)

*Proof.* We do not give a full proof here, but outline its steps in order to prepare the way for our other examples and at the same time bring out the geometric features of the argument.

1. *Choosing* $\mathrm{fp}_T$*:* An algebra $A$ is finitely presentable in the categorical sense (that $\hom(A, -)$ preserves filtered colimits) iff it is finitely presentable in the algebraic sense (it can be presented with finitely many generators and relations). Though this result is not geometric, it suggests the use of $\mathrm{fp}_T$ and the rest of the proof validates this choice. This category is required to be small, and that means in a geometric sense: its sets of objects and arrows are geometrically definable. This is so provided we use the ordinal sense of finiteness, and follows from the fact that finite ordinals essentially form a set, $\mathbb{N}$. Strictly speaking, $\mathrm{fp}_T$ is to be the category not of "finitely presented" $T$-algebras, but of "finite presentations of" $T$-algebras. However, the morphisms in $\mathrm{fp}_T$ are defined so as to make the two categories equivalent: we have a full and faithful functor from $\mathrm{fp}_T$ to the category of $T$-algebras.

2. *A map* $\beta : [T] \to \hat{}\,\mathrm{fp}_T$*:* For each algebra $A$ we need a flat presheaf $\beta(A) = F_A$ over $\mathrm{fp}_T$. By Yoneda's lemma, $F_A(B)$ will have to be the set $\hom(B, A)$ of homomorphisms from $B$ to $A$, and this must be geometric. Although exponentiables $A^B$ are not geometric in general, in this particular case we can deal with $A^G$ where $G$ is the set of generators for $B$. $G$ is finite decidable, and $A^G$ is geometric by Proposition 2.7; it turns out that $\hom(B, A)$ is geometric as required. It is now clear that defining $F_A$ in this way gives a presheaf, but flatness is not automatic. To prove it one uses the fact that $\mathrm{fp}_T$ has all finite colimits, and a useful result in, e.g., (Mac Lane and Moerdijk 1992) that, for such categories, flatness of a presheaf is equivalent to its transforming finite colimits to limits. Then flatness of $F_A$ follows directly from the fact that the colimits in $\mathrm{fp}_T$ are preserved in the category of all algebras.

At this point we have the following:

— A geometrically definable full subcategory $\mathrm{fp}_T$ of models (algebras) of $T$.
— If $A$ is a model of $T$, then the presheaf $F_A$ of $\mathrm{fp}_T$, defined by $F_A(B) = \hom(B, A)$, is geometrically definable and flat.

For the moment, we can continue the proof using just these properties.

3. *A map* $\alpha : \hat{}\,\mathrm{fp}_T \to [T]$*:* By Theorem 2.12 (6) it suffices to define the corresponding functor from $\mathrm{fp}_T$ to the points of $[T]$, and that is immediate because the objects of $\mathrm{fp}_T$ already are (or, rather, present) models of $T$.

4. *A natural isomorphism* $\phi : \mathrm{Id}_{\hat{}\,\mathrm{fp}_T} \to \beta \circ \alpha$*:* Again, by Theorem 2.12 (6) it suffices to define the natural isomorphism on objects $A$ of $\mathrm{fp}_T$. If $\beta \circ \alpha(A)$ is the flat presheaf $F$ then it is defined by letting $F(B)$ be the set of algebra homomorphisms from $B$ to $A$, and that set is isomorphic to $\mathrm{fp}_T(B, A)$. Hence $F$ is isomorphic to the representable presheaf corresponding to $A$ and that is just what we need.

5. *A natural isomorphism* $\theta : \alpha \circ \beta \to \mathrm{Id}_{[T]}$*:* Suppose $A$ is a $T$-algebra and $F_A = \beta(A)$. An object of $\int_{\mathrm{fp}_T} F_A$ is an algebra homomorphism $p : B \to A$ for some object $B$ of $\mathrm{fp}_T$, and it follows from the definition of $\alpha(F_A)$ as a colimit that we have a homomorphism $\theta_A : \alpha \circ \beta(A) \to A$. By geometricity this gives a natural transformation $\theta$. It follows from flatness of the presheaf $F_A$ that each component of $\theta$ is 1-1. For suppose we have two elements $x_1$ and $x_2$ of $\alpha(F_A)$ with $\theta_A(x_1) = \theta_A(x_2)$. Using the properties of filtered

colimits, for each $i = 1, 2$ there is a finitely presented algebra $B_i$ with an element $x'_i$ and a homomorphism $p_i : B_i \to A$ (so $(B_i, p_i)$ is an object of $\int_{\mathrm{fp}_T} F_A$) such that $x_i$ is the image of $x'_i$ under the colimit injection corresponding to $(B_i, p_i)$. Using more properties of filtered colimits, we can take it that $B_1 = B_2 = B$ (say) and $p_1 = p_2 = p$ (say). We can then impose on $B$ a new relation to make $x'_1 = x'_2$ to get another finitely presented algebra $B'$ through which $p$ factors, and $x_1$ and $x_2$ are the images of the images of $x'_1$ and $x'_2$ in $B'$ and hence already equal in $\alpha(F_A)$.

To proceed further, we must return to our specific knowledge that $T$ is algebraic. Then to show each component of $\theta$ is an isomorphism, we show that it is onto. An element $a$ of $A$ (we are taking $T$ to be a single-sorted algebraic theory, but actually the methods work just as well for many-sorted theories) is in the image of a homomorphism from a free algebra, finitely presented with one generator and no relations. □

**Corollary 2.14.** The object classifier [Set] is equivalent to ˆ Fin, where Fin is the category of finite ordinals and functions between them.

However, it is not only algebraic theories that are classified by topical ind-completions. Some simple examples in (Johnstone and Wraith 1978) are provided by categories $C$ adapted from Fin by keeping the same objects, the finite ordinals, but varying the morphisms. With monic functions $\hat{}C$ classifies decidable sets, and with surjective functions $\hat{}C$ classifies finite sets (note how the morphisms in $C$ match the homomorphisms for decidability or for finiteness). These latter two examples are clearly not algebraic, for in neither of them does $C$ have all finite colimits.

Another example presented in (Mac Lane and Moerdijk 1992) is that of totally ordered sets, bounded with *distinct* top and bottom elements. In effect what is proved there is that their classifier [BTOS] is the topical ind-completion of the category of finite ordinals at least 2, with morphisms the monotone functions preserving top and bottom. They also show that this category is dual to the simplicial category $\Delta$ (this is a very rudimentary, discrete case of Stone duality), so that $\mathcal{S}$[BTOS] is equivalent to $\mathbf{Sets}^{\Delta^{op}}$, the category of simplicial sets.

