Sublocales in formal topology

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Abstract

The paper studies how the localic notion of sublocale transfers to formal topology. For any formal topology (not necessarily with positivity predicate) we define a sublocale to be a cover relation that includes that of the formal topology. The family of sublocales has set-indexed joins. For each set of base elements there are corresponding open and closed sublocales, boolean complements of each other. They generate a boolean algebra amongst the sublocales. In the case of an inductively generated formal topology, the collection of inductively generated sublocales has coframe structure.

Overt sublocales and weakly closed sublocales are described, and related via a new notion of "rest closed" sublocale to the binary positivity predicate. Overt, weakly closed sublocales of an inductively generated formal topology are in bijection with "lower powerpoints", arising from the impredicative theory of the lower powerlocale.

Compact sublocales and fitted sublocales are described. Compact fitted sublocales of an inductively generated formal topology are in bijection with "upper powerpoints", arising from the impredicative theory of the upper powerlocale.

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1 Introduction

When one adopts a localic formulation of topology, sublocales are of central importance because they correspond to subspaces. Constructively, their mathematics is well established in topos-valid mathematics. However, many of the techniques use impredicative reasoning principles. The aim of this paper is to take known results from topos-valid locale theory, and treat them in the predicative setting of formal topology. We thus hope to bring the techniques to the attention of those working with predicative foundations such as Martin-Löf type theory, but at the same time we hope the more uniform methods will hold some interest for those familiar with locale theory.

Technically, one might view formal topology as a predicative treatment of *sites* (such as in [Joh82]) where a predicative treatment of *frames* is impossible.

This site view supports a more logical understanding of the point-free topology, since sites are a particular way of presenting propositional geometric theories (see, e.g., [Vic99]). The base elements of the site are propositional symbols, and each cover $a \lhd U$ is an axiom $a \rightarrow \bigvee U$. Other axioms are also implicit in the site formalism. In [Joh82], where the base is required to be a meet semilattice, those implicit axioms are that the semilattice meet becomes logical conjunction. The models of the theory are then the points of the corresponding locale or formal topology. Adding new axioms (new covers) has the effect of restricting the points, and so we obtain a logical notion of subspace as extra axioms. We shall use the word *sublocale* to refer to this notion, whether with ordinary locales or with formal topologies.

When one transfers sites to predicative mathematics, an important fact of life is that there is a fundamental difference between the full cover relation and a presenting cover relation in the site. The site structure includes a selection of "basic covers" from which the rest may be derived, just as a logical theory presentation has axioms from which theorems may be proved. Impredicatively, the cover base and the full cover relation are on the same foundational footing. Specifically, if the cover base is indexed by a set, then so is the full cover relation – essentially, this derives from the fact that powersets are still sets. Thus although it is good to know that the full cover relation can be derived from a cover base, there is no loss of generality in assuming that the full cover relation is available and is a set.

In predicative mathematics, by contrast, the full cover relation is not in general a set, and indeed there are examples ([Gam06]) where it cannot even be generated by a set. There are therefore two distinct notions of formal topology. In the original one (see [Sam87]), the full cover relation is given, though not of course a set. In an *inductively generated* formal topology ([CSSV03]), the full cover relation is generated by a set of basic covers.

We shall see that the predicative notion of sublocale behaves slightly differently in these two cases. For general formal topologies we have joins of sublocales but not meets in general – though we do have a boolean algebra generated by the open and closed sublocales. For inductively generated formal topologies we have coframe structure (arbitrary meets, and finite joins distributing over them) for the collection of inductively generated sublocales, just as in the localic case.

1.1 Sublocales in frame-based locale theory

We briefly summarize the established locale theory. Details can be found in the standard texts such as [Joh82], [Vic89]; we shall refer particularly to [Vic89]. Suppose X is a locale. The idea of "extra axioms" outlined above corresponds to extra relations used in presenting the frame ΩX. This gives a sublocale Y whose frame ΩY is a quotient frame of ΩX, with an inclusion map $i : Y \hookrightarrow X$ (corresponding to frame homomorphism $i^* : \Omega X \to \Omega Y$). Sublocale inclusions can be characterized as the regular monics in the category of locales, and they correspond to the surjective homomorphisms of frames.

From standard algebra of surjective homomorphisms, one finds that the sublocales of X can be expressed as congruences on the frame ΩX. Because frames are lattices, this can be modified to use instead "congruence preorders" $\leqq$, with $a \leqq b$ in ΩX if $i^*(a) \leq i^*(b)$ in ΩY – we say $a \leq b$ *modulo* Y. The larger the congruence or congruence preorder, the more extra relations there are and so the smaller the sublocale.

Proposition 1 *The congruence preorders on a frame A are those preorders (reflexive and transitive binary relations) $\leqq$ on A that satisfy the following conditions.*

1. *If $a' \leq a \leqq b \leq b'$ then $a' \leqq b'$. (Note that, together with reflexivity, this implies that if $a \leq b$ then $a \leqq b$.)*

2. *If $a_i \leqq b$ ($i \in I$) then $\bigvee_i a_i \leqq b$.*

3. *If $a \leqq b_1$ and $a \leqq b_2$, then $a \leqq b_1 \wedge b_2$.*

Proof. [Vic89, Proposition 6.2.3] ■

The *nucleus* of a sublocale Y is the function $j_Y : \Omega X \to \Omega X$ that takes a to the greatest element b such that $b \leq a$ modulo Y. In other words, $a \leq b$ modulo Y iff $a \leq j_Y(b)$. Again, the larger the nucleus, the smaller the sublocale. The functions that arise as nuclei are those $j : \Omega X \to \Omega X$ that are monotone, inflationary ($j(a) \geq a$) and idempotent ($j(j(a)) = a$), and preserve binary meets.

Finally, the sublocale is also characterized by the set of fixed points of the nucleus – indeed, in [Joh82] and [Vic89] the definition says that this set *is* the sublocale. The larger the set of fixed points, the larger the sublocale.

Some key results are as follows.

1. The set of nuclei on ΩX (i.e. the opposite of the set of sublocales of X) is a frame $N\Omega X$.

2. For each open $a \in \Omega X$ there are corresponding open and closed sublocales (written a and $X - a$), and they are boolean complements of each other.

3. Joins and finite meets of opens are preserved for the open sublocales, and transformed to meets and finite joins for the closed sublocales. (Hence the function $\Omega X \to N\Omega X$, $a \mapsto j_{X-a}$, is a frame homomorphism.)

4. Every sublocale is a meet of sublocales of the form $(X - a) \vee b$ where a and b are open. The sublocale $(X - a) \vee b$ is presented by the single extra relation $a \leq b$.

5. The upper and lower *powerlocales* $P_U X$ and $P_L X$ have points that correspond to certain sublocales of X. The points of $P_U X$ are the compact fitted sublocales, and the points of $P_L X$ are the overt, weakly closed sublocales. (The results are gathered in [Vic97]. For the lower powerlocale it was long known classically that the points are in bijection with closed sublocales; the constructive characterization was first published by [BF96]. For the upper powerlocale, [Vic97] gives a constructive proof of the "Hofmann-Mislove-Johnstone Theorem" used in [Joh85] for the Vietoris powerlocale – see also [Esc03].)

Typical proofs rely on exploiting the different advantages of different representations of sublocales. For instance, one can easily calculate from the "extra relations" that $a \wedge (X-a)$ is the bottom sublocale $\emptyset$; but to show $a \vee (X-a) = X$ it is more convenient to convert to nuclei. [Esc03] surveys some impredicative accounts of joins of nuclei (for meets of sublocales).

In fact, this phenomenon is a big influence on the predicative account. In terms of the full cover relations (which express the same information as the nuclei), joins are easy while meets are problematic. In terms of inductively generating sets of basic covers, meets are easy while joins take a little more work. Thus in the distinct notions of formal topology and inductively generated formal topology, the discussions of sublocales tend to diverge.

1.2 Formal topology

To fix notation and terminology, let us now summarize some known results from formal topology. We do not in general assume a positivity predicate – more will be said about this in Sections 5.1 and 5.3.

We define a *formal topology* to be a triple $(P, \leq, \lhd)$ where P, a set, is the *base*, $\leq$ is a preorder on P, and $\lhd$, the *cover relation*, is a collection of pairs (a, U) $(a \in P, U \subseteq P)$ satisfying —

- $\dfrac{a \in U}{a \lhd U}$ (*reflexivity*)

- $\dfrac{a \leq b \quad b \lhd U}{a \lhd U}$ (*$\leq$-left*)

- $\dfrac{a \lhd V \quad V \lhd U}{a \lhd U}$ (*transitivity*)

- $\dfrac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}$ (*$\leq$-right*)

The notation $V \lhd U$ means, as usual, that $v \lhd U$ for every $v \in V$.

