A Universal Characterization of the Double Powerlocale

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Abstract

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The double powerlocale $\mathbb{P}(X)$ (found by composing, in either order, the upper and lower powerlocale constructions P_U and P_L) is shown to be isomorphic in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ to the double exponential $\mathbb{S}^{\mathbb{S}^X}$ where $\mathbb{S}$ is the Sierpiński locale. Further $P_U(X)$ and $P_L(X)$ are shown to be the subobjects $\mathbb{P}(X)$ comprising, respectively, the meet semilattice and join semilattice homomorphisms. A key lemma shows that, for any locales X and Y, natural transformations from $\mathbb{S}^X$ (the presheaf $\mathbf{Loc}(- \times X, \mathbb{S})$) to $\mathbb{S}^Y$ (i.e. $\mathbf{Loc}(- \times Y, \mathbb{S})$) are equivalent to dcpo morphisms (Scott continuous maps) from the frame ΩX to ΩY. It is also shown that $\mathbb{S}^X$ has a localic reflection in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ whose frame is the Scott topology on ΩX.

The reasoning is constructive in the sense of topos validity.

Keywords: powerdomain, topos, Scott continuity, locale.

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1 Introduction

1.1 Background comment on powerlocales

The convex (Plotkin), lower (Hoare) and upper (Smyth) powerdomains are well established constructions in domain theory, providing tools for the semantics of programming languages [Plo81]. The convex powerdomain [Plo76] is in effect an adaptation of the topological theory of hyperspaces (see [Nad78]), but was found to embed in two more primitive powerdomains, the upper and lower [Smy78]. These two are less familiar in general topology, perhaps because their topologies are almost never Hausdorff. See [Sch93] for a summary.

All three constructions work well in localic form, giving *powerlocales* (Vietoris V [Joh85], and lower P_L and upper P_U [Rob86]). They have been studied in particular in [Vic97].

It has long been known that the upper and lower powerdomain constructions commute [FM90], and in [JV91] this was also proved for the upper and lower powerlocales. Their composite is what we are calling the *double powerlocale* $\mathbb{P}$. Its investigation was advocated in Section 5 of [Vic93], partly with a view to unifying the study of the upper and lower powerlocales. In [Vic95] a number of abstract results for order-enriched categories were proved, and it was shown how when these were interpreted twice in the category **Loc** of locales, once with the specialization order enrichment and once with its opposite, they yielded parallel results involving each powerlocale. However, there was nothing there to show interaction between the two powerlocales, whereas the double powerlocale encompasses both together.

Serious study of the double powerlocale started in [Vic04]. A major result there was that if the locale X is locally compact (hence exponentiable, see [Hyl81] or [Joh82]), then $\mathbb{P}X$ is homeomorphic to $\mathbb{S}^{\mathbb{S}^X}$, where $\mathbb{S}$ is the Sierpiński locale (i.e. the locale whose frame of opens is the free frame on the singleton set). This shows that $\mathbb{P}$, restricted to locally compact locales, is the same as the monad Σ^2 (where $\Sigma X = \mathbb{S}^X$) used extensively in Taylor's Abstract Stone Duality (see e.g. [Tay02]).

1.2 Objective

The objective of this paper is to prove that $\mathbb{P}X$ is homeomorphic to $\mathbb{S}^{\mathbb{S}^X}$ *even when X is not locally compact.* In other words we cover cases where $\mathbb{S}^X$ does not necessarily exist as a locale. We do this by using the Yoneda embedding of **Loc** into $[\mathbf{Loc}^{op}, \mathbf{Set}]$. We have to be careful, because **Loc** is large; in particular we do not assume that $[\mathbf{Loc}^{op}, \mathbf{Set}]$ is a topos, or even cartesian closed. However, Yoneda's Lemma still holds good and we can use it to find exponentials of representable presheaves. The Yoneda embedding represents each locale X as the presheaf $\mathbf{Loc}(\_, X)$, and the Yoneda lemma tells us that Y^X is $\mathbf{Loc}(\_ \times X, Y)$. The main result here is therefore that $\mathbb{S}^{\mathbb{S}^X}$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ and is given by the representable functor $\mathbf{Loc}(\_, \mathbb{P}X)$. We have thus found a characterization of the double power locale that is entirely localic. It is presentation independent, unlike the specific constructions given in [Vic04] by which a presentation for $\mathbb{P}X$ is constructed out of each presentation of X. On the other hand, it is also independent of the underlying lattice theory of frames, and – modulo foundational questions raised by $[\mathbf{Loc}^{op}, \mathbf{Set}]$ itself – this may have some virtue in the context of constructivist doctrines (such as predicative type theory, and the "arithmetic" logic conjectured in the conclusions of [Vic99]) in which frames are not admissible as sets.

We also show that the main result restricts to results about the upper and lower powerlocales. $\mathbb{S}$ has internal distributive lattice structure in $\mathbf{Loc}$ and therefore $\mathbb{S}^X$ is an internal distributive lattice in $[\mathbf{Loc}^{op}, \mathbf{Set}]$. It is shown here that the powerlocales $P_U X$ and $P_L X$ can then be identified with the subobjects of $\mathbb{S}^{\mathbb{S}^X}$ whose generalized points are (respectively) the meet and join semilattice homomorphisms from $\mathbb{S}^X$ to $\mathbb{S}$.

1.3 Proof outline

If the double exponentiation $\mathbb{S}^{\mathbb{S}^X}$ exists as a presheaf then $\mathbb{S}^{\mathbb{S}^X}(W)$ is the class of natural transformations from $\mathbf{Loc}(\_, W) \times \mathbf{Loc}(\_ \times X, \mathbb{S})$ to $\mathbf{Loc}(\_, \mathbb{S})$, and these are equivalent to natural transformations from $\mathbf{Loc}(\_ \times X, \mathbb{S})$ to $\mathbf{Loc}(\_ \times W, \mathbb{S})$. The key technical result shown here is that these are equivalent to the dcpo morphisms between the corresponding frames of opens, i.e. from ΩX to ΩW.

Given this technical observation the main result is relatively straightforward. Recall that the defining universal frame-theoretic characterization of $\mathbb{P}X$ is that its frame is free over ΩX qua dcpo. In other words, there exists $\boxtimes : \Omega X \to \Omega \mathbb{P} X$, a universal dcpo morphism to a frame. Any dcpo morphism $q : \Omega X \to \Omega W$ extends uniquely to a frame homomorphism from $\Omega \mathbb{P} X$:

$$\begin{array}{ccc} \Omega X & \overset{\boxtimes}{\to} & \Omega \mathbb{P} X \\ & \searrow^{q} & \downarrow \exists!\ \Omega f \\ & & \Omega W \end{array}$$

The correspondence between natural transformations (in $[\mathbf{Loc}^{op}, \mathbf{Set}]$) and dcpo morphisms therefore allows this defining universal characterization to translate to:

$$\begin{array}{ccc} \mathbf{Loc}(\_ \times X, \mathbb{S}) & \overset{\boxtimes}{\to} & \mathbf{Loc}(\_ \times \mathbb{P} X, \mathbb{S}) \\ & \searrow^{q} & \downarrow \exists!\ \mathbf{Loc}(\_ \times f, \mathbb{S}) \\ & & \mathbf{Loc}(\_ \times W, \mathbb{S}) \end{array}$$

Hence the generalized points (at stage W) of $\mathbb{P}X$ are exactly the morphisms $\mathbb{S}^X \to \mathbb{S}^W$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, i.e. exactly the maps $W \times \mathbb{S}^X \to \mathbb{S}$, i.e. the points of the double exponential of X at stage W. This proves the main result from the technical observation.

To prove that the natural transformations in question are exactly the dcpo morphisms, some basic observations about dcpo presentations are made. Specifically, a "double coverage" result for dcpos is given, allowing the reduction of frame presentations to dcpo presentations. This combines existing results whereby frame presentations are reduced to presentations of suplattices [AV93] or preframes [JV91]. Suppose L^X, R^X are the generators and relations of a frame presentation for the locale X and moreover (as can always be assumed) L^X is a distributive lattice and R^X satisfies certain "meet and join stability" conditions. (In ordinary frame-based locale theory one can take $L^X = \Omega X$. However, we allow ourselves the flexibility of using an arbitrary lattice of generators. This is partly a concession to other approaches without frames, but it also yields effective procedures for dealing with locales that are presented to us by generators and relations.) Then it is shown by the double coverage result that the data for a dcpo morphism $\Omega X \to \Omega$ is a locale map

$$p : \mathrm{Idl}(L^X) \to \mathbb{S}$$

composing equally with the maps $R^X \rightrightarrows \mathrm{Idl}(L^X)$ given by the presentation. ($\mathrm{Idl}(L^X)$ is the locale whose points are the ideals of L^X.) Carrying this out in sheaves over W (i.e. pulling back to $\mathbf{Loc}/W$) provides a description of any dcpo morphism $\Omega X \to \Omega W$, in terms of a map $\mathrm{Idl}(L^X) \times W \to \mathbb{S}$. Finally, we make the new observation that for any locale X, $\mathrm{Idl}(L^X)$ is a weak exponential $\mathbb{S}^X$. That is, it is an exponential without the uniqueness requirement on the transpose [CR00]. It then becomes routine to check that the locale map $\mathrm{Idl}(L^X) \times W \to \mathbb{S}$ is enough data to define a natural transformation.

