A LOGICAL STUDY OF SOME 2-CATEGORICAL ASPECTS OF TOPOS THEORY

by

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Abstract

There are two well-known topos-theoretic models of point-free generalized spaces: the original Grothendieck toposes (relative to classical sets), and a relativized version (relative to a chosen elementary topos $\mathscr{S}$ with a natural number object) in which the generalized spaces are the bounded geometric morphisms from an elementary topos $\mathscr{E}$ to $\mathscr{S}$, and they form a 2-category $\mathcal{B}\mathfrak{Top}/\mathscr{S}$. However, often it is not clear what a preferred choice for the base $\mathscr{S}$ should be.

In this work, we review and further investigate a third model of generalized spaces, based on the 2-category $\mathfrak{Con}$ of 'contexts for Arithmetic Universes (AUs)' presented by AU-sketches which originally appeared in Vickers' work in [Vic19] and [Vic17].

We show how to use the AU techniques to get simple proofs of conceptually stronger, base-independent, and predicative (op)fibration results in $\mathcal{E}\mathfrak{Top}$, the 2-category of elementary toposes equipped with a natural number object, and arbitrary geometric morphisms. In particular, we relate the strict Chevalley fibrations, used to define fibrations of AU-contexts, to non-strict Johnstone fibrations, used to define fibrations of toposes.

Our approach brings to light the close connection of (op)fibration of toposes, conceived as generalized spaces, with topological properties. For example, every local homeomorphism is an opfibration and every entire map (i.e. fibrewise Stone) is a fibration.

To my parents Sepideh and Hamid

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[1] By far one of the most enjoyable modules to teach.

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[2] I really meant "Gezellig" in Dutch for which English does not have an equivalent word.

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Contents

Introduction

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> *What, then, is the topos-theoretic outlook? Briefly, it consists in rejection of the idea that there is a fixed universe of "constant" sets within which mathematics can and should be developed, and the recognition that a notion of 'variable structure' may be more conveniently handled within a universe of 'continuously variable' sets than by the method, traditional since the rise of abstract set theory, of considering separately a domain of variation (i.e. a 'topological space') and a succession of constant structures attached to the points of its domain.*
>
> — **Peter Johnstone**
> From the introduction of Topos Theory
> [Joh77]

At the heart of a historical evolution, both in understanding and formalization, of the notion of *space* lies the generalizing move to study spaces not only by their open parts but also by bundles over that space. This had already appeared, one could argue, in Riemann's work on Riemann surfaces in the 19th century.

Moving to the 20th century, it was one of Brouwer's critical ideas that checking equality of two real numbers, represented by their decimal expansions, is problematic and indeed for constructive reasons one has to work with open intervals instead since it is possible to verify belonging to open intervals by an algorithmic process. Equality of two real numbers is the limiting case achieved only by infinite non-constructive means and thus it is illegitimate. This lucid viewpoint led to further development by H. Weyl in *Das Kontinuum* and later by A. Heyting, a student of Brouwer.

The further formalization of this idea led to discovery that open sets of a topological space, being a special case of what is called a Heyting algebra, form a model of intuitionistic propositional logic. In this view propositions are modelled as open parts of a topological space. This is one of the most significant early examples of mathematical trinitarianism. (See [Shu18] for recent categorified and homotopified analogue.) This discovery should be regarded in the sequel of an older discovery by Boole and Venn in the 19th century that a proposition can be seen as "linear manifold" and implication of propositions as the incidence of linear manifolds ([Car01]).

In the context of algebraic geometry, the generalization from open parts of a topological space to sheaves (aka bundles) over the space appears in Grothendieck's work on étale cohomology. It was later shown that this move corresponds to generalizing propositional geometric logic (internal logic of locales) to predicate geometric logic (internal logic of Grothendieck toposes) ([MR77], [Vic07]). In type theory (e.g. MLTT even without proof relevance i.e. without identity types), a similar phenomenon occurs: the paradigm of "types as propositions" is insufficient, and dependent types are modelled by fibrations (a particular kind of bundles).

Toposes were first conceived as kinds of "generalised spaces" which would provide a foundational frameworks for unifying various cohomology theories, most notably sheaf cohomology ([AGV72]). It is therefore no surprise that the first definition of topos was 'topos as a category of sheaves'. For nice spaces (more precisely 'sober' spaces) this topos is as good as the space itself, from topological point of view. According to its creators the notion of a topos "arose naturally from the perspective of sheaves in topology, and constitutes a substantial broadening of the notion of a topological space, encompassing many concepts that were once not seen as part of topological intuition . . . As the term 'topos' itself is specifically intended to suggest, it seems reasonable and legitimate to the authors of this seminar to consider the aim of topology to be the study of toposes.” ([AGV72])

Although the intended models of axiomatic framework of Grothendieck toposes were all geometrical, workers in category theory made further abstractions which in retrospect happened to be extremely fruitful. As the historical narrative goes William Lawvere worked on the axiomatic of the category of categories and he collaborated with M. Tierney on finding new axioms for toposes.

Having introduced the sub-object classifier, Lawvere discovered the notion of elementary topos and Tierney discovered that a Grothendieck topology is the same thing as a closure operator on the sub-object classifier. The idea that topology can be formulated by the algebraic notion of closure operator was a new understanding that was achieved by a logical formalization of toposes which had geometric roots and came from geometric intuitions. Moreover, once the notion of topos was axiomatized, out of these axioms the new notion of *elementary topos* was born. It was observed their internal logic of elementary topos is higher order intuitionistic. [1]

It was understood that the notion of elementary topos abstracts from the structure of the category of sets; each elementary topos can be though of as a universe of *set-like* objects [MR77], and elementary toposes can be assigned an internal language (Mitchell–Bénabou language) which enables one to reason about the objects and morphisms of a topos as if they were sets and functions.

Through study of various models of theory of elementary toposes it became clear that the abstraction is sufficiently general that elementary toposes encompass not only all Grothendieck toposes (such as the Zariski topos, the topos of quasi-coherent sheaves, Crystalline topos, petit topos and gros topos, Nisnevich topos, etc.) but also structured categories from mathematical logic (e.g. effective toposes in connection with the theory of realizability).

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[1] Only in retrospect by reflecting on the history of the subject and tracing back the original ideas of Brouwer, Weyl, and Grothendieck this can be seen natural!

However, elementary toposes set to depart from the main intuition of 'continuity as geometricity' of toposes. If we take the notion of elementary topos as a kind of structured category (i.e. a cartesian closed category with power object) then the a structure-preserving morphism of elementary toposes is not geometric morphisms, but rather what is known as a 'logical morphism'. This obstructs the essence of toposes as generalized spaces.

One of the main ideas of toposes as *generalized point-free* spaces is that toposes have natural inherent topologies and toposical constructions are performed in continuous fashion. The discontinuities arise precisely from replacing the space by its set of points. Note that by 'point-free' we do not mean ignoring points, but rather to give them a refined meaning. It means that the points are defined as models of a geometric theory, not as elements of a set. Therefore the constraints of geometricity takes the centre stage of dealing with spaces through the mediation of their point. A great number of classical spatial construction, based on *elements-of-a-set* view of points, via arbitrary transformations of sets of points are deemed illegitimate in our way of conceiving points of spaces.

For example, some type theoretic constructions such as function types and Π-types, corresponding respectively to the categorical notions of exponentials and dependent products, are intrinsically discontinuous if understood as constructions on sets (discrete spaces). The technical issue in the internal logic of toposes is that these constructions are not geometric, that is they are not preserved by inverse image functors of geometric morphisms (See §2.1). When performed fibrewise on dependent discrete spaces they are unfortunately not preserved by substitution which is a real drawback particularly when it comes to formulating principles such as induction.

Topos theory also provide a *relative* and *local* foundation for mathematics. In relative topos theory we see a presenting structure in an elementary topos $\mathscr{E}$ as a bounded geometric morphism $p\colon \mathscr{F} \to \mathscr{E}$, where $\mathscr{F}$ is the topos of sheaves over $\mathscr{E}$ for the space presented by the structure. Indeed, for such p, one obtains a canonical $\mathscr{E}$-indexed topos $\mathbb{F}$ whose underlying topos is $\mathscr{F}$ and the indexed category is given by $\mathbb{F}(I) := \mathscr{F}/p^*I$, for each object I in $\mathscr{E}$. Therefor, p makes

$\mathscr{F}$ into an $\mathscr{E}$-topos. This is crucial in Johnstone's approach in development of relative topos theory ([Joh02a]).

Moreover, fixing any elementary topos $\mathscr{S}$, geometric theories give rise to spaces relative to $\mathscr{S}$.[2] The way it works is that one associates to every geometric first order theory $\mathbb{T}$ the classifying[3] topos $\mathscr{S}[\mathbb{T}]$ whose category of points is the category of $\mathscr{S}$-models of $\mathbb{T}$. There is a *generic* (unique up to canonical isomorphism) model of $\mathbb{T}$ in $\mathscr{S}[\mathbb{T}]$ which is *universal*: any model M of $\mathbb{T}$ in an $\mathscr{S}$-topos $\mathcal{E}$ is classified, up to a unique equivalence, by a unique geometric morphism $g_M \colon \mathcal{E} \to \mathscr{S}[\mathbb{T}]$ over $\mathscr{S}$.

The reader familiar with universal algebra may recognize the similarity to the construction of free algebra (which also yields the presentation of algebras by generators and relations). A well-known example is the Lindebaum-Tarski algebra (in this case a frame) $\mathcal{L}_{\mathbb{T}}$ of a propositional geometric theory $\mathbb{T}$. A frame morphisms $\mathcal{L}_{\mathbb{T}} \to A$ is exactly a model of $\mathbb{T}$ in A, and therefore the point of locale $[\mathbb{T}]$ corresponding to $\mathcal{L}_{\mathbb{T}}$ are models of $\mathbb{T}$. Conversely, any locale X is the classifying locale of some propositional geometric theory. The same is true for any Grothendieck topos $\mathscr{E}$ over $\mathscr{S}$: there is a geometric theory $\mathbb{T}$ which classifies $\mathscr{E}$, that is $\mathscr{E} \simeq \mathscr{S}[\mathbb{T}]$ over $\mathscr{S}$. We usually call such a theory, the "theory of points of $\mathscr{E}$". This is in line, for instance, with taking the geometric propositional theory of completely prime filters of a locale as the theory of its points. Indeed, any propositional geometric theory presents a locale by generators and relations. Other examples are theory of groups, theory of rings, theory of local rings, theory of torsors, etc.

This spells out the meaning of word 'generalized' when we view toposes as *generalized spaces*, that the theory of their points is first order geometric as opposed to merely propositional (i.e. no sorts, and therefore, no variables, terms or quantifiers, no function symbols, and the predicate symbols are all

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[2] If we take $\mathscr{S}$ to be the Boolean topos of sets, then we recover classical mathematics in which the axiom of choice and the law of excluded middle are valid. However, for non-Boolean toposes, such as toposes of sheaves, the situation is more interesting: the internal logic of generic topos is intuitionistic. In this light one can see classical mathematics as the limiting case of intuitionistic mathematics, and the law of excluded middle as a unifying principle.

[3] It classifies models of $\mathbb{T}$ in all $\mathscr{S}$-toposes by the geometric morphisms landing in $\mathscr{S}[\mathbb{T}]$.

nullary). What in set theory appears as various proper classes (e.g. of sets, or of groups) become here generalized spaces (object classifier topos, the group classifier topos), and as such universes of various kinds appear.

A crucial fact is that two theories $\mathbb{T}$ and $\mathbb{T}'$ are $\mathscr{S}$-equivalent[4] precisely when the categories of their models are equivalent in that their classifying toposes $\mathscr{S}[\mathbb{T}]$ and $\mathscr{S}[\mathbb{T}']$ are equivalent. For example, consider the geometric theory $\mathbb{T}_{nat}$ consisting of only one sort N, a nullary function symbol $\mathsf{z}\colon \mathsf{N}$ (i.e. a constant symbol) and a unary function symbol $\mathsf{s}\colon \mathsf{N} \to \mathsf{N}$ subject to the following (geometric) axioms:

$$\mathsf{z} = \mathsf{s}(\mathsf{n}) \vdash_{\mathsf{n}\colon \mathsf{N}} \bot$$

$$\mathsf{s}(\mathsf{m}) = \mathsf{s}(\mathsf{n}) \vdash_{\mathsf{m},\mathsf{n}\colon \mathsf{N}} \mathsf{m} = \mathsf{n}$$

$$\top \vdash_{\mathsf{n}\colon \mathsf{N}} \bigvee_{n \in \mathbb{N}} \mathsf{n} = \mathsf{s}^n(\mathsf{z})$$

where $\mathsf{s}^n(\mathsf{z})$ stands for the term $\mathsf{s}(\ldots(\mathsf{s}(\mathsf{z}))\ldots)$ with n occurrences of s.

Relative to any base elementary topos $\mathscr{S}$, equipped with the natural number object (nno) N, the theory above and the empty theory are equivalent: In any model of $\mathbb{T}$ in any $\mathscr{S}$-topos $p\colon \mathscr{E} \to \mathscr{S}$, the sort N is interpreted as an object that is isomorphic to the nno p^*N in $\mathscr{E}$, by a unique isomorphism under which the constant z corresponds to the natural number 0, and the function symbol s corresponds to the successor operation of p^*N.

This indeed shows that the notion of equivalence of theories depend on the kind of infinite structures the base topos supports, and therefore, the equivalence of theories is ‘relative’ to the base topos.[5]

Therefore, Grothendieck toposes (i.e. $\mathcal{S}$et-valued sheaf toposes over sites) and relative toposes (i.e. the 2-category $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ of bounded toposes over a fixed base $\mathscr{S}$ with nno) offer two models of point-free generalized spaces. $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ is studied in [Joh02a, §B.4].

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[4] Or to put it differently, as far as $\mathscr{S}$ is concerned.

[5] Whereas this observation seems to go against the formal/definability account of structural properties, it does yield support to the invariance account of structural properties, first proposed by Felix Klein.

A third model is put forward in [Vic19] and [Vic17] is *contexts for Arithmetic Universes*. They form a (strict) 2-category $\mathfrak{Con}$.

In what sense are 'contexts for Arithmetic Universes' models of generalized point-free spaces? Well, the structures of AUs parallel (relativized) Giraud's characterization of relative Grothendieck toposes, except that AUs have only finitary fragment of geometric logic, and instead of infinitary disjunctions being supplied extrinsically by a base topos (e.g. the strucuture of small-indexed coproducts), we have sort constructors for parametrized list object that allow some[6] infinities to be expressed intrinsically. The goal is to see to what extent AUs can replace Grothendieck toposes as models of spaces. In this approach, geometric theories are replaced by AU-contexts, kind of thought of as *types* of *type theory* of AUs, presented by sketches ([Vic19]), and geometric morphisms are replaced by AU-functors, corresponding to the inverse image functors. AU-contexts provide a base-independent model for generalized point-free spaces in the sense that they form a 2-category $\mathfrak{Con}$ which gets embedded into $\mathcal{G}\mathfrak{Top}$, the 2-category of all relative toposes over all bases, via their classifying AUs.

We emphasize that throughout this dissertation all elementary toposes are assumed to have nno, and we rely on it in a crucial way. Without nno, we would not be able to construct the object classifier topos, a key player in making the model of AU-context of point-free generalized spaces work. Note that existence of nno is sometimes referred to as "axiom of infinity" for toposes analgous to the same axiom in ZF set theory ([Bla89]).

In Chapter 4, we show how to use the arithmetic universe (AU) techniques of [Vic17] to get simple proofs the stronger, base-independent (op)fibration results in $\mathcal{E}\mathfrak{Top}$, the 2-category of elementary toposes with nno, and arbitrary geometric morphisms.

More precisely, for an extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ in $\mathfrak{Con}$, and a model M of $\mathbb{T}_0$ in $\mathscr{S}$, an elementary topos with nno, there is a geometric theory $\mathbb{T}_1/M$,

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[6] But not all! Nonetheless, we have enough infinities to develop point-free continuum for the purposes of calculus and real analysis.

of models of $\mathbb{T}_1$ whose $\mathbb{T}_0$-reduct is M, and so we get a classifying topos $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ ([Vic17]). The main result of [HV19] then states

if U is an (op)fibration in $\mathfrak{Con}$, using the Chevalley criterion,

then p is an (op)fibration is $\mathcal{E}\mathfrak{Top}$, using the Johnstone criterion.

The main novelties of our approach from other previous work are manifold: first, avoiding the use of impredicative structures of toposes (because of the subobject classifier Ω and the power-objects) which makes our methods compatible with arithmetic universes.

Secondly, achieving the results for all toposes uniformly and independent of their base. This guarantees that the results are valid for all toposes over all bases including non-Boolean bases and thus they are full constructive. This approach promises a way to develop a rich theory of fibrations and opfibrations of toposes over various elementary toposes which are not classical such as the effective topos.

Third, the fibrations of contexts are much easier to work with since they enjoy certain strictness property at the level of models and also are all finitary in terms of their construction. All existing 2-limits and colimits in $\mathfrak{Con}$ are strict whereas they are weak (i.e. they are bicategorical limits) in $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ and $\mathcal{G}\mathfrak{Top}$.

Above all, we argue that our approach is conceptually stronger than [Joh02a]: if we are to prove a geometric morphism $p\colon \mathscr{E} \to \mathscr{S}$ in $\mathcal{E}\mathfrak{Top}$ is a fibration (resp. opfibration) we have to show the existence of a lifting structure for every geometric morphism from $\mathscr{A}$ to $\mathscr{S}$, and for every geometric transformation between any such two geometric morphisms. However, if p arises from a fibration of AU-contexts $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ (as in Theorem 4.2.2) we only need to check the (strict) lifting structure along the generic codomain (resp. domain) map $\mathbb{T}_0^{\to} \to \mathbb{T}_0$. Crucially, this lifting structure is strict which in practice makes the problem of verification of tracking coherence data (of the involved pullbacks) much easier.

The results presented in this thesis should be seen in a bigger context of the programme of adapting classical reasoning to constructive reasoning, while at the same time reducing a priory impredicative principles in certain systems to predicative ones (See [Mai05a], [Mai10a], [MV12]).

0.1 The outline of the thesis

We end this introduction by giving a road map of chapters.

The first chapter is a self-contained and sufficiently general introduction to the well-established theory of 2-categories and bicategories. Although it is written in an expository manner, certain points were emphasized as they serve a foundation for the later developments for the next chapters. It serves to provide the concepts and structures needed in the rest of the thesis. However, for our expert reader the only essential parts to the story of the thesis are §1.6, and the Construction of 'display sub-2-category' in §1.4.