All these examples, as well as the strongly algebraic information systems that are our primary interest, have the particular feature, not shared with algebraic theories in general, that $C$ comprises the (ordinally) *finite* models (in fact, the finitely presentable models are all finite). Our general techniques of Section 3 are restricted to such cases, but can be applied (in Proposition 3.7) to all the examples just mentioned.

A final example is that of algebraic dcpos. When $C$ is a poset then its ind-completion is equivalent to its ideal completion Idl$(C)$, and $\mathcal{S}\hat{}C$, equivalent to $\mathbf{Sets}^{C}$, is the category of sheaves over Idl$(C)$ with its Scott topology.

## 3. Theories for which finitely presentable models are finite

In this section we develop new techniques showing how, for certain geometric theories $T$, we can find a small category $C$ such that $\hat{}C$ classifies $T$: in traditional terms, that $\mathcal{S}[T]$ is equivalent to $\mathbf{Set}^{C}$, but we are able to sidestep these categories of sheaves by using a geometric proof that the models of $T$ are equivalent to flat presheaves over $C$.

Our methods do not assume finite colimits of finitely presentables, but are restricted to cases where finitely presentable models are actually finite (in a strong sense).

Referring back to the proof outlined in Theorem 2.13, the big gap lies in the lack of finite colimits in $C$. These are used in proving flatness of the presheaves $\beta(A)$ and also in proving that the natural transformation $\theta$ is an isomorphism. What we see there is that any finite piece of structural information about an algebra $A$ – a finite selection of elements and a finite number of relations involving them – can be expressed in a universal way as a finitely presented algebra which then maps homomorphically to $A$. However, this depends crucially on the finite colimits in $\mathrm{fp}_T$, for collecting generators corresponds to coproducts and imposing relations corresponds to coequalizers. We shall work with theories where though there is no *universal* way of using a finitely presentable model to express each finite piece of structural information, nonetheless for each finite piece of structural information in a given model there is *some* finitely presentable model (in fact a finite model) embodying that information and mapping homomorphically to the given model. We call this, suitably formulated (in Theorem 3.6), a *finite structure condition (FSC)* on the theory. Our main result in this Section is that if a theory $T$ is sufficiently finitary and has the finite structure condition, then it is classified by $\hat{\ }\mathbf{of}_T$ where $\mathbf{of}_T$ is a category of models carried by finite ordinals (Definition 3.2).

It is important to bear in mind that our presentations of geometric theories allow sort constructors such as $\times$, $+$ and $\mathcal{F}$. The sorts declared explicitly in the theory are the *base* sorts $\sigma$, but the argument of a predicate may be of a *derived* sort $\tau$ (possibly a product, to allow for $n$-ary predicates) and the same goes for the argument and result of a function, and for the free variable in an axiom. For a predicate $R(x)$ or an axiom $\forall x.\ (\phi(x) \to \psi(x))$, with $x$ of sort $\tau$, we call $\tau$ the *arity* of the predicate or axiom.

**Definition 3.1.** A geometric theory presentation $T$ is *$\mathcal{F}$-finitary* iff –

— It has only finitely many base sorts, predicate symbols and axioms.
— It has no function symbols.
— The derived sorts used explicitly as arities are constructed from the base sorts using only 1, $\times$ and $\mathcal{F}$.
— In each axiom $\forall x : \tau.\ (\phi(x) \to \psi(x))$, $\phi$ uses only finitary logic and finitely bounded universal quantification.

Some of these conditions have been made unnecessarily strong for the sake of expediency. In restricting our constructors, our main concern is to ensure that if the base sorts and predicates are given strongly finite interpretations, then so also are the derived sorts and premisses of the axioms.

The prohibition of function symbols makes little difference to the scope of the methods. Function symbols can always be eliminated by replacing them by predicate symbols for their graphs, together with functionhood axioms for totality and single-valuedness, and doing this makes no difference to the other conditions. However, it will make a difference to the Finite Structure Condition in Theorem 3.6, where it is important with functions to separate out structure (the graph predicate) from properties (the functionhood axioms), and it is convenient to make the separation now rather than later.

**Definition 3.2.** Let $T$ be an $\mathcal{F}$-finitary geometric theory. We say that a model of $T$ is *ordinally finite* iff the carriers of its base sorts are all finite ordinals and its interpretations of the predicate symbols are all finite (and hence strongly finite). Note that finiteness for the predicate symbols also implies that they are decidable as predicates, i.e. complementable: for $\neg P(x)$ can be defined as $\forall y \in P. x \neq y$.

We write $\mathbf{of}_T$ for the category of ordinally finite models of $T$.

**Proposition 3.3.** Let $T$ be an $\mathcal{F}$-finitary geometric theory.

1 If $\kappa$ and $X$ are models of $T$, with $\kappa$ ordinally finite, then the set of homomorphisms from $\kappa$ to $X$ is geometrically definable.

2 The category $\mathbf{of}_T$ is geometrically small (i.e. it is small and its structure is geometrically definable).

*Proof.* 1. A homomorphism $p : \kappa \to X$ has a vector of carrier functions, $p_\sigma : \kappa_\sigma \to X_\sigma$ for each base sort $\sigma$, and the set of such vectors is geometric (this uses finiteness of the set of sorts, and Proposition 2.7 to show that each $X_\sigma^{\kappa_\sigma}$ can be constructed geometrically). If $R$ is a predicate, with arity $\tau$, then its preservation can be expressed geometrically as $\forall n \in R_\kappa.\ R_X(p_\tau(n))$. (Any geometric construction is functorial, so the notation $p_\tau$ for the application of such a functor to the $p_\sigma$s makes sense.) Since there are only finitely many predicates, the conjunction of these properties is geometric.

