Fundamental Theorem of Calculus, point-free – applications to exp and log.

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Details in [Vic23].

Point-free analysis

Work within constraints of geometricity (colimits, finite limits) – see Vickers "Generalized point-free spaces, pointwise" [Vic22].

Strengths

- Point-free maps can be defined pointwise.
- All maps automatically continuous.
- Topos-valid.
- Deals with generalized spaces (toposes as well as locales).
- Fibrewise topology of bundles.

Weaknesses

- Lack of Π-types: non-trivial to extract geometric content from established arguments, even constructive ones.

Point-free analysis

Opportunities

- Much simpler manipulation of bundles: work fibrewise.
- Dependent type theory of spaces: dependent type = bundle.

Threats

- Don't know how deep it goes, eg into original applications of toposes such as algebraic geometry. Finding out takes effort.

Typical features

- Careful to distinguish 1-sided reals (lower or upper) vs. 2-sided (Dedekind).
- Prominent use of hyperspaces (powerlocales).
- Currying/uncurrying without cartesian closure.

Point-free analysis: Currying

To define $f\colon X \times Y \to Z$:

1. Say "fix x:X".
2. Define $f(x,-)\colon Y \to Z$ (necessarily continuous).
3. Uncurry to get f.

With cartesian closure:
Step (2) defines curry $\mathrm{cur}f\colon X \to (Y \to Z)$.

Geometrically:
Step (1) declares we're working in sheaves over X
— so reasoning must be at least topos-valid!
Then Y and Z interpreted as bundles $X \times Y$, $X \times Z$ over X.
Step (2) defines bundle map $X \times Y \to X \times Z$. First component must be projection to X, second component is $f\colon X \times Y \to Z$.

Point-free analysis: Real exponentials and logarithms

Ming Ng's thesis ([Ng22]; see also [NV22]) geometrically develops $\gamma^x, \log_\gamma x$, and usual algebraic rules.

Proof:

- For rational x, γ^x defined using powers γ^n, reciprocals $\gamma^{-n} = 1/\gamma^n$, and radicals $\gamma^{1/n} = \sqrt[n]{\gamma}$.
- For x a 1-sided real use sups or infs.
- Combine these for 2-sided reals.
- $\log_\gamma$ is inverse of $x \mapsto \gamma^x$.

We now deal with differentiation, integration, e, and natural logs.

A possible standard path

1. Define natural log $\ln x = \int_1^x dt/t$.
2. Show $\ln(\gamma\gamma') = \ln\gamma + \ln\gamma'$, deduce (with a little work) $\ln(\gamma^x) = x\ln\gamma$, so $\ln x = \ln\gamma^{\log_\gamma x} = \log_\gamma x \ln\gamma$.
3. Hence $\log_\gamma x$ is an integral, $\int_1^x 1/(t\ln\gamma)dt$.
4. By the Fundamental Theorem of Calculus (FTC), $\log_\gamma$ is differentiable, with derivative $x \mapsto 1/(x\ln\gamma)$.
5. Writing $\exp_\gamma x = \gamma^x$, the inverse of $\log_\gamma$, use the chain rule to show $x \mapsto \gamma^x$ is differentiable, with derivative $x \mapsto 1/\left(1/(\gamma^x \ln\gamma)\right) = \ln\gamma\gamma^x$.

Along the way, can define e as unique value such that $\ln e = 1$, so $\ln x = \log_e x$ and $x \mapsto e^x$ is its own derivative.

Geometricity: Differentiation

$f(x)$ differentiable if ...?

Most common definition

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \text{ exists.}$$

Then $f'(x)$ is that limit.

Geometrically: use Carathéodory derivative

There is a *slope* map $f^{\langle 1 \rangle}(x, y)$ (necessarily continuous and unique) such that

$$f(y) - f(x) = f^{\langle 1 \rangle}(x, y)(y - x).$$

Then $f'(x) = f^{\langle 1 \rangle}(x, x)$.

Essentially equivalent, but [Vic09] Carathéodory good geometrically, where all maps are continuous and limits harder to discuss.

Geometricity: Integration

1-sided integrals (lower, upper) established geometrically [Vic08].

- Lower integrals $\underline{\int}_X f d\mu$ have f valued in non-negative lower reals, μ a valuation on X – like a measure, but only on opens, with μU a non-negative lower real.
- Upper integrals $\overline{\int}_X f d\nu$, using upper reals instead of lower, and ν a covaluation on X. Think of νU as the measure of $X - U$.
- Riemann integrals $\int_x^y f(t)dt$ constructed as $\int_{[x,y]} f d\lambda_{xy}$: λ_{xy} is Lebegue valuation (with complement covaluation $(\neg\lambda_{xy})U = (y - x) - \lambda_{xy}(U)$). Combines lower and upper integrals of the lower and upper parts of f.

