Domain Theory in Topical Form

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In his short story "Pierre Menard Author of the Quixote", Jorge Luis Borges tells the story of a French author who sets out to compose *Don Quixote* – not, you understand, as a mechanical transcription or copy of Cervantes' original, but as a re-creation, word for word and line for line, of fragments of it.

In 1999 I had a similar experience with my paper "Topical Categories of Domains" [3], based on some results from Samson's thesis [1] that also appeared in his "Domain Theory in Logical Form" [2]. My aim was to give a presentation of Samson's results that recreated, as closely as possible, Samson's own presentation.

Needless to say, it was not exactly the same, but why should such a re-creation have been a worthy aim? The answer is one of foundations: I had an idea for refounding the work using toposes, technically by replacing categories of domains by toposes classifying them (or their compact bases). One aim from this was to use the topos theory to give canonical answers to questions of continuity. When solving domain equations $D \cong F(D)$, F needs certain continuity properties that have been formulated in a special purpose way in domain theory. The canonical answer from topos theory would be to require F to be represented by a geometric morphism. As an unexpected bonus, the toposes also recreate the trick of "embedding-projection pairs", introduced in domain theory to deal with the fact that some important constructions F are not functorial with respect to Scott continuous maps. They reappear – in the case of SFP domains – as homomorphisms between the domains as models of a geometric theory.

Topos machinery can be heavy and untransparent, and Samson for one was not persuaded of the benefits. Why would anyone put themselves to the trouble of using toposes? Seeking answers to such objections was the start of my Menardian quest to recreate parts of his thesis: to leave the essential mathematics of his presentation undisturbed, but by logical means have it reinterpreted in terms of the toposes. The measure of success was to be the similarity to what Samson actually wrote.

With regard to the logical means (using geometric logic), I was by then beginning to understand the topos-theoretic techniques better – particularly through some collaboration with Peter Johnstone. However, it still took me a few years to find a narrative form for my paper. As it finally appeared, the Menard part was wrapped in a broad outline of a geometrization programme by which one might seek to apply the techniques quite generally in mathematics. It involved topologizing everything, either as point-free spaces or more generally as toposes, and using geometric logic to deal with them in terms of their points.

It included bundle ideas to deal with particular families of spaces (for example, SFP domains as a bundle over a topos that classifies their compact bases), and also included a proposal to avoid formal problems arising from the infinitary joins in geometric logic by replacing toposes with Joyal's "arithmetic universes".

In fact, this single paper explicitly sets out the essence of much of my work since. But it is inconceivable that it could have been written without Samson, and I want to mention some of the various ways in which I owe him some gratitude.

The first is obvious from the Menardian nature of my paper. It could no more have been written without Samson than Menard's Quixote could have been written without Cervantes. Without Samson's thesis I would not have had a model to refound.

The second is gratitude to Samson as teacher. My mathematical background was not domain theory, and Samson patiently taught me huge amounts about its different aspects, semantic, logical and topological. I particularly remember a time when I expressed some doubts regarding the importance of powerdomains. No, said Samson, they are the single most successful part of domain theory. He was right. In their localic form (which, of course, is also present in Samson's thesis) of powerlocales, I have since repeatedly found them to be a deep and vital part of geometric reasoning.

The third is gratitude to Samson as employer. After the failure of my computer company (Jupiter Cantab Ltd) in Cambridge, I wanted to return to mathematics and my former director of studies Ken Moody put me in touch with Samson. Samson quickly found me a research post on his project "Formal Semantics for Declarative Languages". I'm not sure I ever really found any worthwhile results in the topic of the project, being at heart a pure mathematician. Yet I was allowed to pursue my real interest, point-free topology and toposes. Its successor project, "Foundational Structures in Computer Science", was in fact typical of the style of serious mathematical research in a context of computer science that Samson did so much to foster in Britain.

So, thank you Samson!

References

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3. Vickers, S.: Topical categories of domains. Mathematical Structures in Computer Science 9, 569–616 (1999)