2. The structure of an ordinally finite model is given by a tuple of natural numbers, for the carriers, and a tuple of finite subsets of the appropriate derived carriers. In each axiom $\forall x : \tau.\ (\phi(x) \to \psi(x))$, $\phi$ denotes a complementable and hence finite subset of the derived carrier for $\tau$, and so the axiom can be expressed as a geometric formula $\forall x \in \phi.\ \psi(x)$. That accounts for the objects. The morphisms come from part (1). □

**Definition 3.4.** Let $T$ be an $\mathcal{F}$-finitary geometric theory presentation. $T$ is *axiom-free* iff it has no axioms, so it has just sorts and predicates. In general, the *structure theory* $T'$ of $T$ is got from $T$ by dropping its axioms. In accordance with Definition 2.2, the models of $T'$ are *structures* for $T$: they interpret the symbols of $T$ but don't necessarily make the axioms true.

Note that every model of $T$ is also a structure. Also, between any two models, the model homomorphisms are the same as the structure homomorphisms.

*Beware!* The structure theory is presentation-dependent. Suppose, for instance, that $T_1$ has two sorts and nothing else. It is already axiom-free. On the other hand, it is equivalent to a theory $T_2$ with one sort, two unary predicates $P$ and $Q$, and axioms to make them Boolean complements of each other. This is not axiom-free, and its structure theory is not equivalent to $T_1$.

**Proposition 3.5.** Let $T$ be an $\mathcal{F}$-finitary *axiom-free* geometric theory. Then

1 $\mathbf{of}_T$ is finitely cocomplete ("finitely" in a strong sense: the diagram must be ordinally finite) and its inclusion in the category of all models of $T$ preserves finite colimits;

2 $[T]$ is equivalent to $\hat{}\mathbf{of}_T$.

*Proof.* 1. Given an ordinally finite diagram of ordinally finite models, at each base sort $\sigma$ we can take a finite ordinal for the colimit $\kappa_\sigma$ (say) of the corresponding carriers. (These colimits are still strongly finite.) Now consider a predicate $P(x)$ of arity $\tau$, say. The sort constructor for $\tau$ can be applied to the colimit cocones to give another cocone (though not necessarily a colimit cocone) to $\kappa_\tau$. We interpret $P$ for $\kappa$ as the union of the images in $\mathcal{F}\kappa_\tau$ of the interpretations for $P$ in the original models. It is not hard to see that $\kappa$ is ordinally finite, and in the category of all models it is still the colimit of the original diagram.

2. The proof of Theorem 2.13 can be followed as far as showing that the components of $\theta$ are 1-1. They are also clearly onto, for if $X$ is a model of $T$, $\sigma$ is a base sort and $x \in X_\sigma$, then we can form an ordinally finite model $\kappa$ carried by a singleton at sort $\sigma$ and the empty set elsewhere and with all predicates interpreted emptily, and then $x$ is in the image of a homomorphism from $\kappa$ to $X$. Once we know that $\theta$ is a bijection at the base sorts, it follows at the derived sorts too. Finally, we must show that $\theta$ reflects the predicates. Suppose $P$ has arity $\tau$ and $x \in X_\tau$ with $P(x)$. There is some ordinally finite model $\kappa$ with $y \in \kappa_\tau$ and $x = \theta_\tau(y)$. We can define a new model $\kappa'$ with the same carriers but $P_\tau$ extended with $\{y\}$, and it follows that $P(y)$ holds in $\alpha \circ \beta(X)$. □

**Theorem 3.6.** Let $T$ be an $\mathcal{F}$-finitary geometric theory satisfying the following *Finite Structure Condition (FSC)*:

If $\kappa$ is an ordinally finite *structure* for $T$, $X$ is a model of $T$, and $p : \kappa \to X$ is a homomorphism, then there is some ordinally finite *model* $\lambda$ of $T$ such that $p$ factors as homomorphisms $\kappa \to \lambda \to X$.

Then $[T]$ is equivalent to $\hat{}\mathbf{of}_T$.

Note the geometric nature of the existence claim in the Finite Structure Condition. Once $X$ is given, we have a geometrically definable set of diagrams $\kappa \to \lambda \to X$, with a function to the geometrically definable set of diagrams $\kappa \to X$. The requirement is that this function should be surjective.

*Proof.* We show geometrically that models of $T$ are equivalent to flat presheaves over $\mathbf{of}_T$, following the route of Theorem 2.13. Examining the details, we find that we can define $\alpha : \hat{}\mathbf{of}_T \to [T]$ without trouble and for $\beta : [T] \to \hat{}\mathbf{of}_T$ we can define the presheaves $\beta(X)$ but must still prove flatness. Once that is done, the constructions of $\phi : \mathrm{Id}_{\hat{}\,\mathrm{fp}_T} \to \beta \circ \alpha$ and $\theta : \alpha \circ \beta \to \mathrm{Id}_{[T]}$ both go through, as does the proof that $\phi$ is an isomorphism, but we still need to prove that $\theta$ is an isomorphism. For both of the parts still to be proved, we need FSC to make up for the loss of finite cocompleteness of $\mathbf{of}_T$. Let $T'$ be the structure theory for $T$.

Writing $F_X$ for $\beta(X)$, its flatness is equivalent to $\int_{\mathbf{of}_T} F_X$ being filtered, and that is equivalent to every (ordinally) finite diagram in $\int_{\mathbf{of}_T} F_X$ having a cocone over it. The flatness conditions correspond to the three special cases where the shape of the diagram is empty, the discrete 2-object category, or a parallel pair of morphisms. Now a finite diagram in $\int_{\mathbf{of}_T} F_X$ is equivalent to a finite diagram in $\mathbf{of}_T$ with a cocone to $X$ in the category of models of $T$. This cocone factors via the colimit in $\mathbf{of}_{T'}$, and by FSC that then factors via an object in $\mathbf{of}_T$.

