ASPECTS OF TOPOLOGY

Steve Vickers

Pure maths ⟷ Computer Science

continuity

computability, semantics

Logic

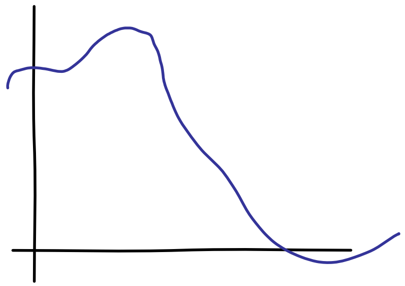
Physics

Quantum theory

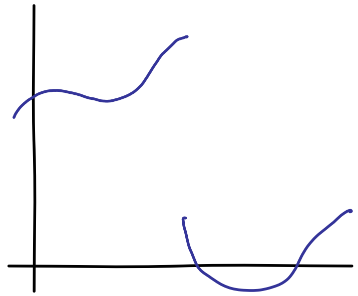
What is continuity?

Continuous function $\mathbb{R} \to \mathbb{R}$: no breaks

Continuous



Discontinuous



"Rubber band geometry" – can stretch, but no breaks

Conceptually

$f : \mathbb{R} \to \mathbb{R}$

If x varies "infinitessimally" then $f(x)$ doesn't make a finite jump

Formally:

For any neighbourhood $(f(x)-\varepsilon, f(x)+\varepsilon)$, however small, round $f(x)$,

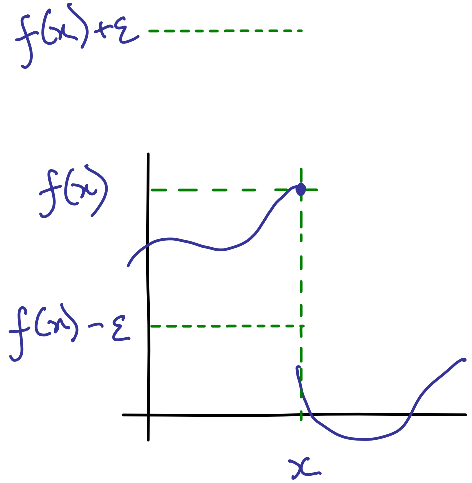
has $f(x)$ and a little to spare all round

$\varepsilon, \delta > 0$

$\exists$ neighbourhood $(x-\delta, x+\delta)$ round x

f maps $(x-\delta, x+\delta)$ into $(f(x)-\varepsilon, f(x)+\varepsilon)$

NOT continuous –



Formal definition

Usual motivation: it turns out to work

X a topological space:

- certain subsets designated "open"
- Any union or finite intersection of opens is still open $\cup \; \cap$

e.g. $X = \mathbb{R}$

$U \subseteq \mathbb{R}$ open if for every $x \in U$, have some neighbourhood $(x-\varepsilon, x+\varepsilon) \subseteq U$

"U doesn't contain any of its boundary points"

Continuity

X, Y topological spaces

$f : X \to Y$ continuous if —

V open in $Y \Rightarrow f^{-1}(V)$ open in X

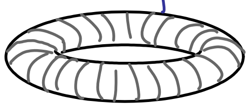
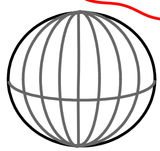
Equivalently:

"for every neighbourhood N of $f(x)$ have neighbourhood M of x such that f maps M into N"

Rubber sheet geometry

e.g. — classify surfaces.

Deform one surface into another? No tearing!

e.g. torus  into sphere  ?

No! Count holes etc.

From this point of view: most toplogical spaces are outlandish.

Scott

Denotational semantics:

Uses domains — outlandish spaces

Computable $\Rightarrow$ continuous

e.g. space D of "information states"

Say $x \sqsubseteq y$ if y is "more information" than x

subset U open if —

- if $x \in U$, $x \sqsubseteq y$ then $y \in U$
- if $x \in U$ then some "finite" $x_0 \sqsubseteq x$, $x_0 \in U$

Continuous $\Leftrightarrow$ definable in terms of finite information

Topology & Logic

Direct conceptual content of defⁿ of topology

Subset of space = property of points

Intuition: open = finitely observable property

union, intersection = or, and

$\cup$ $\cap$ $\vee$ $\wedge$

Logical connectives that have observational meaning

unlike $\rightarrow, \neg$

e.g. computationally

open – observed by watching program run

topology = not having source code access

Spaces of models

Geometric theory (propositional) –

- set of propositional symbols
- axioms $\phi \rightarrow \psi$

ϕ, ψ geometric formulae, built from using $\vee, \wedge$.