One of the underlying principles of this chapter is that categorical notions and constructions are best expressed in the language of 2-categories; this principle is known as *formal category theory*.

However, there is another principle which is dominant in the later chapters, particularly in Chapters 3 and 4: in many situations, the correct way to organize a collection of mathematical objects is not as objects of a category but as points of a generalized space. Notions from category theory can be transferred to objects of a more general kind, and in particular generalized spaces, by collecting the generalized spaces into 2-categories.

These two principles are actually not in conflict for the abstraction involved in the definition of 2-category is general enough so that the "formal study of categories" can be applied to structures other than pure categories, for instance toposes (as generalized spaces). This idea is a vital part of the main results.

Another important motif in writing this chapter has been the observation that the two models of generalized spaces, namely the 2-category $\mathfrak{Con}$ of AU-contexts (Chapter 3) and the 2-category $\mathfrak{GTop}$ of Grothendieck toposes (§1.6) exhibit different 2-dimensional properties: the former is strict and the latter has interesting bicategorical properties (§1.6). For us, the delineation of the 2-categorical and bicategorical features has been crucial in discussing various notions of 2-limits in §1.9.

In Chapter2, following the principle of formal category theory, we review two distinct styles to study Grothendieck (op)fibrations in 2-categories and bicategories. We call them respectively *Chevalley-style* and *Johnstone-style*. Using the construction of display sub-2-category from Chapter 1 we give a cogent and novel reformulation of Johnstone-style fibrations in terms of fibrational objects. The utility of this reformulation is that it repackages lots of coherence data in the definition of Johnstone-style fibrations, arising from bipullbacks involved in that definition, into the universal property of cartesian morphisms of a certain fibration of bicategories.

For the reader already familiar with the theory of Grothendieck fibrations, we suggest to skip most parts except §2.4, §2.5, §2.6. In Chapter 3, we present the third model of generalized spaces, that is the 2-category $\mathfrak{Con}$ of AU-contexts (§3.3) and study its features. We quickly review the main aspects of the theory of AU-contexts, our AU analogue of geometric theories in which the need for infinitary disjunctions in many situations has been satisfied by a type-theoretic style of sort constructions that include list objects (and an nno). The contexts are “sketches for arithmetic universes” [Vic19], and we review the principal syntactic constructions on them that are used for continuous maps and 2-morphisms. We also introduce the notion of fibration of contexts (§3.4) and in the next chapter we prove that they beget fibrations of toposes.

As an original contribution, we shall use this reformulation in obtaining fibrations and opfibrations in the 2-category $\mathfrak{ETop}$ of elementary toposes from Chevalley-style fibrations of AU-contexts in Chapter 4.

Finally, in Chapter 5 we shall consider some further examples, potential applications, and few conjectures concerning new avenues for future research. We shall state these conjectures and give a sketch of a potential proof. We warn that the discussion will be more impressionistic than scientific. One such application concerns *bag toposes*. Bag spaces originally appeared as "bagdomains", was in [Vic92] in the context of directed complete posets (dcpos). In a series of papers ([Joh92], [Joh93], [Joh94]) Johnstone gave a characterization of a bag topos[7] $\mathscr{B}ag(\mathscr{E})$ as a 2-categorical partial product of $\mathscr{E}$ and the opfibration $\mathscr{S}[\mathbb{O}_\bullet] \to \mathscr{S}[\mathbb{O}]$ of object classifier, among other things. Indeed, to take a proper account of specialization (already essential in the dcpo case) it relies on the fact that sets (discrete spaces) are opfibrations. Some colimits of toposes (e.g. coproducts, lifting, scones) can be then be constructed from bag toposes. We state few conjectures which put a research path forward to construct partial products of AUs from bag context.

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[7] Given a space $\mathscr{E}$, $\mathscr{B}ag(\mathscr{E})$ is the space whose points are bags of points (i.e. set-indexed families of points) of $\mathscr{E}$. To use type theoretic notations, it would roughly be expressed as $\Sigma_{I:\mathfrak{U}}\Pi_{i:I}\mathscr{E}$, where $\mathfrak{U}$ is a universe of discrete spaces. In this sense it is an analogue of powerdomain. When $\mathscr{E}$ has one point $\mathscr{B}ag(\mathscr{E})$ is equivalent to the object classifier. Furthermore, Johnstone's 2-categorical generalization made it possible to vary the type of the indexing object; initially, it was considered a set, but it could very well be a a category, or a spectral space.

2-Categorical preliminaries

1

In this chapter we give a concise and self-contained review of the theory of 2-categories and bicategories which constitutes a scaffolding of the next chapters. In particular, §1.2 explains the passage from 2-categories to bicategories which involves a certain weakening of unit and composition structures.

Elementary toposes and Grothendieck toposes (over a fixed base or otherwise), which are the main objects of our interest, actually form 2-categories but a mixed 2-categorical and bicategorical approach is most suitable to them. The need for such an approach is discussed in §1.6 at a greater length: one such need is that the existing limits and colimits of diagrams of toposes are bicategorical. In §1.9, we give a comprehensive and self-contained review of 2-categorical and bicategorical limits (aka weak limits) with a special focus on the delineation between the two. Most significant for us is the well-known class of PIE limits; the 2-category $\mathfrak{Con}$ of AU-contexts[1] (the most significant 2-category for us in Chapter 3) has PIE limits. In §1.4, we introduce the construction of 'display sub-2-category' which shall be essential in later developments in our new characterization of Johnstone fibrations in terms of fibrational objects of the *codomain 2-functor* in §2.6.

We begin in §1.0 introducing the ideas behind the definition of 2-category by explaining the link to *formal category theory*. In §1.8 (and also in §A.7) we shall give a flavour of the view of 2-categories as a framework for formal category theory in action. Few basic concepts of category theory and facts about them are done intrinsically to 2-categories. These section are not meant to serve as an encyclopedia, but rather as a keyhole perspective as an opening to the vast playground of formal category theory within 2-categories.

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[1] AU is short for Arithmetic Universe.

The main references that have been consulted for writing this chapter are [Bén67], [Gra74], [Str72], [BKP89], [Kel89], [GPS95], [PR91], [Joh02a], [Lac10a], [Lac10b], and [Gur11]. There are handful others to which we shall refer in the relevant individual sections.

1.0 Introduction

Before getting into the business of defining 2-categories, bicategories and their morphisms in the next sections, we would like to engage the reader with a broader picture of higher category theory which as its building block includes 2-categories and bicategories but it paints much more. Although this thesis does not need higher categories other than 2-categories and bicategories, a short discussion of higher categories in below sheds light on 2-categories and bicategories themselves.

Higher category theory can be seen under two different lights: first as a generalization of homotopy types of spaces, and second, as a higher analogue of the notion of category. In the first case, the inspiring force has been *the homotopy hypothesis*, originally due to Grothendieck (e.g. in Pursuing Stacks) which roughly asserts that (weak) higher groupoids should classify homotopy types. The weak higher structures in fact has been the hardest part in providing a fully algebraic definition of higher groupoids which model homotopy types of spaces. Higher categories generalize higher groupoids in that the paths (or better known as morphisms) between objects and higher paths between paths have a direction and are not necessarily invertible. If we regard morphisms as physical processes of some kind, it is quite natural to not require their invertibility; after all some processes lose information and are not revertible. That is essentially why categories are more commonly found than groupoids in mathematics, and in applications to sciences.

Another way to arrive at higher categories from categories is the idea of *proof relevance*. To make this clear, we give an example here. In a certain category (i.e. a model of first order theory of categories), we can reason about equality of morphism. For instance, we have the following rules:

- $f = g, g = l \vdash_{f,g,l} f = l$.
- $f = g$, $\mathrm{dom}(h) = \mathrm{cod}(g) \vdash_{f,g,h} h \circ f = h \circ g$
- $h = k$, $\mathrm{dom}(f) = \mathrm{cod}(h) \vdash_{f,h,k} f \circ h = f \circ k$

From these we can deduce

$$f = g, h = k, \mathrm{dom}(h) = \mathrm{cod}(f) \vdash_{f,g,h,k} h \circ f = k \circ g$$

We can go beyond the *mere* fact of equality of two morphisms, and also keep track of process of proving equality of morphisms. For instance two morphisms f and g can be proved to be equal by knowing that $f = f_2 \circ h$, $h = f_1 \circ f_0$, $g = k \circ f_0$, and $k = f_2 \circ f_1$. The proof of equality of f and g uses the associativity law of category where all this morphisms are situated. If we update our knowledge by getting extra data that f_0 is an identity morphism, then we get a *different* proof using the unit law of the category and the last rule above. The main idea of proof relevance applied to this situation is that we should go beyond the strucuture of categories to be able to speak about different proof of equality of morphisms. An equality proof $f = g$ can be regarded as a (bidirectional and invertible) morphism from f to g. The proof-relevance view leads one to go beyond groupoid and to the realm of higher groupoids, and in fact this move is at the core of conception of h-level of types in homotopy type theory (HoTT).

However, to be more general, we might not want to impose the condition that the proofs of equality of morphisms are either bidirectional or invertible. In fact, we might even think of these morphisms as reduction processes than proofs. So, if morphisms are conceived of as general processes, then the reduction processes might be regarded as processes between processes. In the parlance of higher 2-category theory they are called *2-morphisms*. We can think of 2-categories as categorification of categories. The 2-categories can be *weak* in that the unit and associativity laws of morphisms hold only up to invertible 2-morphisms (aka iso-2-morphisms). Following Bénabou, they are referred to as bicategories in the literature of higher category theory. We shall reserve the term 2-category for strict 2-categories where the unit and associativity laws of morphisms hold strictly.

Of course, there is nothing that stops us here: similarly, we might be interested in keeping track of reduction (or equality) of certain 2-morphisms from other ones. Pursuing this idea to its end, we get 3-categories which additionally possess 3-morphisms between 2-morphisms.

Repeating the process leads to the concept of n-categories and as a "colimit" of this process we obtain ∞-categories which consist of k-morphisms for every $k = 0, 1, 2, \ldots$. However, the simplicity of this picture is deceiving and the details have been omitted. In general, it not straight-forward to replace the "structural equalities" which are part of the theory of categories by higher morphisms.

In this chapter we shall give an expository account of 2-category and bicategory theory. By no means, our account will be comprehensive. For the most part, we shall include what is essential for the plan of thesis. As such, we emphasize on the issues of strictness, pseudoness, and laxness, and the corresponding notions of representability to which they give rise. Accordingly, we review construction of weighted limits and colimits with several important examples; they are primarily viewed as 2-dimensional generalizations of ordinary limits and colimits of category theory.

In §1.3, it is argued that strict 2-functors are the most well-behaved morphisms of 2-categories when it comes to existence of various limits and colimits. However, it is sometimes useful to have pseudo functors between various 2-categories of toposes. Also, the essential tool of relative topos theory is that of indexed categories which are essentially pseudo functors to the 2-category $\mathfrak{Cat}$ of locally small categories. As such we shall be concerned with pseudo functors in this chapter.

In §A.6 we review the well-known facts that every bicategory is biequivalent to a 2-category, and that every pseudo functor is pseudo naturally equivalent to a strict 2-functor. What's more, many 2-categories of toposes are indeed strict in that they are strictly unital and associative. So, a natural question is that why do we need to talk about bicategories in this chapter?

The reason is, and this is particularly crucial for us, that many phenomena, such as limits and colimits, in various 2-categories of toposes are bicategorical. The analogue of categorical limits and colimits for bicategories is given by the notion of weighted limits and colimits. They are only determined up to equivalence, but in the 2-category $\mathfrak{Cat}$ there is a canonical choice.

We ocassionally make use of the theory of enriched categories, especially in the cases where enriched definitions and constructions are more cogent and concise than the elementary description in terms of objects, morphisms, and 2-morphisms. Although, the important point to bear in mind is that all enriched notion used in this chapter with regard to bicategories can be carried out in elementary terms. This means we are not bothered by size issues (e.g. that the 2-category of categories is not $\mathcal{C}$at-enriched).

A word on notations: throughout the rest of this paper and particularly in this chapter, we organize categories and 2-categories themselves into various categories and 2-categories (of larger size) based on different notions of morphism between them which will be defined in §1.3. The table 1.1 can be used as a notation guide.

We have not *explicitly* imposed size constraints on categories as objects of $\mathfrak{Cat}$. Note that in absence of any smallness conditions, categories, functors, and natural transformations do not form a 2-category (defined as a $\mathcal{C}$at-category) since for categories $\mathcal{C}$ and $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ is not necessarily small, e.g. take $\mathcal{C} = \mathbf{1}$ and $\mathcal{D} = \mathcal{S}$et. Indeed, we have a meta 2-category $\mathfrak{CAT}$ of (possibly large) categories, functors, and natural transformations. The genuine 2-category $\mathfrak{Cat}$ in the table above is in fact the 2-category consisting of small categories, and by 'small' here we mean internal to an elementary base topos $\mathscr{S}$, e.g. $\mathcal{S}$et. We apply the same standard for all other terms in the table above. In few places, we will allow ourselves to use the cartesian closed structure of $\mathfrak{Cat}$, and we will be explicit about that. However, $\mathfrak{CAT}$ does not admit such a structure.

Symbol	Meaning
$\mathcal{C}\mathrm{at}$	Category of categories and functors
$\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}_{str}$	Category of bicategories and strict 2-functors
$\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}$	Category of bicategories and pseudo functors
$\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}_{lax}$	Category of bicategories and lax functors
$\mathfrak{Cat}$	2-Category of categories, functors and natural transformations
$\mathbf{2}\,\mathcal{C}\mathrm{at}$	2-Category of 2-categories, strict 2-functors, and strict 2-natural transformations
$\mathbf{2}\,\mathcal{C}\mathrm{at}_{psd}$	2-Category of 2-categories, pseudo functors, and strict 2-natural transformations
$\mathbf{2}\,\mathcal{C}\mathrm{at}_{lax}$	Sesquicategory of 2-categories, strict 2-functors, and lax natural transformations
$\mathbf{Icon}$	2-Category of 2-categories, lax functors, and icons
$\mathbf{2}\,\mathfrak{Cat}$	3-Category of 2-categories, strict 2-functors, strict 2-natural transformations, and modifications
$\mathbf{Gray}$	Tricategory of 2-categories, strict 2-functors, pseudo natural transformations, and modifications
$\mathbf{2}\,\mathfrak{Cat}_{psd}$	3-Category of 2-categories, pseudo-functors, pseudo natural transformations, and modifications
$\mathbf{Hom}$	Tricategory of bicategories, pseudo functors, pseudo natural transformations, and modifications

Fig. 1.1.: A notation guide to various (weak) n-categories of (weak) k-categories

We shall use $||(-)||_1$ to denote the truncation of a 2-category to its underlying category by forgetting all 2-morphisms (See 1.4.3). For instance $||\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}||_1$ is the category of (small) strict 2-categories and strict 2-functors between them, and $||\mathbf{2}\,\mathcal{C}\mathrm{at}_{psd}||_1$ is the category of strict 2-categories and pseudo functors between them. For a relationship of various categories of (small) bicategories see 1.7.

A closer look at the table above shows several interesting irregularities:

- There is no 2-category or even a bicategory having bicategories as its objects. This is not accidental and the reason for it appears in Remark 1.3.6.

- Passing from $\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}$ to $\mathbf{2}\,\mathcal{C}\mathrm{at}_{lax}$ we do not get a 2-category but a weaker strucuture of 'sesquicategory' ([Ehr63], [Str96]). Like a 2-category, a sesquicategory has objects, morphisms, and 2-morphisms. Like a 2-category, it possesses a strictly associative and unital composition of morphisms, a strictly associative and unital vertical composition of 2-morphisms, and whiskering of 2-morphisms with 1-morphisms on both sides. *Unlike* a 2-category, this whiskering does not satisfy the *exchange law* (See Appendix A.4).

- Passing from $\mathbf{2}\,\mathcal{C}\mathrm{at}_{lax}$ to $\mathbf{Icon}$ we do get a 2-category again, but we are forced to consider not all 'lax natural transformations', but special kinds of them called 'icons'. We shall see more icons in §1.3.

1.1 What is a 2-category?

Whereas category theory provides a framework to organize collection of mathematical objects into categories and study them within those category, purely in terms of objects, morphisms, and their compositions, 2-category theory gives us a framework to study categories themselves in a formal manner. Along this idea, the first essential observation is that whatever definition of 2-categories we propose, one thing is clear: categories, functors, and natural transformations should form the archetypal example of such a definition.

The theory of 2-categories has three sorts: a sort for objects, a sort for 1-morphisms, and finally a sort for 2-morphisms. It also has partial operators for various compositions of 1-morphisms and 2-morphisms together with unit and associativity axioms which ensure these compositions are coherent. In order to *formally* study categories, we should abstract away from their definitions as categories and treat them purely as objects of the 2-category of categories with certain essential properties which have to be distilled into laws or axioms to ensure that a certain 2-category behaves in those essential ways like $\mathfrak{Cat}$.

This view is memorably summarized by Gray in [Gra74] which states that

> The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well as to other parts of mathematics.

As it is the case with the study of categories, we do not study a 2-category in isolation, but rather we put the real importance on morphisms of 2-categories, that is the ways in which a certain 2-category relates to other 2-categories.

To give a concrete example consider the theorem concerning the uniqueness of adjoints up to a unique isomorphism. A standard categorical proof of this fact goes as follows: suppose $R\colon \mathcal{A} \to \mathcal{X}$ is a functor which has a left adjoint. We want to show that any two left adjoints of R are (naturally) isomorphic. Assume $L, L'\colon \mathcal{X} \to \mathcal{A}$ are both left adjoints of R. Then

$$\mathcal{A}(LX, A) \cong \mathcal{X}(X, RA) \cong \mathcal{A}(L'X, A)$$

and these bijections are natural in $X \in \mathcal{X}$ and $A \in \mathcal{A}$. By Yoneda lemma, L and L' are naturally isomorphic. A 2-categorical proof should be expressed only by objects (categories), 1-morphisms (functors), and 2-morphisms (natural transformations). As such, we should not really be using objects of categories like above. Recall that an adjunction of categories can be purely expressed in terms of unit, counit, and two equations (known as *the triangle equations*); for any object X of $\mathcal{X}$, the left hand side diagram commutes and for any object A of $\mathcal{A}$ the right hand side diagram commutes.

$$\begin{tikzcd}
L(X) \arrow[r, "L(\eta_X)"] \arrow[dr, "1"'] & LRL(X) \arrow[d, "\epsilon_{LX}"] & & R(A) \arrow[d, "\eta_{R(A)}"'] \arrow[dr, "1"] & \\
& L(X) & \text{and} & RLR(A) \arrow[r, "R(\epsilon_A)"'] & R(A)
\end{tikzcd} \qquad (1.1)$$

One can express these equations without reference to the objects of $\mathcal{X}$ and $\mathcal{A}$ and only by equations involving natural transformations.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, equal] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow\eta", near start] & \mathcal{X} \arrow[d, "L"] \\
\mathcal{A} \arrow[r, equal] \arrow[ur, "R" description] \arrow[ur, phantom, "\Downarrow\epsilon", near end, shift right=3] & \mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[d, bend right, "L"'] \arrow[d, bend left, "L"] \arrow[d, phantom, "="] \\
\mathcal{A}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{A} \arrow[r, equal] \arrow[d, "R"'] \arrow[dr, phantom, "\epsilon\Uparrow", near start] & \mathcal{A} \arrow[d, "R"] \\
\mathcal{X} \arrow[r, equal] \arrow[ur, "L" description] \arrow[ur, phantom, "\eta\Uparrow", near end, shift right=3] & \mathcal{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A} \arrow[d, bend right, "R"'] \arrow[d, bend left, "R"] \arrow[d, phantom, "="] \\
\mathcal{X}
\end{tikzcd}
\tag{1.2}$$

Therefore, for left adjoints (L, η, ϵ) and (L', η', ϵ') of functor $R \colon \mathcal{A} \to \mathcal{X}$, one readily checks that the natural transformations $(\epsilon L') \circ (L\eta')$, from L to L', and $(\epsilon' L) \circ (L'\eta)$, from L' to L, are inverses of each other and therefore, L and L', are isomorphic.