It remains to show that $\theta$ is an isomorphism. To do this, we also apply the same constructions to the structure theory to get a flat presheaf $F'_X$ over $\mathbf{of}_{T'}$, and a structure $\alpha'(F'_X)$ with homomorphism $\theta' : \alpha'(F'_X) \to X$. Since every model is a structure, the elements $p : \kappa \to X$ of $F_X$ are also in $F'_X$ and so we get a homomorphism $\psi : \alpha(F_X) \to \alpha'(F'_X)$. The FSC then shows that this is an isomorphism. For each structure homomorphism $p' : \kappa' \to X$ factors via a model $\kappa$, hence giving a homomorphism from $\kappa'$ to $\alpha(F_X)$, which, it turns out, is independent of the choice of $\kappa$. For if we have two factorizations via models $\kappa_i$ $(i = 1, 2)$ then by flatness we can factor them via a single $\kappa$. Suppose, then, we have $q_i : \kappa' \to \kappa$ $(i = 1, 2)$ and $p : \kappa \to X$ with $q_i; p = p'$. Then $p$ factors via the coequalizing *structure* $\kappa''$ (say) for $q_1$ and $q_2$, and then by FSC that factors via a finite model $\kappa'''$:

$$\begin{array}{ccccccc} \kappa' & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & \kappa & \overset{q}{\to} & \kappa'' & \overset{q''}{\to} & \kappa''' \\ \downarrow_{p'} & & \downarrow_{p} & & \downarrow_{p''} & & \downarrow_{p'''} \\ X & = & X & = & X & = & X \end{array}$$

If $\mathrm{in}_p : \kappa \to \alpha(F_X)$ and $\mathrm{in}_{p'''} : \kappa''' \to \alpha(F_X)$ are the colimit injections, then $\mathrm{in}_p = q; q''; \mathrm{in}_{p'''}$ and it follows that $q_1; \mathrm{in}_p = q_2; \mathrm{in}_p$. These homomorphisms $\kappa' \to \alpha(F_X)$ give a homomorphism $\alpha'(F'_X) \to \alpha(F_X)$ which, it is clear, is an inverse for $\psi$.

Now it suffices to show that $\theta'$ is an isomorphism, and this follows from Proposition 3.5. □

The result is enough to prove the known results mentioned at the end of Section 2.5.

**Proposition 3.7.**

1 (Johnstone and Wraith 1978) [FinSet], classifying finite sets, is equivalent to $\hat{}\,\mathrm{Fin}_{onto}$, where $\mathrm{Fin}_{onto}$ is the category of finite ordinals and surjective functions.

2 (Johnstone and Wraith 1978) [DecSet], classifying sets with decidable equality, is equivalent to $\hat{}\,\mathrm{Fin}_{\text{1-1}}$, where $\mathrm{Fin}_{\text{1-1}}$ is the category of finite ordinals and injective functions.

3 (See (Mac Lane and Moerdijk 1992)) [BTOS], classifying bounded totally ordered sets with distinct top and bottom, is equivalent to $\hat{}\,\Delta^{op}$, where $\Delta$ is the simplicial category.

*Proof.* 1. The theory FinSet of Section 2.4 must first be adapted to convert the constant $T$ into a predicate:

$$\begin{array}{l} \mathrm{FinSet}[X] \\ \left[\ T \subseteq \mathcal{F}X \right. \\ \left[ \begin{array}{l} \exists S : \mathcal{F}X. T(S) \\ \forall S_1, S_2 : \mathcal{F}X.\ (T(S_1) \wedge T(S_2) \to S_1 = S_2) \\ \forall S : \mathcal{F}X, x : X.\ (T(S) \to x \in S) \end{array} \right. \end{array}$$

The ordinally finite models are the finite ordinals, and their homomorphisms are the surjections. An ordinally finite structure is a finite ordinal $\kappa$ equipped with a finite subset $T$ of $\mathcal{F}\kappa$. Suppose we have a homomorphism $p$ from $\kappa$ to a finite set $X$. If $T$ is nonempty then we have some $S \in \mathcal{F}\kappa$ with $T(S)$. Since $p$ is a homomorphism, $\mathcal{F}p(S) = X$. If the only $S$ for which $T$ holds is $\kappa$ itself, then $\kappa$ is a model of FinSet and $p$ is a homomorphism. But suppose we have $T(S)$ and some $x$ not in $S$ – i.e. $\forall y \in S.\ x \neq y$. Then there is some $y$ in $S$ for which $p(x) = p(y)$. We can form a quotient of $\kappa$ in which $x$ becomes equal to $y$, with still a homomorphism to $X$. Iterating this we eventually get $\kappa$ a model. If $T$ is empty, $p$ might not be surjective and we proceed slightly differently. We know there is some surjection $p' : \kappa' \to X$ where $\kappa'$ is a finite ordinal; then we can take $\lambda$ to be $\kappa + \kappa'$ with $f : \kappa \to \lambda$ the coproduct injection.

2. DecSet is somewhat easier. An ordinally finite structure is a finite ordinal $\kappa$ equipped with a binary predicate $\neq$. If we have a homomorphism $p$ from $\kappa$ to a decidable set $X$, then $\{(x, y) \in \kappa \times \kappa | p(x) = p(y)\}$ is a decidable equivalence relation on $\kappa$ disjoint from $\neq$. Quotienting it out gives us our $\lambda$.

3. Each model of BTOS is a set $X$ equipped with a total order $\leq$ and distinct top and bottom elements $\top$ and $\bot$. To show the Finite Structure Condition, let $\kappa$ be a finite ordinal equipped with a finite binary predicate $A$ (for $\leq$) and two finite unary predicates $B_\top$ and $B_\bot$, and let $p : \kappa \to X$ be a homomorphism to a bounded totally ordered set. Without loss of generality we can assume that $B_\top$ and $B_\bot$ are non-empty, for otherwise we can add extra elements and extend $p$ to map them to $\top$ and $\bot$. Let $A_1 = A \cup \{(x, y) \in \kappa \times \kappa | B_\bot(x) \vee B_\top(y)\}$. For each pair $(x, y)$ in $\kappa \times \kappa$ we have either $p(x) \leq p(y)$ or the other way round, and we can add $(x, y)$ or $(y, x)$ to $A_1$ accordingly to get eventually a finite relation $A_2 \supseteq A_1$ such that for every $(x, y) \in A_2$ we have $p(x) \leq p(y)$ and for every $(x, y) \in \kappa \times \kappa$ either $(x, y)$ or $(y, x)$ is in $A_2$. Let $A_3$ be the reflexive transitive closure of $A_2$ (also finite, and a preorder) and let $\lambda$ be the poset reflection of $(\kappa, A_3)$, an ordinally finite bounded total order. Then $p$ factors via $\lambda$. As was mentioned at the end of Section 2.5, the category of ordinally finite models is dual to the simplicial category $\Delta$ and it follows that $\mathcal{S}[\text{BTOS}]$ is equivalent to the category $\mathbf{Set}^{\Delta^{op}}$ of simplicial sets. □

## 4. The main result

We now turn to Plotkin's result (Theorem 2.5) and use Theorem 3.6 to show that [IS], the classifier for strongly algebraic information systems (Definition 2.4), is equivalent to the topical ind-completion of $\mathbf{of}_{\text{IS}}$, the category of ordinally finite information systems. First we elucidate $\mathbf{of}_{\text{IS}}$.