The notation $U \downarrow V$ here means $\downarrow U \cap \downarrow V$, where $\downarrow U = \{a \in P \mid (\exists u \in U) a \leq u\}$. We have followed [CSSV03] in requiring P to be a preorder, and using its order to define $U \downarrow V$ rather than (as is common) the singleton cover relation $a \lhd \{b\}$. This decouples the use of $U \downarrow V$ in ($\leq$-right) from the cover relation, and will be convenient when for sublocales we consider a variety of cover relations on the same base P. They will all share the same preorder $\leq$.

The reason this works is that, essentially, the role of ($\leq$-right) is to express that P is a base for the topology, and not just a subbase. This means that each

binary meet $(\bigvee U) \wedge (\bigvee V)$ has to be a join of basics, and ($\leq$-right) says that $\bigvee(U \downarrow V)$ is enough. Once this is known for X, it follows also for the sublocales.

Given a formal topology $X = (P, \leq, \lhd)$, we write $\lhd U = \{a \in P \mid a \lhd U\}$ for each $U \subseteq P$, and we say that U is a *formal open* if $U = \lhd U$. The collection ΩX of formal opens has the structure of a frame, though predicatively it is not carried by a set. Meets are given by intersection, since any (even infinite) intersection of formal opens is again a formal open. Joins are expressed by $\bigvee_i U_i = \lhd (\bigcup_i U_i)$. Frame distributivity, $V \wedge \bigvee_i U_i \leq \bigvee_i(V \wedge U_i)$, follows from ($\leq$-*right*). For if $a \in V$ and $a \lhd \bigcup_i U_i$ then $a \lhd V \downarrow \bigcup_i U_i = \bigcup_i(V \downarrow U_i) = \bigcup_i(V \cap U_i)$.

We shall also consider *inductively generated formal covers*, as in [CSSV03]. However, we shall follow the notation of [Vic06], in terms of "flat sites". (The word "flat" is taken from the notion of flat functor, a certain kind of functor from a category $\mathcal{C}$ to **Set**. We do not need to examine the flatness property here, but it specializes to preservation of all finite limits in the case where $\mathcal{C}$ has them. In a similar manner, the flat sites generalize the sites of [Joh82], in which the base is assumed to have finite meets.)

Definition 2 *A* flat site *is a structure* $(P, \leq, \lhd_0)$ *where* $(P, \leq)$ *is a preorder (i.e. transitive and reflexive), and* $\lhd_0 \subseteq P \times \mathcal{P}P$ *has the following* flat stability *property: if* $a \lhd_0 U$ *and* $b \leq a$, *then there is some* $V \subseteq b \downarrow U$ *such that* $b \lhd_0 V$.

Predicatively, our statement "$\lhd_0 \subseteq P \times \mathcal{P}P$" is an abuse of notation since $\mathcal{P}P$ is not a set. What we mean is that $\lhd_0$ is a set of (indices for) covers, and for each cover there is an element of P and a set indexing elements of P. This is made explicit in the equivalent *localized axiom-sets* of [CSSV03].

Theorem 3 *([CSSV03]) Let* $(P, \leq, \lhd_0)$ *be a flat site. Let* $\lhd$ *be generated by rules reflexivity,* $\leq$-*left and*

- $\dfrac{a \lhd_0 V \quad V \lhd U}{a \lhd U}$ *(infinity)*

Then $\lhd$ *is a cover relation on* $(P, \leq)$, *and is the least that includes* $\lhd_0$.

The way we shall exploit the Theorem is that if, given U, we wish to show $a \lhd U$ implies some property $\Phi(a)$, then we shall verify three rules –

- $\dfrac{a \in U}{\Phi(a)}$

- $\dfrac{a \leq b \quad \Phi(b)}{\Phi(a)}$

- $\dfrac{a \lhd_0 V \quad (\forall v \in V)\Phi(v)}{\Phi(a)}$

These will then show that any proof of $a \lhd U$ can be transformed into a proof of $\Phi(a)$.

We shall treat the cover relation $\lhd$ as an implicit part of the structure of any flat site $X = (P, \leq, \lhd_0)$, treating X as also a formal topology with frame ΩX of formal opens.

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Proposition 4 *(Impredicatively) Let $(P, \leq, \lhd_0)$ be a flat site, and let A be the frame presented by generators and relations as*

$$\mathbf{Fr}\langle P \text{ (qua preorder)} \,|\, 1 \leq \bigvee P$$

$$a \wedge b \leq \bigvee(a \downarrow b) \quad (a, b \in P)$$

$$a \leq \bigvee U \quad (a \lhd_0 U)\rangle.$$

This denotes the frame freely generated by the preorder P (preserving the order) modulo the given inequations. (See [Vic89].)

1. If $a \lhd U$ then $a \leq \bigvee U$ in A.

2. $A \cong \Omega X$.

Proof. 1. We use Theorem 3. Define $\Phi(a)$ if $a \leq \bigvee U$ in A. If $a \lhd_0 V$ and $(\forall v \in V)\Phi(v)$, then $a \leq \bigvee V \leq \bigvee U$.

2. Define $\alpha : \Omega X \to A$ by $\alpha(U) = \bigvee U$. This is a frame homomorphism. For binary meets,

$$\alpha(U) \wedge \alpha(V) = \bigvee\{u \wedge v \mid u \in U, v \in V\} = \bigvee\{\bigvee(u \downarrow v) \mid u \in U, v \in V\}$$

$$= \bigvee(U \downarrow V) = \alpha(U \wedge V).$$

For joins,

$$\alpha(\bigvee_i U_i) = \alpha(\lhd (\bigcup_i U_i)) = \bigvee\{a \mid a \lhd \bigcup_i U_i\}$$

$$= \bigvee(\bigcup_i U_i) \qquad \text{by (1)}$$

$$= \bigvee_i \bigvee U_i = \bigvee_i \alpha(U_i).$$

Now define $\beta : A \to \Omega X$ by $\beta(a) = \lhd a$. It follows from the properties of $\lhd$ that β respects the relations presenting A, and so induces a frame homomorphism. We have $\alpha \circ \beta(a) = \bigvee(\lhd a) = a$ using (1), and so $\alpha \circ \beta = \mathrm{Id}_A$. Also, $\beta \circ \alpha(U) = \lhd (\bigcup_{u \in U} \lhd u) = U$, so $\beta \circ \alpha = \mathrm{Id}_{\Omega X}$. ∎

It follows that, impredicatively, the full coverage $a \lhd U$ can also be defined as $a \leq \bigvee U$ in the frame A.

From this presentation we see directly what are the points of the corresponding locale. A point is a function from P to the frame Ω of truth values that preserves the order and respects the inequational relations. The function from P to Ω corresponds to a subset x of P, preservation of order says x is upper closed, the first inequation says it is inhabited and the second that any two elements of x have a lower bound in x. In short, "qua preorder" and the first two inequations say that x is a filter in P (a down-directed upset). Then the third inequation says that if $a \lhd_0 U$ and $a \in x$, then U meets x (i.e. U and x have an element in common).

We shall use the notation $U \between x$ for U meets x.

Impredicatively, we can always use the full cover $\lhd$ as a cover base $\lhd_0$, and the points are the filters x for which if $a \lhd U$ and $a \in x$ then $U \between x$. This matches the usual definition in formal topology of formal point.

2 Sublocales of a formal topology

Impredicatively, sublocales correspond to congruence preorders on the frame. We now show how those relate to cover relations.

Proposition 5 *Let $X = (P, \leq, \lhd)$ be a formal topology. Then (impredicatively) there is a bijection between congruence preorders on ΩX and cover relations on $(P, \leq)$ that include $\lhd$.*

If congruence preorder $\trianglelefteq$ corresponds to cover relation $\lhd^Y$, then ΩY (the frame for $(P, \leq, \lhd^Y)$) is isomorphic to the quotient frame $\Omega X / (\trianglelefteq \cap \trianglerighteq)$.

Proof. First, suppose $\lhd^Y$ is a cover relation on $(P, \leq)$ that includes $\lhd$. Since $U \subseteq (\lhd U) \lhd U$, we have $V \lhd^Y U$ iff $(\lhd V) \lhd^Y (\lhd U)$ and so $\lhd^Y$ (as a binary relation on subsets) is determined by its restriction to formal opens. That restriction satisfies the criteria of Proposition 1 for a congruence preorder on ΩX. If $V_i \lhd^Y U$ $(i \in I)$ then $\bigvee_i V_i \lhd \bigcup_i V_i \lhd^Y U$, so $\bigvee_i V_i \lhd^Y U$; and if $V \lhd^Y U_i$ $(i = 1, 2)$ then $V \lhd^Y U_1 \downarrow U_2 = U_1 \wedge U_2$.