1.4 Notation

For notation our references are [Joh82] and [Vic89].

If X is a locale then we write ΩX for its frame and $\sqsubseteq$ for its specialization order. If f is a map (i.e. a continuous map) between locales then we write Ωf for its inverse image function. We write **Loc** for the category of locales and maps.

We write **Top** for the category of Grothendieck toposes and geometric morphisms.

For the standard "qua" notation, which generally indicates an implied application of a forgetful functor, consult, e.g. [JV91]. If "qua" is used in a presentation it means "add in the equations true of the algebraic structure", so "qua $\wedge$-SemiLat" means include all the meet semilattice equations. We use **Fr, Sup, dcpo, PreFr**, **DL**, $\vee$-**SemiLat**, $\wedge$-**SemiLat** and **Pos** for the categories of frames, suplattices (complete lattices; morphisms preserve all joins), directed complete partial orders (morphisms preserve directed joins), preframes (dcpos with finite meets distributing over directed joins; morphisms preserve finite meets and directed joins), distributive lattices, join semilattices, meet semilattices and posets respectively. In all semilattices and lattices we assume top and bottom elements as appropriate. For instance, join semilattices always have bottom elements (nullary joins), preserved by homomorphisms.

2 Frames via Dcpo Presentations

The idea of presentation by generators and relations is well known from universal algebra in the case of finitary algebraic theories. It does not automatically apply to frames, because of the unbounded arities of the join operators. However, the existence of free frames makes it work ([Joh82]; and for a more detailed description see [Vic89]).

We shall make extensive use of "coverage theorems", by which we mean results that enable us to convert presentations of objects as ("qua") one kind of algebra into presentations of the same objects as a different kind of algebra. The prototype is Johnstone's coverage theorem of [Joh82, II.2.11]. This requires a particular form of presentation of a frame: the generators form a meet semilattice G, and the relations R use only joins and are meet stable. Then the construction has the universal property of $\mathbf{Fr}\langle G \text{ (qua } \wedge\text{-SemiLat)} \mid R\rangle$. In the original papers the relations (expressed by a coverage) were always of the form $a \leq \bigvee U$, but it is not hard to see that the discussion still holds with relations of the general form $\bigvee U = \bigvee V$. Then the meet stability requirement is that given such a presenting relation, and a generator b, then $\bigvee\{u \wedge b \mid u \in U\} = \bigvee\{v \wedge b \mid v \in V\}$ is also one of the presenting relations. These restrictions on the presentation are not significant as any presentation can be manipulated into this form (details omitted; but see [Vic04] for an extensive discussion).

In [AV93] it is observed that Johnstone's construction (via C-ideals) is in fact a construction of $\mathbf{Sup}\langle G \text{ (qua poset)} \mid R\rangle$ and so there it is suggested that the essence of the coverage theorem is that

$$\mathbf{Fr}\langle G \text{ (qua } \wedge\text{-SemiLat)} \mid R\rangle \cong \mathbf{Sup}\langle G \text{ (qua poset)} \mid R\rangle. \qquad \text{(CovThm)}$$

(More precisely, the obvious suplattice homomorphism from right to left that preserves the generators is an isomorphism.) Hence the theorem can be used to transform frame presentations into suplattice presentations.

Dually (replacing finite meets with finite joins) the analogous "preframe coverage theorem" in [JV91] revolves around join semilattice structure and transforms *join* stable frame presentations into preframe presentations. (A *preframe* has finite meets and directed joins, with distributivity of the former over the latter.) It states that for any join stable set of relations R on a join semilattice G of generators,

$$\mathbf{Fr}\langle G \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \cong \mathbf{PreFr}\langle G \text{ (qua poset)} \mid R\rangle.$$

There is in fact a whole family of such coverage theorems, and they can be described by reference to a cubical diagram

$$\begin{array}{ccccccc}
\vee\text{-SemiLat} & & & \longleftarrow & & & \text{DL} \\
& \nwarrow & & & & \nearrow & \\
& & \text{SupLat} & \longleftarrow & \text{Fr} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \text{dcpo} & \longleftarrow & \text{PreFr} & & \\
& \swarrow & & & & \searrow & \\
\text{Poset} & & & \longleftarrow & & & \wedge\text{-SemiLat}
\end{array}$$

Each arrow here represents a forgetful functor, but we know also that it has a left adjoint, a free algebra functor, with the adjunction being monadic. (cf. [Joh02, C1.1], but we have replaced Set by Poset as well as renaming several of the categories.)

Numerous coverage theorems exist in this diagram, though we do not know of any general unifying account. The general proof technique is that once one knows that the presentation for the theory with less structure does indeed present, then it is possible to use its universal property and the meet or join stability to define the extra structure needed for the other theory and to prove the universal property needed for that. For example, in Johnstone's original theorem, once one has presented the suplattice then it is routine to use that presentation to define binary meet on it and show that it makes the required frame.

The main aim of this section is to prove a double coverage result that combines both the preframe coverage and the original suplattice coverage result and transforms frame presentations to dcpo presentations. The presentation must be of the form of join and meet stable relations on a distributive lattice.

We shall also give an explicit description of locale product in terms of a dcpo presentation. This description is necessary to prove the main result.

2.1 The Double Coverage Theorem

We shall define the notion of *DL-site* which is a type of presentation for a frame. In a DL-site the generators form a distributive lattice (DL) and the relations, involving only directed joins, must have both meet and join stability. The double coverage result is that

$$\begin{aligned}&\mathbf{Fr}\langle G \text{ (qua DL)} \mid R\rangle \\ &\qquad\qquad \cong \mathbf{dcpo}\langle G \text{ (qua poset)} \mid R\rangle.\end{aligned}$$

To express the meet and join stability properties succinctly we use the idea of an L-set for any distributive lattice L. This is simply a set with two actions by L, for the monoids $(L, 0, \vee)$ and $(L, 1, \wedge)$.

Example 1 *The set* $\mathrm{idl}(L)$ *of all ideals (lower closed directed subsets) of* L *is* *an L-set with actions*

$$(l, I) \longmapsto \{l \wedge m \mid m \in I\}$$
$$(l, I) \longmapsto \downarrow \{l \vee m \mid m \in I\}$$

Definition 2 *1. A* DL-site *comprises a distributive lattice L, an L-set R and a pair of L-set homomorphisms $e_1, e_2 : R \rightrightarrows \mathrm{idl}(L)$.*

2. A dcpo presentation *comprises a poset P and a set R together with a pair of functions $e_1, e_2 : R \rightrightarrows \mathrm{idl}(P)$*

Given a DL-site (L, R, e_1, e_2) then we write

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle$$

as abbreviation for

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid \bigvee\nolimits^{\uparrow} e_1(r) = \bigvee\nolimits^{\uparrow} e_2(r)\ (r \in R)\rangle$$

and similarly for a dcpo presentation (P, R, e_1, e_2).

Example 3 *Any frame has a presentation by a DL-site. Given a frame ΩX, take $L^X = \Omega X$ and $R^X = \mathrm{idl}(\Omega X)$. Our two L^X-set morphisms from R^X to $\mathrm{idl}(L^X)$ are the identity and $\downarrow \circ \bigvee^{\uparrow}$. Such a presentation is referred to as the* standard presentation *for the frame. More generally,* any *frame presentation can be manipulated into a DL-site presenting the same frame (see [Vic04], though we shall not need the details of this in what follows).*

By definition every DL-site can also be used as a dcpo presentation; the double coverage result is that they present the same poset. To prove this it must first be checked that dcpo presentations present. That this is so seems to be folklore, though we have not found a good reference in the literature. It uses the fact that coequalizers of dcpos exist, and this has probably been known at least since [Mar77]. We give a proof that reapplies the techniques of [JV91].