Geometricity: FTC(1)

Suppose $f(x) = \int_{x_0}^{x} g(t)dt$. Then f is differentiable with $f'(x) = g(x)$

Proof

For $x \neq y$, slope map is

$$f^{\langle 1 \rangle}(x, y) = \frac{\int_x^y g(t)dt}{y - x} = \int_{[x,y]} gd\left(\frac{\lambda_{xy}}{y - x}\right) = \int_{[x,y]} gd\upsilon_{xy},$$

where υ_{xy} is the uniform probability valuation on $[x, y]$.
But υ_{xy} is defined even if $x = y$, and then get $f^{\langle 1 \rangle}(x, x) = g(x)$.

Suffices to define υ_{xy} on rational open intervals (a, b) –

$q < \upsilon_{xy}(a, b)$ if either $q(y - x) < \lambda_{xy}(a, b)$,
or $a < x$, $y < b$ and $q < 1$.

Geometricity: FTC(2)

If f is differentiable, then

$$f(y) - f(x) = \int_x^y f'(t)dt.$$

Proof

Fixing x_0, define $g_{x_0}(x) = \int_{x_0}^x f'(t)dt$.
By FTC(1), g_{x_0} is differentiable, and $g'_{x_0} = f'$, so $(f - g_{x_0})' = 0$.
Using Rolle's Theorem, $f - g_{x_0}$ is constant $f(x_0) - g_{x_0}(x_0) = f(x_0)$, and the result follows.

Geometricity

- Rolle's Theorem: already known geometrically – see [Vic09].
- Riemann integral $\int_x^y g(t)dt$ for $y < x$, and equation

$$\int_x^z g(t)dt = \int_x^y g(t)dt = \int_y^z g(t)dt$$

  Not that hard, and proved in my new notes [Vic23].
- 2-sided integrals $\int_X f d\mu$: non-trivial [Vic23].

2-sided integrals

For 2-sided integrals with respect to uniform probability valuations:

Theorem

Let X be a compact, overt space, let $f: X \to [0,\infty)$, and let μ be a valuation on X that is finite (ie μX is Dedekind).
Then the pair

$$\int_X f d\mu = \left(\underline{\int}_X L(f) d\mu, \overline{\int}_X R(f) d(\neg\mu) \right)$$

is a Dedekind section. Here L and R extract the lower and upper parts.

Can then generalize to signed $f: X \to \mathbb{R}$:

$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu$, where
$f_+ = \max(f, 0), f_- = \max(-f, 0)$.

2-sided integrals: Proof

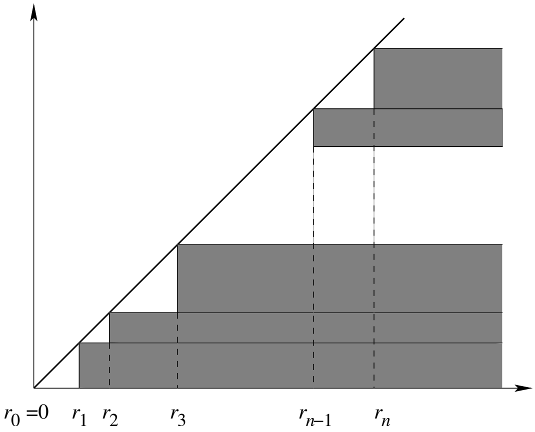
X compact, overt

These imply that X, as subspace of itself, corresponds to a point in its Vietoris hyperspace VX.
Image under f is $(Vf)(X)$ in $V\mathbb{R}$, a compact, overt subspace of $\mathbb{R}$.
It has a sup K, say, in $\mathbb{R}$.
See [Vic09] for details.

2-sided integrals: Lower integral $\underline{\int}_X L(f)d\mu$

Supremum over rational sequences $0 = r_0 < \cdots < r_n$ $(n \geq 1)$ of

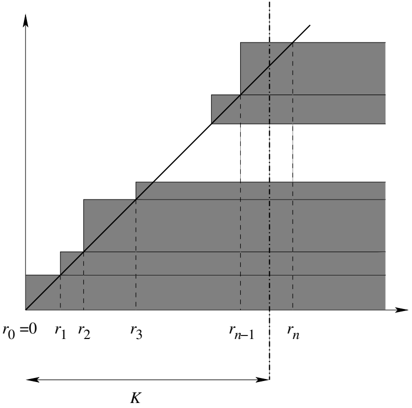
$$\underline{I}(r_i)_i = \sum_{i=1}^{n} (r_i - r_{i-1}) \mu f^*(r_i, \infty)$$



2-sided integrals: Upper integral $\overline{\int}_X R(f)d(\neg\mu)$

Infimum over rational sequences $0 = r_0 < \cdots < r_n$ ($n \geq 1$, $X \leq f^*[0, r_n)$) of

$$\overline{I}(r_i)_i = \sum_{i=1}^{n} (r_i - r_{i-1})\,(\mu X - \mu f^*[0, r_{i-1}))$$



2-sided integrals: Locatedness

For rationals $q < r$, want either $q < \underline{I}(r_i)_i$ for some $(r_i)_i$, or $\overline{I}(r_i)_i$ for some $(r_i)_i$.