Conversely, if $\trianglelefteq$ is a congruence preorder on ΩX, define $a \lhd^Y U$ iff $\lhd a \trianglelefteq \lhd U$. Then $V \lhd^Y U$ iff $\lhd v \trianglelefteq \lhd U$ for all $v \in V$, i.e. $\lhd V = \bigvee_{v \in V} \lhd v \trianglelefteq \lhd U$, so $\trianglelefteq$ is regained from $\lhd^Y$ as above. $\lhd^Y$ is a cover relation that includes $\lhd$. If $a \lhd^Y U$ and $a \lhd^Y V$, then $\lhd a \trianglelefteq \lhd U \wedge \lhd V = \lhd (U \downarrow V)$, so $a \lhd^Y U \downarrow V$.

For the frames, consider the function $\pi : \Omega X \to \Omega Y$ defined by $\pi(U) = \lhd^Y U$. This is a surjective frame homomorphism, and $\pi(V) \leq \pi(U)$ iff $V \lhd^Y U$, i.e. $V \trianglelefteq U$. ∎

We therefore *define* a sublocale Y of a formal topology $(P, \leq, \lhd)$ to be a cover relation $\lhd^Y$ on $(P, \leq)$ that includes $\lhd$.

The (formal) points of Y are the filters x of P such that if $a \lhd^Y U$ and $a \in x$ then U meets x. Each point of Y is already a point of X, but the instances of $\lhd^Y$ act as constraints, restricting the points that are allowed. It follows that the larger the cover relation, the more constraints there are, and hence the fewer the points. We therefore order sublocales by reverse inclusion: $Y \leq Z$ if $\lhd^Y \supseteq \lhd^Z$.

Given a point x, we can define a sublocale $\{x\}$ by

$$a \lhd^{\{x\}} U \text{ iff } a \in x \to U \between x.$$

Proposition 6 *Let $X = (P, \leq, \lhd)$ be a formal topology, and let Y be a sublocale. Then a point x of X is in Y iff $\{x\} \leq Y$.*

Clearly, the family of cover relations that include $\lhd$ is closed under arbitrary intersections, and so the family of sublocales has arbitrary joins.

Note a special case. The empty join $\emptyset$ has $V \lhd^{\emptyset} U$ for all V and U, and in particular $a \lhd^{\emptyset} \emptyset$ for all $a \in P$. It has no points, and its frame is a singleton.

Classically, finite joins of sublocales correspond to unions of subspaces, for the following reason. Suppose Y and Z are sublocales, and x is a point of $Y \vee Z$ but not of Y. Then we can find $a \lhd^Y U$ such that $a \in x$ but U does not meet x. Now suppose $b \lhd^Z V$ with $b \in x$. Note that $a \downarrow b \lhd^{Y \vee Z} U \cup V$. Since x is a filter with respect to $\leq$, we can find $c \in x \cap (a \downarrow b)$, and then since x is a point of $Y \vee Z$ it follows that $U \cup V$ meets x. Since U does not meet x, it follows that V meets x, and we deduce that x is a point of Z. Constructively, however, this argument will not hold. See also Proposition 12.

We shall also examine meets of sublocales. However, it is not clear (at least to me) whether they always exist in predicative mathematics. The problem is that for a meet of sublocales Y_i we should be looking for the cover relation generated by $\bigcup_i \lhd^{Y_i}$. On the face of it, we should want to apply rules such as

$$\frac{a \lhd^{Y_i} V \qquad V \lhd^Y U}{a \lhd^Y U}$$

where $Y = \bigwedge_i Y_i$. This is similar to the attempted use of transitivity as a generating rule, as discussed in [CSSV03]. In their words, "The use of the arbitrary subset V creates an unbounded branching in the tree of possible premisses to conclude $a \lhd^Y U$."

Nonetheless, we shall see particular cases where meets can be found. Some important examples involve open and closed sublocales (Section 3). Another general case is that of inductively generated formal topologies (Section 4).

When a sublocale meet $\bigwedge_i Y_i$ does exist, we see from Proposition 6 that a point is in $\bigwedge_i Y_i$ iff it is in every Y_i: meet of sublocales corresponds to intersection of subspaces.

3 Open and closed sublocales

Let $(P, \leq, \lhd)$ be a formal topology. Since P is a base, an arbitrary open is given by a join of a subset A of P. The formal opens give a canonical representative for each open, but we can also work with the arbitrary subsets modulo the preorder induced by the inclusion order on formal opens: $A \leq B$ if $A \lhd B$. The opens have arbitrary set-indexed joins, given by union, and finite meets given by $A \wedge B = A \downarrow B$. A top open is P.

For each open we shall define an open sublocale and a complementary closed sublocale. To prepare for that, we first define a generalization of both.

Definition 7 *Let $X = (P, \leq, \lhd)$ be a formal topology, and let $A, B \subseteq P$. Then the* crescent *sublocale $(X - A) \wedge B$ is defined by*

$$c \lhd^{(X-A) \wedge B} U \ \ if \ c \downarrow B \lhd A \cup U.$$

The use of $\wedge$ in the notation will be justified in Proposition 10.

This is indeed a cover relation that includes $\lhd$. For transitivity, if $c \downarrow B \lhd A \cup V$ and $V \downarrow B \lhd A \cup U$, then $c \downarrow B \lhd (A \cup V) \downarrow B \lhd A \cup (V \downarrow B) \lhd A \cup U$. For ($\leq$-right), if $c \downarrow B \lhd A \cup U$ and $c \downarrow B \lhd A \cup V$ then $c \downarrow B \lhd (A \cup U) \downarrow (A \cup V) \lhd A \cup (U \downarrow V)$.

Proposition 8 *Let $X = (P, \leq, \lhd)$ be a formal topology, and let $A, B \subseteq P$. Then $(X - A) \wedge B$ is the greatest sublocale Y such that $P \lhd^Y B$ and $A \lhd^Y \emptyset$.*

Proof. $(X - A) \wedge B$ does have these properties. Now suppose Y does too, and $c \downarrow B \lhd A \cup U$. Then $c \lhd^Y c \downarrow B \lhd A \cup U \lhd^Y U$, hence $Y \leq (X - A) \wedge B$. ∎

Definition 9 *Let $X = (P, \leq, \lhd)$ be a formal topology and $A, B \subseteq P$.*

1. *The* open *sublocale B corresponding to B is given by*

$$c \lhd^B U \ \ if \ c \downarrow B \lhd U.$$

2. *The closed sublocale $X - A$ corresponding to A is given by*

$$c \lhd^{X-A} U \ \textit{if} \ c \lhd A \cup U.$$

These are crescents $(X - A) \land B$ in the special cases $A = \emptyset$ and $B = P$.

The intuition for $\lhd^B$ is that $c \downarrow B$ represents the intersection of opens c and $\bigvee B$, so that c is covered by U modulo $\bigvee B$ iff $c \downarrow B$ is covered by U. For $\lhd^{X-A}$, c is covered by U modulo $X - \bigvee A$ iff c is covered by A and U together.

Here are some special cases. The least sublocale $\emptyset$ ($c \lhd^\emptyset U$ for all c, U, including $U = \emptyset$) is open $\emptyset$ and closed $X - P$. The greatest sublocale X is open P and closed $X - \emptyset$.

Proposition 10 *Let $X = (P, \leq, \lhd)$ be a formal topology and let $A, B \subseteq P$.*

1. *The open sublocale B is the greatest sublocale Y for which $P \lhd^Y B$.*

2. *The closed sublocale $X - A$ is the greatest sublocale Y for which $A \lhd^Y \emptyset$.*

3. *The crescent sublocale $(X - A) \land B$ is the sublocale meet of $X - A$ and B.*

Proof. For (1) and (2), apply Proposition 8 in the special cases. Then (3) follows from the same Proposition. ■

Lemma 11 *Let $X = (P, \leq, \lhd)$ be a formal topology and let $A, B \subseteq P$. Then $(X - A) \lor B$ is the greatest sublocale Y for which $A \lhd^Y B$.*

Proof. We have both $A \lhd^{X-A} B$ and $A \lhd^B B$, since $A \lhd A \cup B$ and $A \downarrow B \lhd B$. Now suppose Y has $A \lhd^Y B$, and suppose $c \lhd A \cup U$ and $c \downarrow B \lhd U$. Then $c \lhd^Y B \cup U$, so $c \lhd^Y c \downarrow (B \cup U) \lhd (c \downarrow B) \cup U \lhd U$. Hence $Y \leq (X - A) \lor B$. ■

We call a sublocale of the form $(X - A) \lor B$ a *cocrescent*.