Lemma 4 *If A is a dcpo, then the free suplattice over it is provided by the set of Scott closed subsets. The injection of generators is monic.*

Proof. By *Scott closed subset* of A we understand a subset that is lower closed and closed under directed joins (in A). (Constructively this is different from being the complement of a Scott open set.) Let us write $F(A)$ for the set of Scott closed subsets of A. Any intersection of Scott closed subsets is clearly Scott closed and so $F(A)$ is a complete lattice. Note that the joins are not unions, but the Scott closures of unions. (Constructively, not even finitary joins of Scott closed subsets are Scott closed.) $\downarrow: A \to F(A)$ preserves directed joins and is monic, and this map will prove to be the injection of generators – the unit of the free suplattice monad on **dcpo**. To see this first note that for any $B \in F(A)$, $B = \bigvee\{\downarrow b \mid b \in B\}$ since the join always contains the set-theoretic union. So, given any dcpo morphism $\phi : A \to M$ with M a suplattice, the assignment $q(B) = \bigvee_M \{\phi(b) \mid b \in B\}$ is therefore necessary if ϕ is to factor as $q \circ \downarrow$. But $r : M \to F(A)$ given by $r(m) = \{b \mid \phi(b) \leq m\}$ provides a right adjoint to q so we know that q is a suplattice homomorphism, and therefore $F(A)$ provides the correct universal properties. ■

Theorem 5 (dcpo presentations present) *For any dcpo presentation* $(P, R, ...)$, $\mathbf{dcpo}\langle P \text{ (qua poset)} \mid R\rangle$ *is well defined.*

Proof. First note that the problem reduces to a proof of the existence of **dcpo** coequalizers since the ideal completion of any poset is the free dcpo on that poset. The relevant dcpo coequalizer is of $e_1, e_2 : R \rightrightarrows \mathrm{idl}(P)$. [JV91] shows the existence of preframe coequalizers given the existence of frame coeqalizers, and the same technique shows the existence of dcpo coequalizers given the existence [JT84] of suplattice coequalizers. ■

Before proving the main double coverage theorem, we first prove a result that uses the techniques of the preframe coverage theorem [JV91].

Proposition 6 *Let* L *be a join semilattice and* R *a join-stable set of directed relations on it. Then*

$$\mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid R\rangle.$$

Proof. The standard technique applies. The RHS is known to exist by Theorem 5. Then from its universal property and join stability we can define $\vee$, show that it is a suplattice and prove the suplattice universal property required by the left-hand side. ■

Theorem 7 (Double Coverage Theorem) *If* $(L, R, ...)$ *is a DL-site, then*

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle \cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid R\rangle$$

Proof. We have

$$\begin{aligned} &\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle \\ &\qquad \cong \mathbf{Fr}\langle L \text{ (qua } \wedge\text{-SemiLat)} \mid \text{(qua } \vee\text{-SemiLat)}, R\rangle \\ &\qquad \cong \mathbf{Sup}\langle L \text{ (qua poset)} \mid \text{(qua } \vee\text{-SemiLat)}, R\rangle \\ &\qquad \cong \mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle. \end{aligned}$$

where the middle step is an application of the original coverage theorem CovThm. The relations "qua $\vee$-SemiLat" and R are meet stable, the former by the distributivity of L and the latter by definition of DL-site. Finally, apply Proposition 6 to get the result. ■

Remark 8 *Given a DL-site* $(L, R, ...)$ *presenting* X*, we already know from the suplattice and preframe coverage theorems ([AV93], [JV91]) that*

$$\begin{aligned} \mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle &\cong \mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \\ &\cong \mathbf{PreFr}\langle L \text{ (qua } \wedge\text{-SemiLat)} \mid R\rangle. \end{aligned}$$

Suppose then that the Double Coverage Theorem is used to define a dcpo morphism $q : \Omega X \to \Omega Y$ from a monotone function $q' : L \to \Omega Y$. It follows that q is a suplattice homomorphism iff q' preserves finite joins, and a preframe homomorphism iff q' preserves finite meets.

2.2 Semilattice tensor product

It is a standard part of locale theory that $\Omega(X \times Y)$, the coproduct of the frames for X and Y, can also be understood as a tensor product of those frames in two different ways: as suplattices [JT84] and as preframes [JV91]. It is immediate from universal algebra that tensor products of semilattices exist in a similar way [Fra76]. For instance, if A, B and C are join semilattices then we define a *join bimorphism* to be a function $\theta : A \times B \to C$ such that, if you fix one argument, it preserves finite joins in the other. The join semilattice tensor $A \otimes_{\vee\text{-SemiLat}} B$ is equipped with a universal join bimorphism from $A \times B$. Similarly, the meet semilattice tensor $A \otimes_{\wedge\text{-SemiLat}} B$ is equipped with a universal meet bimorphism.

If A and B are both distributive lattices, then the two tensor products are isomorphic to each other and provide a distributive lattice coproduct; in this case we normally write $(a, b) \mapsto a \times b$ and $(a, b) \mapsto a \odot b$ for the universal join and meet bimorphisms to match the notation in frames of product locales. These are related (just as in frames) by

$$
\begin{aligned}
a \odot b &= a \times 1 \vee 1 \times b \\
a \times b &= a \odot 0 \wedge 0 \odot b.
\end{aligned}
$$

We shall need a more explicit construction of the tensor products.

Proposition 9 *Let A and B be two join semilattices. Then their join semilattice tensor $(A \otimes_{\vee\text{-}SemiLat} B)$ is given by*

$$
\begin{aligned}
&\mathbf{Pos}\langle \mathcal{F}(A \times B) \text{ (qua poset)} \mid \\
&\quad \{(\textstyle\bigvee_{i \in I} a_i, \bigvee_{j \in J} b_j)\} \cup U = \{(a_i, b_j) \mid i \in I, j \in J\} \cup U \\
&\qquad\qquad (U \in \mathcal{F}(A \times B))\rangle.
\end{aligned}
$$

($\mathcal{F}$ here denotes the (Kuratowski) finite powerset; the relations are over all Kuratowski finite indexing sets I and J.)

Proof. Let C be the poset presented above, with universal monotone function $\gamma : \mathcal{F}(A \times B) \to C$ satisfying the relations. Because of the join stability of the relations, binary union on $\mathcal{F}(A \times B)$ defines a binary operation $\vee$ on C, $\gamma(U) \vee \gamma(V) = \gamma(U \cup V)$. This is binary join, and in fact C is a $\vee$-semilattice with γ a homomorphism. (The nullary join is $\gamma(\emptyset)$.)

Now suppose $\theta : A \times B \to D$ is a bimorphism for some join semilattice D. The mapping $U \longmapsto \bigvee_{(a,b) \in U} \theta(a, b)$ respects the relations that define C, since

$$
\theta(\textstyle\bigvee_{i \in I} a_i, \bigvee_{j \in J} b_j) \vee \bigvee_{(a,b) \in U} \theta(a, b) = \bigvee_{i \in I} \bigvee_{j \in J} \theta(a_i, b_j) \vee \bigvee_{(a,b) \in U} \theta(a, b)
$$

The monotone map defined by this mapping clearly commutes with the construction of join on C and so there is a (necessarily unique) join semilattice from C to D extending θ. ■

Note that, as expected, these relations tell us that if $a \leq a'$ and $b \leq b'$ then $(a, b) \leq (a', b')$. For

$$\begin{aligned}(a', b') &= (a \vee a', b \vee b') \\ &= (a, b) \vee (a, b') \vee (a', b) \vee (a', b').\end{aligned}$$

Remark 10 *An exactly dual construction shows how to exhibit* $A \otimes_{\wedge\text{-}SemiLat} B$.

2.3 Dcpo presentations for product locales

We can now describe locale product via a dcpo presentation.

Proposition 11 *Suppose* X *and* Y *are locales with DL sites* (L^X, R^X) *and* (L^Y, R^Y)*. Then*

$$\begin{aligned}\Omega(X \times Y) \cong \mathbf{dcpo}\langle L^X \otimes_{\vee\text{-}SemiLat} L^Y \ \textit{(qua poset)} \mid \\ \bigvee\nolimits^{\uparrow}_{t \in e_1(r)} (t \times b \vee u) = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} (t \times b \vee u) \\ \textit{(}r \in R^X, b \in L^Y, u \in L^X \otimes_{\vee\text{-}SemiLat} L^Y\textit{)} \\ \bigvee\nolimits^{\uparrow}_{t \in e_1(r)} (a \times t \vee u) = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} (a \times t \vee u) \\ \textit{(}r \in R^Y, a \in L^X, u \in L^X \otimes_{\vee\text{-}SemiLat} L^Y\textit{)} \ \rangle\end{aligned}$$

Proof. We have

$$\begin{aligned}\Omega(X \times Y) &\cong \mathbf{Fr}\langle L^X, L^Y \text{ (qua DLs)} \mid R^X, R^Y\rangle \\ &\cong \mathbf{Fr}\langle L^X, L^Y \text{ (qua } \wedge\text{-SemiLats)} \mid (L^X, L^Y \text{ qua } \vee\text{-SemiLats}), R^X, R^Y\rangle \\ &\cong \mathbf{Fr}\langle L^X \times L^Y \text{ (qua } \wedge\text{-SemiLat)} \mid \vee\text{-bilinearity}, R^X \otimes L^Y, L^X \otimes R^Y\rangle\end{aligned}$$

where $R^X \otimes L^Y$ denotes the set of relations

$$\bigvee\nolimits^{\uparrow}_{t \in e_1(r)} t \times b = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} t \times b$$

for $r \in R^X, b \in L^Y$ and similarly for $L^X \otimes R^Y$. For the moment, we are writing $t \times b$ in $L^X \times L^Y$ for the pair (t, b). Of course, this really becomes $t \times b$ when we map $L^X \times L^Y$ to $L^X \otimes L^Y$. We see that the relations obtained are all meet stable, and so

$$\begin{aligned}\Omega(X \times Y) &\cong \mathbf{Sup}\langle L^X \times L^Y \text{ (qua poset)} \mid \vee\text{-bilinearity}, R^X \otimes L^Y, L^X \otimes R^Y\rangle \\ &\cong \mathbf{Sup}\langle L^X \otimes_{\vee\text{-SemiLat}} L^Y \text{ (qua } \vee\text{-SemiLat)} \mid R^X \otimes L^Y, L^X \otimes R^Y\rangle\end{aligned}$$