In fact, as we shall see the adjoint situation $f \dashv u$ in 2-categories are in a sense one of the most general form of expressing universal properties of morphisms: liftings, extensions, cartesian properties, fibrations, etc. can be expressed in terms of adjoint pairs.

Consider the example of a category equipped with terminal object. In standard category theory, a category $\mathcal{C}$ is equipped with a terminal object 1 is expressed by the universal property of the limit over the empty diagram. How do express this purely 2-categorically? We observe the structure of a terminal object T of $\mathcal{C}$ is equivalent to a (fully faithful) right adjoint T of the unique functor $! \colon \mathcal{C} \to \mathbf{1}$ (where $\mathbf{1}$ is the terminal category.) In the above discussion we showed how the structure of adjunction is inherently 2-categorical. Therefore, in any 2-category $\mathfrak{K}$ with a terminal object 1 (which is in here representably defined by the equivalence $\mathfrak{K}(X, 1) \simeq \mathbf{1}$, for every object X in $\mathfrak{K}$), we define an *object equipped with a terminal point* to be a right adjoint t of $!_X \colon X \to 1$. The left equation in (1.2) gives no new information and the right equation simply says that $\eta \centerdot t = \mathrm{id}_t$.

So we conclude that in a 2-category with a terminal object 1, an object equipped with a terminal point consists of $(X, t \colon 1 \to X, \eta \colon 1_X \Rightarrow t \circ !_X)$ satisfying $\eta \centerdot t = \mathrm{id}_t$. In $\mathfrak{K} = \mathfrak{Cat}$ this is exactly a category equipped with a terminal object. In $\mathfrak{K} = \mathcal{B}\mathfrak{Top}$ this is a pointed topos. Of course the dual structure gives the notion of an *object equipped with an initial object*: it is a left adjoint $i \dashv !_X$, and therefore, it can be described by the triple $(X, i\colon 1 \to X, \epsilon\colon i \circ !_X \Rightarrow 1_X)$ satisfying $\epsilon \centerdot i = \mathrm{id}_i$

The main lesson of this and many other similar observations is that by writing the constructions of category theory in the language of 2-categories, not only do we get useful generalization to other, sometimes vastly different, 2-categories than $\mathfrak{Cat}$, but also we understand the essence of the very same categorical constructions in a deeper and more categorical way.

In the presentation of this chapter, we shall rely on a modicum of enriched category theory. For an extensive treatment of enrichment see [Kel82]. The idea is that an enriched category is a category in which the hom-functors take their values in some monoidal category $(\mathcal{V}, \otimes, I)$ instead of $(\mathcal{S}\mathrm{et}, \times, \{\star\})$, and composition is formulated by the monoidal structure of $\mathcal{V}$. A concise account of all which we shall assume about enriched category theory can be found in [Lur09, Appendix A.1.4]. Although in this thesis we only need enrichment in the monoidal category of (small) categories, the use of enrichment in general goes much further beyond than that. To give but one example, *graph-enriched* categories (whereby hom-sets are graphs instead of sets) are extensively studies in the theory of rewriting. The objects are types, the vertices of hom-graphs are terms, and the edges of hom-graphs are term-rewrites which describe the process of computation ([SM17], [BW19]).

Definition 1.1.1. A **2-category** is a $\mathcal{C}$at-enriched category, where $\mathcal{C}$at is the cartesian closed monoidal category of small categories and functors. A **2-functor** between 2-categories is a $\mathcal{C}$at-enriched functor.

If $\mathfrak{K}$ is a 2-category and x and y are two objects of $\mathfrak{K}$ (i.e. elements of the underlying class of objects of $\mathfrak{K}$), then we depict an object f of the hom-category $\mathfrak{K}(x, y)$ by a 1-cell $f\colon x \to y$, and a morphism α of the hom-category $\mathfrak{K}(x, y)$ by a 2-cell

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, shorten=2pt, "\delta"] \end{tikzcd} \quad .$$

However, we call f a **1-morphism** and α a **2-morphism** instead of calling them 1-cell and 2-cell respectively, as is customary in some of the literature of higher category theory. We follow the principle of not naming concepts based on a certain model in which those objects are represented especially when there are other models whereby those same concepts get different names: For 2-categories, other than *pasting diagrams* pictured by cells of various dimensions, there are *string diagrams* which are planar dual to cellular pasting diagrams. Objects are depicted as regions, 1-morphisms as lines/wires separating regions, and 2-morphisms as nodes (or boxes) separating (or connecting) lines (or wires). For more on string diagrams we refer the reader to the appendix A.5.

1.2 From 2-categories to bicategories

It happens that the structure of 2-categories and $\mathcal{C}$at-enriched categories and particularly 2-functors is too strict and fails to deal with many interesting practical cases. For example, algebras, bimodules, and bimodule morphisms form a bicategory, not a 2-category, because tensor product is associative and unital only up to a non-trivial isomorphism.

Notice that this situation is the categorified version of strict monoidal categories and monoidal categories. Even though strict monoidal categories are easier to work with they often are too strict and non-interesting in practice; for instance the monoidal category $\mathbf{Vec}_{\mathbb{C}}^{fin}$ of complex finite dimensional vector spaces over the field of complex numbers $\mathbb{C}$ is a monoidal category which is not strict monoidal. Nonetheless, by the coherence theorem of Mac Lane we know that every monoidal category is equivalent to a strict monoidal category. (For formulation and proof see [ML98] and [JS91].) A similar coherence theorem exists for 2-categories and bicategories.

The notion of bicategory is a weakening of notion of 2-category; we have *weak* unital and associativity of 1-morphisms. To see this more clearly, suppose $\mathfrak{K}$ is a 2-category. By definition, the diagram

$$\begin{tikzcd}
\mathfrak{K}(x,y,z,w) \arrow[r, "1\times c_{x,y,z}"] \arrow[d, "c_{y,z,w}\times 1"'] & \mathfrak{K}(x,z,w) \arrow[d, "c_{x,z,w}"] \\
\mathfrak{K}(x,y,w) \arrow[r, "c_{x,y,w}"'] & \mathfrak{K}(x,w)
\end{tikzcd} \tag{1.3}$$

commutes[2], and this precisely expresses the associativity law of composition of 1-morphisms and horizontal composition of 2-morphisms. It means that for any 1-morphisms $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to w$, we have $(h \circ g) \circ f = h \circ (g \circ f)$ and, furthermore, for any 2-morphisms ϕ, γ, and χ of the form

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\phi"] & y & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\gamma"] & z & z \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\chi"] & w
\end{tikzcd}$$

we have $\chi \centerdot (\gamma \centerdot \phi) = (\chi \centerdot \gamma) \centerdot \phi$. The structure of a bicategory requires that the strict equality in the associativity law of 1-morphisms above to be weakened to an (specified) iso-2-morphism natural in arguments f, g, h. This can be done by requiring that diagram (1.3) commutes up to a natural isomorphism $\alpha_{x,y,z,w}$ for all objects x, y, z, w. Therefore, we have $\alpha(f,g,h)\colon (h \centerdot g) \centerdot f \cong h \centerdot (g \centerdot f)$ and also, the diagram below of iso-2-morphisms commutes.

$$\begin{tikzcd}
(h \centerdot g) \centerdot f \arrow[r, "\alpha(f,g,h)"] \arrow[d, "(\gamma\centerdot\beta)\centerdot\alpha"'] & h \centerdot (g \centerdot f) \arrow[d, "\gamma\centerdot(\beta\centerdot\alpha)"] \\
(h' \centerdot g') \centerdot f' \arrow[r, "\alpha(f',g',h')"'] & h' \centerdot (g \centerdot f')
\end{tikzcd} \tag{1.4}$$

Similarly, one weakens the unital law so that for any 1-morphism $f\colon x \to y$ there exists an iso-2-morphism $\rho_{x,y}(f)\colon f \circ 1_x \cong f$ and $\lambda_{x,y}(f)\colon 1_y \circ f \cong f$, naturally in x, y, f. In the literature the 2-morphism α is referred to as the "associator", ρ as the "right unitor", and λ as the "left unitor". They are required to satisfy the familiar coherence conditions. For a full list of coherence laws of bicategories see Appendix §A.1. For external references we refer the reader

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[2] We use the shorthand notation $\mathfrak{K}(x_1, x_2, \ldots, x_n)$ for $\mathfrak{K}(x_{n-1}, x_n) \times \ldots \times \mathfrak{K}(x_0, x_1)$.

to [Bén67] and [Lei98]. A historical discussion of bicategories appears at the final section of this chapter.

A good exercise, which helps one to parse the list of coherence axioms of a bicategory, is to show that the notion of bicategory is a categorification of the notion of monoidal category, i.e. a bicategory with one object is the same thing as a monoidal category, and moreover, for every object A in a bicategory $\mathfrak{K}$, the endomorphism category $\mathrm{End}_{\mathfrak{K}}(A) = \mathfrak{K}(A, A)$ is a monoidal category. The following are the paradigmatic instances of bicategories which we use again and again to justify certain bicategorical formalizations of various categorical concepts.

EXAMPLE 1.2.1. For a monoidal category $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ there is an associated bicategory $\Sigma\mathcal{V}$ which has only one object $*$ and $\Sigma\mathcal{V}(*, *) := \mathcal{V}$. The identity morphism is given by the unit $* \xrightarrow{I} *$, and the composition of $* \xrightarrow{X} * \xrightarrow{Y} *$ is given by $* \xrightarrow{X \otimes Y} *$. The 2-morphisms are morphisms of $\mathcal{V}$, the vertical composition is given by their categorical composition and their horizontal composition is given by tensoring. The bicategory $\Sigma\mathcal{V}$ is referred to as *delooping* (and sometimes suspension) of $\mathcal{V}$. In this way, bicategories naturally generalize monoidal categories.

EXAMPLE 1.2.2. From any topological space X we can extract a bicategory, indeed a *bigroupoid* $\Pi_{\leq 2}X$. An object is a point x of X, a 1-morphisms is a path $p \colon x \to y$ in X (i.e. a map $p \colon I \to X$ where I is the unit interval with its standard Euclidean topology.) and a 2-morphism is a homotopy class of paths (i.e. a class $\alpha = [h]$ where $h \colon I \times I \to X$ is continuous with $h(0,0) = h(0,1)$ and $h(1,0) = h(1,1)$. The equivalence class above is defined with respect to the homotopy relation: $h_0 \sim h_1$ iff there exists a homotopy $H \colon I \times I \times I \to X$ with $H(-,-,0) = h_0$ and $H(-,-,1) = h_1$). Paths can be composed, however, as we do not quotient by the relation of homotopy, such composition is not associative. Associativity is only up to isomorphism: for paths α, β, γ we have $\gamma \circ (\beta \circ \alpha) \simeq (\gamma \circ \beta) \circ \alpha$ by continuous re-parametrization. We note that 1-morphisms in $\Pi_{\leq 2}X$ are equivalences (weakly invertible) and all 2-morphisms are (strictly) invertible. Any bicategory in which all 1-morphisms are equivalences and all 2-morphisms are invertible is called a bigroupoid. A bigroupoid is *strong* if 1-morphisms are strictly invertible. Bigroupoids are groupoid-enriched (aka *track* categories). [Rob16] shows that $\Pi_{\leq 2}X$ is indeed a topological bicategory.

EXAMPLE 1.2.3. There is a bicategory $\mathcal{T}op_{\leq 2}$ of topological spaces. Here the objects are topological space, 1-morphisms are continuous maps, and 2-morphisms are equivalence classes of homotopies. In a similar way, one constructs the bicategory of pointed-topological spaces.

Some 2-categorical definitions go through bicategories without much change. For example the definition of an adjoint pair defined in §1.1 in 2-categories can be defined in bicategories. An adjoint pair of morphism $f \dashv u \colon y \to x$ in a bicategory $\mathfrak{B}$ is defined by the following triangle equations (of 2-morphisms)

$$\begin{tikzcd}
f \arrow[r, Rightarrow, "\rho_f^{-1}"] \arrow[drrrr, Rightarrow, "\mathrm{id}_f"'] & f1_x \arrow[r, Rightarrow, "f \centerdot \eta"] & f(uf) \arrow[r, Rightarrow, "\alpha^{-1}"] & (fu)f \arrow[r, Rightarrow, "\epsilon \centerdot f"] & 1_y f \arrow[d, Rightarrow, "\lambda_f"] \\
 & & & & f
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, Rightarrow, "\lambda_u^{-1}"] \arrow[drrrr, Rightarrow, "\mathrm{id}_u"'] & 1_x u \arrow[r, Rightarrow, "\eta \centerdot u"] & (uf)u \arrow[r, Rightarrow, "\alpha"] & u(fu) \arrow[r, Rightarrow, "u \centerdot \epsilon"] & u(1_y) \arrow[d, Rightarrow, "\rho_u"] \\
 & & & & u
\end{tikzcd}$$

1.3 Morphisms of bicategories

For any particular mathematical structure, there is a category whose objects are instances of that structure, and whose morphisms are the structure preserving maps (aka homomorphisms) from one instance to the other. Examples are the category $\mathcal{M}on$ of monoids, the category $\mathcal{G}rp$ of groups, the category $\mathcal{C}$Ring of commutative rings, the category $\mathcal{D}$ist$\mathcal{L}$at of distributive lattices, the category $\mathcal{M}an$ of smooth manifolds, etc.

Similarly for the structure of category (with a cartesian first order theory consisting of two sorts), we have the category of categories and functors which is the underlying category of a 2-category, namely the 2-category of categories, functors, and natural transformations. If the mathematical structure we begin with is itself 2-dimensional, such as the structure of bicategory, then again we can make the category of instances of that structure and structure *preserving* maps. However we should take care in what we mean by preservation here. Since the notion of structural identity between 1-morphisms of a bicategory is iso-2-morphism rather than strict equality it is unreasonable to ask for a morphism of bicategories to preserve compositions of 1-morphisms up to equality. In our paradigmatic examples 1.2.1 and 1.2.2, neither a monoidal functor nor a continuous map of spaces preserve the composition of 1-morphisms in bicategories $\Sigma\mathcal{V}$ and $\Pi_{\leq 2}(X)$ up to equality. In both cases the compositions are preserved up to a canonical iso-2-morphism. This is the main intuition behind the concept of *pseudo functor*. The details of its definition is deferred to the appendix.

In this section, we shall also look at the contrast with strict and lax morphism of bicategories. However, for good reason which we will mention, pseudo functors are the structure preserving morphisms of bicategories. It turns out that bicategories and pseudo functors form a tricategory whose 2-morphisms and 3-morphisms are respectively *pseudo natural transformations* and *modifications*.

Still we would like to have strict and particularly lax functors around. For any structure, weaker notion of morphisms of structures than homomorphisms are occasionally useful. For instance, any two elementary toposes can be glued together along a cartesian functor to obtain another topos. Similarly, any two 2-categories of algebras of monads can be glued together along lax functors to obtain a 2-category of algebras.

It is useful to continue our analogy between bicategories and monoidal categories. There are various notions of morphisms between monoidal categories: strict monoidal functors, pseudo monoidal functors, and lax monoidal functors. similarly, between bicategories, there are *strict 2-functors*, *pseudo functors*, and *lax functors*.

A pseudo-functor of bicategories is a weaker notion than strict 2-functors of bicategories in the sense that a pseudo-functor preserves composition of morphisms only up to a chosen iso-2-morphism. A pseudo-functor $F\colon \mathfrak{B} \to \mathfrak{C}$ of bicategories assigns to any identity morphism $1_x\colon x \to x$ in $\mathfrak{B}$ an iso-2-morphism $\iota_x\colon 1_{Fx} \cong F(1_x)$ and to every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$ in $\mathfrak{B}$, an iso-2-morphism $\phi_{f,g}\colon F(g) \circ f(f) \cong F(gf)$. These assignments are natural and they cohere with bicategorical structures of

$\mathfrak{B}$ and $\mathfrak{C}$. See §A.3 for a complete definition of pseudo-functors including a full list of coherence conditions. We shall refer to iso-2-morphisms τ_x and ϕ_x as **comparison** (aka **constraints**) 2-morphisms.

A *strict* 2-functor (cf. Definition 1.1.1) can be viewed as a pseudo-functor whereby comparison 2-morphisms ι and ϕ are identity 2-morphisms. 1 Pseudo-functors of bicategories are generalized to **lax functors** by dropping the condition of invertibility of $\iota_x \colon 1_{Fx} \Rightarrow F(1_x)$ and $\phi \colon F(g) \circ F(f) \Rightarrow F(gf)$ for all x in $\mathfrak{B}_0$ and all (composable) morphisms f and g. Reversing the direction of comparison 2-morphisms yield the notion of **oplax functors**[3] of bicategories. An oplax functor of the type $\mathfrak{B} \to \mathfrak{C}$ is the same thing as a lax functor of the type $\mathfrak{B}^{\mathrm{co}} \to \mathfrak{C}^{\mathrm{co}}$. An (op)lax functor of bicategories is **normal** (resp. **strictly normal**) whenever the comparison 2-morphisms τ_x are all iso (resp. identity) 2-morphisms .

Remark 1.3.1. We recall two well-known observations on lax functors ([Bén67], [Lac10a]):

(i) A monad in a bicategory $\mathfrak{B}$ is precisely a lax functor $\mathbf{1} \to \mathfrak{B}$.