**Proposition 4.1.**

1 Any ordinally finite poset $\kappa$ is already, and uniquely, an ordinally finite information system, with

$$\text{CUB}(S, T) \equiv_{\text{def}} (\forall s \in S.\ \forall t \in T.\ s \sqsubseteq t) \wedge (\forall n \in \kappa.\ (\exists s \in S.\ s \not\sqsubseteq n \vee \exists t \in T.\ t \sqsubseteq n))$$

2 Let $\kappa$ and $\lambda$ be ordinally finite posets, and let $f : \kappa \to \lambda$ be a function. Then $f$ preserves CUB iff it is monotone and has a right adjoint.

*Proof.* 1. Note that under Definition 3.2 an ordinally finite poset has two unrelated order structures. Being carried by a finite ordinal (hence strongly finite) it has a decidable total order $\leq$; as an ordinally finite poset it has an additional partial order $\sqsubseteq$, also decidable.

It is already known (Vickers 1999) that $\mathrm{CUB}(S,T)$ has to be defined by the intuitionistic formula

$$(\forall s \in S.\ \forall t \in T.\ s \sqsubseteq t) \wedge (\forall n : \kappa.\ (\forall s \in S.\ s \sqsubseteq n \rightarrow \exists t \in T.\ t \sqsubseteq n))$$

In the present case, this can be made geometric. Finiteness of $\kappa$ allows us to replace "$\forall n : \kappa$" by the geometric "$\forall n \in \kappa$". By decidability of the order $\sqsubseteq$, we can replace "$\forall s \in S.\ s \sqsubseteq n \rightarrow \cdots$" by "$\exists s \in S.\ s \not\sqsubseteq n \vee \cdots$". CUB is a complementable and hence finite subset of $\mathcal{F}\kappa \times \mathcal{F}\kappa$. To show that this does give an information system, the only part of any difficulty is to show that for each $S$ there is some $T$ with $\mathrm{CUB}(S,T)$. We can define $T = \{t \in \kappa | \forall s \in S.\ s \sqsubseteq t\}$.

2. (We know already from (Vickers 1999) that $f$ preserves CUB iff the corresponding map from $\mathrm{Idl}\,\kappa$ to $\mathrm{Idl}\,\lambda$ has a right adjoint. Classically, the present result is a special case of that.)

$\Leftarrow$: Suppose $f$ has right adjoint $g$, and suppose $\mathrm{CUB}(S,T)$ in $\kappa$. If $n$ is an upper bound for $\mathcal{F}f(S)$, then $g(n)$ is an upper bound for $S$ and hence $t \sqsubseteq g(n)$, so $f(t) \sqsubseteq n$, for some $t \in T$. It follows that $\mathrm{CUB}(\mathcal{F}f(S), \mathcal{F}f(T))$.

$\Rightarrow$: $f$ is monotone, for we have $m \sqsubseteq n$ iff $\mathrm{CUB}(\{m,n\},\{n\})$. Given $n \in \lambda$, define $S = \{m \in \kappa | f(m) \sqsubseteq n\}$. This is a complementable subset of $\kappa$, hence finite. Choose $T$ with $\mathrm{CUB}(S,T)$. Then $\mathrm{CUB}(\mathcal{F}f(S), \mathcal{F}f(T))$. We have that $n$ is an upper bound for $\mathcal{F}f(S)$, so there is some $t \in T$ with $f(t) \sqsubseteq n$. It follows that $t \in S$, and hence that $t$ is a (unique) greatest element of $S$. Defining $g(n) = t$ we get a function $g : \lambda \rightarrow \kappa$ right adjoint to $f$. □

By analogy with Plotkin's result, one might perhaps have expected embedding-projection pairs for the morphisms. They are slightly stronger, being the adjunctions for which the unit is the identity, and in fact they do arise (Theorem 4.8) when we consider information systems for which the order $\sqsubseteq$ is decidable.

**Definition 4.2.** We write $\mathbf{ofPos}_{\mathrm{adj}}$ for the category whose objects are ordinally finite posets and whose morphisms are adjunctions, i.e. CUB-preserving functions. By Proposition 4.1, $\mathbf{ofPos}_{\mathrm{adj}}$ is isomorphic to $\mathbf{of}_{\mathrm{IS}}$.

By Proposition 3.3, $\mathbf{ofPos}_{\mathrm{adj}}$ is geometrically small. Explicitly, its objects are those elements $(N, \sqsubseteq)$ of $\mathbb{N} \times \mathcal{F}(\mathbb{N} \times \mathbb{N})$ satisfying the geometric formulae

$$\forall (m,n) \in \sqsubseteq .\ (m < N \wedge n < N)$$
$$\forall n < N.\ n \sqsubseteq n$$
$$\forall (m,n) \in \sqsubseteq .\ \forall (m',n') \in \sqsubseteq .\ (n \neq m' \vee m \sqsubseteq n')$$
$$\forall (m,n) \in \sqsubseteq .\ \forall (m',n') \in \sqsubseteq .\ (n \neq m' \vee n' \neq m \vee m = n)$$

The first of these says that $\sqsubseteq$ is a relation on $\kappa_N$, and the second that it is reflexive there. The last two say that it is transitive and antisymmetric. Note that we have in effect used the equivalence in classical logic between $P \rightarrow Q$ and $\neg P \vee Q$, justified here by finite decidability. Similarly, the morphisms of $\mathbf{ofPos}_{\mathrm{adj}}$ are elements $(M, \sqsubseteq, N, \sqsubseteq', f, g)$

of $\mathbb{N} \times \mathcal{F}(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \times \mathcal{F}(\mathbb{N} \times \mathbb{N}) \times \mathcal{F}(\mathbb{N} \times \mathbb{N}) \times \mathcal{F}(\mathbb{N} \times \mathbb{N})$ satisfying a geometric formula to say that $(M, \sqsubseteq)$ and $(N, \sqsubseteq')$ are ordinally finite posets and $(f, g)$ is an adjoint pair.

The only remaining ingredient is the Finite Structure Condition.