Now we can make the relations join-stable by joining u for all $u \in L^X \otimes_{\vee\text{-SemiLat}} L^Y$, as in the statement, and we can apply Proposition 6. ■

Combining Propositions 11 and 9, we obtain —

Proposition 12 *Suppose X and Y are locales with DL sites (L^X, R^X) and (L^Y, R^Y). Then*

$$\begin{aligned}
\Omega(X \times Y) \cong \mathbf{dcpo}\langle & \mathcal{F}(L^X \times L^Y) \text{ (qua poset) } | \\
& \{(\textstyle\bigvee_{i\in I} a_i, \bigvee_{j\in J} b_j)\} \cup U = \{(a_i, b_j) \mid i \in I, j \in J\} \cup U \\
& \qquad (U \in \mathcal{F}(L^X \times L^Y)) \\
& \textstyle\bigvee^{\uparrow}_{t\in e_1(r)} (\{(t,b)\} \cup U) = \bigvee^{\uparrow}_{t\in e_2(r)} (\{(t,b)\} \cup U) \\
& \qquad (r \in R^X, b \in L^Y, U \in \mathcal{F}(L^X \times L^Y)) \\
& \textstyle\bigvee^{\uparrow}_{t\in e_1(r)} (\{(a,t)\} \cup U) = \bigvee^{\uparrow}_{t\in e_2(r)} (\{(a,t)\} \cup U) \\
& \qquad (r \in R^Y, a \in L^X, U \in \mathcal{F}(L^X \times L^Y)) \ \rangle.
\end{aligned}$$

If $\theta : \mathcal{F}(L^X \times L^Y) \to D$ satisfies these relations (with D a dcpo), then the Proposition allows us to define a dcpo morphism $q : \Omega(X \times Y) \to D$ such that

$$q(\bigvee_{i\in I} a_i \times b_i) = \theta(\{(a_i, b_i) \mid i \in I\})$$

for I finite, $a_i \in L^X$, $b_i \in L^Y$. (All other elements of $\Omega(X \times Y)$ can be got as directed joins of such elements $\bigvee_{i\in I} a_i \times b_i$.)

There is also a dual result that we shall need.

Proposition 13 *Suppose X and Y are locales with DL sites (L^X, R^X) and (L^Y, R^Y). Then*

$$\begin{aligned}
\Omega(X \times Y) \cong \mathbf{dcpo}\langle & \mathcal{F}(L^X \times L^Y) \text{ (qua poset) } | \\
& \{(\textstyle\bigwedge_{i\in I} a_i, \bigwedge_{j\in J} b_j)\} \cup U = \{(a_i, b_j) \mid i \in I, j \in J\} \cup U \\
& \qquad (U \in \mathcal{F}(L^X \times L^Y)) \\
& \textstyle\bigvee^{\uparrow}_{t\in e_1(r)} (\{(t,b)\} \cup U) = \bigvee^{\uparrow}_{t\in e_2(r)} (\{(t,b)\} \cup U) \\
& \qquad (r \in R^X, b \in L^Y, U \in \mathcal{F}(L^X \times L^Y)) \\
& \textstyle\bigvee^{\uparrow}_{t\in e_1(r)} (\{(a,t)\} \cup U) = \bigvee^{\uparrow}_{t\in e_2(r)} (\{(a,t)\} \cup U) \\
& \qquad (r \in R^Y, a \in L^X, U \in \mathcal{F}(L^X \times L^Y)) \ \rangle.
\end{aligned}$$

Proof. For this we use the formal dual of Proposition 9 together with a preframe version of Proposition 11. In this the relations become, for instance,

$$\bigvee^{\uparrow}_{t\in e_1(r)} (t \odot b \wedge u) = \bigvee^{\uparrow}_{t\in e_2(r)} (t \odot b \wedge u)$$

and the proof has to use the preframe coverage theorem [JV91]. ■

This result, given $\theta : \mathcal{F}(L^X \times L^Y) \to D$ satisfying the relations, allows us to define a dcpo morphism $q : \Omega(X \times Y) \to D$ such that

$$q(\bigwedge_{i\in I} a_i \odot b_i) = \theta(\{(a_i, b_i) \mid i \in I\}).$$

3 The ideal completion as a locale

Definition 14 *Let P be a poset. The locale* $\mathrm{Idl}(P)$ *is defined by*

$$\begin{aligned}\Omega\,\mathrm{Idl}(P) = \mathbf{Fr}\langle \uparrow p \ (p \in P) \mid & \\ & \uparrow p \leq \uparrow q \ (p \geq q) \\ & 1 \leq \textstyle\bigvee_{p \in P} \uparrow p \\ & \uparrow p \wedge \uparrow q \leq \bigvee\{\uparrow r \mid p \leq r, q \leq r\} \quad \rangle\end{aligned}$$

Its points are the ideals of P.

We write $\mathrm{idl}(P)$ for the discrete reflection of $\mathrm{Idl}(P)$, i.e. its set of global points (locale maps $1 \to \mathrm{Idl}(P)$). $\Omega\,\mathrm{Idl}(P)$ is equivalent to the set of Scott opens on $\mathrm{idl}(P)$. In fact, $\mathrm{Idl}(P)$ is constructively spatial and there is a bijection between locale maps $\mathrm{Idl}(P_1) \to \mathrm{Idl}(P_2)$ and dcpo maps $\mathrm{idl}(P_1) \to \mathrm{idl}(P_2)$. If $e : \mathrm{idl}(P_1) \to \mathrm{idl}(P_2)$ is a dcpo map then the corresponding locale map e' : $\mathrm{Idl}(P_1) \to \mathrm{Idl}(P_2)$ is defined by $\Omega e'(\uparrow p_2) = \bigvee\{\uparrow p_1 \mid p_2 \in e(\downarrow p_1)\}$. See [Vic93] or [Tow96, Ch. 1 Sect. 1.6] for further details. Note that if P is a discrete poset (i.e. a set) then $\mathrm{Idl}(P)$ is the discrete locale P.

Given a DL-site(L, R, e_1, e_2), the locale $\mathrm{Idl}(L)$ will play an important role in our development, providing the connection between the frame-theoretic discussions of presentations and a more purely localic one. Let us immediately note that because $\mathrm{idl}(L)$ is the discrete reflection of $\mathrm{Idl}(L)$,the functions $e_i : R \to \mathrm{idl}(L)$ are equivalent to maps $e'_i : R \to \mathrm{Idl}(L)$. (We abuse notation slightly: R denotes both a set and a discrete locale.)

The following result is a fragment of a more general (and well known) topos-theoretic conclusion which states that if C is a small category and E and F are two toposes, then there is a bijection between functors $C \to \mathbf{Top}(E, F)$ and geometric morphisms $[C, \mathbf{Set}] \times E \to F$.

Proposition 15 *Let $(P, \leq)$ be a poset and X, Y two locales. Then the following are equivalent.*

1. *Monotone functions $P \to \mathbf{Fr}(\Omega Y, \Omega X)$.*
2. *dcpo maps $\mathrm{idl}(P) \to \mathbf{Fr}(\Omega Y, \Omega X)$.*
3. *Locale maps $\mathrm{Idl}(P) \times X \to Y$.*

Furthermore, the bijection (2)$\Longleftrightarrow$(3) is natural with respect to dcpo maps $\mathrm{idl}(P_1) \to \mathrm{idl}(P_2)$.

Note that the naturality proved here is what is needed to deal with the functions $R \to \mathrm{idl}(L)$ (with the discrete order on R) that arise in DL-sites.

Proof. The equivalence between (1) and (2) is immediate since $\mathrm{idl}(P)$ is the free dcpo on P qua poset. ($\mathbf{Fr}(\Omega Y, \Omega X)$ is always a dcpo, the directed joins being calculated elementwise – e.g. [Joh82, Lemma 1.11 of Ch. 2].)

(2) $\Longleftrightarrow$(3). Given a dcpo morphism $f : \mathrm{idl}(P) \to \mathbf{Fr}(\Omega Y, \Omega X)$ define $F : \mathrm{Idl}(P) \times X \to Y$ by

$$\Omega F(b) = \bigvee_{p \in P} \uparrow p \times f(\downarrow p)(b).$$

Using the presentation of $\mathrm{Idl}(P)$ one can check directly that this is a frame homomorphism.