(ii) For a monoidal category $\mathcal{V}$ and a set C_0, a lax functor $C \colon {C_0}^{\mathrm{ind}} \to \Sigma\mathcal{V}$ is a $\mathcal{V}$-enrichment structure on elements of C_0. Note that ${C_0}^{\mathrm{ind}}$ is the indiscrete category of C_0 so that the unique functor ${C_0}^{\mathrm{ind}} \to \mathbf{1}$ is fully faithful.

$$\begin{tikzcd}
{C_0}^{\mathrm{ind}} \arrow[r, "C"] \arrow[d, tail] & \Sigma\mathcal{V} \\
\mathbf{1} &
\end{tikzcd} \tag{1.5}$$

Note that $C(x) = *$ for all elements $x \in C_0$, and we write $C(x, y) \in \mathcal{V}$ for the value of C at the unique morphism from x to y in X^{ind}. The lax constraints give the (enriched) composition $C(y, z) \otimes C(x, y) \to C(x, z)$ and the unit $I \to C(x, x)$. In particular a lax functor $C \colon {C_0}^{\mathrm{ind}} \to \Sigma\,\mathcal{S}\mathrm{et}$, where $\mathcal{S}\mathrm{et}$ is

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[3] Lax and oplax functors of bicategories generalize lax and oplax functors of monoidal categories.

the cartesian monoidal category of sets, is just a small category whose set of objects is exactly C_0.

Consider two functors $F, G \colon \mathcal{B} \rightrightarrows \mathcal{C}$ of categories and a natural transformation $\theta \colon F \Rightarrow G$. For any morphism $f \colon b \to b'$ in $\mathcal{B}$ we get an identity of morphisms in $\mathcal{D}$, namely $G(f) \circ \theta_b = \theta_{b'} \circ F(f)$. In passing from categories to 2-categories, we can weaken this condition by replacing the above identity with an iso-2-morphism or even just any 2-morphism which places $G(f) \circ \theta_c$ and $\theta_{c'} \circ F(f)$ in the same connected component. Of course this weakening must be compatible with 2-categorical structures of domain and codomain of F and G and also how strictly F and G preserve these structures. Detailed definitions of various well-known notions of **2-transformation of functors** of bicategories with their coherence conditions are given in Appendix A.4. We have the following classes of natural transformation between morphisms of bicategories:

$$\{\text{strict}\} \subset \{\text{pseudo}\} \subset \{\text{normal (op)lax}\} \subset \{\text{(op)lax }\} \tag{1.6}$$

DEFINITION 1.3.2. A 2-transformation (strict, pseudo, lax) $\theta \colon F \Rightarrow G \colon \mathfrak{B} \to \mathfrak{C}$ is an **equivalence 2-transformation** whenever every morphism $\theta_x \colon Fx \to Gx$ is an equivalence in $\mathfrak{C}$.

REMARK 1.3.3. We remark that there is quite some confusion in literature in using prefixes "op" and "co". For instance, 'lax' and 'oplax' as attributes of functors of 2-categories and bicategories are occasionally used in exactly opposite way we just defined. Same goes for their use as attributes of natural transformations (e.g. [Joh02a]). The terms "left lax" (for what we called lax) and "right lax" (for what we called oplax) were introduced in [Str72]. Adding to this confusion, some people have used 'colax' instead of oplax particularly in the context of monoidal categories. However, our main concern is not to introduce yet new terminology, but to maintain consistency throughout the thesis. So, in our terminology we follow Benabou's original choice ([Bén67]), as well as Leinster ([Lei98]), Borceaux, and most of Australian writings (e.g. [Kel74b]).

A pseudo functor from a 2-category $\mathfrak{K}$ to $\mathfrak{Cat}$ can be strictified to a strict 2-functor up to a (pseudo) natural equivalence.

REMARK 1.3.4. Suppose $F\colon \mathcal{K} \to \mathfrak{Cat}$ is a pseudo functor where $\mathcal{K}$ is a 2-category. For each object c of $\mathcal{K}$, let $F_{\mathrm{s}}(c)$ be the category whose objects are pairs (f, x) where $f\colon d \to c$ is a morphism in $\mathcal{K}$ and x is an object of $F(d)$, and a morphism $u\colon (f, x) \to (g, y)$ in $F_{\mathrm{s}}(c)$ is given by a morphism $u\colon f_*(x) \to g_*(y)$ in category $F(c)$, where $f_*(x) := F(f)(x)$. The identity morphism and composition in $F_{\mathrm{s}}(c)$ are trivially given by the identity and composition structure in $F(c)$. Also, F_{s} extends to a *strict* 2-functor on $\mathfrak{K}$: its action on a morphism $h\colon c \to c'$ is given by $F_{\mathrm{s}}(h)(f, x) = (hf, x)$, and $F_{\mathrm{s}}(h)((f, x) \xrightarrow{u} (g, y)) = (hf, x) \xrightarrow{\phi_{g,h} h_*(u) \phi_{f,h}} (hg, y)$. The action of F_{s} on 2-morphism $\theta\colon h_0 \Rightarrow h_1$ is given at the component (f, x) by $(\alpha \bullet f)_*(x)$. Finally, $\eta\colon F \Rightarrow F_{\mathrm{s}}$ with $\eta_c(x) = (id_c, x)$ establishes a pseudo natural equivalence with quasi-inverse $\eta_c^{-1}(f, x) = f_*(x)$.

There is a really powerful and more general approach which covers a wide range of strictification results about bicategories and pseudo functors, and in particular covers the case of remark above ([BKP89], [Pow89]). See appendix A.6

REMARK 1.3.5. Any normal lax functor $F\colon \mathfrak{B} \to \mathfrak{C}$ of bicategories can naturally be modified to a strictly normal lax functor $\widetilde{F}\colon \mathfrak{B} \to \mathfrak{C}$. The functor $\widetilde{F}$ is defined exactly as F on objects and non-identity morphisms. We define $\widetilde{F}(1_x) := 1_{Fx}$, and accordingly modify definition of F on 2-morphisms using invertible $\iota\colon 1_{Fx} \Rightarrow F(1_x)$. Thus, we get an equivalence pseudo natural transformation $\delta\colon \widetilde{F} \overset{\simeq}{\Rrightarrow} F$ where $\delta_x = \mathrm{id}_x$ for all objects x of $\mathfrak{B}$, and

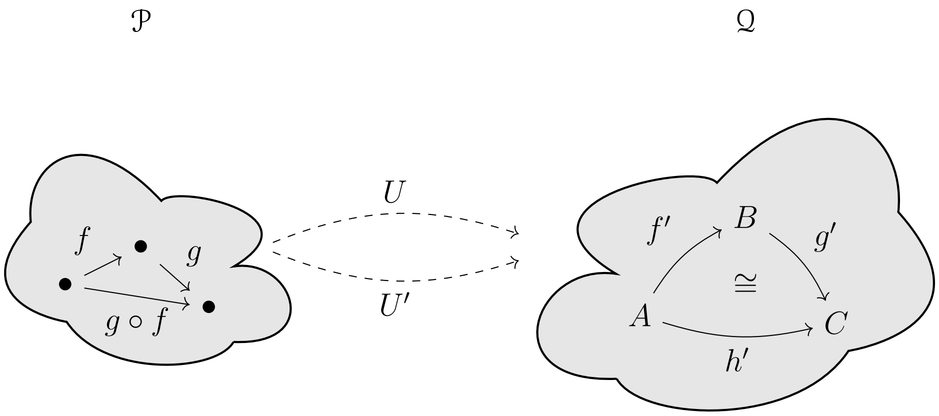
$$\delta_f = \begin{cases} \lambda^{-1}_{F(1_x)} \circ \rho_{F(1_x)} \circ (\iota_{Fx} \bullet 1_{Fx}) & \text{if } f = 1_x \\ \lambda^{-1}_{F(1_x)} \circ \rho_{F(1_x)} & \text{otherwise.} \end{cases}$$

Evidently $\widetilde{F}$ is strictly normal.

From the structural point of view, the more appropriate morphisms of bicategories are pseudo functors. For observe that $\mathfrak{B}$ has the structure of a bicategory iff the representable $\mathfrak{B}(X, -)\colon \mathfrak{B} \to \mathfrak{Cat}$ has the structure of a pseudo functor, and for a morphism $f\colon X' \to X$ in $\mathfrak{B}$, there is an induced pseudo natural transformation $f^*\colon \mathfrak{B}(X, -) \Rightarrow \mathfrak{B}(X', -)$. For this reason, we shall sometimes refer to pseudo functors of bicategories as **homomorphism** of bicategories.

Moreover, once we introduce the notion of limit for bicategories (at the appropriate generality they are weighted bilimits of §1.9) it is straightforward to see that the representable homomorphism $\mathfrak{B}(X, -)$ preserves bilimits.

However, we are still interested in strict 2-functors of 2-categories and bicategories and they play an important role in Chapters 2 and 4. Additionally, The strict 2-functors are generally better behaved than pseudo-functors and lax functors with respect to (strict) limits and colimits. For instance, in the category $\|\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}\|_1$, the pushout of span $\mathbf{2} \xleftarrow{0} \mathbf{1} \xrightarrow{1} \mathbf{2}$ exists and is isomorphic to the category $\mathbf{3}$. However, this does not hold in the category $\|\mathbf{2}\,\mathcal{C}\mathrm{at}_{psd}\|_1$: any such pushout $\mathcal{P}$ must contain two arrows and their composite and it is in general not uniquely decidable where to send the composite in some other cocone categories: the cocone $\mathcal{Q}$ in below has three 1-morphisms and an iso-2-morphism $\varphi\colon g' \circ f' \cong h'$. Now, there is no *unique* pseudo-functor from $U\colon \mathcal{P} \to \mathcal{Q}$ with $U \circ g = g'$ and $U \circ f = f'$: we can choose $U\colon \mathcal{P} \to \mathcal{Q}$ with $U(g \circ f) = g' \circ f'$ and iso 2-morphism $\phi_{f,g}$ being id, or U' with $U'(g \circ f) = h'$ and iso 2-morphism $\phi_{f,g}$ being φ.



Pseudo functors (resp. lax functors) of bicategories are composed strictly: given pseudo functors $(F, \phi, \iota)\colon \mathfrak{B} \to \mathfrak{C}$ and $(G, \psi, \kappa)\colon \mathfrak{C} \to \mathfrak{D}$, we define the composition $G \circ F\colon \mathfrak{B} \to \mathfrak{D}$ on objects and morphisms by successive actions of F and G, that is $G \circ F(x) = G(F(x))$, $G \circ F(f) = G(F(f))$, and $G \circ F(\alpha) = G(F(\alpha))$. The unit comparison is given by $(\kappa \circ \iota)_x := G(\iota_x) \circ \kappa_{F(x)}$ and the composition comparison is given by $(\psi \circ \phi)_{f,g} := G(\phi_{f,g}) \circ \psi_{F(f),F(g)}$.

Hence, we write $(G \circ F, \kappa \circ \iota, \psi \circ \phi)$ for the composite pseudo (resp. lax) functor. With this composition we get a category $\mathcal{B}i\mathcal{C}at$ (resp. $\mathcal{B}i\mathcal{C}at_{lax}$) of small bicategories and pseudo (resp. lax) functors. We have the following chain of (non-full) subcategories:

$$\mathcal{B}i\mathcal{C}at_{str} \subsetneq \mathcal{B}i\mathcal{C}at \subsetneq \mathcal{B}i\mathcal{C}at_{nlax} \subsetneq \mathcal{B}i\mathcal{C}at_{lax} \tag{1.7}$$

REMARK 1.3.6. We note that unlike the situation with 2-categories of categories, bicategories and pseudo functors do not from a 2-category or even a bicategory. The reason is that independent of the choice of the kind of 2-transformation, be it strict, pseudo, or lax, one fundamental issue persists and that is they do not have a strict vertical composition. For any 2-transformations

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, bend left=80, "K"] \arrow[r, bend left=25, "H"] \arrow[r, bend right=25, "G"'] \arrow[r, bend right=80, "F"'] \arrow[r, phantom, bend left=55, "\Uparrow\gamma"] \arrow[r, phantom, "\Uparrow\beta"] \arrow[r, phantom, bend right=55, "\Uparrow\alpha"] & \mathfrak{C}
\end{tikzcd}$$

and for any object x of $\mathfrak{B}$, we have $\gamma_x \circ (\beta_x \circ \alpha_x) \cong (\gamma_x \circ \beta_x) \circ \alpha_x)$ in $\mathfrak{C}$. Therefore, the vertical composition of 2-transformations is weakly associative and as such this forces us to arrive at the *tricategory* $\mathbf{Hom}$ of bicategories, homomorphisms, pseudo natural transformations, and modifications (See [Str80], [Lac10b]). $\mathbf{Hom}$ constitutes the archetypal instance of tricategory structure. However, we shall not define this structure. We refer the interested reader to [GPS95] and [Gur06]. $\mathbf{Hom}$ is enriched over bicategories. Observe that for any 2-category $\mathfrak{K}$, the bicategory $\mathbf{Hom}(\mathfrak{B}, \mathfrak{K})$ is actually a 2-category even if $\mathfrak{B}$ is a bicategory.

REMARK 1.3.7. There is a sub-tricategory $\mathbf{Gray}$ of $\mathbf{Hom}$ which consists of strict 2-categories, strict 2-functors, pseudo transformations, and modifications. In $\mathbf{Gray}$ the composition of morphisms is strictly associative and unital as well as vertical composition of 2-morphisms. However, the interchange law holds only up to an invertible modification. Indeed, $\mathbf{Gray}$ is a prototypical example of $\mathcal{G}ray$-enriched category where $\mathcal{G}ray$ is the closed monoidal category of 2-categories and strict 2-functors with monoidal structure given by the Gray tensor product $\otimes_{psd}$. The underlying category of $\mathcal{G}$ray is given by $||\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}||_1$. Recall that for 2-categories $\mathfrak{J}$ and $\mathfrak{K}$, the *Gray tensor product* $\mathfrak{J} \otimes_{psd} \mathfrak{K}$ is a "fattened up" deformation of the cartesian product $\mathfrak{J} \times \mathfrak{K}$ in which the equality $(f,1)(1,g) = (1,g)(f,1)$ is replaced by an invertible[4] 2-morphism for any pair of morphisms $f\colon x \to x'$ in $\mathfrak{J}$ and $g\colon y \to y'$ in $\mathfrak{K}$. The tensor product is given by the universal property expressed by the following bijection

$$||\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}||_1(\mathfrak{J} \otimes_{psd} \mathfrak{K}, \mathfrak{L}) \cong ||\mathbf{2}\,\mathcal{C}\mathrm{at}_{str}||_1(\mathfrak{J}, \mathbf{2}\,\mathfrak{Cat}_{psd}(\mathfrak{K}, \mathfrak{L}))$$

The closed structure $[\mathfrak{K}, \mathfrak{L}]$ of $\mathcal{G}$ray, as used in the bijection above, is given by the hom 2-category $\mathbf{2}\,\mathfrak{Cat}_{psd}(\mathfrak{K}, \mathfrak{L})$. Note that analogous to the case of bicategories, for every object X of a tricategory $\mathbf{T}$, $\mathbf{T}(X,X)$ is a $\mathcal{G}$ray-monoid [5].

In [Lac07] it is proved that the tricategory $\mathbf{Gray}$ is not equivalent to $\mathbf{Hom}$. However, any tricategory, including $\mathbf{Hom}$, is equivalent to some $\mathcal{G}$ray-category [GPS95]. We also note that there is an embedding $\mathbf{Gray}(\mathfrak{K}, \mathfrak{L}) \hookrightarrow \mathbf{2}\,\mathfrak{Cat}_{psd}(\mathfrak{K}, \mathfrak{L})$ of 2-categories, and for a strict 2-functor $H\colon \mathfrak{L} \to \mathfrak{L}'$, post-composition by H induces a strict 2-functor $H_*\colon \mathbf{Gray}(\mathfrak{K}, \mathfrak{L}) \to \mathbf{Gray}(\mathfrak{K}, \mathfrak{L}')$. This observation is also true in any $\mathcal{G}$ray-enriched category. The same observation also shows that why $\mathbf{2}\,\mathfrak{Cat}_{psd}$ can not be a $\mathcal{G}$ray-category.

Our interest in lax functors of bicategories comes directly through the way we arrived at bicategories as a generalization (in this case *oidification*) of monoidal categories. In fact, strong monoidal functors $F\colon (\mathcal{V}, \otimes, I) \to (\mathcal{V}', \otimes', I')$ are in one-to-one correspondence with pseudo functors $\Sigma F\colon \Sigma\mathcal{V} \to \Sigma\mathcal{V}'$ of bicategories. However, not the strong monoidal but the lax (and colax) monoidal are the prevalent functors of monoidal categories. For instance, in the context of monoidal Dold-Kan correspondence, the Moore chain complex functor and the nerve functor are lax functors ([nLa19a]). Also, note that lax monoidal functors transfer monoids to monoids: if $\langle M, \mu\colon M \otimes M \to M, \eta\colon I \to M\rangle$ is

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[4] We remark that the original version of Gray tensor product ([Gra74]) did not require invertibility condition and introduced the concept using a general 2-morphism

[5] i.e. a monoid object in $\mathcal{G}$ray or equivalently a one-object $\mathcal{G}$ray-category

a monoid (resp. comonoid) in a monoidal category $(\mathcal{V}, \otimes, I)$ and (F, ϕ, ι), as above) is a lax (resp. colax) monoidal functor then

$$\langle F(M), F(\mu) \circ \phi_{M,M} \colon F(M) \otimes F(M) \to F(M), F(\eta) \circ \iota_I \colon I \to F(M) \rangle$$

is a monoid in $\mathcal{V}'$.

Same is true when we generalize from monoidal categories to bicategories: a lax (resp. oplax) functor $(F, \phi, \iota) \colon \mathfrak{B} \to \mathfrak{C}$ of bicategories take any monad $\langle t \colon X \to X, \mu, \eta \rangle$ to the monad $\langle F(t) \colon F(X) \to F(X), F(\mu) \circ \phi_{t,t}, F(\eta) \circ \iota_X \rangle$. This can be observed from the aforementioned fact of lax monoidal functors and the observation that $F_{X,X} \colon \mathfrak{B}(X, X) \to \mathfrak{C}(FX, FX)$ is a lax monoidal functor of monoidal categories. Another way to reach to the same conclusion is to realize that a monad in $\mathfrak{B}$ is exactly a lax functor from the terminal bicategory to $\mathfrak{B}$ and that lax functors are stable under composition.