Get topological space of models

opens defined by geometric formulae

Point-free topology

Start from logic side

– reconstruct points afterwards

not always possible

"Point-free space" = logical gadget:

Lindenbaum algebra

= geometric formulae modulo provable equivalence

algebraic embodiment of geometric theory

Frames

= abstract algebra for Lindenbaum algebras

$\vee, \wedge$ become algebraic operations

- give a complete lattice
- distributivity $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$

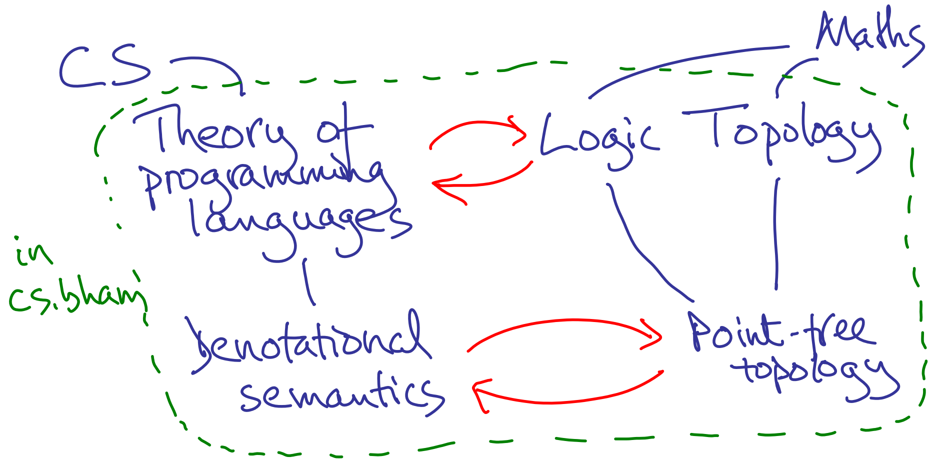
Point-free space = frame

Continuous from A to B

= function $B \rightarrow A$ preserving $\vee \wedge$

frame homomorphism

Research communities



BUNDLES

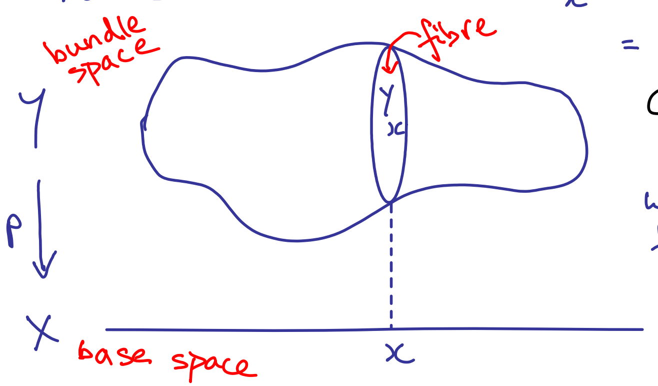
Variable spaces

$p : Y \longrightarrow X$ a (continuous) map

For each $x \in X$ write $Y_x = p^{-1}(\{x\})$

= fibre over x

Call p a bundle when think of fibre Y_x as variable space



Variable sets

Bundles $p : Y \to X$ in which –

- fibres have discrete topology (fibres "are" sets) — all subsets are open
- fibre Y_x varies "continuously" with x

Technically – p a local homeomorphism.

$\simeq$ sheaf over x

Mathematics of sheaves over X

"Set theory parametrized by $x \in X$"

Some mathematics (geometric) unscathed
— works fibrewise
e.g. products, quotients, free algebras ...

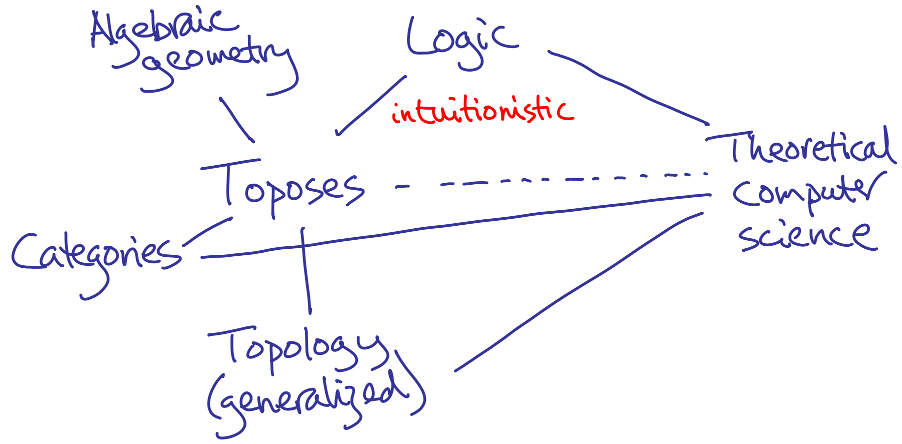
Some (intuitionistic) affected by neighbouring fibres
e.g. powersets, sets of functions