In the other direction (given $F : \mathrm{Idl}(P) \times X \to Y$), define f by $I \longmapsto \Omega(I \times X) \circ \Omega(F)$ where we are considering the ideal I as a point of $\mathrm{Idl}(P)$. To show this preserves directed joins, it suffices to show that $\Omega(I \times X) = \bigvee^{\uparrow}_{p \in I} \Omega(\downarrow p \times X)$. This follows because

$$\begin{aligned}\Omega(I \times X)(\uparrow q \times a) &= \bigvee \{a \mid q \in I\} \\ &= \bigvee\nolimits^{\uparrow}_{p \in I} \{a \mid q \leq p\} = \bigvee\nolimits^{\uparrow}_{p \in I} \Omega(\downarrow p \times X)(\uparrow q \times a).\end{aligned}$$

(Note that the join $\bigvee\{a \mid q \in I\}$ is of a subsingleton set, with at most one element a, and that only if $q \in I$.)

Note also that

$$\bigvee_{p \in P} (\uparrow p \times \Omega(\downarrow p \times X)(\uparrow q \times a)) = \bigvee_{q \leq p} (\uparrow p \times a) = \uparrow q \times a,$$

and so $\bigvee_{p \in P} \uparrow p \times \Omega(\downarrow p \times X)(u) = u$ for every $u \in \Omega(\mathrm{Idl}(P) \times X)$.

One can then show that the correspondence $f \leftrightarrow F$ is a bijection. Starting from F, we have

$$\bigvee_{p \in P} \uparrow p \times \Omega(\downarrow p \times X) \circ \Omega(F)(b) = \Omega(F)(b).$$

Starting from f,

$$\begin{aligned}\Omega(I \times X) \circ \Omega(F)(b) &= \Omega(I \times X)(\bigvee\nolimits_{p \in P} \uparrow p \times f(\downarrow p)(b)) \\ &= \bigvee\nolimits^{\uparrow}_{p \in I} f(\downarrow p)(b) = f(I)(b).\end{aligned}$$

For naturality, suppose we have a dcpo map $g : \mathrm{idl}(P_1) \to \mathrm{idl}(P_2)$. Then g corresponds to $g' : \mathrm{Idl}(P_1) \to \mathrm{Idl}(P_2)$ defined by

$$\Omega g'(\uparrow q) = \bigvee \{\uparrow p \mid q \in g(\downarrow p)\}.$$

Suppose we have $F_2 : \mathrm{Idl}(P_2) \times X \to Y$ corresponding to $f_2 : \mathrm{idl}(P) \to \mathbf{Fr}(\Omega Y, \Omega X)$. Then $F_2 \circ (g' \times X)$ corresponds to

$$f_1(I) = \Omega(I \times X) \circ \Omega(g' \times X) \circ \Omega F_2 = \Omega(g(I) \times X) \circ \Omega F_2 = f_2 \circ g(I).$$

■

Corollary 16 *Let P be a set, and X and Y two locales. Then there is a bijection between functions $f : P \to \mathbf{Fr}(\Omega Y, \Omega X)$ and locale maps $F : P \times X \to Y$.*

Proof. P is discrete and so $\mathrm{Idl}(P) \cong P$. ■

4 Dcpo morphisms as locale maps

The results of the previous Section are now considered in conjunction with the Double Coverage Theorem 7, and this enables a localic characterization of dcpo morphisms between frames to be given.

Recall that in a DL-site (L, R, e_1, e_2), the two functions $e_i : R \to \mathrm{idl}(L)$ correspond to two maps $e'_i : R \to \mathrm{Idl}(L)$.

Proposition 17 *Let (L, R, e_1, e_2) be a DL-site presenting a locale X, and let W be a locale. Then there is a bijection between dcpo morphisms $\Omega X \to \Omega W$ and maps $\mathrm{Idl}(L) \times W \to \mathbb{S}$ that compose equally with the two maps $e'_i \times W : R \times W \to \mathrm{Idl}(L) \times W$. Moreover this bijection is natural in W.*

Proof. Since the frame $\Omega\mathbb{S}$ is free on one generator, ΩW is isomorphic to $\mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$. By the Double Coverage Theorem (7), dcpo morphisms $\Omega X \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ are equivalent to monotone functions $L \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ respecting the relations, and so to dcpo morphisms $f : \mathrm{idl}(L) \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ composing equally with the e_is. On the other hand, by Proposition 15 the dcpo morphisms f are equivalent to maps $F : \mathrm{Idl}(L) \times W \to \mathbb{S}$. We must show that f composes equally with the e_is iff F composes equally with the maps $e'_i \times X$. This is a consequence of the naturality part of Proposition 15, with P_1 and P_2 specialized as R (with its discrete order) and L. ■

4.1 Weak exponentiation

To complete the proof of the main result we shall need to check that $\mathbb{S}^X$ exists weakly in **Loc**, and this is an interesting fact in itself. Recall (e.g. [CR00]) that the definition of weak exponentiation is the same as true exponentiation, but without the uniqueness requirement on the exponential transpose. In other words, a weak exponential for Y^X is a locale W equipped with a map $ev : W \times X \to Y$ such that for any map $z : Z \times X \to Y$ there exists a (not necessarily unique) map $\overline{z} : Z \to W$ such that $ev \circ (\overline{z} \times X) = z$.

Proposition 18 *For any locale X presented by DL-site $(L^X, R^X, ...)$, the ideal completion locale $\mathrm{Idl}(L^X)$ is a weak exponential $\mathbb{S}^X$.*

Proof. By Proposition 17 with $W = X$, the identity function on ΩX corresponds to a map $ev : \mathrm{Idl}(L^X) \times X \to \mathbb{S}$ that composes equally with the maps $e'_i \times X : R \times X \to \mathrm{Idl}(L^X) \times X$. This is the evaluation map. As an open in $\mathrm{Idl}(L^X) \times X$, it is $\bigvee_{l \in L^X} \uparrow l \times l$.

Given $c : Y \times X \to \mathbb{S}$ define $\overline{c} : Y \to \mathrm{Idl}(L^X)$ by

$$\Omega\overline{c}(\uparrow l) = \bigvee\{b \in \Omega Y \mid b \times l \leq c\}.$$

Then, as an open in $Y \times X$, $ev \circ (\overline{c} \times X)$ is

$$\begin{aligned} &\bigvee_{l \in L^X} (\bigvee\{b \in \Omega Y \mid b \times l \leq c\}) \times l \\ &\qquad = \bigvee\{b \times l \mid b \times l \leq c\} = c \end{aligned}$$

$\blacksquare$

The weak exponential $\mathbb{S}^X$ can also be found via the spectrum $Spec(L^X)$ of L^X, whose frame is $\mathrm{idl}(L^X)$. This locale is spectral and so is locally compact and hence exponentiable. But there is a locale inclusion $i : X \hookrightarrow Spec(L^X)$ and $\mathbb{S}$ is injective with respect to locale inclusions (in particular with respect to $Z \times i$ for any Z) and so $\mathbb{S}^{Spec(L^X)}$ is a weak exponential with weak evaluation map $ev \circ (Z \times i)$, where ev is the true evaluation map at $Spec(L^X)$. We thank Martín Escardó for this description of the weak exponential. It can be verified that $\mathbb{S}^{Spec(L^X)} \cong \mathrm{Idl}(L^X)$.

5 Dcpo morphisms as natural transformations

The results so far have, to use the language of the set-class distinction, concerned *sets*. (However, in Section 8.1 we shall qualify this in a topos-theoretic interpretation, and indeed we have taken care to use reasoning that is valid in the internal logic of toposes.) We now turn to issues that require more care regarding classes. **Loc** is a large category, and the presheaf category $[\mathbf{Loc}^{op}, \mathbf{Set}]$ cannot be assumed to have small hom-classes. Our main result in this section is to show that if X and W are locales, then the hom-class from $\mathbb{S}^X = \mathbf{Loc}(\_\times X, \mathbb{S})$ to $\mathbb{S}^W = \mathbf{Loc}(\_ \times W, \mathbb{S})$ is in fact (in bijection with) a set.

Theorem 19 *Let X be a locale. Then there are bijections Φ_W, natural in locales W, between –*

- *natural transformations $\mathbf{Loc}(\_\times X, \mathbb{S}) \to \mathbf{Loc}(\_\times W, \mathbb{S})$, and*
- *dcpo morphisms $\Omega X \to \Omega W$.*

Proof. Suppose X is presented by a DL-site (L^X, R^X, e_1, e_2), and let $ev : \mathrm{Idl}(L^X) \times X \to \mathbb{S}$ be the weak evaluation map.

Let $\alpha : \mathbf{Loc}(\_\times X, \mathbb{S}) \stackrel{\cdot}{\to} \mathbf{Loc}(\_\times W, \mathbb{S})$ be a natural transformation. Since ev composes equally with the maps $e'_i \times X$, it follows that $\alpha_{\mathrm{Idl}(L^X)}(ev) : \mathrm{Idl}(L^X) \times W \to \mathbb{S}$ composes equally with the maps $e'_i \times W$ and hence (by Proposition 17) corresponds to a dcpo morphism $\Omega X \to \Omega W$. Define $\Phi_W(\alpha)$ to be this dcpo morphism.