**Lemma 4.3 (Finite Structure Condition for IS).** Suppose –

— $X$ is a strongly algebraic information system;
— $\kappa$ is a finite ordinal (note that its natural order $\leq$ does not interact with the poset structure of $X$);
— $p : \kappa \to X$ is a function;
— $A$ is a finite binary relation on $\kappa$ such that whenever $A(m, n)$ then $p(m) \sqsubseteq p(n)$;
— $B$ is a finite binary relation on $\mathcal{F}\kappa$ such that whenever $B(S, T)$ then $\mathrm{CUB}_X(\mathcal{F}p(S), \mathcal{F}p(T))$.

Then there is a factorization as $p = f; q : \kappa \to \lambda \to X$ where $\lambda$ is an ordinally finite poset, $q$ is a homomorphism of information systems and $f$ is a function such that whenever $A(m, n)$ then $f(m) \sqsubseteq f(n)$, and whenever $B(S, T)$ then $\mathrm{CUB}_\lambda(\mathcal{F}f(S), \mathcal{F}f(T))$.

If the image of $p$ is CUB-closed, then $f$ can be chosen to be a surjection.

*Proof.* The image of $p$ is finite and so included in a finite CUB-closed subset $X_0$ of $X$. Let $p' : \kappa' \to X$ be a surjective function from a finite ordinal $\kappa'$ *onto* $X_0$. We can find a finite total relation $g$ from $\kappa$ to $\kappa'$ such that $p = g; p'$, and by choosing for each $m \in \kappa$ the least element of $\kappa'$ related to it by $g$, we can assume without loss of generality that $g$ is a function. Let $A' = \mathcal{F}(g \times g)(A)$ and $B' = \mathcal{F}(\mathcal{F}g \times \mathcal{F}g)(B)$ be the images of $A$ and $B$ under $g$. Then $(\kappa', p', A', B')$ has the same properties as were hypothesized for $(\kappa, p, A, B)$, but in addition the image of $p'$ is CUB-closed. By replacing $(\kappa, p, A, B)$ by $(\kappa', p', A', B')$, we can henceforth assume without loss of generality that the image of $p$ is CUB-closed.

Our aim now is to put an ordinally finite *pre*order structure on $\kappa$, and obtain $\lambda$ by factoring out the corresponding equivalence relation (hence $f$ will be a surjection).

Because the image of $p$ is CUB-closed, for every $S \in \mathcal{F}\kappa$ we can find $T \in \mathcal{F}\kappa$ such that $\mathrm{CUB}(\mathcal{F}p(S), \mathcal{F}p(T))$, and so we can find a finite, total, binary relation cub on $\mathcal{F}\kappa$ such that whenever $\mathrm{cub}(S, T)$ we have $\mathrm{CUB}(\mathcal{F}p(S), \mathcal{F}p(T))$. By taking the union with $B$, we can assume that $B \subseteq \mathrm{cub}$. Our aim is to construct a finite preorder $R$ on $\kappa$, including $A$, for which

$$\text{if } \mathrm{cub}(S, T) \text{ then } T \text{ is a complete set of upper } R\text{-bounds for } S \quad (1)$$

$$\text{if } mRn \text{ then } p(m) \sqsubseteq p(n) \quad (2)$$

Suppose we can achieve this. Then we form an ordinally finite poset $\lambda = \kappa/R$ by factoring out the equivalence relation corresponding to $R$. (We get $\lambda$ to be strongly finite by using the fact that every equivalence class has a canonical representative, namely its least element with respect to the natural total order on $\kappa$.) Let $f : \kappa \to \lambda$ be the quotient map and $h : \lambda \to \kappa$ its canonical splitting. Then $p$ factors via $q : \lambda \to X$ and this is monotone by (2) above. Suppose $\mathrm{CUB}_\lambda(S, T)$, so that $\mathcal{F}h(T)$ is a complete set of upper $R$-bounds for $\mathcal{F}h(S)$. Choose $T' \subseteq \kappa$ such that $\mathrm{cub}(\mathcal{F}h(S), T')$, so $\mathrm{CUB}_X(\mathcal{F}q(S), \mathcal{F}p(T'))$. If $n' \in T'$ then there is some $n \in T$ such that $h(n)Rn'$ and so $q(n) \sqsubseteq p(n')$. It follows that $\mathrm{CUB}_X(\mathcal{F}q(S), \mathcal{F}q(T))$, and so $q$ is a homomorphism as required.

It remains to find our $R$. Clearly it must contain the finite relation $R_0$ on $\kappa$ defined by

$$mR_0n \equiv_{\mathrm{def}} m = n \vee \exists S, T : \mathcal{F}\kappa.\ (m \in S \wedge n \in T \wedge \mathrm{cub}(S,T)) \vee A(m,n)$$

In addition it must be transitive and satisfy the following formula, which is geometric by decidability.

$$\forall (S,T) \in \mathrm{cub}\,.\ \forall k \in \kappa.\ (\forall m \in S.\ mRk \to \exists n \in T.\ nRk) \tag{3}$$

We shall attain $R$ by starting with $R_0$ and adding in more and more pairs to give transitivity and (3), subject always to the constraint that if $mRn$ then $p(m) \sqsubseteq p(n)$. First note that transitivity is not a problem. If $R$ is any finite binary relation on a finite decidable set of cardinality $N$, then the transitive closure $R^*$ is also finite and in fact is $R \cup R^2 \cup \cdots \cup R^N$. This is because if we have a chain $x_0 R x_1 R \cdots R x_n$ with $n > N$, then by the pigeonhole principle there must be duplicate elements $x_i = x_j$ with $1 \leq i < j \leq n$, allowing the chain to be shortened. Using this, terminating algorithms for computing $R^*$ are well-known. (Actually, we just need to know that $R$ is a finite binary relation on a finite set with at most $N$ elements.) Moreover, if (2) holds for $R$, it also holds for $R^*$.

Suppose, then, we have a finite transitive relation $R \supseteq R_0$ satisfying (2). If $R$ satisfies (3) then we are done. Otherwise (and (3) is decidable), we have $\mathrm{cub}(S,T)$ and $k$ an $R$-upper bound for $S$ but with $\forall n \in T. n \not\!R k$. Applying $p$, we find that $\mathrm{CUB}_X(\mathcal{F}p(S), \mathcal{F}p(T))$ and $p(k)$ is an upper bound for $\mathcal{F}p(S)$, so for some $n \in T$ we have $p(n) \sqsubseteq p(k)$. Let $R' = (R \cup \{(n,k)\})^*$. By induction on the cardinality of $(\kappa \times \kappa) \setminus R$ we see that this process terminates. □

**Corollary 4.4.** Let $X$ be a strongly algebraic information system whose token set $(X)$ is finite. Then there is an ordinally finite poset $\lambda$ and a surjective information system homomorphism $q : \lambda \to X$.