Strategy

- Seek a single sequence $(r_i)_i$ for which $\underline{I}(r_i)_i$ and $\overline{I}(r_i)_i$ are sufficiently close together.
- Choose $r_n > K$.
- Error of $\overline{I}(r_i)_i$ is bounded by sizes of squares along diagonal (diagram previous slide).
- Aim for $r_i = ir_n/n$, n large to make squares small.
- $$\overline{I}(r_i)_i = r_n \mu X - r_1 \sum_{i=1}^{n} \mu f^*[0, r_{i-1})$$
- Let $a = r_1 \sum_{i=1}^{n} \mu f^*[0, r_{i-1})$, and seek r_1 so that $\underline{I}(r_i)_i + a$ is close to $r_n \mu X$.

2-sided integrals: Locatedness

$$\begin{aligned}
\underline{I}(r_i)_i + a &= r_1 \left(\sum_{i=1}^{n-1} \mu f^*(r_i, \infty) + \sum_{i=1}^{n} \mu f^*[0, r_{i-1}) \right) \\
&\geq r_1 \sum_{i=2}^{n-1} (\mu f^*(r_{i-1}, \infty) + \mu f^*[0, r_i)) \\
&= r_1 \sum_{i=2}^{n-1} (\mu X + \mu f^*(r_{i-1}, r_i)) \\
&\geq (r_n - 2r_1)\mu X
\end{aligned}$$

We can choose n to make $2r_1\mu X$ as small as we like.

2-sided integrals: Locatedness

How small is small enough?

- Can find $q' < q'' < r_n\mu X < r'$ with $r' - q' \leq r - q$.
- Choose n so that $2r_1\mu X < q'' - q'$.
- Then $q' < \underline{I}(r_i)_i + a$, so $q' = s + t$ with $s < \underline{I}(r_i)_i$, $t < a$.
- If $q \leq s$ then $q < \underline{I}(r_i)_i$.
- Otherwise, $s < q$, have

$$r_n\mu X - r < r' - r \leq q' - q = s + t - q < t < a,$$

so $\overline{I}(r_i)_i = r_n\mu X - a < r$.

2-sided integrals: Disjointness

Show we cannot have $\overline{I}(r_i)_i < q < \underline{I}(r'_{i'})_{i'}$.

- By refining sequences, can assume use same one for $\underline{I}$ and $\overline{I}$.
- Get $q = \sum_{i=1}^{n} q'_i = \sum_{i=1}^{n} q''_i$ with

$$q'_i < (r_i - r_{i-1})\mu f^*(r_i, \infty), \quad (r_i - r_{i-1})(\mu X - \mu f^*[0, r_{i-1})) < q''_i.$$

- Can't have $q''_i > q'_i$ for all i, so $q''_i \leq q'_i$ for some i. Then

$$\mu X - \mu f^*[0, r_{i-1}) < \mu f^*(r_i, \infty)$$

and we get a contradiction from

$$\begin{aligned} \mu X &< \mu f^*[0, r_{i-1}) + \mu f^*(r_i, \infty) \\ &= \mu f^*([0, r_{i-1}) \vee (r_i, \infty)) \leq \mu X. \end{aligned}$$

Conclusions

- With the right fundamental results, geometric reasoning can be reasonably pain-free.
- The result on 2-sided integrals encapsulates some detailed constructive reasoning about approximating reals.
- Further work: integration and differentiation for vectors and complex numbers.
- Work in progress: trigonometry. Extract cos, sin from group homomorphisms $\mathbb{R} \to S^1$ (use $[-1, 1]$ as Escardó-Simpson interval object, [Vic17]). Use FTC to get π and differentiation.
- Further work: power series.
- Related work: I have results describing projective space $k\mathrm{P}^n$ as subspace of the lower hyperspace $P_L k^{n+1}$, and showing that the real and complex projective lines $\mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^1$ are the circle S^1 and the (Riemann) sphere S^2.