Now suppose we have $a : Y \times X \to \mathbb{S}$ with weak transpose $\overline{a} : Y \to \mathrm{Idl}(L^X)$. We have $a = ev \circ (\overline{a} \times X)$, so, by naturality of α,

$$\alpha_Y(a) = \alpha_{\mathrm{Idl}(L^X)}(ev) \circ (\overline{a} \times W)$$

Therefore, α is uniquely determined by $\Phi_W(\alpha)$.

In the other direction, say $q : \Omega X \to \Omega W$ is given as a dcpo morphism. To define $\alpha^q : \mathbf{Loc}(\_\times X, \mathbb{S}) \stackrel{\cdot}{\to} \mathbf{Loc}(\_\times W, \mathbb{S})$ we define, for every locale Y, a dcpo morphism $q_Y : \Omega(Y \times X) \to \Omega(Y \times W)$. Suppose that Y is presented by a DL-site (L^Y, R^Y, e_1, e_2). We shall use Proposition 12, defining

$$q_Y(\bigvee_{i \in I} b_i \times a_i) = \bigvee_{I' \in \mathcal{F} I} \bigwedge_{i \in I'} b_i \times q(\bigvee_{i \in I'} a_i).$$

(Here I is finite. For the more general infinite I we take the directed join over finite subsets of I. In this case, however, that gives the same formula.) It will be convenient to alter the notation slightly. If $U \in \mathcal{F}(L^Y \times L^X)$, let us write

$$\theta(U) = \bigvee_{U' \in \mathcal{F}U} \bigwedge_{(b,a)\in U'} b \times q(\bigvee_{(b,a)\in U'} a)$$

so that $q_Y(\bigvee_{i\in I} b_i \times a_i) = \theta(\{(b_i, a_i) \mid i \in I\})$.

The first relation we have to check is that

$$\theta(\{(\bigvee_{i\in I} b_i \times \bigvee_{j\in J} a_j)\} \cup U) = \theta(\{(b_i, a_j) \mid i \in I, j \in J\} \cup U).$$

The left hand side gives us

$$\begin{aligned}
&\bigvee_{U'\in\mathcal{F}U} (\textstyle\bigwedge_{(b,a)\in U'} b \times q(\bigvee_{(b,a)\in U'} a) \\
&\qquad\qquad \vee ((\textstyle\bigvee_{i\in I} b_i) \wedge \bigwedge_{(b,a)\in U'} b) \times q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)) \\
&= \bigvee_{U'\in\mathcal{F}U} (\textstyle\bigwedge_{(b,a)\in U'} b \times q(\bigvee_{(b,a)\in U'} a) \\
&\qquad\qquad \vee (\textstyle\bigvee_{i\in I} (b_i \wedge \bigwedge_{(b,a)\in U'} b)) \times q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)) \\
&= \theta(U) \vee \bigvee_{U'\in\mathcal{F}U} \bigvee_{i\in I} ((b_i \wedge \textstyle\bigwedge_{(b,a)\in U'} b) \times q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)),
\end{aligned}$$

while on the right we have

$$\bigvee_{U'\in\mathcal{F}U} \bigvee_{K\in\mathcal{F}(I\times J)} (\textstyle\bigwedge_{(i,j)\in K} b_i \wedge \bigwedge_{(b,a)\in U'} b) \times q(\bigvee_{(i,j)\in K} a_j \vee \bigvee_{(b,a)\in U'} a).$$

First we show that LHS $\leq$ RHS. By taking $K = \emptyset$, we get $\theta(U) \leq$ RHS. For a disjunct on the left with U' and i, we take the same U' on the right and $K = \{i\} \times J$.

Next we show RHS $\leq$ LHS. Consider a disjunct on the right with U' and K. For Kuratowski finite sets, emptiness is a decidable property (see e.g. [JL78]), so we can argue by cases for K empty or inhabited. If K is empty, then the disjunct $\leq \theta(U)$. On the other hand, suppose $(i', j') \in K$. Then

$$\text{disjunct } \leq (b_{i'} \wedge \textstyle\bigwedge_{(b,a)\in U'} b) \times q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a) \leq \text{ LHS}.$$

Next we must check the relations that arise from the relations R^X and R^Y. Suppose $r \in R^X$. Then

$$\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{t\in e_1(r)} \theta(\{(b', t)\} \cup U) \\
&= \textstyle\bigvee^{\uparrow}_{t\in e_1(r)} (\theta(U) \vee \displaystyle\bigvee_{U'\in\mathcal{F}U} ((b' \wedge \textstyle\bigwedge_{(b,a)\in U'} b) \times q(t \vee \bigvee_{(b,a)\in U'} a))) \\
&= \theta(U) \vee \bigvee_{U'\in\mathcal{F}U} ((b' \wedge \textstyle\bigwedge_{(b,a)\in U'} b) \times q(\bigvee^{\uparrow} e_1(r) \vee \bigvee_{(b,a)\in U'} a))
\end{aligned}$$

because everything in sight preserves directed joins. But by now it is clear that we get the same answer from $e_2(r)$. The argument is similar for relations in R^Y.

It can be shown that $\Phi_W(\alpha^q) = q$, by comparing their corresponding elements in $\mathbf{Loc}(\mathrm{Idl}(L^X) \times W, \mathbb{S})$. On the one hand q gives us the element $\bigvee_{l \in L} \uparrow l \times q(l)$. On the other, $\alpha^q_{\mathrm{Idl}(L^X)}$ gives us

$$
\begin{aligned}
&\alpha^q_{\mathrm{Idl}(L^X)}(\textstyle\bigvee_{l \in L^X} \uparrow l \times l) \\
&\qquad = \bigvee^{\uparrow}_{L' \in \mathcal{F}L^X} \textstyle\bigvee_{L'' \in \mathcal{F}L'} (\bigwedge_{l \in L''} \uparrow l \times q(\bigvee_{l \in L''} l)) \\
&\qquad = \bigvee^{\uparrow}_{L' \in \mathcal{F}L^X} \textstyle\bigvee_{L'' \in \mathcal{F}L'} (\uparrow (\bigvee L'') \times q(\bigvee L'')) \\
&\qquad = \bigvee_{l \in L} \uparrow l \times q(l).
\end{aligned}
$$

Naturality in W is clear from the statement of Proposition 17. ■

Remark 20 *There is also a dual proof that will be needed and sketched in Theorem 23.*

6 The main results

Given the characterization of dcpo morphisms in terms of natural transformations, the main result is immediate:

Theorem 21 *If X is a locale then the exponential $\mathbb{S}^{\mathbb{S}^X}$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ and is naturally isomorphic to the representable functor $\mathbf{Loc}(\_, \mathbb{P}X)$.*

Proof. $\Omega\mathbb{P}X$ is the free frame on ΩX qua dcpo. But the dcpo morphisms $\Omega X \to \Omega W$ have been characterized, naturally in W, as the natural transformations $\mathbb{S}^X \to \mathbb{S}^W$, i.e. exactly the natural transformations $\mathbf{Loc}(\_, W) \times \mathbb{S}^X \to \mathbf{Loc}(\_, \mathbb{S})$ (by the definition of the exponential $\mathbb{S}^W$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$). This set is exactly $\mathbb{S}^{\mathbb{S}^X}(W)$ and therefore it has been shown that $\mathbb{S}^{\mathbb{S}^X}(W) \cong \mathbf{Loc}(W, \mathbb{P}X)$ naturally in W. ■

6.1 The upper and lower powerlocales

The main result specializes to the upper and lower powerlocale constructions (P_U and P_L).

First note [Vic04] that any $\mathbb{P}X$ is an internal distributive lattice in $\mathbf{Loc}$, and hence (because the Yoneda embedding preserves finite limits) in $[\mathbf{Loc}^{op}, \mathbf{Set}]$. In particular this includes $\mathbb{S}$, which is $\mathbb{P}\emptyset$. It follows (because $Y \mapsto Y^X$ preserves all limits, being right adjoint to $Z \mapsto Z \times X$) that $\mathbb{S}^X$ is an internal distributive lattice in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ for any locale X. The lattice structure on each component $\mathbf{Loc}(Y \times X, \mathbb{S})$ is inherited straightforwardly from the localic lattice structure of $\mathbb{S}$. Note also that if M and N are two internal lattices (or indeed internal algebras of any kind) in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, then a morphism $\alpha : M \to N$ is a homomorphism iff every component $\alpha_X : M(X) \to N(X)$ is a homomorphism.