Proposition 12 *As sublocales, $X - A$ is a boolean complement of A.*

Proof. $(X - A) \lor A$ is the greatest sublocale Y for which $A \lhd^Y A$. But that is just X. By Proposition 10, $c \lhd^{(X-A) \land A} U$ iff $c \downarrow A \lhd A \cup U$, which holds for all c and U. Hence $(X - A) \land A = \emptyset$. ■

Using Proposition 6, we can now deduce that a point x is in $(X - A) \lor B$ iff, if A is a neighbourhood, then so is (some element of) B. In particular, x is in the open B iff x meets B, and it is in $X - A$ iff it is impossible for x to meet A.

Recall from Section 2 that, classically, binary join of sublocales corresponds to union of subspaces. We can now see that this must imply excluded middle. Take $P = \{*\}$, with $* \lhd U$ if $* \in U$, and let $A_\phi = \{* \mid \phi\}$ for each proposition ϕ. If $*$ is either in A or $X - A$ then we get $\phi \lor \neg\phi$.

The following result allows us to identify the opens with their corresponding sublocales.

Theorem 13   1. *The assignment of open sublocales to opens is an order embedding (with respect to the preorders on opens and on sublocales) that preserves joins and finite meets.*

2. *The assignment of closed sublocales to opens is an order reversing embedding that transforms joins to meets and finite meets to joins.*

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Proof. (1) Let A and B be subsets of P. We have $A \leq B$ as sublocales iff $P \lhd^A B$, i.e. iff $P \downarrow A \lhd B$, i.e. $A \lhd B$, i.e. $A \leq B$ as opens.

To show the embedding preserves joins, let A_i ($i \in I$) be a family of subsets of P and let A be their union. We must show that $b \lhd^A U$ iff for every i we have $b \lhd^{A_i} U$ and this is obvious from the definition.

To show the embedding preserves finite meets, suppose A and B are subsets of P. It suffices to show that, for any sublocale Y, we have $P \lhd^Y A \downarrow B$ iff $P \lhd^Y A$ and $P \lhd^Y B$, but this is obvious. Also, it has already been remarked that the open sublocale for P is the top sublocale.

(2) Let A and B be subsets of P. We have $X - A \leq X - B$ as sublocales iff $B \lhd^A \emptyset$, i.e. iff $B \lhd A \cup \emptyset = A$.

To show the embedding transforms joins to meets, let A_i ($i \in I$) be a family of subsets of P and let A be their union. If $Y \leq X - A_i$ for every i, i.e. $A_i \lhd^Y \emptyset$, then $A \lhd^Y \emptyset$ and so $Y \leq X - A$.

To show the embedding transforms finite meets to joins, suppose A and B are subsets of P. We have $c \lhd^{(X-A) \vee (X-B)} U$ iff $c \lhd A \cup U$ and $c \lhd B \cup U$, iff $c \lhd A \downarrow B \cup U$, iff $c \lhd^{X-(A \downarrow B)} U$. ∎

We now describe a class of sublocales for which finite meets always exist – indeed, the class has boolean algebra structure. It comprises the finite joins of crescents or, equivalently, the finite meets of cocrescents.

In the following Lemma, as elsewhere in this paper, "finite" means *Kuratowski finite*, i.e. the set can be represented by a finite list of its elements (possibly with repetitions).

For a finite set S, the set of finite partitions $S = I \cup J$ with I and J both finite is itself finite ([Vic06]). Note that the Kuratowski finiteness of I and J is *not* implied by the fact that they are subsets of S.

Lemma 14 *Let $X = (P, \leq, \lhd)$ be a formal topology, let $A_i, B_i \subseteq P$ ($1 \leq i \leq n$), and let Y be a sublocale of X. Then the sublocale meet $\bigwedge_{i=1}^{n}(Y \vee (X - A_i) \vee B_i)$ exists, and is equal to*

$$Y \vee \bigvee \{(X - \bigcup_{i \in I} A_i) \wedge \downarrow_{j \in J} B_j \mid \{1, \ldots, n\} = I \cup J, \text{ with } I, J \text{ both finite}\}.$$

Proof. Let Y' be the sublocale join displayed. This is a lower bound of the sublocales $Y \vee (X - A_i) \vee B_i$. For suppose $1 \leq k \leq n$ and $\{1, \ldots, n\} = I \cup J$. If $k \in I$ then $X - \bigcup_{i \in I} A_i \leq X - A_k$, while if $k \in J$ then $\downarrow_{j \in J} B_j \leq B_k$.

To show it is a greatest lower bound, we use induction on n. In the base case $n = 0$, the empty meet is the top sublocale X and $Y' = Y \vee ((X - \emptyset) \wedge P) = X$.

For $n > 0$, suppose $Z \leq Y \vee (X - A_i) \vee B_i$ for $1 \leq i \leq n$, in other words if $c \lhd^Y U$ and $c \lhd A_i \cup U$ and $c \downarrow B_i \lhd U$ then $c \lhd^Z U$. This is equivalent to if $c \lhd^Y U$ then $c \downarrow A_i \lhd^Z B_i \cup U$. We must show $Z \leq Y'$. If $c \lhd^{Y'} U$ then $c \lhd^Y U$, and for each partition $\{1, \ldots, n-1\} = I \cup J$ we have two partitions $(I \cup \{n\}) \cup J$ and $I \cup (J \cup \{n\})$ of $\{1, \ldots, n\}$, giving

$$c \downarrow (\downarrow_{j \in J} B_j) \lhd \bigcup_{i \in I} A_i \cup A_n \cup U,$$

$$c \downarrow B_n \downarrow (\downarrow_{j \in J} B_j) \lhd \bigcup_{i \in I} A_i \cup U.$$

By induction we deduce $c \lhd^Z A_n \cup U$ and $c \downarrow B_n \lhd^Z U$ as well as $c \downarrow A_n \lhd^Z B_n \cup U$. It follows that $c \lhd^Z c \downarrow A_n \cup U \lhd^Z B_n \cup U$, so $c \lhd^Z c \downarrow B_n \cup U \lhd^Z U$. ∎

Corollary 15 *Let* $X = (P, \leq, \lhd)$ *be a formal topology, and let* $A_i, B_i \subseteq P$ *($1 \leq i \leq n$).*

    1. *The sublocale* $\bigwedge_{i=1}^{n}((X - A_i) \vee B_i)$ *exists, and is equal to*

$$\bigvee \{(X - \bigcup_{i \in I} A_i) \wedge \downarrow_{j \in J} B_j \mid \{1, \ldots, n\} = I \cup J, \text{ with } I, J \text{ both finite}\}.$$

    2. *Join with any sublocale* Y *distributes over this meet.*

    Proof. (1) is Lemma 14 in the case $Y = \emptyset$. (2) then is Lemma 14 with (1) substituted. ∎

Proposition 16 *Let* $X = (P, \leq, \lhd)$ *be a formal topology. Then a sublocale* Y *is a finite join of crescents iff it is a finite meet of cocrescents.*

    Proof. Corollary 15 shows that every finite meet of cocrescents is a finite join of crescents. For the converse, consider $\bigvee_{i=1}^{n}((X - A_i) \wedge B_i)$. We use induction on n. The base case, for $n = 0$, is the sublocale $\emptyset$, which is a cocrescent. Now suppose we have the result for n. Then $\bigvee_{i=1}^{n+1}((X - A_i) \wedge B_i) = ((X - A_{n+1}) \wedge B_{n+1}) \vee$ (finite meet of cocrescents). By Corollary 15 (2) this is a finite meet of sublocales of the form

$$((X - A_{n+1}) \wedge B_{n+1}) \vee (X - C) \vee D$$
$$= ((X - A_{n+1}) \vee (X - C) \vee D) \wedge (B_{n+1} \vee (X - C) \vee D)$$
$$= ((X - A_{n+1} \downarrow C) \vee D) \wedge ((X - C) \vee (B_{n+1} \cup D)),$$

a binary meet of cocrescents. ∎

Theorem 17 *Let* $X = (P, \leq, \lhd)$ *be a formal topology. Then the family of finite meets of cocrescents (equivalently, finite joins of crescents) has the structure of a boolean algebra (though not necessarily carried by a set) under sublocale meet and join.*

    Proof. As finite meets of cocrescents they are closed under finite meets, while as finite joins of crescents they are closed under finite joins. Hence the family is a lattice. By Corollary 15 (2) it is distributive. From the fact that every open or closed sublocale has a boolean complement, it follows that $\bigwedge_{i=1}^{n}((X - A_i) \vee B_i)$ has boolean complement $\bigvee_{i=1}^{n}(A_i \wedge (X - B_i))$. ∎

4   Inductively generated sublocales

Suppose $X = (P, \leq, \lhd_0)$ is a flat site. We define an *inductively generated sublocale* of X to be a flat site $Y = (P, \leq, \lhd_0^Y)$ such that $\lhd_0 \subseteq \lhd_0^Y$. Clearly then $\lhd \subseteq \lhd^Y$ (as usual, we denote the full cover relation by omitting the subscript 0 that signifies basic covers), so we get a sublocale in the previous sense. We relate these by the sublocale ordering, $Y \leq Z$ if $\lhd^Y \supseteq \lhd^Z$ (or, equivalently, $\lhd^Y \supseteq \lhd_0^Z$). What was a partial order on sublocales becomes a preorder on inductively generated sublocales, so we say that Y and Z are *equal* if $Y \leq Z$ and $Z \leq Y$, i.e. if $\lhd^Z = \lhd^Y$.