*Proof.* There is a finite ordinal $\kappa$ with a surjective function $p : \kappa \to X$. Apply Lemma 4.3 with $A$ and $B$ both empty. □

**Lemma 4.5.** Let $f : X \to Y$ be a surjective homomorphism of strongly algebraic information systems. Suppose $S \in \mathcal{F}X$ and $y \in Y$ is an upper bound of $\mathcal{F}f(S)$. Then $y = f(x)$ for some upper bound $x$ of $S$.

*Proof.* We have $y = f(x')$ for some $x' \in X$. Suppose $\mathrm{CUB}(S \cup \{x'\}, T)$. Then

$$\mathrm{CUB}(\mathcal{F}f(S) \cup \{y\}, \mathcal{F}f(T))$$

and it follows that $f(x) \sqsubseteq y$ for some $x \in T$. We have that $x$ is an upper bound for $S$. Because $x' \sqsubseteq x$, we deduce that $y = f(x') \sqsubseteq f(x)$ and so $y = f(x)$. □

**Corollary 4.6.** Let $X$ be a strongly algebraic information system whose token set is finite. Then its order $\sqsubseteq$ is also finite.

*Proof.* Let $q : \lambda \to X$ be as in Corollary 4.4. We show that $\sqsubseteq = \mathcal{F}(q \times q)(\sqsubseteq_\lambda)$. If $x \sqsubseteq y$, then choose $m \in \lambda$ such that $q(m) = x$. Using Lemma 4.5 with $S = \{m\}$, we find there is some $n \sqsupseteq m$ with $q(n) = y$. □

**Theorem 4.7 (Main Theorem).** The topos [IS] classifying strongly algebraic information systems is equivalent to $\hat{}\mathbf{ofPos}_{\mathrm{adj}}$.

*Proof.* In view of the Finite Structure Condition, Lemma 4.3, this is now a direct application of Theorem 3.6. □

By similar methods it is possible to prove a number of similar results for strongly algebraic information systems with additional properties or structure. Note how in part (1) below, for decidable ordering, we see the appearance of embedding-projection pairs, familiar from Plotkin's result. Classically all orderings are decidable: so, classically, every strongly algebraic information system can be obtained as a filtered colimit of finite posets either with adjunctions between, or with embedding-projection pairs between. Nonetheless we still obtain two distinct classical results, for while the two ind-completions have the same objects they have different morphisms.

**Theorem 4.8.**

1 *Decidable ordering:* Let decIS be the theory in which IS is augmented with a binary relation $\not\sqsubseteq$, constrained by axioms to be the complement of $\sqsubseteq$. [decIS] is equivalent to the topical ind-completion of $\mathbf{ofPos}_{\mathrm{ep}}$, the category of ordinally finite posets with morphisms being the embedding-projection pairs – i.e. adjunctions $(f, g)$ with $f; g =$ Id.

2 *Bottom:* Existence of bottom can be imposed by adding an axiom $\exists x : X.\ \mathrm{CUB}(\emptyset, \{x\})$, making a topos $[\mathrm{IS}_\perp]$, an open subtopos of [IS]. This is equivalent to the topical ind-completion of $\mathbf{ofPos}_{\perp,\mathrm{adj}}$, the full subcategory of $\mathbf{ofPos}_{\mathrm{adj}}$ whose objects have bottom element.

3 *Decidable bottom:* (Abramsky 1991) uses a "termination predicate" which in our context is equivalent to adding not only the axiom of existence of bottom (as above) but also a unary predicate $\mathbf{T}(x)$ constrained by axioms to be the complement of the formula $\mathrm{CUB}(\emptyset, \{x\})$. The classifying topos $[\mathrm{IS}_{\perp\mathbf{T}}]$ is equivalent to the topical ind-completion of the category $\mathbf{ofPos}_{\perp\mathbf{T},\mathrm{adj}}$ of ordinally finite posets with bottom, with the morphisms being bottom-reflecting homomorphisms: functions preserving $\sqsubseteq$, CUB and non-bottomness.

4 *Bottom and decidable order:* If we add both bottom and decidable order (which makes bottom decidable too: $\mathbf{T}(x)$ is equivalent to $\exists y.x \not\sqsubseteq y$) then the classifying topos $[\mathrm{decIS}_\perp]$ is equivalent to the topical ind-completion of $\mathbf{ofPos}_{\perp,\mathrm{ep}}$, the full subcategory of $\mathbf{ofPos}_{\mathrm{ep}}$ whose objects have bottom.

*Proof.*

1 In Proposition 4.1 (2) it is easy to see that $f$ reflects order (i.e. it preserves $\not\sqsubseteq$) iff $f; g =$ Id. Hence the morphisms in $\mathbf{ofPos}_{\mathrm{ep}}$ have been chosen correctly. It remains only to show the Finite Structure Condition. Suppose $(\kappa, A, A', B)$ is an ordinally finite structure, with $A$, $A'$ and $B$ interpreting $\sqsubseteq$, $\not\sqsubseteq$ and CUB, and $p : \kappa \to X$ a homomorphism to a decidable information system. By Lemma 4.3 we can factor $p$ through an ordinally finite poset $\lambda$ with an IS-homomorphism $q : \lambda \to X$. $\not\sqsubseteq$ can be interpreted in $\lambda$ by the direct image of $A'$. Let $\equiv$ be the congruence on $\lambda$ defined by

$m \equiv n$ iff $q(m) = q(n)$ (which, by decidability of $X$, is finite) and let $\mu$ be $\lambda/\equiv$. $q$ factors via a decIS-homomorphism $r : \mu \to X$, and $\kappa \to \mu$ is a decIS-structure homomorphism.