We shall also need the fact that if L is a distributive lattice, then $\mathrm{Idl}(L)$ is an internal distributive lattice in **Loc**. This follows because Idl provides a functor from **Pos** to **Loc** that preserves products, $\mathrm{Idl}(P \times Q)$ being homeomorphic to $\mathrm{Idl}(P) \times \mathrm{Idl}(Q)$ by $\uparrow (p,q) \longleftrightarrow \uparrow p \times \uparrow q$. On monotone functions $f : P \to Q$, the functor Idl acts by $\Omega\,\mathrm{Idl}(f)(\uparrow q) = \bigvee\{\uparrow p \mid q \leq f(p)\}$, and this enables us to calculate the inverse image functions for meet and join on $\mathrm{Idl}(L)$:

$$\Omega(\wedge)(\uparrow l) = \bigvee\{\uparrow m \times \uparrow n) \mid l \leq m \wedge n\} = \uparrow l \times \uparrow l$$
$$\Omega(\vee)(\uparrow l) = \bigvee\{\uparrow m \times \uparrow n \mid l \leq m \vee n\}.$$

Since $\mathbb{S}$ is $\mathrm{Idl}(\{\bot, \top\}, \bot \leq \top)$, we can use this to calculate inverse image functions for meet and join on $\mathbb{S}$. Expressing them as opens of $\mathbb{S} \times \mathbb{S}$, meet is $\uparrow\top \times \uparrow\top$ and join is $\uparrow \top \times 1 \vee 1 \times \uparrow \top = \uparrow \top \odot \uparrow \top$ where $(a,b) \mapsto a \odot b$ is the universal preframe bimorphism ([JV91]; there $\odot$ is written as an upside down &). Note that $\uparrow\top$ is the free generator of $\Omega\mathbb{S}$.

Lemma 22 *Let L be a distributive lattice and W a locale. By Proposition 15 there is a bijection between monotone functions $f : L \to \Omega W \cong \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ and maps $F : \mathrm{Idl}(L) \times W \to \mathbb{S}$. Then f preserves finite meets (respectively joins) iff F preserves finite meets (respectively joins) on* $\mathrm{Idl}(L)$*.*

Proof. As explained in Proposition 17, the bijection is a consequence of Proposition 15. F, considered as an open of $\mathrm{Idl}(L) \times W$, is $\bigvee_{l \in L} \uparrow l \times f(l)$.

Preservation by F of n-ary meets or joins on $\mathrm{Idl}(L)$ means equality of two maps $\mathrm{Idl}(L^n) \times W \to \mathbb{S}$. We shall present the argument for binary meets and joins. For binary meets, the first map is

$$F \circ (\mathrm{Idl}(\wedge_L) \times W) : \mathrm{Idl}(L^2) \times W \to \mathrm{Idl}(L) \times W \to \mathbb{S}.$$

The second,

$$\begin{aligned}&\wedge_{\mathbb{S}} \circ F^2 \circ \langle \pi_1 \times W, \pi_2 \times W\rangle \circ \cong : \\ &\qquad \mathrm{Idl}(L^2) \times W \cong \mathrm{Idl}(L)^2 \times W \to (\mathrm{Idl}(L) \times W)^2 \to \mathbb{S}^2 \to \mathbb{S},\end{aligned}$$

can be expressed using the lattice operations in $\mathbf{Loc}(\mathrm{Idl}(L^2) \times W, \mathbb{S})$ as

$$(F \circ (\mathrm{Idl}(\pi_1) \times W)) \wedge (F \circ (\mathrm{Idl}(\pi_2) \times W)).$$

(We are writing π_i for the product projections.) Writing $\uparrow\top$ for the generator of $\Omega\mathbb{S}$, we find the inverse image for $F \circ (\mathrm{Idl}(\wedge) \times W)$ takes

$$\begin{aligned}\uparrow\top &\mapsto \bigvee\nolimits_l \uparrow l \times f(l) \\ &\mapsto \bigvee\nolimits_l \bigvee\{\uparrow (m,n) \mid l \leq m \wedge n\} \times f(l) \\ &= \bigvee\nolimits_{mn} \uparrow (m,n) \times f(m \wedge n)\end{aligned}$$

which corresponds to the function $(m, n) \mapsto f(m \wedge n)$.

The inverse image for $\bigwedge_{i=1}^{2}(F \circ (\mathrm{Idl}(\pi_i) \times W))$ takes

$$\begin{aligned}
\uparrow\top \mapsto& \uparrow\top \times \uparrow\top \\
&\mapsto \bigvee\nolimits_{mn} \uparrow m \times f(m) \times \uparrow n \times f(n) \\
&\mapsto \bigvee\nolimits_{mn} \uparrow m \times \uparrow n \times f(m) \wedge f(n) \\
&\mapsto \bigvee\nolimits_{mn} \uparrow (m, n) \times f(m) \wedge f(n)
\end{aligned}$$

which corresponds to the function $(m, n) \mapsto f(m) \wedge f(n)$ from L^2 to ΩW.

It follows that F preserves binary meets in $\mathrm{Idl}(L)$ iff f preserves binary meets.

For joins, we find the first map takes

$$\begin{aligned}
&\uparrow\top \mapsto \bigvee\nolimits_{l} \uparrow l \times f(l) \\
&\mapsto \bigvee\nolimits_{l} \bigvee\{\uparrow (m, n) \mid l \leq m \vee n\} \times f(l) \\
&= \bigvee\nolimits_{mn} \uparrow (m, n) \times f(m \vee n)
\end{aligned}$$

which corresponds to the function $(m, n) \mapsto f(m \vee n)$. The second map takes

$$\begin{aligned}
\uparrow\top \mapsto& \uparrow\top \times 1 \vee 1 \times \uparrow\top \\
&\mapsto \bigvee\nolimits_{l} (\uparrow l \times f(l) \times \uparrow 0 \times 1) \vee \bigvee\nolimits_{l} (\uparrow 0 \times 1 \times \uparrow l \times f(l)) \\
&\mapsto \bigvee\nolimits_{l} ((\uparrow l \times \uparrow 0 \times f(l)) \vee (\uparrow 0 \times \uparrow l \times f(l))) \\
&\mapsto \bigvee\nolimits_{l} (\uparrow (l, 0) \vee \uparrow (0, l)) \times f(l) \\
&= \bigvee\nolimits_{mn} \uparrow (m, n) \times f(m) \vee f(n)
\end{aligned}$$

which corresponds to the function $(m, n) \mapsto f(m) \vee f(n)$. ■

Theorem 23 *Let X be a locale.*

1. *There is a bijection, natural in W, between locale maps $W \to P_L(X)$ and join semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*

2. *There is a bijection, natural in W, between locale maps $W \to P_U(X)$ and meet semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*

3. *There is a bijection, natural in W, between locale maps $W \to X$ and lattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*

Proof. A map $W \to \mathbb{P}X$ factors via $P_L X$, $P_U(X)$ or X iff its dcpo morphism q between the frames preserves finite joins, finite meets or both. Suppose, in the context of Theorem 19, that a dcpo morphism $q : \Omega X \to \Omega W$ corresponds to a natural transformation $\alpha : \mathbb{S}^X \to \mathbb{S}^W$ What we have to show is that q preserves finite joins or meets iff α is a join or meet semilattice homomorphism. This will prove (1) and (2), and then (3) follows immediately.

(1): First, suppose q preserves finite joins (so it is a suplattice homomorphism). Then the dcpo morphism of Theorem 19, $q_Y : \Omega(Y \times X)$ to $\Omega(Y \times W)$, assigns

$$\begin{aligned}\bigvee_{i\in I} b_i \times a_i \longmapsto & \bigvee_{I' \in \mathcal{F}I} \bigwedge\nolimits_{i\in I'} b_i \times q(\bigvee\nolimits_{i\in I'} a_i) \\ &= \bigvee_{I' \in \mathcal{F}I} \bigwedge\nolimits_{i\in I'} b_i \times \bigvee\nolimits_{i\in I'} q(a_i) \\ &= \bigvee_{i\in I} b_i \times q(a_i)\end{aligned}$$

and hence preserves finite joins. It follows that $\alpha_Y : \mathbb{S}^X(Y) \to \mathbb{S}^W(Y)$ preserves finite joins.

Now suppose that we are given a join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^W$. We suppose as usual that X is presented by a DL-site $(L, R, ...)$. By Remark 8 it suffices to show that the composite function $L \to \Omega X \to \Omega W$ preserves finite joins, and then by Lemma 22 it suffices to show that $\alpha_{\mathrm{Idl}(L)}(ev)$ preserves finite joins in $\mathrm{Idl}(L)$. Lemma 22 already tells us that ev does this, because it corresponds to the identity morphism on ΩX. For n-ary joins, we have

$$\begin{aligned}\alpha_{\mathrm{Idl}(L)}(ev) \circ (\mathrm{Idl}(\vee) \times W) &= \alpha_{\mathrm{Idl}(L^n)}(ev \circ (\mathrm{Idl}(\vee) \times X)) \\ &= \alpha_{\mathrm{Idl}(L^n)}(\bigvee\nolimits_{i=1}^n (ev \circ (\mathrm{Idl}(\pi_i) \times X))) \\ &= \bigvee\nolimits_{i=1}^n \alpha_{\mathrm{Idl}(L^n)}(ev \circ (\mathrm{Idl}(\pi_i) \times X)) \\ &= \bigvee\nolimits_{i=1}^n \alpha_{\mathrm{Idl}(L)}(ev) \circ (\mathrm{Idl}(\pi_i) \times W)\end{aligned}$$

as required.