Some (classical) doesn't work any more
e.g. excluded middle, axiom of choice

My own work: How much is geometric?

Topos theory

Abstracted from mathematics of sheaves



Idea

- Get used to non-classical maths of sheaves
- "Do topology" in it (parametrized by $x \in X$)
- Is that the maths of bundles?

YES! But ...

Topology must be point-free
— bundles (variable spaces) can't always be approximated well enough by local homeomorphisms (variable sets (of points))

Example

X = Sierpinski space

Two points $\{\bot, \top\}$
Three opens $\{\emptyset, \{\top\}, \{\bot, \top\}\}$

$Y = \{*\}$, $p(*) = \bot$

fibres $Y_\bot = \{*\}$, $Y_\top = \emptyset$

"Variable set of points"
(= best approximation by local homeomorphism)
has both fibres empty.

Theorem
Joyal, Tierney, Fourman, Scott

X a point-free space

Equivalence between:

- Point-free spaces in mathematics of sheaves (variable sets) over X
- Point-free bundles over X, i.e. maps $Y \longrightarrow X$

Fibrewise hyperspaces

One focus of my own work

If Y a space:
a hyperspace is a space of its subspaces

cf. powerdomains – used for non-deterministic computation

– various kinds

- works point-free (powerlocales) & for bundles
- construction is geometric ∴ works fibrewise

Mathematics of sheaves over X

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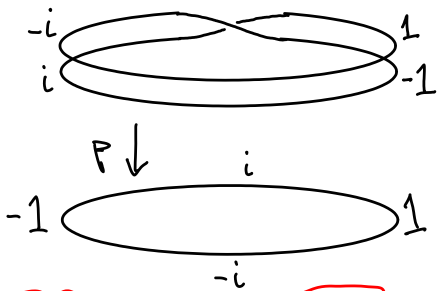
Very simple example

$X = \{z \in \mathbb{C} \mid |z| = 1\}$ = circle

$Y = X$, $p: Y \to X$ $p(z) = z^2$

p is a local homeomorphism



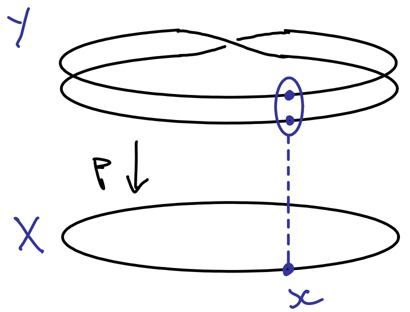
- Every fibre – 2 elements
- But can't continuously choose one everywhere
- p is surjective
- But can't demonstrate this by choosing preimages $\sigma: X \to Y$, $p(\sigma(z)) = z$

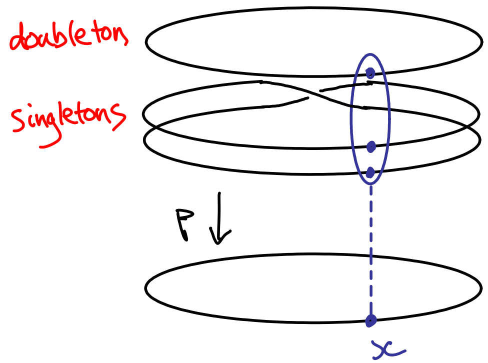
Axiom of choice fails

Using powerlocale

Continuously choose non-empty set of preimages – enough to show p surjective

Non-empty powerlocale





QUANTUM MECHANICS

- Isham, Butterfield, Döring
- Landsman, Spitters, Heunen

Classical physics

Measurable quantity a takes values.

Physical state determines what value.

$a : \Sigma \longrightarrow \mathbb{R}$ Σ = set of states

Often incomplete knowledge of state.

Replace Σ by $\mathrm{Dist}(\Sigma)$ distributions

a becomes probabilistic

Quantum physics

Hilbert space H of quantum states.

a is a linear operator on H.