However there are some aspects of lax monoidal functors which do not generalize to lax functors of bicategories and may be regarded as unpleasant properties of lax functors. There is a 2-category $\mathfrak{MonCat}_{lax}$ of monoidal categories, lax monoidal functors, and monoidal transformations. This 2-category has a sub-2-category $\mathfrak{MonCat}_{strong}$ where the 1-morphisms are restricted to the strong monoidal functors. Although $\mathfrak{MonCat}_{strong}$ is not a full sub-2-category it has some nice properties: by doctrinal adjunction, any left adjoint in $\mathfrak{MonCat}_{lax}$ is automatically strong monoidal. Since any equivalence in a 2-category can be improved to an adjoint equivalence, any equivalence in $\mathfrak{MonCat}_{lax}$ consists of strong monoidal functors. Thus, the notion of "equivalence of monoidal categories" doesn't depend on what kind of functor one chooses to work with, and the notion of "lax monoidal functor" is invariant under this equivalence.

For a start, we can not make a 2-category out of bicategories, lax functors, and any kind of natural transformation of 2-functors (See 1.6). The reason is simple: were they to form a 2-category we would be able to whisker 1-morphisms with 2-morphisms. To the contrary suppose we can. Let $(G, \psi, \kappa) \colon \mathfrak{C} \to \mathfrak{D}$ be a lax functor of bicategories and $\alpha \colon (F, \phi, \iota) \Rightarrow (F', \phi', \iota')$ a lax natural transformation of lax functors $F, F' \colon \mathfrak{B} \rightrightarrows \mathfrak{C}$. Form the whiskered lax natural transformation $G \centerdot \alpha\colon GF \Rightarrow GF'$. For a morphism $f\colon x \to y$ in $\mathfrak{B}$, we have the 2-morphisms

$$GF'(f) \circ G(\alpha_x) \xRightarrow{G(\psi_{F'(f),\alpha_x})} G(F'(f) \circ \alpha_x) \xRightarrow{G(\alpha_f)} G(\alpha_y \circ F(f)) \xLeftarrow[G(\psi_{\alpha_y,F(f)})]{} G(\alpha_y) \circ GF(f)$$

in $\mathfrak{D}$. But we see that the most right arrow goes in the wrong direction and there is no chance we can form the component of $G \centerdot \alpha$ at f.

REMARK 1.3.8. Under two special circumstances such whiskering in above would be possible: first, if the functors of bicategories are pseudo instead of lax. In this scenario, for our desired whiskering, we could use the inverse of the troublesome 2-morphism $G(\psi_{\alpha_y,Ff})$ in $\mathfrak{D}$. Although whiskering is possible it does not satisfy the exchange law, even for strict 2-functors, for there is no reason that the pasting of the diagrams

$$\begin{tikzcd}
GFx \arrow[r, "G(\alpha_x)"] \arrow[d, "GF(f)"'] \arrow[dr, phantom, "\overset{G(\alpha_f)}{\Swarrow}"] & GF'x \arrow[r, "\beta_{F'x}"] \arrow[d, "GF'(f)"] \arrow[dr, phantom, "\underset{\beta_{F'(f)}}{\Swarrow}"] & G'F'x \arrow[d, "G'F'f"] \\
GFy \arrow[r, "G(\alpha_y)"'] & GF'y \arrow[r, "\beta_{F'y}"'] & G'F'y
\end{tikzcd}
\qquad
\begin{tikzcd}
GFx \arrow[r, "\beta_{Fx}"] \arrow[d, "GF(f)"'] \arrow[dr, phantom, "\overset{\beta_{Ff}}{\Swarrow}"] & G'Fx \arrow[r, "G'(\alpha_x)"] \arrow[d, "G'F(f)"] \arrow[dr, phantom, "\underset{G'(\alpha_f)}{\Swarrow}"] & G'F'x \arrow[d, "G'F'(f)"] \\
GFy \arrow[r, "\beta_{Fy}"'] & G'Fy \arrow[r, "G'(\alpha_y)"'] & G'F'y
\end{tikzcd}$$

on the two sides should be equal unless either $\alpha\colon F \Rightarrow F'$ or $\beta\colon G \Rightarrow G'$ is identity. Therefore, we still cannot form a 2-category with lax transformations even if we restricted to strict 2-categories and strict functors.

Indeed, there is only one good way of getting a 2-category of bicategories and lax functors with non-strict natural transformations as its 2-morphisms. The 2-morphisms are restricted forms of lax natural transformations called "icons"[6] ([Lac10b]). An **icon** between lax functors $F, G\colon \mathfrak{B} \rightrightarrows \mathfrak{C}$ of bicategories is an oplax transformation α with extra constraints that all components α_x are identity morphisms for all objects x in $\mathfrak{B}$.

$$\begin{tikzcd}
Fx \arrow[r, equal] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\alpha_f}{\Nearrow}"] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, equal] & Gy
\end{tikzcd}$$

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[6] Short for Identity Component Oplax Natural-transformation

In the case of one-object bicategories the icons are precisely the monoidal natural transformations. This shows that, from a certain perspective, the reason icons are to be preferred as generalization of monoidal transformations. Another observation in this direction is to look at the structure of an oplax transformation of lax functors $C, C' \colon S^{\mathrm{ind}} \rightrightarrows \Sigma\,\mathcal{S}\mathrm{et}$. By remark (ii), C, C' are categories with a common set of objects S. An oplax transformation $\alpha\colon C \Rightarrow C'$ provides us with a family of sets $\{X(c)\}_{c\in S}$ and a family $\{\alpha_{c,d}\colon C(c,d)\times X(d) \to X(c) \times C'(c,d)\}$ of functions, satisfying evident identity and composition constraints. The data of α can be elegantly packaged into a bundle $\alpha_S\colon X \to S$ together with a bundle map $\alpha_{S\times S} C \times \pi_1^* X \to \pi_0^* X \times C'$ over $S \times S$ satisfying a unit and a composition law. When α is an icon the bundle $\alpha_S\colon X \to S$ is isomorphic to the trivial bundle $\mathrm{Id}_S\colon S \to S$, and the bundle map $\alpha_{S\times S}$ is a functor $C \to D$. Therefore, icons between lax functors $C, C'\colon S^{\mathrm{ind}} \rightrightarrows \Sigma\,\mathcal{S}\mathrm{et}$ correspond exactly to the functors $C \to C'$ which are identity on objects.

However, icons have their downsides as well. The requirement the the components α_x must be strict equalities is unsatisfactory in many situations. For instance as we shall see in chapter 2 that a cloven Grothendieck prefibration $P\colon \mathcal{E} \to \mathcal{B}$ of categories correspond to a lax functor $P\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$, and a map of prefibrations to a pseudo transformation. However, an icon between any two such lax functors would require strict equality of fibres of corresponding prefibrations, i.e. an equality of categories.

At any rate, The additional constraints of icons make the obstructions in Remark 1.3.6 and Remark 1.3.8 in forming a 2-category of bicategories disappear. Indeed, we can form the 2-category $\mathbf{Icon}$ of bicategories, lax functors, and icons. The same paper introduces a 2-monad on the 2-category of $\mathcal{C}$at-enriched graphs for whose algebras are 2-categories, and pseudo (resp. lax, resp. oplax) functors are the pseudo (resp. lax, resp. oplax) morphisms of algebras, and icons are the transformations of algebra morphisms.

Another serious problem with the lax functors of bicategories is that they are not invariant under equivalence or biequivalence of bicategories. Again, consider an inhabited category C as a lax functor $C\colon {C_0}^{\mathrm{ind}} \to \Sigma\,\mathcal{S}\mathrm{et}$ For an inhabited set C_0. We have the equivalence ${C_0}^{\mathrm{ind}} \simeq \mathbf{1}$. But composing C with this equivalence does not yield a lax functor if C, for instance, has more than one connected components.

There are stronger notions than equivalence of bicategories and although we shall not make a great use of them, we define them here for the sake of completeness and, more importantly, contrast.

DEFINITION 1.3.9. A pseudo functor (resp. lax) $(F, \phi, \iota)\colon \mathfrak{B} \to \mathfrak{C}$ is an **isomorphism of bicategories** if it has an inverse pseudo functor (resp. lax) $(G, \psi, \kappa)\colon \mathfrak{C} \to \mathfrak{B}$, i.e. $(G \circ F, \psi \circ \phi, \kappa \circ \iota) = (\mathrm{Id}_{\mathfrak{B}}, \mathrm{id}, \mathrm{id})$ and $(F \circ G, \phi \circ \psi, \iota \circ \kappa) = (\mathrm{Id}_{\mathfrak{C}}, \mathrm{id}, \mathrm{id})$.

Recall that a functor $U\colon \mathcal{C} \hookrightarrow \mathcal{D}$ exhibits $\mathcal{C}$ as a full subcategory of $\mathcal{D}$ if U is a fully faithful functor, that is $U_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(Ux, Uy)$ is an equivalence of sets for all objects x, y of $\mathcal{C}$. In a similar fashion

DEFINITION 1.3.10. A homomorphism (resp. 2-functor) $U\colon \mathfrak{B} \to \mathfrak{C}$ exhibits $\mathfrak{B}$ as a **sub-bicategory** (resp. sub-2-category) of $\mathfrak{C}$ if the functor $U_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(Ux, Uy)$ is an equivalence of categories for all objects x, y of $\mathfrak{B}$.

This means that any morphism $g\colon Ux \to Uy$ in $\mathfrak{C}$ is isomorphic to Uf for some morphism $f\colon x \to y$ in $\mathfrak{B}$, and any 2-morphism $\beta\colon Uf \Rightarrow Uf'$ in $\mathfrak{C}$ is equal to $U(\alpha)$ for a unique 2-morphism $\alpha\colon f \Rightarrow f'$ in $\mathfrak{B}$.

As a non-example of full subbicategory consider the embedding of bigroupoids $\Pi_{\leq 2}(S^1) \to \Pi_{\leq 2}(S^1 \vee S^1)$ induced by the inclusion of, say, the left component.

THEOREM 1.3.11. The category $\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}$ of (small) bicategories is bicategory-enriched: for any bicategories $\mathfrak{B}$ and $\mathfrak{C}$, pseudo functors, pseudo natural transformations and modification between them form a bicategory $\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}(\mathfrak{B}, \mathfrak{C})$.

For important examples of categories enriched in a bicategory see [Wal81], [Wal82], [Bet+83].

1.4 Constructions on bicategories

Construction 1.4.1 (The symmetries of bicategories). The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on the class of bicategories.[7] This action yields four 3-functors:

- $(-)^{(0,0)} = \mathbf{Id}\colon\ \mathbf{Hom} \to \mathbf{Hom}$
- $(-)^{(1,0)} = (-)^{\mathrm{op}}\colon \mathbf{Hom}^{\mathfrak{co}} \to \mathbf{Hom}$
- $(-)^{(0,1)} = (-)^{\mathrm{co}}\colon\ \mathbf{Hom} \to \mathbf{Hom}$
- $(-)^{(1,1)} = (-)^{\mathrm{coop}}\colon\ \mathbf{Hom} \to \mathbf{Hom}$

For any bicategory $\mathfrak{B}$, the bicategory $\mathfrak{B}^{\mathrm{op}}$ is obtained by reversing the 1-morphisms only, $\mathfrak{B}^{\mathrm{co}}$ by reversing the 2-morphisms only, and $\mathfrak{B}^{\mathrm{coop}}$ by reversing both 1-morphisms and 2-morphisms. Since the definition of a bicategory $\mathfrak{B}$ was given in terms of its local hom-categories, we remark that

$$\mathfrak{B}^{\mathrm{co}}(X^{\mathrm{co}}, Y^{\mathrm{co}}) := \mathfrak{B}(X,Y)^{\mathrm{op}}$$

The operation $(-)^{\mathrm{co}}$ is sometimes referred to as *conjugation*. Note that if $f \dashv g$ with unit η and counit ϵ in $\mathfrak{K}$, then $g^{\mathrm{co}} \dashv f^{\mathrm{co}}$ with unit ϵ^{co} and counit η^{co} in $\mathfrak{K}^{\mathrm{co}}$.

Moreover, we have $\mathbf{Hom}(\mathfrak{B}^{\mathrm{op}}, \mathfrak{C}^{\mathrm{op}}) \simeq \mathbf{Hom}(\mathfrak{B}, \mathfrak{C})^{\mathrm{op}}$ an also $\mathbf{Hom}(\mathfrak{B}^{\mathrm{co}}, \mathfrak{C}^{\mathrm{op}}) \simeq \mathbf{Hom}(\mathfrak{B}, \mathfrak{C})^{\mathrm{co}}$

Remark 1.4.2. We seriously warn the reader to *not* carry the logic in construction above to its conclusion. The terminological inconsistency mentioned in Remark 1.3.3 is not accidental. As we have said, in a 2-category 'op' refers to reversing the 1-morphisms and in a category it denotes reversing the 1-morphisms. If we use the terminology of 'op' and 'co' strictly consistently, doesn't it follow that we should call colimits in a 1-category or in a 2-category 'oplimits' and yet, no one does that.

---

[7] In general the group $(\mathbb{Z}/2\mathbb{Z})^n$ acts on the (meta) n-category of (weak) n-categories and every element $g = (g_1, \dots, g_n)$ of the group determined a meta n-functor $\mathrm{rs}(g)\colon (\mathbf{n}\,\mathfrak{CAT})^{\mathfrak{g}} \to \mathbf{n}\,\mathfrak{CAT}$ where $\mathrm{rs}\colon (\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$ is the "right shift" group homomorphism. In particular $\mathrm{rs}(0,1) = (0,0)$ and $\mathrm{rs}(1,0) = (0,1)$.

In category theory we use 'co' for most dualizations. Furthermore, later in Chapter 2, we shall see that an opfibration internal to a 2-category $\mathfrak{K}$ is indeed a fibration in $\mathfrak{K}^{\mathrm{co}}$. In fact, what is nowadays called opfibrations was originally called 'cofibrations' in [Gra66]. However, this clashed with the use of the term 'cofibration' in topology, so it was avoided in the category theory literature quite consistently thereafter. One of the root reasons for recurring inconsistencies is the fact that categorical structures can be generalized to 2-categories in many ways different ways: through the realization of a category as a discrete 2-category, as the delooping bicategory of a monoidal category (See Example 1.2.1), and through representational approach (See §1.7). Each of these axes of generalization gives its own account of arriving at "op-concepts" and "co-concepts".

CONSTRUCTION 1.4.3 (The underlying category of a 2-category). We can throw away all 2-morphisms of a 2-category and get a category. More precisely, this is done by the transport of enrichment structure. Suppose $F\colon \mathcal{V} \to \mathcal{V}'$ is a lax-monoidal functor and $\mathcal{C}$ is a $\mathcal{V}$-enriched category. We transport the enrichment structure of $\mathcal{C}$ along F: we construct a $\mathcal{V}'$-enriched category $\mathcal{C}_F$ where

- $\mathrm{Ob}(\mathcal{C}_F) := \mathrm{Ob}(\mathcal{C})$
- $\mathcal{C}_F(c,d) := F(\mathcal{C}(c,d))$ for any pair of objects c, d of $\mathcal{C}$.
- The composite morphism $I_{\mathcal{V}'} \to F(I_{\mathcal{V}}) \to F(\mathcal{C}(c,c))$ in $\mathcal{V}'$ defines the unit map of $\mathcal{C}_F$.
- The composite morphism $F(\mathcal{C}(c,c')) \otimes F(\mathcal{C}(c',c'')) \to F(\mathcal{C}(c,c') \otimes \mathcal{C}(c',c'')) \to F(\mathcal{C}(c,c''))$ in $\mathcal{V}'$ defines the composition map of $\mathcal{C}_F$.

Transporting the enrichment structure of a 2-category $\mathfrak{K}$ along the representable cartesian monoidal functor $\mathrm{Hom}(\mathbf{1}, -) : \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}$, which sends a small category C to the set of objects of C, yields a category $||\mathfrak{K}||_1$ which is called the **underlying category** of $\mathfrak{K}$. We have:

- $\mathrm{Ob}(\mathfrak{K}_0) = \mathrm{Ob}(\mathfrak{K})$
- $\mathfrak{K}_0(x,y) := \mathrm{Hom}(\mathbf{1}, \mathfrak{K}(x,y)) \cong \mathrm{Ob}(\mathfrak{K}(x,y))$

Obviously, $(\mathfrak{K}^{\mathrm{op}})_1 \cong (||\mathfrak{K}||_1)^{op}$ and $(\mathfrak{K}^{\mathrm{co}})_1 \cong ||\mathfrak{K}||_1$.

CONSTRUCTION 1.4.4. Any bicategory $\mathfrak{B}$ has a **classifying category** $\Pi_{\leq 1}(\mathfrak{B})$ associated to it. The objects remain the same while the morphisms of the classifying category are isomorphism classes of morphisms of $\mathfrak{B}$. This construction gives us a homomorphism $\Pi_{\leq 1}\colon\ \mathcal{B}i\mathcal{C}at \to \mathfrak{Cat}$. This construction originally appeared in [Bén67, page 56]. Of course the Construction 1.4.3 cannot be carried out in the same way for bicategories since we cannot discard 2-morphisms of a bicategory and get a category. However, we can regard the classifying category of a bicategory as its *homotopical underlying 1-category*. This view is justified by the observation that the classifying category of the bigroupoid $\Pi_{\leq 2}X$ of a topological space X (Example 1.2.2) is precisely the fundamental groupoid of X.