Note that we are requiring the inductive generation in two places: in X itself, and also in the sublocales. Conceivably it might be possible to consider "inductively generated sublocales of an arbitrary formal topology", but we do not know whether this is possible in general. On the face of it there are problems similar to those mentioned in connection with meets of sublocales.

Suppose $\lhd_1$ is an arbitrary set (i.e. set-indexed family) of pairs (a, U) with $a \in P$ and $U \subseteq P$. Then we can form an inductively generated sublocale Y by defining $b \lhd_0^Y V$ if $b \lhd_0 V$ or $b \leq a \lhd_1 U$ and $V = b \downarrow U$. Now suppose we have a cover relation $\lhd$ on P that includes $\lhd_0$. Then $\lhd_0^Y \subseteq \lhd$ iff $\lhd_1 \subseteq \lhd$. For the $\Rightarrow$ direction, if $a \lhd_1 U$ then $a \lhd_0^Y a \downarrow U$, hence $a \lhd a \downarrow U \lhd U$. For $\Leftarrow$, if $b \leq a \lhd_1 U$ then $b \lhd \{a\} \lhd U$ and $b \lhd \{b\}$ so $b \lhd b \downarrow U$. Hence although we defined inductively generated sublocale to include a localization condition, we can still cope with unlocalized axiom sets. For an arbitrary such axiom set, we can find an inductively generated sublocale that generates the same cover.

Example 18 *It follows from Proposition 10 that open and closed sublocales are inductively generated. The open sublocale A is generated by $P \lhd A$; localized, this becomes $a \lhd_0^A a \downarrow A$ for every $a \in P$. The closed sublocale $X - A$ is generated by $A \lhd \emptyset$; localized this becomes $b \lhd_0^{X-A} \emptyset$ for all $b \leq a \in A$.*

Our main result here is that the family of inductively generated sublocales is – modulo equality – the opposite of a frame. Finite joins are still inductively generated, and set-indexed meets exist and are inductively generated. Binary joins distribute over meets. This is a predicative analogue of the well-known localic result.

Proposition 19 *Let $X = (P, \leq, \lhd_0)$ be a flat site and Y_i an inductively generated sublocale for each $i \in I$. Then they have a sublocale meet $Y = \bigwedge_i Y_i$, given as an inductively generated sublocale by*

$$\lhd_0^Y = \lhd_0 \cup \bigcup_i \lhd_0^{Y_i}.$$

Proof. It is clear that this is an inductively generated sublocale. It is also a sublocale meet, since for any sublocale Z, we have $\lhd_0^Y \subseteq \lhd^Z$ iff $\lhd_0^{Y_i} \subseteq \lhd^Z$ for every i. Note that the part "$\lhd_0 \cup$" is included only to deal with the case where I is empty. ∎

Next we deal with finite joins. The nullary join $\emptyset$, the least sublocale, can be presented by $a \lhd_0^\emptyset \emptyset$ for all a. More work is required for the binary join.

Theorem 20 *Let $(P, \leq, \lhd_0)$ be a flat site, and let Y and Z be two inductively generated sublocales. Then their join $Y \vee Z$ is also inductively generated, by $c \lhd_0^{Y \vee Z} U \cup V$ if there are a and b with $c \in a \downarrow b$, $a \lhd_0^Y U$ and $b \lhd_0^Z V$.*

Proof. It is clear that $Y \vee Z$ is an upper bound of Y and Z. To show it is the sublocale join, we must show that if $a \lhd^Y W$ and $a \lhd^Z W$, then $a \lhd^{Y \vee Z} W$.

Fixing $W \subseteq P$, let us define a predicate $\Phi(a)$ as $(\forall b)(b \lhd^Z W \to a \downarrow b \lhd^{Y \vee Z} W)$. Our result will follow if we can show that $(\forall a)(a \lhd^Y W \to \Phi(a))$. As usual, we use induction on the proof of $a \lhd^Y W$.

If $a \in W$, then for all b we have $a \downarrow b \lhd^{Y \vee Z} W$ and so $\Phi(a)$ holds.

If $\Phi(a)$ and $a' \leq a$ then for all b we have $a' \downarrow b \subseteq a \downarrow b$, and again it is clear that $\Phi(a')$ holds.

Now suppose we have $a \lhd_0^Y U$ and $(\forall u \in U)\Phi(u)$. We must show $\Phi(a)$, and now we use induction on the proof of $b \lhd^Z W$. Again, the first two parts of the induction are easy. What we are left with is to show that if $b \lhd_0^Z V$ and $a \downarrow V \lhd^{Y \vee Z} W$ then $a \downarrow b \lhd^{Y \vee Z} W$.

Suppose $c \in a \downarrow b$. We can then find U' and V' such that $c \lhd_0^Y U' \subseteq c \downarrow U$ and $c \lhd_0^Z V' \subseteq c \downarrow V$. Then $c \lhd_0^{Y \vee Z} U' \cup V'$, so it suffices to show that $U' \cup V' \lhd^{Y \vee Z} W$. Now $V' \subseteq a \downarrow V \lhd^{Y \vee Z} W$. Note also from this that $V' \lhd^Z W$, and so $c \lhd^Z W$. Next, suppose $u' \in U'$. Then $u' \le u \in U$ for some u, and so $\Phi(u)$ holds. Since $c \lhd^Z W$ we get $u' \in u \downarrow c \lhd^{Y \vee Z} W$. ∎

Note that our proof does not extend to infinitary joins. Joins of infinite families of inductively generated sublocales exist as sublocales, but we have not been able to show that they are inductively generated.

It is easy to see that the binary join of sublocales distributes over arbitrary meets. Hence although the collection of inductively generated sublocales modulo $\le$ does not necessarily form a set in predicative mathematics, it does have the structure needed for a *coframe* (the opposite of a frame).

Proposition 21 *Let $X = (P, \le, \lhd_0)$ be a flat site. Then a sublocale Y is inductively generated iff it is a set-indexed meet of cocrescents.*

Proof. Since open and closed sublocales are inductively generated, so are cocrescents and set-indexed meets of them.

Conversely, Y is the greatest sublocale Z such that $\lhd_0^Y \subseteq \lhd^Z$, i.e. such that whenever $a \lhd_0^Y U$ we have $a \lhd^Z U$, i.e. $Z \le (X - \{a\}) \vee U$. In other words, Y is the meet of the family of sublocales $(X - \{a\}) \vee U$ for $a \lhd_0^Y U$. ∎

5 Examples of sublocales

We now look at examples of different kinds of sublocales.

The first two (overt and compact) are properties of formal topologies that we characterize in terms of the sublocale structure for general sublocales. The other two (weakly closed and fitted) are classes of sublocales presented in particular ways, and for these our results are mainly in the inductively generated case.

"Weakly closed" generalizes the classical notion of closed in a way that includes two constructively distinct notions: closed as already defined in Section 3, and a different notion which we call "rest closed", following the notion of "rest" in [Sam03]. Fitted sublocales are a localic analogue of saturated subspaces (up-closed under the specialization order).

Two fundamental representation theorems show a bijection between overt weakly closed sublocales and certain subsets of the base, and between compact fitted sublocales and certain sets of finite subsets of the base.

5.1 Overt sublocales

An overt locale is the same as an open locale, which in formal topological terms means having a positivity predicate satisfying certain conditions ([Neg02]). We follow Paul Taylor in using the word "overt" so that we can talk about an overt sublocale without ambiguity. This has also been described as a sublocale "with open domain".