Roughly:
- possible results are eigenvalues of a
- state determines their probabilities
- after measurement, state changes to corresponding eigenstate

Kochen-Specker Theorem

Are there underlying classical states, over which quantum states are distributions?

NO! Problem: non-commuting operators

e.g. position v. momentum cf. Heisenberg uncertainty principle

If all operators commute:
- have a basis of shared eigenstates
- can use those as spectrum of classical states

"representation th^m for commutative C^*-algebras"

Topos approach

Isham Butterfied Döring; Landsman Spitters Heunen

A a non-commutative algebra of operators.

Make space $\mathcal{C}(A)$:
each commutative subalgebra of A is a point of $\mathcal{C}(A)$

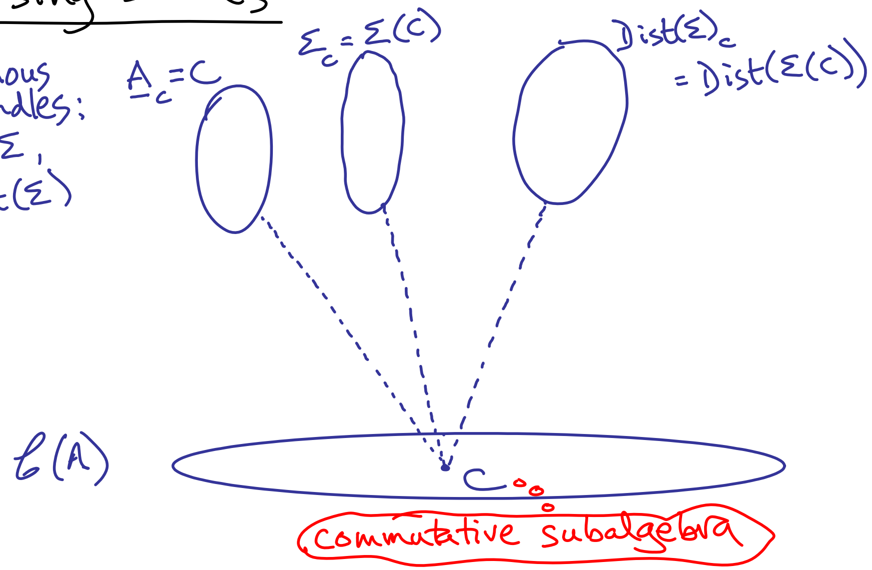
In sheaves over $\mathcal{C}(A)$ (internal mathematics of topos)
- find spectrum
- construct distribution space (geometric)
- etc. - internally "A is commutative"

Then recover external properties.

Using bundles

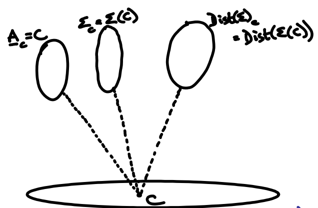
Matches Landsman et al.

Various bundles: $\underline{A}$, Σ, $\mathrm{Dist}(\Sigma)$



Kochen-Specker

$\Rightarrow$ Σ has no cross-sections
(continuous choice of point from every fibre = external point of bundle)



However — $\mathrm{Dist}(\Sigma)$ does have cross-sections
Each quantum state produces one.

Internally: quantum states are distributions over classical states

Externally: this is impossible

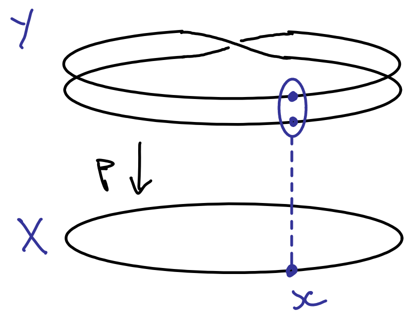
Does topos support "neoclassical" reasoning? Isham

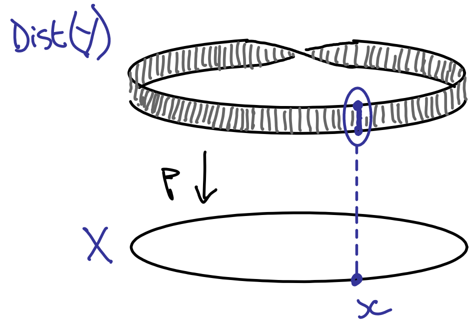
Analogy

(Though not a quantum system)