Appendix: Trigonometry (work in progress)

- Seek group homomorphisms $\mathrm{E}_a\colon (\mathbb{R}, +) \to (S^1, \times)$, $\mathrm{E}_a(\theta) = e^{2\pi i a\theta}$.
- S^1 = unit circle in $\mathbb{C}$
- θ is in angular units of a revolutions
- cos and sin are real and imaginary parts of E_a
- Seek E_1 (angular unit = revolution) first.

To get group homomorphism:

Define *midpoint* homomorphism from $\mathbb{I} = [-1, 1]$ to part of S^1, then extend.

Angular midpoint structure on $S^1 - \{-1\}$

- $\sqrt{z}$ = square root with positive real part
- $m(z_1, z_2) = \sqrt{z_1}\sqrt{z_2}$

$\mathbb{I}$ is an Escardó-Simpson interval object [ES01, Vic17]

Midpoint hom $\mathbb{I} \to A$ uniquely determined by images of ± 1
– provided A is *iterative*.

For every X, h and t, there is a unique M making the diagram commute.

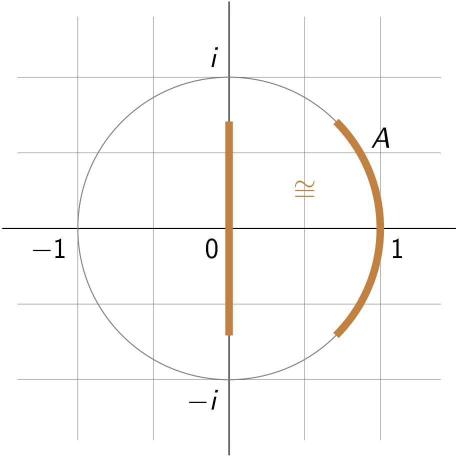
$$\begin{array}{ccc} A \times X & \xrightarrow{A \times M} & A \times A \\ \uparrow{\scriptstyle \langle h,t \rangle} & & \downarrow{\scriptstyle m} \\ X & \xrightarrow[M]{} & A \end{array}$$

Proof for compact metric space A:
T operator on maps $X \to P_U A$,
where $m'(a, -) = P_U(m(a, -))$.
Let M be least fixpoint of T.
Suppose $\mathrm{r}\, T(f)(x) \leq \gamma \mathrm{r} f(x)$, $\gamma < 1$.
(r = radius)
Then M factors via A.

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times f} & A \times P_U A \\ \uparrow{\scriptstyle \langle h,t \rangle} & & \downarrow{\scriptstyle m'} \\ X & \overset{T(f)}{\underset{f}{\rightrightarrows}} & P_U A \end{array}$$

For $\mathbb{I}$, $\gamma = 1/2$: $m(a, K)$ has half the radius of K.

Arcs as iterative midpoint algebras



- Arc A inherits midpoints from $S^1 - \{-1\}$.
- A homeomorphic to interval on y-axis
- – hence inherits metric.
- But not midpoint isomorphic, because of curvature.
- Approximate flatness $\Rightarrow$ can adjust γ in previous proof.

A iterative $\Rightarrow$ midpoint isomorphism $\mathbb{I} \to A$.
Can scale to larger and smaller arcs, and to group homomorphism $\mathrm{E}_1 : \mathbb{R} \to S^1$.

Calculus

- We expect $\frac{d}{dy} \arcsin y = 1/\sqrt{1-y^2}$ ...
- Hence *define*

$$\arcsin y = \int_0^y \frac{dt}{\sqrt{1-t^2}}; \quad \text{also } \pi = 4 \arcsin \frac{1}{\sqrt{2}}.$$

- Lemma: Suppose $z_1, z_2, z_1 z_2$ all in arc A. Then

$$\arcsin \Im(z_1 z_2) = \arcsin \Im z_1 + \arcsin \Im z_2 .$$

- $\arcsin \circ \Im$ is locally the inverse of a homomorphism $\mathrm{E}_a \colon \mathbb{R} \to S^1$, and $a = 1/2\pi$.
- Write $\mathrm{E}_{1/2\pi}(\theta) = \cos\theta + i \sin\theta$. ($\theta$ in radians.)

Calculus - derivatives

- FTC $\Rightarrow \frac{d}{dy} \arcsin y = 1/\sqrt{1-y^2}$ as expected.
- On a restricted interval, chain rule $\Rightarrow$

$$\frac{d}{d\theta} \sin\theta = \sqrt{1-\sin^2\theta} = \cos\theta,$$

and then

$$\frac{d}{d\theta}\cos\theta = \frac{d}{d\theta}\sqrt{1-\sin^2\theta} = \left(\frac{1}{2}\right)\frac{-2\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}} = -\sin\theta.$$

- Extend these to the whole of $\mathbb{R}$ using homomorphism property of $E_{2\pi}$: $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$, etc.

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