For later use, we define a notion of "lower powerpoint" that is more general than that of positivity predicate. In the inductively generated case, these are equivalent to points of the lower powerlocale $P_L X$ (see e.g. [Vic97]). This follows from [Vic06, Coverage theorem for flat sites].

Definition 22 *Let $X = (P, \leq, \lhd)$ be a formal topology. Then a subset $F \subseteq P$ is a* lower powerpoint *iff it satisfies the following two conditions.*

1. *F is upper closed under $\leq$.*

2. *If $a \lhd U$ and $a \in F$, then $U \between F$.*

The condition can also be expressed directly in terms of flat sites.

Proposition 23 *Let $X = (P, \leq, \lhd_0)$ be a flat site. Then an upper closed subset $F \subseteq P$ is a lower powerpoint iff whenever $a \lhd_0 U$ and $a \in F$, then $U \between F$.*

Proof. An induction on $a \lhd U$ based on Theorem 3. ∎

Proposition 24 *A formal topology $(P, \leq, \lhd)$ is overt iff there is a lower powerpoint $\mathrm{Pos} \subseteq P$ such that $a \lhd \{a\} \cap \mathrm{Pos}$ for every $a \in P$.*

Such a subset Pos is uniquely determined by those conditions. It then necessarily comprises the basic opens that are positive *in the sense that any set covering them must be inhabited.*

Proof. [Neg02]; or see e.g. [Vic06]. ∎

Given a sublocale Y, it follows that Y is overt iff there is a subset Pos^Y satisfying the same conditions with respect to $(P, \leq, \lhd^Y)$. That is to say, Pos^Y is upper closed; and if $a \lhd^Y U$ and $a \in \mathrm{Pos}^Y$ then $U \between \mathrm{Pos}^Y$; and $a \lhd^Y \{a\} \cap \mathrm{Pos}^Y$ for each a.

5.2 Compact sublocales

The compactness property of locales or formal topologies is well known: every cover has a finite subcover. In the following characterization of it we write $\mathcal{F}P$ for the finite powerset of P, i.e. the set of all Kuratowski finite subsets. We also write $\sqsubseteq_L$ for the *lower order* (a preorder) on finite subsets, $S \sqsubseteq_L T$ if for every $s \in S$ there is some $t \in T$ with $s \leq t$.

Proposition 25 *A formal topology $X = (P, \leq, \lhd)$ is compact iff there is a subset $\mathrm{Cov} \subseteq \mathcal{F}P$ satisfying the following conditions.*

1. *Cov is upper closed with respect to $\sqsubseteq_L$.*

2. *Cov is inhabited.*

3. *If $S \lhd U$ and $S \in \mathrm{Cov}$, then $U_0 \in \mathrm{Cov}$ for some $U_0 \in \mathcal{F}U$.*

4. *If $S \in \mathrm{Cov}$ then $P \lhd S$.*

Such a subset Cov is uniquely determined by those conditions. It then necessarily comprises the finite sets of basic opens that cover the whole of P, and is a filter with respect to $\sqsubseteq_L$.

Proof. If X is compact, then the family Cov of finite subsets of P that cover X has those properties.

Conversely, suppose Cov has those properties. Choose $S \in \text{Cov}$. If $P \lhd U$ then $S \lhd S \downarrow U$, so there is some $T \in \mathcal{F}(S \downarrow U) \cap \text{Cov}$. Then $T \sqsubseteq_L U_0$ for some $U_0 \in \mathcal{F}U$, so $U_0 \in \text{Cov}$. It follows that X is compact. By the same argument in the case when U is finite, we find that every finite subset of P that covers X must be in Cov. To see that Cov is a filter with respect to $\sqsubseteq_L$, suppose $S, T \in \text{Cov}$. Then $P \lhd S \downarrow T$, so $P \lhd U_0 \in \mathcal{F}(S \downarrow T)$ for some U_0 which is then a lower bound for S and T in Cov. ∎

Given a sublocale Y, it follows that Y is compact iff there is a subset Cov^Y satisfying the same conditions with respect to $(P, \leq, \lhd^Y)$.

Our aim (Theorem 35) is to give a predicative version of the topos-valid result ([Vic97]) that certain compact sublocales are in bijection with points of the *upper powerlocale*. A predicative treatment of the upper powerlocale for the inductively generated case has been given in [Vic05], and we shall prove a limited selection of results here in the general case to show the relation with Cov.

Definition 26 *Let $X = (P, \leq, \lhd)$ be a formal topology. Then a subset $F \subseteq \mathcal{F}P$ is an* upper powerpoint *iff it satisfies the following three conditions.*

1. *F is upper closed with respect to $\sqsubseteq_L$.*

2. *F is inhabited.*

3. *If $a \lhd U$ and $\{a\} \cup T \in F$, then $U_0 \cup T \in F$ for some $U_0 \in \mathcal{F}U$.*

4. *If $\{a\} \cup T$ and $\{b\} \cup T$ are both in F, then there is some $U_0 \in \mathcal{F}(a \downarrow b)$ such that $U_0 \cup T \in F$.*

Lemma 27 *Let $X = (P, \leq, \lhd)$ be a formal topology and $F \subseteq \mathcal{F}P$ upper closed with respect to $\sqsubseteq_L$. Then the following are equivalent.*

1. *F satisfies condition (3) of Definition 26.*

2. *If $S, T \in \mathcal{F}P$, $S \lhd U$ and $S \cup T \in F$, there is some $U_0 \in \mathcal{F}U$ such that $U_0 \cup T \in F$.*

3. *If $S \in F$ and $S \lhd U$, then there is some $U_0 \in \mathcal{F}U$ such that $U_0 \in F$.*

In the case where X is inductively generated by a flat site $(P, \leq, \lhd_0)$, it suffices to replace $\lhd$ by $\lhd_0$ in (1).

Proof. $(1 \Rightarrow 2)$: Induction on the size of S. (Really, this is induction on the length of the list of elements used to describe S. S itself does not have a well defined cardinality unless P has decidable equality.)

$(2 \Rightarrow 3)$: Take $T = \emptyset$.

$(3 \Rightarrow 1)$: If $a \lhd U$ and $\{a\} \cup T \in F$, then $\{a\} \cup T \lhd U \cup T$ and so $V \in F$ for some $V \in \mathcal{F}(U \cup T)$. We can now write $V = U_0 \cup T_0$ with $U_0 \in \mathcal{F}U$ and $T_0 \in \mathcal{F}T$, and it follows that $U_0 \cup T \in F$ because F is upper closed under $\sqsubseteq_L$ (which includes $\subseteq$).

Now suppose in the inductively generated case that (1) holds in the weaker form with $\lhd_0$ instead of $\lhd$. We show (1) by induction on the proof of $a \lhd U$.

Fixing U, define $\Phi(a)$ as for every $T \in \mathcal{F}P$, if $\{a\} \cup T \in F$ then $U_0 \cup T \in F$ for some $U_0 \in \mathcal{F}U$. The main part to be proved is that if $a \lhd_0 V$, and $\Phi(v)$ for every $v \in V$, then $\Phi(a)$. Suppose $\{a\} \cup T \in F$. We have some $V_0 \in \mathcal{F}V$ with $V_0 \cup T \in F$. Now by a similar induction (on V_0) to that of $(1 \Rightarrow 2)$, we find $U_0 \in \mathcal{F}U$ such that $U_0 \cup T \in F$. $\blacksquare$

In [Vic05, Proposition 14.3] it is proved, in the inductively generated case, that points of the upper powerlocale are equivalent to upper powerpoints as in Definition 26 here, except that condition (3) has $a \lhd_0 U$ instead of $a \lhd U$. From Lemma 27 we now see that that weakening makes no difference.

Lemma 28 *Let $X = (P, \leq, \lhd)$ be a formal topology and $F \subseteq \mathcal{F}P$ upper closed with respect to $\sqsubseteq_L$. Then the following are equivalent.*

1. *F satisfies condition (4) of Definition 26.*

2. *If $A \cup T$ and $B \cup T$ are both in F, then there is some $U_0 \in \mathcal{F}(A \downarrow B)$ such that $U_0 \cup T \in F$.*

3. *If A and B are both in F, then there is some $U_0 \in \mathcal{F}(A \downarrow B)$ such that $U_0 \in F$. In other words, any pair of elements of F is bounded below in F with respect to $\sqsubseteq_L$.*

Proof. $(1 \Rightarrow 2)$: If $A, B \in \mathcal{F}P$, let us write $\Psi(A, B)$ for the property that for all T, if $A \cup T$ and $B \cup T$ are both in F, then so is $U_0 \cup T$ for some $U_0 \in \mathcal{F}(A \downarrow B)$. Obviously this is symmetric, and holds if either A or B is empty, or if both are singletons. Also, if $\Psi(A, B_1)$ and $\Psi(A, B_2)$ hold then so does $\Psi(A, B_1 \cup B_2)$. For suppose $A \cup T$ and $B_1 \cup B_2 \cup T$ are both in F. Then also $A \cup B_2 \cup T \in F$ and so $U_1 \cup B_2 \cup T \in F$ for some $U_1 \in \mathcal{F}(A \downarrow B_1)$. Then $A \cup U_1 \cup T \in F$ and so $U_2 \cup U_1 \cup T \in F$ for some $U_2 \in \mathcal{F}(A \downarrow B_2)$. Then we can take $U_0 = U_1 \cup U_2 \in \mathcal{F}(A \downarrow (B_1 \cup B_2))$. From this we can deduce that $\Psi(\{a\}, B)$ for every B, and then the result.