Space $2 = \{0,1\}$, all subsets open

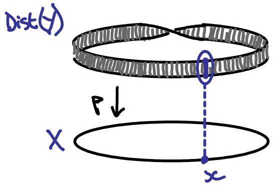
$\mathrm{Dist}(2) = \{(p_0, p_1) \in [0,1] \mid p_0 + p_1 = 1\}$



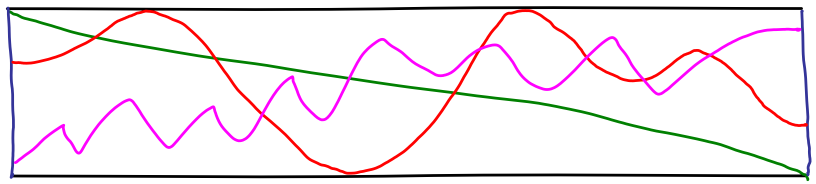


No cross-sections

Cross-sections of Dist(Y)



Map $\sigma : [0,1] \longrightarrow [0,1]$
with $\sigma(0) + \sigma(1) = 1$



Twist & make Möbius band: coloured lines join up.

Quantum systems – are more complicated.

Still, aim –
Exploit – bundle picture & geometricity of internal topos reasoning
to clarify the topos approach to quantum physics.

Qubit system

Quantum bits

A = algebra of 2x2 complex matrices $M_2(\mathbb{C})$

Commutative subalgebras C? For $\dim C > 1$:

- Take reals a, b, c with $a^2 + b^2 + c^2 = 1$
- Let $C = \{ \lambda E_1 + \mu E_2 \mid \lambda, \mu \in \mathbb{C} \}$,

$$E_1 = \frac{1}{2}\begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 1-a & -b-ic \\ -b+ic & 1+a \end{pmatrix}$$

$E_1^2 = E_1$, $E_2^2 = E_2$, $E_1 E_2 = E_2 E_1 = 0$, $E_1 + E_2 = I$

$C \cong \mathbb{C} \times \mathbb{C} \cong \mathbb{C}^2$ (as algebra)

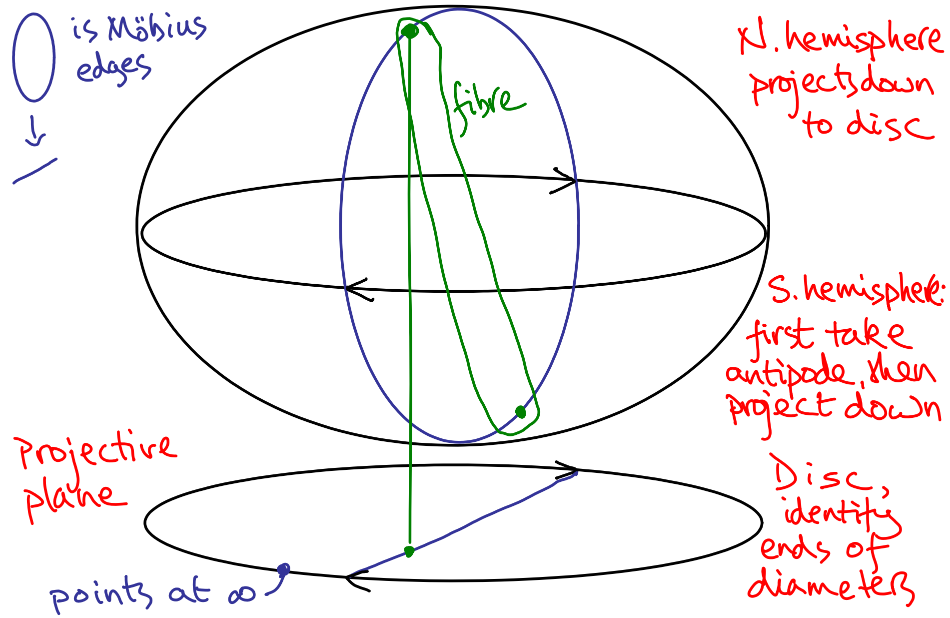
$\therefore$ Spectrum $= 2 = \{0,1\}$

Space $\mathcal{C}(A)$

{Ignore $C = \mathbb{C}$

Triples (a,b,c) are points of sphere S^2
BUT –
$(-a,-b,-c)$ define same subalgebra as (a,b,c)
$\mathcal{C}(A) = S^2$ with antipodal points identified
$\cong$ projective plane

Get $S^2 \longrightarrow \mathcal{C}(A)$, fibres $\cong 2$



Summary

Computer science
contributes to
point-free topology
toposes
new quantum formulation