2 The category $\mathbf{ofPos}_{\perp,\mathrm{adj}}$ is isomorphic to that of ordinally finite models. Let $p : \kappa \to X$ be a homomorphism from an IS-structure to an information system with bottom. By Lemma 4.3, $p$ factors via an ordinally finite poset $\lambda$, with a homomorphism $q : \lambda \to X$. Choose $S \in \mathcal{F}\lambda$ such that $\mathrm{CUB}(\emptyset, S)$ and let $\perp$ be the bottom element of $X$ (i.e. the unique element such that $\mathrm{CUB}(\emptyset, \{\perp\})$). Then $\mathrm{CUB}(\emptyset, \mathcal{F}p(S))$, and since $\perp$ is an upper bound of $\emptyset$ it follows that $\perp \in \mathcal{F}p(S)$, so there is some $s \in S$ with $p(s) = \perp$. Now define $A' = \sqsubseteq_\lambda$, $B' = \mathrm{CUB}_\lambda \cup \{(\emptyset, \{s\})\}$ and apply Lemma 4.3 to $\lambda$ with $A'$ and $B'$. We get that $q$ factors via an ordinally finite poset with bottom (the image of $s$).

3 The category $\mathbf{ofPos}_{\perp\mathbf{T},\mathrm{adj}}$ is isomorphic to that of ordinally finite models. Let $p : \kappa \to X$ be a homomorphism from an $\mathrm{IS}_{\perp\mathbf{T}}$-structure to an information system with decidable bottom; $\kappa$ is now equipped not only with relations $A$ and $B$ as in Lemma 4.3, but also with a unary predicate $C$ to interpret $\mathbf{T}$. As in (2), we can now factor $p$ as $f; q : \kappa \to \lambda \to X$. Now define $A' = \sqsubseteq_\lambda$, $B' = \mathrm{CUB}_\lambda \cup \{(\emptyset, \{s\}) | q(s) = \perp\}$ (which is finite because the condition $q(s) = \perp$ is decidable). Applying Lemma 4.3, $q$ factors as $g; r : \lambda \to \mu \to X$ and moreover we can take $g$ to be a surjection (because the image of $q$ is CUB-closed). By the same argument as in (2), $\mu$ has a bottom element. If $x$ in $\kappa$ has $C(x)$, then $p(x) \neq \perp$ and so $g(f(x)) \neq \perp$ in $\mu$. Hence $g \circ f$ preserves the $\mathbf{T}$-structure as required. Finally, $r$ reflects bottomness, for if $r(x) = \perp$ we can find $y$ with $x = g(y)$ and then $q(y) = \perp$ so by construction $g(y) = x = \perp$.

4 The category $\mathbf{ofPos}_{\perp,\mathrm{ep}}$ is isomorphic to that of ordinally finite models. Let $p : \kappa \to X$ be a homomorphism from a decIS-structure to a decidable information system with bottom. To get the required factorization, we carry out first the construction of (2) and then that of (1).

□

Finally, recall that [IS][idl] classifies the theory of an information system $X$ equipped with an ideal $I$ (a subset of $X$, represented by a unary predicate, that is directed and lower closed). Then the geometric morphism from [IS][idl] to [IS] that forgets the ideal is localic and is in effect the generic strongly algebraic domain.

**Theorem 4.9.** [IS][idl] is equivalent to the topical ind-completion of $\mathbf{ofPos}_{*\,\mathrm{adj}}$, the category whose objects are ordinally finite posets equipped with distinguished elements $*$, and in which a morphism from $(\kappa, *_\kappa)$ to $(\lambda, *_\lambda)$ is a homomorphism $f : \kappa \to \lambda$ such that $f(*_\kappa) \sqsubseteq *_\lambda$.

*Proof.* In an ordinally finite model $(\kappa, I)$ of [IS][idl], $I$ is finite and hence contains an upper bound for itself, i.e. a greatest element: so $I$ is a principal ideal. Let $*$ be its generator. A homomorphism $f$ must satisfy $I_\kappa(x) \Rightarrow I_\lambda(f(x))$, which translates to the given conditions on $*$'s.

For the Finite Structure Condition, suppose we are given $\kappa$, $A$ and $B$ as in Lemma 4.3, and also a unary predicate $C$ for $I$, a model $(X, I)$ of [IS][idl] and a homomorphism

$p$ from $(\kappa, A, B, C)$ to $(X, I)$. By Lemma 4.3, we can assume that $(\kappa, A, B)$ is already an ordinally finite poset $(\kappa, \sqsubseteq, \mathrm{CUB})$. Choose $T \in \mathcal{F}\kappa$ so that $\mathrm{CUB}(C, T)$. The image of $C$ is a finite subset of $I$, and hence has an upper bound $x$, say, in $I$, and then there is some $* \in T$ such that $p(*) \sqsubseteq x$, so $p(*) \in I$. □

## 5. Conclusions

(Vickers 1999) served as a case study in a much wider programme of using geometric logic to "topologize" mathematics in the generalized sense, replacing classes by classifying toposes. A crucial test is how much mathematics can be handled this way, and even in the limited field of domain theory the paper was only a start. There remained much domain theory for which topical methods were not obvious. The present work shows how to bring in just a little more in the form of Plotkin's SFP result and can thus be seen as another case study in the applicability of topical methods. From that point of view, it can be taken as an indication of the topical approach to other questions of representability by filtered colimits.

However, it also provides novel techniques for handling some instances of the wider question of whether a topos is a presheaf topos, applicable in circumstances where all finitely presentable models of a geometric theory are finite. The "Finite Structure Condition" focuses attention on the technical questions of the matter to hand, and although it may (as in our main result) take a little work to verify, it is work that does not get concerned with underlying category theory.

Regarding constructive domain theory, the presentation of IS is complicated, and it was not obvious from the start how to choose between classically equivalent variants. The success of the results of (Vickers 1999) suggests that IS does present a sensible topos, but it is gratifying to have the more directly simple description provided here. In addition, Theorem 4.8 gives a natural account of constructive bifurcations of the classical theory and classifies constructive differences of structure (many already evident in (Abramsky 1991)) by classically visible differences between the finite morphisms.

Finally, let us mention a particular consequence of Plotkin's SFP result (Plotkin 1976). The class of finite posets is closed under many domain constructors including function space and the Plotkin power domain, and these are functorial with respect to embedding-projection pairs and to adjunctions. One can use these facts in proving that the class of SFP domains is closed under the same constructors. It would be interesting to know how this works in the topical setting. For instance, the category $\mathbf{ofPos}_{\mathrm{mon}}$ of ordinally finite posets with monotone functions is Cartesian closed, and the exponentiation of objects is functorial on $\mathbf{ofPos}_{\mathrm{adj}}$. This then easily gives a map from $[\mathrm{IS}]^2$ to [IS]. Can the laborious calculation of function spaces in (Abramsky 1991; Vickers 1999) be reconstructed from this?

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