CONSTRUCTION 1.4.5. Recall that to each category $\mathcal{C}$, one associates the maximal sub-groupoid $\mathcal{C}\mathrm{ore}(\mathcal{C})$ whose morphisms are invertible morphisms of $\mathcal{C}$. Indeed, $\mathcal{C}\mathrm{ore}$ is the right adjoint to the forgetful embedding $\mathcal{G}rpd \to \mathcal{C}\mathrm{at}$ whose left adjoint in turn is the reflective localization $L\colon\ \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}[\eta^{-1}]$, where $\eta\colon\ \mathbf{2} \to \mathbb{I}$ is the inclusion of the free walking arrow category into the walking isomorphism interval. Indeed, L adds formal inverses to categories to make them into groupoids. Similarly, to each bicategory $\mathfrak{B}$, we associate the maximal sub-bigroupoid $\mathfrak{Core}(\mathfrak{B})$ whose 2-morphisms are invertible 2-morphisms of $\mathfrak{B}$. For instance, $\mathfrak{Core}(\mathfrak{Cat})(\mathbf{1}, \mathcal{C}) \cong \mathcal{C}\mathrm{ore}(\mathcal{C})$. All pseudo weighted limits and colimits (1.9) in a bicategory $\mathfrak{B}$ are indeed lax weighted limits and colimits in $\mathfrak{Core}(\mathfrak{B})$. Also, to any bicategory $\mathfrak{B}$, we associate the full sub-bicategory $\mathfrak{Grpd}(\mathfrak{B})$ whose objects are bigroupoidal objects (Definition 1.7.5) of $\mathfrak{B}$. For instance $\mathfrak{Grpd}(\mathfrak{Cat}) = \mathfrak{Grpd}$. Obviously $\mathfrak{Grpd}(\mathfrak{Core}(\mathfrak{B})) \cong \mathfrak{Core}(\mathfrak{B})$. Finally, we have an adjunction

$$\begin{tikzcd}
(\mathbf{2},\mathbf{1})\,\mathfrak{Cat}_{str} \arrow[r, "\text{Inc}" description] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \mathbf{2}\,\mathfrak{Cat}_{str} \arrow[l, bend right=70, "\mathfrak{L}"'] \arrow[l, bend left=70, "\mathfrak{Core}"]
\end{tikzcd}$$

CONSTRUCTION 1.4.6 (The pseudo-functor of points). Suppose $\mathfrak{B}$ is a bicategory with the terminal object 1. For every object $X \in \mathfrak{B}_0$, a **point** x of X is a morphism

$x\colon 1 \to X$. The points of X form a category, namely $pt_{\mathfrak{B}}(X) \simeq \mathfrak{B}(1, X)$. The homomorphism $pt_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{Cat}$ is represented by the terminal object 1 of $\mathfrak{B}$.

For instance, in the bicategory $\mathcal{T}op_{\leq 2}$ from Example 1.2.3, the groupoid $pt(\mathrm{D}^2)$ of points of 2-dimensional disk D^2 is discrete with uncountable many objects and the groupoid $pt(\mathrm{S}^1 \coprod S^1)$ has two connected components and in each component any two objects are isomorphic in exactly $\mathbb{Z}$ ways.

The 2-category $\mathfrak{Cat}$ is very special: any of its objects (i.e. a category) is completely determined by its category of points that is, for every category $\mathcal{C}$, the functor category $\mathcal{F}\mathrm{un}(\mathbf{1}, \mathcal{C}) \cong \mathcal{C}$.

PROPOSITION 1.4.7. The 2-functor $pt_{\mathfrak{Cat}}\colon \mathfrak{Cat} \to \mathfrak{Cat}$ is 2-isomorphic to the identity 2-functor $\mathrm{Id}\colon \mathfrak{Cat} \to \mathfrak{Cat}$ and therefore, $pt_{\mathfrak{Cat}}$ is a 2-equivalence.

For a bicategory $\mathfrak{B}$, equipped with a terminal object, and for any pair of objects X, Y of $\mathfrak{B}$, we have the action functor

$$\mathfrak{B}(X, Y) \times pt(X) \to pt(Y)$$

which can be transposed to the functor

$$\mathfrak{B}(X, Y) \to \mathfrak{Cat}(pt(X), pt(Y)) \tag{1.8}$$

DEFINITION 1.4.8. A bicategory $\mathfrak{B}$ (equipped with a terminal object) is called **well-pointed** whenever the homomorphism $pt_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{Cat}$ is faithful, that is the action functors (1.8) are faithful for all objects X and Y of $\mathfrak{B}$.

Note that the above definition of well-pointedness for a bicategory generalizes the usual definition of well-pointedness for categories. Of course, a well-pointed category $\mathfrak{B}$ is in particular a *concrete* bicategory with the faithful functor to $\mathfrak{Cat}$ being $pt_{\mathfrak{B}}$. Proposition 1.4.7 shows that the 2-category $\mathfrak{Cat}$ is indeed well-pointed. The 2-category $\mathfrak{Cat}(\mathcal{S})$ from Example 1.5.1 is well-pointed if category $\mathcal{S}$ is well-pointed. On the other hand, the bicategory $\mathcal{T}op_{\leq 2}$ is not well-pointed.

Remark 1.4.9. The concrete 2-categories $\mathfrak{Loc}$, $\mathcal{E}\mathfrak{Top}$, $\mathcal{B}\mathfrak{Top}$ are not well-pointed.

The construction below of 'Display sub-2-category' requires explaining the notion of bipullback in 2-categories. We shall later give a precise intrinsic definition based on weighted limits in §1.9. Nonetheless, for the sake of readers unfamiliar with or uninterested in weighted limits, we give a concrete definition of bipullback listing the required data and axioms. The latter definition is equivalent to the intrinsic one.

Definition 1.4.10. A **bipullback** of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a bicategory $\mathfrak{B}$ is the universal isocone over f and g, i.e. an object P together with 1-morphisms $d_0 \colon P \to A, d_1 \colon P \to B$ and an iso-2-morphism $\pi \colon fd_0 \cong gd_1$ satisfying the following universal properties

(BP1) Given another iso-cone $(l_0, l_1, \lambda \colon fl_0 \cong gl_1)$ over f and g (with apex X), there exist a 1-morphism u and two iso-2-morphisms γ_0 and γ_1 such that the pasting diagrams below are equal.

$$\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & {} \arrow[d, phantom, "\cong_{\gamma_1}"] & \\
{} \arrow[r, phantom, "\cong_{\gamma_0}"] & P \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_{\pi}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & & \\
& \cong_{\lambda} & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
\tag{1.9}$$

(BP2) Given 1-morphisms $u, v \colon X \rightrightarrows P$ and 2-morphisms $\alpha_i \colon d_i u \Rightarrow d_i v$ $(i = 0, 1)$ such that the diagram

$$\begin{tikzcd}
fd_0u \arrow[r, "f \centerdot \alpha_0"] \arrow[d, "\pi \centerdot u"'] & fd_0v \arrow[d, "\pi \centerdot v"] \\
gd_1u \arrow[r, "g \centerdot \alpha_1"'] & gd_1v
\end{tikzcd}$$

commutes in $\mathfrak{K}(X, C)$, there is a unique $\beta \colon u \Rightarrow v$ such that each $\alpha_i = d_i \centerdot \beta$.

The two conditions (BP1) and (BP2) together are equivalent to saying that the functor

$$\mathfrak{B}(X,P) \xrightarrow{\simeq} \Big(\mathfrak{B}(X,f) \downarrow_{\cong} \mathfrak{B}(X,g)\Big)$$

is an equivalence where the right hand side is the pseudo pullback of functors $\mathfrak{B}(X,f)\colon \mathfrak{B}(X,A) \to \mathfrak{B}(X,C)$ and $\mathfrak{B}(X,g)\colon \mathfrak{B}(X,B) \to \mathfrak{B}(X,C)$. One direction of the equivalence is obtained from whiskering by the iso-cone (d_0, d_1, π).

Note the distinction from pseudopullbacks, for which the equivalence is an isomorphism of categories. And of course a strict pullback has similar condition of universality as in above except that they are with regard to strict cones instead of iso cones.

DEFINITION 1.4.11. A 1-morphism in $\mathfrak{K}$ is **bicarrable** (resp. **carrable**, **pseudo-carrable**) whenever a bipullback (resp. strict pullback, pseudo pullback) of it along any other 1-morphism (with the same codomain) exists in $\mathfrak{K}$.

Of course, bipullbacks are defined up to equivalence and the class of bicarrable 1-morphisms is closed under bipullback.

Two important facts that we are going to deploy in chapters 3 and 4 are:

- All extension maps in the 2-category $\mathfrak{Con}$ of AU-contexts are carrable. (See [Vic19])
- In the 2-category $\mathcal{E}\mathfrak{Top}$ of elementary toposes all bounded geometric morphisms are bicarrable. (See [Joh02a, B3.3.6]).

CONSTRUCTION 1.4.12 (Display sub-2-category). Suppose $\mathfrak{K}$ is a 2-category. Let $\mathcal{D}$ be a chosen class of bicarrable 1-morphisms in $\mathfrak{K}$, which we shall call "display 1-morphisms", with the following properties:

- Every identity 1-morphism is in $\mathcal{D}$.
- If $x\colon \overline{x} \to \underline{x}$ is in $\mathcal{D}$, and $\underline{f}\colon \underline{y} \to \underline{x}$ in $\mathfrak{K}$, then there is some bipullback y of x along $\underline{f}$ such that $y \in \mathcal{D}$.

We form the **display** 2-category $\mathfrak{K}_{\mathfrak{D}}$ as follows. We use a systematic "upstairs-downstairs" notation with 'overbars' (e.g. $\overline{f}$) and 'underbars' (e.g. $\underline{f}$) to help navigate diagrams.

($\mathfrak{K}_{\mathfrak{D}} : 0$) Objects are $x \colon \overline{x} \to \underline{x}$ in $\mathfrak{D}$.

($\mathfrak{K}_{\mathfrak{D}} : 1$) For any objects x and y, the 1-morphisms from y to x are given by the triples $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$ where $\underline{f} \colon \underline{y} \to \underline{x}$ and $\overline{f} \colon \overline{y} \to \overline{x}$ are 1-morphisms in $\mathfrak{K}$, and $\overset{\blacktriangledown}{f} : x\overline{f} \Rightarrow \underline{f}y$ is an iso-2-morphism in $\mathfrak{K}$.

($\mathfrak{K}_{\mathfrak{D}} : 2$) If f and g are 1-morphisms from y to x, then 2-morphisms from f to g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} \colon \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ are 2-morphisms in $\mathfrak{K}$ so that the obvious diagram of 2-morphisms commutes.

$$\begin{tikzcd}
\overline{x} \arrow[d, "x"] \\
\underline{x}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{y} \arrow[rr, bend left, "\overline{g}"] \arrow[rr, bend right, "\overline{f}"'] \arrow[rr, phantom, "\overline{\alpha}\Nearrow"] \arrow[d, "y"'] \arrow[drr, phantom, "\overset{\blacktriangledown}{f}\Downarrow \quad \overset{\blacktriangledown}{g}\Downarrow"] & & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[rr, bend left, "\underline{g}"] \arrow[rr, bend right, "\underline{f}"'] \arrow[rr, phantom, "\underline{\alpha}\Nearrow"] & & \underline{x}
\end{tikzcd}$$

Composition of 1-morphisms $k \colon z \to y$ and $f \colon y \to x$ is given by pasting them together, more explicitly it is given by $fk := \langle \overline{fk}, \overset{\blacktriangledown}{k} \boxbar \overset{\blacktriangledown}{f}, \underline{f}\underline{k} \rangle$ where $\overset{\blacktriangledown}{k} \boxbar \overset{\blacktriangledown}{f} := (\underline{f} \centerdot \overset{\blacktriangledown}{k}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{k})$. Vertical composition of 2-morphisms consists of vertical composition of upper and lower 2-morphisms. Similarly, horizontal composition of 2-morphisms consists of horizontal composition of upper and lower 2-morphisms. Identity 1-morphisms and 2-morphisms are defined trivially from those of $\mathfrak{K}$.

Notice that $\mathfrak{K}_{\mathfrak{D}}$ is a sub-2-category of the 2-category $\mathbf{cyl}_{\cong}(\mathfrak{K}) := \mathbf{Gray}(\mathbf{2}, \mathfrak{K})$, where the latter consists of strict 2-functors, pseudo-natural transformations, and modifications from the free walking arrow category $\mathbf{2}$. Indeed, $\mathbf{cyl}_{\cong}(-)$ construction is a 2-dimensional generalization of the construction of arrow category. There is a (strict) 2-functor $\mathbf{cod}\colon \mathbf{cyl}_{\cong}(\mathfrak{K}) \to \mathfrak{K}$ which takes object x to its codomain $\underline{x}$, a 1-morphism f to $\underline{f}$ and a 2-morphism $(\overline{\alpha}, \underline{\alpha})$ to $\underline{\alpha}$. The 2-category $\mathbf{cyl}_{\cong}(\mathfrak{K})$ has a universal property: Any pseudo natural transformation $\theta\colon F \to G\colon \mathfrak{L} \rightrightarrows \mathfrak{K}$ lifts to a strict 2-functor $\widetilde{\theta}\colon \mathfrak{L} \to \mathbf{cyl}_{\cong}(\mathfrak{K})$ with $\mathbf{dom} \circ \widetilde{\theta} = F$ and $\mathbf{cod} \circ \widetilde{\theta} = G$. The relationship between $\mathfrak{K}$, $\mathfrak{K}_{\mathfrak{D}}$, and $\mathbf{cyl}_{\cong}(\mathfrak{K})$ is illustrated in the following commutative diagram of 2-categories and 2-functors:

$$\begin{tikzcd}
\mathfrak{K}_{\mathfrak{D}} \arrow[rr, hook] \arrow[dr, "\mathbf{cod}"'] & & \mathbf{cyl}_{\cong}(\mathfrak{K}) \arrow[dl, "\mathbf{cod}"] \\
& \mathfrak{K} &
\end{tikzcd} \tag{1.10}$$

We can generalize the construction above to bicategories although some care has to be taken with regard to weak unitality and weak associativity when we paste squares and cylinders both horizontally and vertically. Depending on whether we drop the invertibility condition of the 2-morphisms inside squares of 1-morphisms we obtain **cylinder bicategory** $\mathrm{cyl}(\mathfrak{B})$ or **iso-cylinder bicategory** $\mathrm{cyl}_{\cong}(\mathfrak{B})$ of a bicategory $\mathfrak{B}$ ([Bén67]). We would instead obtain a homomorphism $\mathbf{cod}\colon \mathbf{cyl}(\mathfrak{B}) \to \mathfrak{B}$ defined in the same way and a display sub-bicategory $\mathfrak{B}_{\mathfrak{D}}$.

In passing from categories to 2-categories, the construction of slices of categories is bifurcated into four versions: strict, pseudo, lax, and oplax slice 2-categories.

CONSTRUCTION 1.4.13. For an object B of a 2-category $\mathfrak{K}$, there is a **lax slice 2-category** $\mathfrak{K} \swarrow B$: the objects of $\mathfrak{K} \swarrow B$ are morphisms $q\colon E \to B$ in $\mathfrak{K}$, the morphisms of $\mathfrak{K} \swarrow B$ are pairs $\langle f, \varphi \rangle\colon q \to p$ such that $\varphi\colon pf \Rightarrow q$ is a 2-morphism in $\mathfrak{K}$, and the 2-morphisms of $\mathfrak{K} \swarrow B$ are of the form $\alpha\colon \langle f, \varphi \rangle \Rightarrow \langle f', \varphi' \rangle$ where α is a 2-morphism from f to f' in $\mathfrak{K}$ which is compatible with φ and φ', i.e. $\varphi' \circ (p \centerdot \alpha) = \varphi$. The composition of morphisms $\langle g, \psi \rangle\colon q' \to q$ $\langle f, \phi \rangle\colon q \to p$ is given by the morphism $\langle fg, \psi \circ (\phi \centerdot g) \rangle$. A morphism $\langle f, \varphi \rangle\colon q \to p$ is an isomorphism in $\mathfrak{K} \swarrow B$ iff both f and α are invertible. It is an equivalence iff f is an equivalence of and α is an iso-2-morphism.

The construction of **oplax slice 2-category** $\mathfrak{K} \nearrow B$ is similar except that in the definition of morphism $\langle f, \varphi \rangle$, the 2-morphism φ goes in the opposite direction, i.e. $\varphi \colon q \Rightarrow pf$. If all φ are invertible (and therefore their direction does not matter), then we obtain the notion of **pseudo slice 2-category** (or sometimes simply a slice 2-category) which we shall denote by $\mathfrak{K} /\!\!/ B$. If all φ are identity, then we get the notion of **strict slice 2-category** which is denoted by $\mathfrak{K}/B$. There is a strict 2-functor $\mathfrak{K}/B \hookrightarrow \mathfrak{K} /\!\!/ B$ which is identity on objects and sends a morphism f to $\langle f, \mathrm{id} \rangle$, and is identity on 2-morphisms. It is locally full and faithful, however, it is not necessarily an embedding of 2-categories. Also, it is not locally *replete* (Recall that a subcategory is replete if the property of belonging to it respects the principle of equivalence of categories, i.e. if $f \colon x \xrightarrow{\cong} y$ and $x \in \mathcal{D} \hookrightarrow \mathcal{C}$ then $y \in \mathcal{D}$ and f lies in $\mathcal{D}$ as well). Similarly, there are 2-functors $\mathfrak{K} /\!\!/ B \hookrightarrow \mathfrak{K} \nearrow B$, and $\mathfrak{K} /\!\!/ B \hookrightarrow \mathfrak{K} \swarrow B$ which are identity on objects, morphisms, and 2-morphisms. They are locally fully faithful and replete, but not necessarily emebedding of 2-categories.

The embeddings of slice 2-categories above lie over $\mathfrak{K}$, i.e. the following triangles of 2-functors commute.

$$\begin{tikzcd}
\mathfrak{K}/B \arrow[r, hook] \arrow[dr, "\mathbf{dom}"'] & \mathfrak{K} /\!\!/ B \arrow[r, hook] \arrow[d, "\mathbf{dom}"] & \mathfrak{K} \swarrow B \arrow[dl, "\mathbf{dom}"] \\
& \mathfrak{K} &
\end{tikzcd}$$

Any morphism $\langle f, \alpha \rangle \colon q \to p$ in $\mathfrak{K} /\!\!/ B$ factors as $\mathbf{dom}$-vertical morphism (i.e. a morphism whose image under $\mathbf{dom}$ is identity) followed by a strict morphism (i.e. a morphism in the strict slice $\mathfrak{K}/B$). The same is true for morphisms in $\mathfrak{K} \swarrow B$ and $\mathfrak{K} \nearrow B$.

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "q"'] & {} \arrow[d, phantom, "\overset{\alpha}{\Longleftarrow}" near start] & F \arrow[dl, "p"] \\
& B &
\end{tikzcd}
=
\begin{tikzcd}
E \arrow[r, "1"] \arrow[dr, "q"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}", bend left=20] & E \arrow[r, "f"] \arrow[d, "pf" description] \arrow[d, phantom, "=" {xshift=2em}] & F \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

Therefore, we may write

$$\langle f, \alpha \rangle = \langle f, \mathrm{id} \rangle \circ \langle 1, \alpha \rangle$$

Also, any object $p \colon E \to B$ of $\mathfrak{K} \swarrow E$ induces a 2-functor $\Sigma_p \colon \mathfrak{K} \swarrow E \to \mathfrak{K} \swarrow B$ which takes object $x \colon X \to E$ to $px \colon X \to B$, morphism $\langle f, \phi \rangle \colon y \to x$ to $\langle f, p \centerdot \phi \rangle \colon py \to px$, and it acts identically on 2-morphisms.