$(2 \Rightarrow 3)$: Take $T = \emptyset$.

$(3 \Rightarrow 1)$: Suppose $\{a\} \cup T$ and $\{b\} \cup T$ are both in F. Then there is some $V \in F$ with $V \in \mathcal{F}((\{a\} \cup T) \downarrow (\{b\} \cup T))$. We can now write $V = U_0 \cup T_0$ with $U_0 \in \mathcal{F}(a \downarrow b)$ and $T_0 \in \mathcal{F}(\downarrow T)$, and it follows that $U_0 \cup T \in F$. $\blacksquare$

Hence every upper powerpoint is a filter with respect to $\sqsubseteq_L$.

Putting the preceding results together, we get the following.

Proposition 29 *A flat site $(P, \leq, \lhd_0)$ is compact if there is an inhabited subset $\mathrm{Cov} \subseteq \mathcal{F}P$, upper closed under $\sqsubseteq_L$, such that:*

1. *if $a \lhd_0 U$ and $\{a\} \cup T \in \mathrm{Cov}$ then $U_0 \cup T \in \mathrm{Cov}$ for some $U_0 \in \mathcal{F}U$, and*

2. *if $S \in \mathrm{Cov}$ then $P \lhd S$.*

Such a subset Cov is uniquely determined by those conditions and is an upper powerpoint. It then necessarily comprises the finite sets of basic opens that cover the whole of P.

5.3 Weakly closed sublocales

In the setting of topos-valid frame-based locale theory, [Joh02, C1.1.22, C1.2.14] defines a sublocale Y to be *weakly closed* if its nucleus j_Y is least in its equivalence class with respect to a certain equivalence relation $\sim$ on nuclei. We write $j \sim k$ if $j(p) = k(p)$ for all $p \in \Omega$, the set of truth values (i.e., in a topos, the subobject classifier). Here we are identifying p with its image under the unique frame homomorphism from Ω to ΩX. Writing more carefully, we should say p here denotes $\bigvee\{1 \mid p\}$. (Translated into our predicative setting, it denotes the open corresponding to the set $\{a \in P \mid p\}$ where p is a proposition.)

Now suppose Y is any sublocale. (We stay in the topos-valid setting.) We present another sublocale Y' by the relations $a \leq p$ such that $a \in \Omega X$, $p \in \Omega$ and $a \leq j_Y(p)$ in ΩX. Clearly Y' depends only on the equivalence class (under $\sim$) of j_Y. Also $Y \leq Y'$, since $a \leq j_Y(p)$ in ΩX iff the relation $a \leq p$ holds modulo Y. Now we claim $j_Y \sim j_{Y'}$. If $p \in \Omega$ then $j_{Y'}(p) \leq j_Y(p)$ because $Y \leq Y'$. Then $j_Y(p) \leq j_{Y'}(p)$ holds because by the presentation $j_Y(p) \leq p$ holds modulo Y'. It follows that $j_{Y'}$ is the least nucleus in the equivalence class of j_Y, so Y is weakly closed iff $Y' = Y$. Let us call Y' the *weak closure $wc(Y)$* of Y.

We can now conclude that Y is weakly closed iff it can be presented by relations in the form $a \leq p$. The $\Rightarrow$ direction follows from the form of presentation of $wc(Y)$. For $\Leftarrow$, if Y is presented in that form then $wc(Y) \leq Y$ and so we have equality.

Now let us translate that into the context of a flat site $X = (P, \leq, \lhd_0)$. Clearly for a set of relations $a \leq p$ we can assume without loss of generality that a is basic. A set of relations $a_i \leq p_i$ will be described by a set I, a function $i \mapsto a_i$ from I to P, and, to represent $i \mapsto p_i$, a subset $I_0 \subseteq I$.

Definition 30 *Let $X = (P, \leq, \lhd_0)$ be a flat site. Then a* weakly closed *sublocale Y is one that can be described as follows. Let a_i ($i \in I$) be a family of elements of P and let $I_0 \subseteq I$. Then for each i and for each $a \leq a_i$ we have $a \lhd_0^Y \{a' \mid a' = a \text{ and } i \in I_0\}$.*

Note that the covering set here is a *subsingleton* – it has at most one element (a), present iff $i \in I_0$.

Every closed sublocale $X - A$ is weakly closed, taking I to be A itself and I_0 to be empty. Classically this is the only possibility – a weakly closed sublocale is $X - A$ where $A = \{a_i \mid i \notin I_0\}$. Constructively, however, we must distinguish between the two.

There is also a third notion which we shall (tentatively) call *rest closed*. This is the weakly closed case where I is the whole of P. Thus a rest closed sublocale is given by a subset $F \subseteq P$ and for each a in P a cover $a \lhd_0 \{a\} \cap F$. A point x is in this sublocale if all its basic neighbourhoods are in F, and so this is the sublocalic analogue of what [Sam03] calls $\mathsf{rest}F$. We shall denote it $\mathsf{Rest}F$.

In closed and rest closed we see analogues of two classically equivalent characterizations of closed subspaces. On the one hand they are the complements of open subspaces, and Theorem 12 displays a similar property for closed sublocales. On the other hand, a subspace Y is closed if it equals its closure, defined as the set of points x for which every neighbourhood meets Y. Now consider the set F of basic opens that meet Y. A point x is in the closure of Y iff it satisfies all the relations used in presenting $\mathsf{Rest}F$: for every basic open a, if

x is in a then a is in F (i.e. x is in some open in $\{a\} \cap F$). Thus rest closed sublocales match more closely the second classical characterization. Papers such as [Sam03], recognizing the constructive bifurcation of the notion of "closed", have tended to use the word in relation to subsets such as F. Outside classical reasoning, we do not know an implication in either direction between closed and rest closed.

It is an appealing connection between, on the one hand, the second classical characterization of closedness and, on the other, the covers in rest closedness. However, the subsets F defined above, comprising the basic opens that meet a subspace, are not arbitrary – they are lower powerpoints. One consequence of this (we shall see) is that the corresponding rest closed sublocale is *overt*, and it turns out that for overt sublocales rest closed is equivalent to weakly closed. In impredicative (frame-based) locale theory we have a presentation independent characterization of weak closedness, as well as a weak closure of sublocales, and I do not know of analogues for rest closedness. Hence it may be that, in general, weak closedness is better behaved. For the present we restrict attention to the overt case where they are equivalent.

Proposition 31 *Let $X = (P, \leq, \lhd_0)$ be a flat site, and let Y be an overt inductively generated sublocale with positivity set Pos^Y.*

1. *Y is weakly closed iff it is equal to $\mathsf{Rest}(\mathrm{Pos}^Y)$.*

2. *If Y is $\mathsf{Rest}F$ then $\mathrm{Pos}^Y \subseteq F$.*

Proof. 1. The $\Leftarrow$ direction is *a fortiori*. For $\Rightarrow$, suppose Y is weakly closed using a_i ($i \in I$) and $I_0 \subseteq I$. From Proposition 24, we see that $Y \leq \mathsf{Rest}(\mathrm{Pos}^Y)$. Now suppose $a \leq a_i$. We must show $a \lhd^{\mathsf{Rest}(\mathrm{Pos}^Y)} \{a' \mid a' = a \text{ and } i \in I_0\}$. Since $a \lhd^{\mathsf{Rest}(\mathrm{Pos}^Y)} \{a\} \cap \mathrm{Pos}^Y$, it follows that we can assume $a \in \mathrm{Pos}^Y$. Then since $a \lhd_0^Y \{a' \mid a' = a \text{ and } i \in I_0\}$ it follows that the covering set meets Pos^Y, i.e. $i \in I_0$. But then $\{a' \mid a' = a \text{ and } i \in I_0\} = \{a\}$, which certainly covers a in $\mathsf{Rest}(\mathrm{Pos}^Y)$.