(2) Half of the argument is dual to that for (1): if α preserves finite meets then so does q.

The other direction requires somewhat more care. The problem is that in the presentation of Proposition 11 the relations are not meet stable, so it does not trivially give a preframe presentation. Instead we use Proposition 13. Theorem 19 has a dual proof in which, given q, $q'_Y : \Omega(Y \times X) \to \Omega(Y \times W)$ is defined by

$$q'_Y(\bigwedge\nolimits_{(b,a)\in U} b \odot a) = \bigwedge_{U' \in \mathcal{F}U} (\bigvee\nolimits_{(b,a)\in U'} b) \odot q(\bigwedge\nolimits_{(b,a)\in U'} a).$$

It is not evident that this gives the same dcpo morphisms as the previous version q_Y, though we conjecture that it does. However, it gives the same natural transformation $\mathbf{Loc}(\_ \times X, \mathbb{S}) \to \mathbf{Loc}(\_ \times W, \mathbb{S})$ because the dual proof shows that it too gives back the original q. Now, dually to part (1), we see that if q preserves finite meets then so does each q'_Y, and hence so does the natural transformation.

(3) Follows by combining the first two parts. ■

7 Applications

7.1 The Strength of the Double Power Monad

As an application, the monad structure on $\mathbb{P}$ can be found fairly easily using this representation as $\mathbb{S}^{\mathbb{S}^X}$ (see e.g. [Tay02]). In particular, the strength $\chi : \mathbb{P}X \times Y \to \mathbb{P}(X \times Y)$ becomes $\chi : \mathbb{S}^{\mathbb{S}^X} \times Y \to \mathbb{S}^{\mathbb{S}^{X\times Y}}$ and can be defined by a λ-term in the style developed in [Esc03]:

$$\chi(\Phi, y) = \lambda U.\ \Phi(\lambda x.\ U(x, y)).$$

Defining the strength direct from the definition of $\mathbb{P}$ is a little intricate, and in fact seems to embody some of the argument of Theorem 19.

7.2 The localic reflection of $\mathbb{S}^X$

As a further application of the methods given here, we show that even though $\mathbb{S}^X$ is not always a locale (because X is not always exponentiable), it nonetheless has a localic reflection.

Proposition 24 *If ΩX and ΩY are two frames, then $\mathbf{dcpo}(\Omega Y, \Omega X)$ is a frame.*

Proof. Conceptually this is because $\mathbb{P}Y$ is a localic distributive lattice, so $\mathbf{Loc}(X, \mathbb{P}Y)$ is a distributive lattice as well as (by the dcpo-enrichment of $\mathbf{Loc}$) a dcpo. Reasoning internally it is easy enough to check that the finite meets and joins (calculated pointwise) of dcpo morphisms between frames are still dcpo morphisms. ■

Proposition 25 *Let X be a locale. Then the presheaf $\mathbb{S}^X = \mathbf{Loc}(\_ \times X, \mathbb{S})$ has a localic reflection Y. It is defined by*

$$\Omega Y = \mathbf{dcpo}(\Omega X, \Omega),$$

in other words the topology on Y is the Scott topology on the frame ΩX.

Proof. From the proof of Theorem 19 we see that if $q : \Omega X \to \Omega$ is a dcpo morphism, then we get a dcpo morphism $q_W : \Omega(W \times X) \to \Omega W$. We therefore get a function $\mathbf{dcpo}(\Omega X, \Omega) = \Omega Y \to \mathbf{dcpo}(\Omega(W \times X), \Omega W)$, and in fact this is a frame homomorphism. It follows that for every W we have a dcpo morphism $\Omega(W \times X) \to \mathbf{Fr}(\Omega Y, \Omega W)$, natural in W, and hence a natural transformation $\gamma : \mathbf{Loc}(\_ \times X, \mathbb{S}) \to \mathbf{Loc}(\_, Y)$, i.e. from $\mathbb{S}^X$ to Y.

Now suppose we have a natural transformation $\beta : \mathbb{S}^X \to Z$ for some locale Z. Again applying the argument of Theorem 19, using $\beta_{\mathrm{Idl}(L^X)}$, we get a map $\mathrm{Idl}(L^X) \to Z$ composing equally with the two maps from R^X. This gives us a dcpo morphism $\Omega X \to \mathbf{Fr}(\Omega Z, \Omega)$ and hence by Proposition 15 a frame homomorphism $\Omega Z \to \mathbf{dcpo}(\Omega X, \Omega) = \Omega Y$, so a locale map $\overline{\beta} : Y \to Z$. We find $\beta = \gamma; \overline{\beta}$, and in fact $\overline{\beta}$ is the unique such locale map. ■

8 Conclusions

We have shown how **Loc** can be embedded in a category ([$\mathbf{Loc}^{op}$, **Set**]) in which $\mathbb{P}X \cong \mathbb{S}^{\mathbb{S}^X}$. This characterizes $\mathbb{P}X$ (and the other powerlocales too) in a way that depends purely on the categorical structure of **Loc**, not on the concrete structure of frames. At the same time we have also displayed techniques for calculating with $\mathbb{P}X$ that depend on presentation rather than on having the entire frame. It is our hope that this will prove useful in developing locale theory in contexts (such as formal topology within the doctrine of predicative type theory) where frames cannot be constructed as sets.

We hope also that the work will provide insight into the problem of axiomatizing a synthetic locale theory (see e.g. [Vic04]). For instance, an abstract category of spaces could be defined as an order enriched category **C** with an internal distributive lattice $\mathbb{S}$ such that $\mathbb{S}^{\mathbb{S}^X}$ exists for any space X. Using the techniques of Theorem 23 the familiar theory of the upper and lower power spaces re-emerges from a single assumption about the existence of a double power space. This is a subject for further work.

8.1 Remark on set-theoretic foundations

We have concealed some topos-theoretic aspects in the exposition, though they have influenced the mathematics in a number of places. In the initial sections (to Section 4), we have reasoned using topos-valid mathematics so that "set" can mean "object in a given topos". From Section 5 there arises the deeper question of external vs. internal sets and this is best understood by reference to Theorem 19. The theorem is stated as though there is simply a (not necessarily classical) category of sets in which we can discuss frames and hence also locales. The proof, however, is designed to yield a more subtle result about locales over toposes. Suppose S is an elementary topos (we believe our proofs do not require a natural number object) and $f : X \to S$ and $g : W \to S$ are two localic geometric morphisms, in other words locales over S. By the known correspondence [JT84] between locales and frames, we have two frames $\Omega^S(X_f)$ and $\Omega^S(W_g)$, internal in S. (The notation X_f denotes X, considered as a locale over S by the morphism f.) They can be calculated as $f_*(\Omega_X)$ and $g_*(\Omega_W)$. The known correspondence shows that locale maps $W_g \to X_f$, i.e. geometric morphisms $W \to X$ making the triangle to S commute, correspond to morphisms $\Omega^S(X_f) \to \Omega^S(W_g)$ that are, internally, frame homomorphisms. What we show is that internal dcpo morphisms $\Omega^S(X_f) \to \Omega^S(W_g)$ are in bijection with natural transformations

$$\mathbf{Loc}/S(\_ \times_S X_f, \mathbb{S}_S) \to \mathbf{Loc}/S(\_ \times_S W_g, \mathbb{S}_S)$$

where $\mathbb{S}_S$ denotes the Sierpiński locale over S. Thus we have a correspondence not only between Scott continuity and naturality, but also between internal and external.

This has some effects on the shape of the proofs. Where the exposition refers to $\mathbf{Loc}(X, \mathbb{S})$ one might imagine this to be identical (or at least isomorphic) to the frame ΩX. However, in a more sophisticated interpretation, $\mathbf{Loc}(X, \mathbb{S})$, the set of locale maps from X to $\mathbb{S}$, is actually the set of global elements of ΩX. For any morphism $\Omega X \to \Omega W$ (i.e. $\Omega^S(X_f) \to \Omega^S(W_g)$) we can find a corresponding function $\mathbf{Loc}(X, \mathbb{S}) \to \mathbf{Loc}(W, \mathbb{S})$ (i.e. $\mathbf{Loc}/S(X_f, \mathbb{S}_S) \to \mathbf{Loc}/S(W_g, \mathbb{S}_S)$) by restricting to global elements, but we cannot necessarily go in the reverse direction. In a couple of places (one in the proof of Theorem 19 and more substantial ones in Theorem 23), a more direct proof can by found by using the component $\alpha_1 : \mathbf{Loc}(X, \mathbb{S}) \to \mathbf{Loc}(W, \mathbb{S})$ of a natural transformation as giving directly the morphism $q : \Omega X \to \Omega W$. In our broader context this is invalid and instead we carry out more explicit calculations using $\alpha_{\mathrm{Idl}(L^X)}$.

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