REMARK 1.4.14. The slice and coslice categories can be realized as oplax and lax limits in the 2-category $\mathfrak{Cat}$, respectively (See Remark 1.9.20). One might be tempted to construct lax (resp. oplax) slice 2-categories as oplax (resp. lax) limits in some 3-category of 2-categories. However, this is not straightforward (if possible at all!) since the construction requires the use of lax (op)lax natural transformations which do not form a 3-category of 2-categories. Nonetheless, similar to the fact the slice and coslice categories are obtained as special cases of comma categories, lax and oplax slice 2-categories are obtained as special cases of Gray's 2-comma categories [Gra74].

1.5 Examples of 2-categories and bicategories

In this section we give few typical examples of 2-categories and bicategories. For more examples we refer the reader to [Lac10a, Section 1].

EXAMPLE 1.5.1. Suppose $\mathcal{S}$ is finitely complete category. There is a 2-category $\mathfrak{Cat}(\mathcal{S})$ of internal (small) categories in $\mathcal{S}$, internal functors and natural transformations. See Definition A.8.1 in Appendix. In Chapter 2, we shall see that it embeds into the 2-category $\mathfrak{Fib}(\mathcal{S})$ of categorical fibrations over $\mathcal{S}$. This embedding though only holds in the bicategorical sense of Section 1.3.

An special case of the above example is the 2-category of (internal) groupoids.

EXAMPLE 1.5.2. Groupoids, functors, and natural transformations between them (necessarily invertible) form a 2-category $\mathfrak{Grpd}$. Consider the delooping 2-functor $\Sigma\colon \mathcal{G}rp^{\mathfrak{d}} \to \mathfrak{Grpd}$ where $\mathcal{G}rp^{\mathfrak{d}}$ is the discrete 2-category of groups. In the theory of groups, one is often concerned only with group homomorphisms *up to conjugacy* (i.e. study of groups by inner automorphisms). We note that the essential image of $\Sigma\colon \mathcal{G}rp^{\mathfrak{d}} \to \mathfrak{Grpd}$ is the 2-category of groups where a 2-morphism $\theta\colon \Sigma(f) \Rightarrow \Sigma(g)\colon G \rightrightarrows H$ is an iso-2-morphism iff it is a conjugacy between group homomorphisms f and g, i.e. an element θ of H such that $g(x) = \theta f(x)\theta^{-1}$ for all x in G. Whiskering θ on the left with a morphism $\Sigma(h)\colon \Sigma G' \to \Sigma G$ is given by the same element $\theta \in H$, while whiskering θ on the right with a morphism $\Sigma(k)\colon \Sigma H \to \Sigma H'$

is given by the element $k(\theta) \in H'$. The vertical composition of 2-morphisms landing at ΣH is given by multiplication of the corresponding elements in H. In particular, for an endomorphism $f\colon G \to G$, a 2-morphism $\theta\colon \Sigma(\mathrm{id}_G) \Rightarrow \Sigma(f)$ exhibits f as the inner automorphism $f = (-)^{\theta}\colon G \to G$. Therefore, the connected component of the groupoid $\mathfrak{Grpd}(\Sigma G, \Sigma G)$ containing $\Sigma(\mathrm{id}_G)$ is precisely the set of all inner automorphisms of G. Finally, the group $\mathfrak{Grpd}(\Sigma G, \Sigma G)(\Sigma(\mathrm{id}_G), \Sigma(\mathrm{id}_G))$ is isomorphic to the central subgroup $Z(G)$ of G.

Example 1.5.3. Locales and locale maps with specialization order form a 2-category $\mathfrak{Loc}$. Recall that for a locale X we have an associated *frame of 'opens'* $\mathcal{O}(X)$ and a map $f\colon Y \to X$ of locales give rise to a map of frames $f^*\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ in the reverse direction. A 2-morphism between such two such maps $f, g\colon Y \rightrightarrows X$ if $f^*(V) \leq g^*(V)$ for any open V in the frame $\mathcal{O}(Y)$. This order is known by the name of "specialization order": we write $f \sqsubseteq g$ Note that there is at most one 2-morphism between any two 1-morphisms. In fact, $\mathfrak{Loc}$ is $\mathcal{D}cpo$-enriched: given a directed family $\{f_i\}$ of maps in $\mathfrak{Loc}(X, Y)$, the directed join of them is given by the formula $(\bigvee^{\uparrow} f_i)^* V = \bigvee^{\uparrow}(f_i)^* V$ The $\mathcal{D}cpo$-enrichment implies that a The construction of frame of opens of a locale gives a 2-functor $\mathcal{O}\colon\ \mathfrak{Loc} \to \mathfrak{Frm}$ which is represented by the Sierpinski space $\mathbb{S}$ whose frame $\mathcal{O}(\mathbb{S})$ is given by the poset $\{0 \leq I \leq 1\}$. Therefore $\mathbb{S}$ has two points $\bot, \top$ with $\bot \sqsubseteq \top$.

Example 1.5.4. For an elementary topos $\mathscr{S}$ (with nno) the object classifier (over $\mathscr{S}$) is a topos $\mathscr{S}[\mathbb{O}]$ whose (generalized) points in other toposes form the underlying category of that topos, i.e.

$$\mathcal{B}\mathfrak{Top}\big/_{\mathscr{S}}(\mathscr{E}, \mathscr{S}[\mathbb{O}]) \simeq \mathscr{E}$$

By underlying category $\mathcal{E}$ of a topos $\mathscr{E}$, we simply mean the category of objects of topos $\mathscr{E}$ which is locally representable. The role of object classifier $\mathscr{S}[\mathbb{O}] \to \mathscr{S}$ generalizes the the role of Sierpinski space $\mathbb{S}$. While $\mathbb{S}$ classifies opens (i.e. subterminals) of locales, $\mathscr{S}[\mathbb{O}] \to \mathscr{S}$ classifies objects of other $\mathscr{S}$-toposes (i.e. $\mathscr{S}$-sheaves). Note that the object classifier represents the pseudo functor

$$\left(\mathcal{B}\mathfrak{Top}\big/_{\mathscr{S}}\right)^{\mathrm{op}} \to \mathfrak{Cat}_{lrp}$$

which takes a geometric morphism (f^*, f_*) of $\mathscr{S}$-toposes to the cocontinuous functor f^* of locally representable categories. [BC95] shows that the 2-category $(\mathcal{B}\mathfrak{Top}/\mathscr{S})^{\mathrm{op}}$

is 2-monadic over the 2-category of locally presentable categories and cocontinuous functors between them internal to $\mathscr{S}$. Therefore, the pseudo functor $\mathfrak{BTop}/\mathscr{S}(-, \mathscr{S}[\mathbb{O}])$ has a left 2-adjoint.

EXAMPLE 1.5.5. The simplex category[8] Δ of finite ordinals can be updated to a 2-category in three ways: first, as a locally discrete 2-category, and second, as a delooping of its monoidal structure (See §A.6), and finally, and perhaps the most interesting way is to consider Δ as a locally posetal 2-category. This insight goes back to [Str80] which uses this 2-category to define the notion of *doctrine* on any bicategory $\mathfrak{B}$: it is a strict monoidal homomorphism from Δ (considered as monoidal 2-category) to the monoidal bicategory $\mathbf{Hom}(\mathfrak{B}, \mathfrak{B})$. A bit of calculation shows that doctrines on bicategories are basically the same thing as pseudomonads, i.e. monads whose associativity and unit laws hold only up to coherent isomorphisms, instead of strict equalities.

More precisely, the objects and morphisms are the same as standard simplex category Δ and 2-morphisms are obtained in virtue of poset-structure of ordinals. For instance, the hom-category $\Delta(\mathbf{1}, \mathbf{2})$ consist of two monomorphisms $\delta_1 \leq \delta_0$, and $\Delta(\mathbf{2}, \mathbf{3})$ consist of three monomorphisms $\delta_2 \leq \delta_1 \leq \delta_0$ where the order is pointwise. Morphisms δ_i are known as *coface* morphisms, and geometrically, they are pictured as follows (but now with the addition of 2-morphisms):

$$\begin{tikzcd}
\emptyset \arrow[r, "\delta_0"] & \{0\} \arrow[r, shift left, "\delta_1"] \arrow[r, shift right, "\delta_0"'] & \{0 \longrightarrow 1\} \arrow[r, shift left=3, "\delta_2"] \arrow[r, "\delta_1" description] \arrow[r, shift right=3, "\delta_0"'] & \left\{ \begin{tikzcd}[ampersand replacement=\&] \& 1 \arrow[dr] \& \\ 0 \arrow[ur] \arrow[rr] \& \Rightarrow \ \Downarrow \ \Uparrow \& 2 \end{tikzcd} \right\} & \cdots
\end{tikzcd}$$

In general in hom-category $\Delta(\mathbf{n}, \mathbf{n+1})$, we have a chain of 2-morphisms

$$\delta_n \Rightarrow \delta_{n-1} \Rightarrow \ldots \Rightarrow \delta_0$$

This is half of the picture; there are epimorphisms σ_i which go in the other direction and they are called *codegeneracy* morphisms. In general in the hom-category $\Delta(\mathbf{n+1}, \mathbf{n})$, we have a chain of 2-morphisms

$$\sigma_0 \Rightarrow \ldots \Rightarrow \sigma_{n-1}$$

---

[8] This is the simplex category of category theorists, not topologists.

The chains of 2-morphisms above are generated by the following string of adjunctions:

$$\delta_n \dashv \sigma_{n-1} \dashv \delta_{n-1} \dashv \ldots \dashv \sigma_0 \dashv \delta_0$$

where the unit of $\delta_k \dashv \sigma_{k-1}$ and the counit of $\sigma_k \dashv \delta_k$ are identities. From these we obtain:

$$\delta_k = \delta_k \circ \sigma_{k-1} \circ \delta_{k-1} \overset{\varepsilon}{\Rightarrow} \delta_{k-1}$$

Similarly,

$$\sigma_{k-1} = \sigma_k \circ \delta_k \circ \sigma_{k-1} \xRightarrow{\sigma_k \bullet \varepsilon} \sigma_k$$

REMARK 1.5.6. Note that in hom-category $\Delta(\mathbf{n}, \mathbf{n+1})$ there are more morphisms than just coface and codegeneracy morphisms. For arbitrary m and n, there are in fact $\binom{n+m-1}{m}$ number of objects in the hom-category $\mathrm{Hom}_\Delta(\mathbf{m}, \mathbf{n})$.

$$\begin{gathered}
|\mathrm{Hom}_\Delta(\mathbf{m}, \mathbf{n})| = \sum_k |\text{mono}(\mathbf{k}, \mathbf{n})| \times |\text{epi}(\mathbf{m}, \mathbf{k})| = \\
\sum_k \binom{n}{k}\binom{m-1}{k-1} = \sum_k \binom{n}{k}\binom{m-1}{m-k} = \\
\binom{n+m-1}{m}
\end{gathered}$$

This uses the well-known canonical decomposition of morphisms into cofaces and codegeneracies, and Vandermonde's identity.

EXAMPLE 1.5.7. For any finitely complete category $\mathcal{S}$ there is an associated bicategory $\mathfrak{Span}(\mathcal{S})$ of *spans* (aka *correspondences*) in $\mathcal{S}$. The objects of $\mathfrak{Span}(\mathcal{S})$ are the same as the objects of $\mathrm{Ob}(\mathcal{S})$, and the morphisms from A to B are spans between A and B, that is diagrams of the form

$$\begin{tikzcd}
& S \arrow[dl, "s_0"'] \arrow[dr, "s_1"] & \\
A & & B
\end{tikzcd}$$

where s_0, s_1 are morphisms in of $\mathcal{S}$. We denote such 1-morphism by $s = \langle s_0, S, s_1 \rangle$. A 2-morphism $\alpha \colon s \Rightarrow s'$ is a morphism $\alpha \colon S \to S'$ in $\mathcal{S}$ which makes both triangles in below commute.

$$\begin{tikzcd}
 & S \arrow[dl, "s_0"'] \arrow[dr, "s_1"] \arrow[dd, "\alpha"] & \\
A & & B \\
 & S' \arrow[ul, "s'_0"] \arrow[ur, "s'_1"'] &
\end{tikzcd}$$

The composition of 1-morphisms is given by pullback.

$$\begin{aligned}
\mathfrak{Span}(\mathcal{S})(A,B) \times \mathfrak{Span}(\mathcal{S})(B,C) &\longrightarrow \mathfrak{Span}(\mathcal{S})(A,C) \\
(\langle s_0, S, s_1 \rangle, \langle t_0, T, t_1 \rangle) &\longmapsto \langle s_0 \circ s_1^*(t_0), S \times_B T, t_1 \circ t_0^*(s_1) \rangle
\end{aligned}$$

The vertical composition of 2-morphisms is given by composition of morphisms in $\mathcal{S}$, and the horizontal composition of 2-morphisms is the induced morphism on the pullbacks obtained by their universal property.

A monad in $\mathfrak{Span}(\mathcal{S})$ is the exactly the same thing as a (small) category internal to $\mathcal{S}$ ([Str74]) and a monad morphism corresponds to a profunctor of internal categories.

There are embedding homomorphism $\langle 1, - \rangle \colon \mathcal{S}^{\mathfrak{d}} \hookrightarrow \mathfrak{Span}(\mathcal{S})$ and $\langle -, 1 \rangle \colon (\mathcal{S}^{\mathfrak{d}})^{\mathrm{op}} \hookrightarrow \mathfrak{Span}(\mathcal{S})$ of bicategories whereby the first embedding takes a morphism $f \colon X \to Y$ in $\mathcal{S}$ to the span $\langle 1_X, X, f \rangle$, and the second embedding takes $f^{\mathrm{op}} \colon Y \to X$ in $\mathcal{S}^{\mathrm{op}}$ to $\langle f, X, 1_X \rangle$. We also have an invertible involution 2-functor $\mathfrak{Span}(\mathcal{S}) \to (\mathfrak{Span}(\mathcal{S}))^{\mathrm{op}}$ which is identity on objects and acts on morphisms and 2-morphisms by switching the legs of spans.

$\mathfrak{Span}(\mathcal{S})$ has a certain 1-dimensional property: any functor F from the underlying category of $\mathfrak{Span}(\mathcal{S})$ to a category $\mathcal{C}$ is uniquely determined by a pair of functors $F^* \colon \mathcal{S}^{\mathrm{op}} \to \mathcal{C}$ and $F_* \colon \mathcal{S} \to \mathcal{C}$ which take the same value on objects of $\mathcal{C}$ and moreover, any pullback in $\mathcal{S}$ on the left is taken to a commutative square in $\mathcal{C}$ on the right:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "k"] & & A \arrow[r, "F_*(g)"] & B \\
C \arrow[r, "h"'] & D & \mapsto & C \arrow[u, "F^*(f)"] \arrow[r, "F_*(h)"'] & D \arrow[u, "F^*(k)"']
\end{tikzcd}$$

EXAMPLE 1.5.8. Suppose $\mathcal{S}$ is a regular category, in particular we need stable epi-mono factorization in $\mathcal{S}$. The bicategory $\mathfrak{Rel}(\mathcal{S})$ of relations internal to $\mathcal{S}$ has the same objects as $\mathcal{S}$, and as morphism spans $r = \langle r_0, R, r_1 \rangle$ for which r_0 and r_1 are jointly monic, and we consider only the 2-morphism h which are monic. This makes $\mathfrak{Rel}(\mathcal{S})$ is a locally posetal bicategory. Note that in any locally posetal bicategory, the 2-dimensional coherence equations become redundant as all parallel 2-morphisms manifestly commute. There is a lax functor $U \colon \mathfrak{Rel}(\mathcal{S}) \to \mathfrak{Span}(\mathcal{S})$ which forgets the jointly monic property of spans. The composition of relations $r \colon A \nrightarrow B$ and $s \colon B \nrightarrow C$ has one more step than composition of their corresponding spans: it is calculated as the image (i.e. the monomorphism of epi-mono factorization in $\mathcal{S}$) of morphism $\langle r_0 \circ r_1^* s_0, s_1 \circ s_0^* r_1 \rangle \colon R \times_B S$. In the internal language of $\mathcal{S}$, the composite relation $s \circ r$ may be described as follows:

$$a(S \circ R)c \iff \exists b \colon B.(aRb) \wedge (bSc)$$

EXAMPLE 1.5.9. The 2-category $\mathfrak{Par}(\mathcal{S})$ is a sub-2-category of $\mathfrak{Span}(\mathcal{S})$; we only consider those 1-morphism $\langle i, D, f \rangle$ for which i is monic, and we consider only the 2-morphism h which are monic. The 2-functor $P \colon \mathfrak{Par}(\mathcal{S}\mathrm{et}) \to \mathcal{S}\mathrm{et}_*$ which takes a object A to the pointed set $(A \coprod \{*\}, *)$ and is furthermore defined on hom-categories by $P_{A,B} \colon \mathfrak{Par}(\mathcal{S})(A, B) \to \mathcal{S}\mathrm{et}_*(A \coprod \{*\}, B \coprod \{*'\})$, where $P_{A,B}(i, f)(x) = f(x)$ if $x \in D$ and $P_{A,B}(i, f)(x) = *'$ otherwise, establishes and equivalence of bicategories.

EXAMPLE 1.5.10. Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category equipped with equalizers and coequalizers which are stable under tensoring (such as the monoidal category of Abelian groups). Then the bimodules in $\mathcal{V}$ form a bicategory $\mathfrak{BiMod}(\mathcal{V})$. This bicategory generalizes bicategories $\mathfrak{Span}(\mathcal{V})$ and $\mathfrak{opSpan}(\mathcal{V})$. (See Construction A.8.11 and Examples A.8.13 and A.8.14 in Appendix.)

EXAMPLE 1.5.11. Suppose $\mathcal{V}$ is a complete and cocomplete closed symmetric monoidal category (i.e. A Bénabou cosmos). There is a bicategory $\mathfrak{Dist}(\mathcal{V})$ of categories, $\mathcal{V}$-distributors (aka profunctors), and $\mathcal{V}$-natural transformations. More precisely, the objects are $\mathcal{V}$-enriched categories $\mathcal{A}$, $\mathcal{B}$, etc., a morphism between objects $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ (here $\mathcal{V}$ considered self-enriched itself via its closed structure), and a 2-morphism between morphisms H and K is a $\mathcal{V}$-natural transformation $\alpha\colon H \Rightarrow K\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \rightrightarrows \mathcal{V}$. The identity morphism on $\mathcal{A}$ is given by $\mathcal{V}$-hom-functor $\mathcal{A}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$: the local $\mathcal{V}$-morphisms

$$(\mathcal{A}^{\mathrm{op}} \times \mathcal{A})((a,b),(a',b')) \to [\mathcal{A}(a,b), \mathcal{A}(a',b')]_{\mathcal{V}}$$

are induced by

$$\mathcal{A}(a',a) \times \mathcal{A}(a,b) \times \mathcal{A}(b,b') \xrightarrow{m\circ(m\times\mathrm{id})} \mathcal{A}(a',b')$$

Distributors can be considered as bimodules of categories.