2. Suppose $a \in \mathrm{Pos}^Y$. We have $a \lhd_0^Y \{a\} \cap F$ and it follows that this covering set is inhabited, and so $a \in F$. ∎

Theorem 32 *Let $X = (P, \leq, \lhd_0^X)$ be a flat site. Then the following are equivalent.*

1. *Lower powerpoints $F \subseteq P$.*

2. *Overt, weakly closed sublocales of X.*

3. *Overt, rest closed sublocales of X.*

Proof. Given F as in (1), F satisfies all the conditions for $\mathrm{Pos}^{\mathsf{Rest}F}$ in Proposition 24, and it follows that $\mathsf{Rest}F$ is overt and $F = \mathrm{Pos}^{\mathsf{Rest}F}$. Proposition 31 now completes the result. ∎

This result is known in locale theory ([BF96], [Vic97]) as a characterization of the points of the lower powerlocale $P_L X$.

We finish this Section by showing some connections with the "binary positivity" predicate of [Sam03]. The connections arise from [MV04] and [Val05].

In [MV04] it is shown by coinduction that (in our terms) for every flat site $X = (P, \leq, \lhd_0)$ there is a greatest lower powerpoint $\mathsf{Pos} \subseteq P$, that (as can also be seen from our Theorem 32) the corresponding rest closed sublocale is overt, and that it is an overt (open) coreflection of X. Subsequently [Val05] generalized this by showing that for every $G \subseteq P$ there is a greatest lower powerpoint $\mathsf{Pos}(G)$ included in G. In fact, this follows directly from [MV04] by considering $\mathsf{Rest}G$ as formal topology in its own right. In [Val05], a binary positivity predicate is defined by saying $\mathsf{Pos}(a, G)$ (or $a \ltimes G$) if $a \in \mathsf{Pos}(G)$.

[Val05] describes the coinductive generation of Pos in a slightly more general setting. In the notation there, the basic covers $a \lhd_0 U$ are expressed by, for each $a \in P$, a set $I(a)$ indexing the basic covers of a, and, for each $i \in I(a)$, a set $C(a, i)$ making $a \lhd_0 C(a, i)$. (This is the notation of [CSSV03] that we mentioned after Definition 2.) However, [Val05] also allows for *positivity* axioms, given by sets $J(a)$ and $D(a, j)$ for $j \in J(a)$, and used in an extra rule Pos-*infinity*:

$$\frac{\mathsf{Pos}(a, G) \qquad j \in J(a)}{\mathsf{Pos}(G) \between D(a, j)}$$

We have applied this in the case where each $J(a)$ is empty so that Pos-infinity is never applied. However, we could equally well take J and D to be the same as I and C, for then Pos-infinity says the same as the condition in Proposition 23 applied to $\mathsf{Pos}(G)$. Moreover, because we are assuming the flat stability property, the condition is also equivalent to a further rule of [Val05] called "compatibility on axioms". Thus although [Val05] introduces the positivity axioms J, D for the sake of symmetry with the cover axioms I, C, one reasonable interpretation is to use the cover axioms in both roles.

5.4 Fitted sublocales

Definition 33 *A sublocale is* fitted *if it is a meet of a set-indexed family of open sublocales.*

Note that we have not been able to show that such meets exist outside the inductively generated case. In that case, a fitted sublocale of a flat site $(P, \leq, \lhd_0)$ is given by a set A_i ($i \in I$) of subsets of P. The basic covers are then $a \lhd_0^A a \downarrow A_i$ ($a \in P$, $i \in I$).

Classically, a subspace is an intersection of open subspaces iff it is *saturated*, i.e. upper closed under the specialization order. (Any saturated subspace $Y \subseteq X$ is the intersection of the open subspaces that include it. For suppose $x \in X - Y$. The topological closure $\mathrm{Cl}\{x\}$ is the down closure of $\{x\}$ under the specialization order, and so is disjoint with Y. Hence $X - \mathrm{Cl}\{x\}$ is an open subspace that includes Y but does not contain x.) Thus "fitted" is a constructive localic analogue of "saturated".

In impredicative locale theory, every sublocale Y has a *fitted hull* $fh(Y)$, namely the meet of all the open sublocales greater than Y. In formal topology one should not expect this collection to be a set in general. However, we can make more progress when we consider the set of *finite* open covers of Y, and this fits naturally with compactness.

Just as weakly closed sublocales are best behaved when also overt, fitted sublocales are best behaved when also compact and in fact the reasoning is in many ways parallel ([Vic95]).

Proposition 34 *Let $X = (P, \leq, \lhd)$ be a formal topology, and let Y be a compact sublocale with Cov^Y the set of finite covers by basics. Then Y is fitted iff it is the meet of the opens corresponding to the elements of Cov^Y.*

Proof. The $\Leftarrow$ direction is obvious. For $\Rightarrow$, by Proposition 10 we see that Y is a lower bound of the open sublocales S for $S \in \mathrm{Cov}^Y$. Suppose Y is fitted as the meet of open sublocales A_i $(i \in I)$. Let Z be a lower bound of the open sublocales for elements of Cov^Y, and suppose $i \in I$. Find $S \in \mathrm{Cov}^Y$. Since $S \lhd^Y A_i$, by Lemma 27 there is some $A' \in \mathcal{F}A_i$ with $A' \in \mathrm{Cov}^Y$. Then $P \lhd^Z A' \subseteq A_i$ so $Z \leq A_i$. Hence $Z \leq Y$. $\blacksquare$

We can now prove the main result about upper powerpoints in the inductively generated case.

Theorem 35 *Let $X = (P, \leq, \lhd_0^X)$ be a flat site. Then the following are equivalent.*

1. *Upper powerpoints $F \subseteq \mathcal{F}P$.*

2. *Compact fitted sublocales of X.*

Proof. Given F as in (1), let Y be its fitted sublocale. Then F satisfies the conditions for Cov^Y in Proposition 29. For suppose $a \lhd_0^Y U$ and $\{a\} \cup T \in F$. We have either $a \lhd_0 U$ or $U = a \downarrow S$ for some $S \in F$. In the first case we use the fact that F is an upper powerpoint. In the second, by Lemma 28, we have $V \in F$ for some $V \in \mathcal{F}((\{a\} \cup T) \downarrow S)$. We can now write $V = U_0 \cup T_0$ where $U_0 \in \mathcal{F}(a \downarrow S) = \mathcal{F}U$ and $T_0 \in \mathcal{F}(T \downarrow S)$, and it follows that $U_0 \cup T \in F$. We deduce that Y is compact and $F = \mathrm{Cov}^Y$. Proposition 34 now completes the result. $\blacksquare$

This result is known in locale theory ([Vic97], based on [Joh85]) as a characterization of the points of the upper powerlocale $P_U X$.

6   Conclusions

The impredicative theory of sublocales is, unsurprisingly, more complicated in predicative mathematics.

The best behaviour is for inductively generated sublocales of inductively generated formal topologies, for which we have meets and finite joins (making coframe structure) just as in the impredicative case. For more general sublocales we have joins, but it is doubtful whether meets can always exist. Nonetheless, we have shown there is boolean algebra structure on a class of sublocales generated by the open and the closed sublocales, and we have described that structure explicitly. The boolean algebra structure is constructive, thus illustrating a constructive difference between joins of sublocales and unions of the corresponding subspaces.

The remaining results derive from the localic theory of lower and upper powerlocales, which have been described in formal topology in the inductively generated case. Known localic results are that the points of the lower powerlocale are equivalent to overt, weakly closed sublocales, while points of the upper powerlocale are equivalent to compact, fitted sublocales. We have proved the corresponding predicative results for inductively generated formal topologies.

In describing the weakly closed sublocales we also identified a class of "rest closed" sublocales related to the binary positivity predicate. These appear to give a sublocalic version of rest$F$ ([Sam03]). Part of the discussion there was the realization that the classical notion of closed subspace bifurcates constructively: the notions of "complement of open" and "equal to its closure" become distinct. Our work has addressed both sides of this. On the one hand, the notion "complement of open" loses its problematic dependence on excluded middle when transferred to sublocales, since corresponding open and closed sublocales are still boolean complements. On the other hand, the rest closed sublocales are directly related to the notion of "equal to its closure" if, in Rest$F$, one takes F to be the family of basic opens that meet a given subspace. We have also proposed that the overt, weakly closed sublocales (which we have proved are a special case of rest closed) are more especially related to this notion, because the F that arises as above should be expected to be a lower powerpoint.

One technical question we have not been able to answer is the general existence of meets of sublocales. Is there, despite our doubts, a predicative construction, or is there an example of a set-indexed family of sublocales for which there is no meet?

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