The composition of morphisms $H\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $K\colon \mathcal{B} \nrightarrow \mathcal{C}$ is given by the coend $\int^{b\in\mathcal{B}} H(b,-) \otimes K(-,b)$ which traces out the middle variable b of $\mathcal{B}$.

A 2-morphism $\theta\colon H \Rightarrow H'$ is a $\mathcal{V}$-enriched natural transformation (in the case of Bimodules a bilinear homomorphism). The vertical and horizontal composition of 2-morphism is performed similarly to that of bimodules (See A.8).

A lax monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ of cosmoi induces a lax 2-functor $\mathfrak{Dist}(F)\colon \mathfrak{Dist}(\mathcal{V}) \to \mathfrak{Dist}(\mathcal{W})$ (all applying F to all the 'hom-objects' a $\mathcal{V}$-category or $\mathcal{V}$-distributor), and a lax monoidal adjunction $F \dashv G\colon \mathcal{W} \to \mathcal{V}$ of cosmoi induces a *local adjunction* $\mathfrak{Dist}(F) \dashv \mathfrak{Dist}(G)$.

A special case of distributors are matrices.

EXAMPLE 1.5.12. The 2-category $\mathfrak{Mat}$ of matrices is formed of (finite) sets (i.e. discrete categories in the context of example above) as objects and 1-morphisms between objects X and Y are $X \times Y$-indexed families of sets. We denote such a family by $(A_{xy})_{x\in X, y\in Y}$. The composition of two 1-morphisms $A \in \mathfrak{Mat}(X,Y)$ and $B \in \mathfrak{Mat}(Y,Z)$ is given by their product $(AB)_{xz} = \sum_y A_{xy} \times B_{yz}$. The 2-morphisms are defined component-wise. Note that $\mathfrak{Mat}$ is a genuine bicategory since for sets A, B, C, we have $(A \times B) \times C \neq A \times (B \times C)$, but are isomorphic via a canonical associator α given by $\alpha((a,b),c) = (a,(b,c))$.

1.6 2-categories of toposes

Elementary and Grothendieck toposes form honest 2-categories and concerning the core of the thesis, we really doe not need bicategories in their full generality. However, there are persisting and essential bicategorical aspects to these 2-categories, such as the use of bilimits of toposes, which require us to have a mixed approach.

Another reason is that a geometric morphism from the classifying topos $\mathcal{S}\mathrm{et}[\mathbb{T}]$ to $\mathcal{S}\mathrm{et}[\mathbb{T}']$ is *up to unique isomorphism* a model of $\mathbb{T}'$ in $\mathcal{S}\mathrm{et}[\mathbb{T}]$, i.e. a model of $\mathbb{T}'$ constructed *geometrically* from the generic model of $\mathbb{T}$. As such, the isomorphism, and not the equality, of 1-morphisms of toposes should be regarded as the correct notion of structural sameness (§A.2) of morphisms of toposes. If the objects are of interest as classifying toposes, then they are defined only up to equivalence. We can only get bipullbacks, not strict or pseudo pullbacks of toposes. These properties of toposes and their morphisms are manifestly bicategorical. Therefore, throughout the thesis we have the bicategorical aspect in mind. The section §1.9 emphasizes the distinction between strict, pseudo, and bilimits on which we shall heavily rely in the next chapters. By contrast as we shall see in Chapter 3 the 2-category $\mathfrak{Con}$, the third model of generalized spaces, is strictly 2-categorical (all exisiting limits and colimits are strict).

The setting for our main result of the thesis (4.2.2) is the 2-category $\mathcal{E}\mathfrak{Top}$ whose objects are elementary toposes (equipped with nno[9]), whose morphisms are geometric morphisms, and whose 2-morphisms are geometric transformations.

However, our concern with generalized spaces means that we must also take care to deal with *bounded* geometric morphisms. Recall that a geometric morphism $p \colon \mathscr{E} \to \mathscr{S}$ is *bounded* whenever there exists an object B in $\mathscr{E}$ (a *bound* for p) such that every A in $\mathscr{E}$ is a subquotient of an object of the form

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[9] natural number object

$(p^*I) \times B$ for some $I \in \mathscr{S}$: that is one can form the following span in $\mathscr{E}$, with the left leg a mono and the right leg an epi.

$$\begin{tikzcd}
 & E \arrow[dl, tail] \arrow[dr, two heads] & \\
(p^*I) \times B & & A
\end{tikzcd}$$

The significance of this notion can be seen in the relativized version of Giraud's Theorem (see [Joh02a, B3.4.4]): p is bounded if and only if $\mathscr{E}$ can be got as the topos of sheaves over an internal site in $\mathscr{S}$. (In the original Giraud Theorem, relative to $\mathcal{S}$et, the bound relates to the small set of generators.) It follows from this that the bounded geometric morphisms into $\mathscr{S}$ can be understood as the generalized spaces, the Grothendieck toposes, relative to $\mathscr{S}$.

Bounded geometric morphisms are closed under isomorphism and composition (see [Joh02a, B3.1.10(i)]) and we get a 2-category $\mathcal{B}\mathfrak{Top}$ of elementary toposes, bounded geometric morphisms, and geometric transformations. It is a sub-2-category of $\mathcal{E}\mathfrak{Top}$, full on 2-morphisms.

Also [Joh02a, B3.1.10(ii)], if a bounded geometric morphism q is isomorphic to pf, where p is also bounded, then so too is f. This means that if we are only interested in toposes bounded over $\mathscr{S}$, then we do not have to consider unbounded geometric morphisms between them. We can therefore take the "2-category of generalized spaces over $\mathscr{S}$" to be the slice 2-category $\mathcal{B}\mathfrak{Top}/\mathscr{S}$, where the 1-morphisms are triangles commuting up to an iso-2-morphism. [Joh02a, B4] examines $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ in detail.

For the (op)fibrational results, [Joh02a, B4] reverts to $\mathcal{B}\mathfrak{Top}$. This is appropriate, since the properties hold with respect to arbitrary geometric transformations, whereas working in $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ limits the discussion to those that are identities over $\mathscr{S}$.

Unbounded geometric morphisms are rarely encountered in practice, and so it might appear reasonable to stay in $\mathcal{B}\mathfrak{Top}$ or $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ [Joh02a, B3.1.14]. However, one notable property of bounded geometric morphisms is that their bipullbacks along *arbitrary* geometric morphisms exist in $\mathcal{E}\mathfrak{Top}$ and are still bounded [Joh02a, B3.3.6]. (Note that where [Joh02a] says pullback in a 2-category, it actually means bipullback – this is explained there in section B1.1.) Thus for any geometric morphism of base toposes $f\colon \mathscr{S}' \to \mathscr{S}$, we have the change of base pseudo functor $f^*\colon \mathcal{B}\mathfrak{Top}/\mathscr{S} \to \mathcal{B}\mathfrak{Top}/\mathscr{S}'$. One might say the '2-category of Grothendieck toposes' is indexed over $\mathcal{E}\mathfrak{Top}_{\cong}$ (where the 2-morphisms in $\mathcal{E}\mathfrak{Top}_{\cong}$ are restricted to isos). [Vic17] develops this in its use of AU techniques to obtain base-independent topos results, and there is little additional effort in allowing change of base along arbitrary geometric morphisms. To avoid confronting the coherence issues of indexed 2-categories it takes a fibrational approach, with a 2-category $\mathcal{G}\mathfrak{Top}$ "of Grothendieck toposes" fibred (in a bicategorical sense) over $\mathcal{E}\mathfrak{Top}_{\cong}$.

We shall take a similar approach, but note that our 2-category $\mathcal{G}\mathfrak{Top}$, which we are about to define, is *not* the same as that of [Vic17] – we allow *arbitrary* geometric transformations "downstairs". We shall write $\mathcal{G}\mathfrak{Top}_{\cong}$ when we wish to refer to the $\mathcal{G}\mathfrak{Top}$ of [Vic17].

DEFINITION 1.6.1. Following the Construction 1.4.12, the 2-category $\mathcal{G}\mathfrak{Top}$ is defined as $\mathcal{E}\mathfrak{Top}_{\mathcal{D}}$, where $\mathcal{D}$ is the class of bounded geometric morphisms of elementary toposes. We call $\mathcal{G}\mathfrak{Top}$ the **2-category of Grothendieck toposes.**

$$\begin{tikzcd}
\mathcal{G}\mathfrak{Top} \arrow[rr, hook] \arrow[dr, "\mathbf{cod}"'] & & (\mathcal{E}\mathfrak{Top} \downarrow \mathcal{E}\mathfrak{Top}) \arrow[dl, "\mathbf{cod}"] \\
& \mathcal{E}\mathfrak{Top} &
\end{tikzcd}$$

To be more explicit, in $\mathcal{G}\mathfrak{Top}$

($\mathcal{G}\mathfrak{Top}$ 0) Objects are Grothendieck toposes $p\colon \mathscr{E} \to \mathscr{S}$ over some elementary topos $\mathscr{S}$.

($\mathcal{G}\mathfrak{Top}$ 1) For objects p and q, the 1-morphisms from q to p are given by the triples $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$ where $\underline{f}\colon \underline{q} \to \underline{p}$ and $\overline{f}\colon \overline{q} \to \overline{p}$ are geometric morphisms, and the geometric transformation $\overset{\blacktriangledown}{f} : p\overline{f} \Rightarrow \underline{f}q$ is an invertible geometric transformation.

($\mathfrak{GTop}$ 2) If f and g are 1-morphisms from q to p, then 2-morphisms from f to g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} \colon \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ are geometric transformations such that $\overline{\alpha}$ lies over $\underline{\alpha}$ (modulo invertible geometric transformations $\overset{\blacktriangledown}{f}$ and $\overset{\blacktriangledown}{g}$).

$$\begin{tikzcd}
\mathscr{E} \arrow[d, "p"] \\
\mathscr{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{F} \arrow[r, "\overline{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\cong"] & \mathscr{E} \arrow[d, "p"] \\
\mathscr{S}' \arrow[r, "\underline{f}"'] & \mathscr{S}
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=large]
\mathscr{F} \arrow[r, bend left, "\overline{g}"] \arrow[r, bend right, "\overline{f}"'] \arrow[r, phantom, "\overline{\alpha}\Uparrow"] \arrow[d, "q"'] \arrow[dr, phantom, "\cong \quad \cong"] & \mathscr{E} \arrow[d, "p"] \\
\mathscr{S}' \arrow[r, bend left, "\underline{g}"] \arrow[r, bend right, "\underline{f}"'] \arrow[r, phantom, "\underline{\alpha}\Uparrow"] & \mathscr{S}
\end{tikzcd}$$

Notice that in particular, $\mathfrak{GTop}(\mathscr{S}) = \mathbb{B}\mathbf{ase}^{-1}\mathscr{S} = \mathfrak{BTop}/\mathscr{S}$.

An important part of the next chapter will focus on the codomain 2-functor

$$\mathbf{cod} \colon \mathfrak{GTop} \to \mathfrak{ETop}.$$

It is important to note that this codomain functor is *not* a fibration in any 2-categorical sense, as it is not well behaved with respect to arbitrary 2-morphisms in $\mathfrak{ETop}$. This will turn out to be easy to see if one takes the point of view of indexed 2-categories (and the corresponding change-of-base functors).

Indeed, it becomes a fibration if one restricts the downstairs 2-morphisms to be isos, as in [Vic17]. However, it will still be interesting to consider its fibrational objects, cartesian 1-morphisms and 2-morphisms, which we shall do in §2.5 §2.6.

1.7 Representability and bicategorical concepts

In this section, we shall discuss the importance of the notion of representability in 1-categorical and 2-categorical setting. Recall that

DEFINITION 1.7.1. A functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ is **representable** whenever there is an object A in the category $\mathcal{C}$ with a natural isomorphism $\phi\colon F \cong \mathrm{Hom}(A,-)$. In this situation, we say F is *represented* by the object A. A presheaf $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is **representable** when there is an object B in the category $\mathcal{C}$ with a natural isomorphism $\psi\colon P \cong \mathrm{Hom}(-,B)$.

NOTE. We usually use notations $y^A = \mathrm{Hom}(A,-)$ and $y_B = \mathrm{Hom}(-,B)$. The functors $y_\_$ and y^- are, respectively, Yoneda and co-Yoneda embeddings. By Yoneda lemma, the representing object is determined uniquely up to canonical isomorphism for a given representable functor (resp. presheaf).

There are many reasons why representable functors and representable presheaves are so important in category theory and higher category theory. Suppose we want to define an object satisfying a universal property, such as a limit, a colimit, an exponential, etc. in a given category $\mathcal{C}$. One elegant approach is to take advanatge of topos structures (e.g. cartesian closedness, completeness, cocompleteness, etc.) of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ and the Yoneda embedding $\mathcal{C} \to \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$: The desired object (satisfying our universal property), provided it exists in $\mathcal{C}$, is the representing object for a presheaf, constructed from representables, which satisfy the same universal property in $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$. The Yoneda lemma ensures us that this object, if it exists, will be unique up to canonical isomorphism.

EXAMPLE 1.7.2. Let $\mathcal{C}$ be a category and A and B objects in $\mathcal{C}$. Take the functor $y_A \times y_B\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. If this functor is represented by an object C in $\mathcal{C}$, then $\mathrm{Hom}(X,C) \cong \mathrm{Hom}(X,A) \times \mathrm{Hom}(X,B)$, naturally in X. The data of these natural isomorphisms is exactly the data of a product of A and B in $\mathcal{C}$, provided that the later exists in $\mathcal{C}$.

An application of the representational approach is found in defining new objects in mathematics with higher structures. Suppose we want to define a group internal to any category with binary product and terminal object. One way is to write down all the data needed for operations of a group plus the group axioms for these operations. This is special case of the definition of an internal category. (See Appendix A.8). For more sophisticated structures such as topological groups and groupoids, bicategories and double categories, Lie groups, spectra, etc. internal to categories (with enough structures), this approach can be tedious. Instead we can use Yoneda embedding again: An object A in $\mathcal{C}$ is a group object iff the representable presheaf y_A has a unique lifting along the forgetful functor $U\colon \mathcal{G}rp \to \mathcal{S}\text{et}$.

$$\begin{tikzcd}
 & \mathcal{G}rp \arrow[d, "U"] \\
\mathcal{C}^{\mathrm{op}} \arrow[r, "y_A"'] \arrow[ur, dashed, "\widetilde{y_A}"] & \mathcal{S}\text{et}
\end{tikzcd}$$

One example of such lifting is the fundamental group of a topological space.

EXAMPLE 1.7.3. Let $\mathcal{T}op_{*,\leq 1}$ be the category consisting of pointed topological spaces with morphisms homotopy classes of base-point preserving maps. The co-representable functor $y^{(S^n,*)}$ computes, for every pointed spaces (X, x_0), the set of n-spheres (loops for $n = 1$), up to homotopy, based at x_0 in X. The lifting of $y^{(S^n,*)}$ along the forgetful functor U gives the n-th fundamental group.

$$\begin{tikzcd}
 & \mathcal{G}rp \arrow[d, "U"] \\
h\mathcal{T}op \arrow[r, "y^{(S^1,*)}"'] \arrow[ur, dashed, "\pi_1"] & \mathcal{S}\text{et}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A}\text{b} \arrow[d, "U"] \\
h\mathcal{T}op \arrow[r, "y^{(S^n,*)}"'] \arrow[ur, dashed, "\pi_n"] & \mathcal{S}\text{et}
\end{tikzcd}$$

Therefore, $(S^n, *)$ is an internal cogroup in the category $\mathcal{T}op_{*,\leq 1}$ whose co-multiplication map is given by the canonical map $S^n \to S^n \vee S^n$.

We can jump one level up from categories (i.e. $\mathcal{S}$et-categories) to 2-categories (i.e. 2-categories) and bicategories. The idea is still the same with the main difference that in the world of 2-categories and bicategories there are two distinct ways to formulate representability: using isomorphism versus equivalence of hom-categories and precisely these different choices account for strict and weak structures of representing objects such as limits, colimits, etc. We shall use the prefix “bi” when we refer to the bicategorical cases.

PRINCIPLE. If $\mathcal{P}$ is a property/structure of categories, then we say that an object X in a bicategory $\mathfrak{B}$ representably satisfies $\mathcal{P}$ (or *is* representably $\mathcal{P}$) if for all objects W of $\mathfrak{B}$, $\mathfrak{B}(W, X)$ satisfies/exhibits $\mathcal{P}$. If $\mathcal{P}$ is a property/structure of functors of categories, then we say that a 1-morphism $f\colon X \to Y$ in a bicategory $\mathfrak{B}$ representably satisfies $\mathcal{P}$ (or *is* representably $\mathcal{P}$) if for all objects W of $\mathfrak{B}$, $f_*\colon \mathfrak{B}(W, X) \to \mathfrak{B}(W, Y)$ satisfies/exhibits $\mathcal{P}$.

REMARK 1.7.4. Recall from category theory that a category is *indiscrete* (aka *codiscrete chaotic*) whenever for any two of its objects there is a unique morphism (necessarily invertible) between them. An indiscrete category is inhabited iff it is equivalent to the terminal category. A typical example of an indiscrete category is the fundamental groupoid of a contractible topological space.

Consider the chain below of (forgetful) functors where Ob forget morphisms, $||{-}||_0 = \mathrm{Und}$ is the underlying category.

$$\begin{tikzcd}
pt(1) \arrow[r, bend left=60, "\bot"] \arrow[r, bend right=60, "\top"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & (pt(\mathbb{S}), =) \arrow[l, "||{-}||_{-2}" description] \arrow[r, bend left=60, "(-)^{\mathrm{d}}"] \arrow[r, bend right=60, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & (\mathcal{S}\mathrm{et}, \cong) \arrow[l, dashed, "||{-}||_{-1}" description] \arrow[r, bend left=60, "(-)^{\mathrm{d}}"] \arrow[r, bend right=60, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & (\mathfrak{Cat}, \simeq) \arrow[l, "||{-}||_{0}" description] \arrow[r, bend left=60, "(-)^{\mathrm{d}}"] \arrow[r, bend right=60, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & (\mathbf{2}\,\mathfrak{Cat}_{psd}, \simeq) \arrow[l, dashed, "||{-}||_{1}" description]
\end{tikzcd}$$

where $\mathbb{S}$ is the Sierpinski space, *pt*($\mathbb{S}$) can be regarded as the category of truth values (aka (-1)-categories) $\bot = \emptyset$ and $\top = \{\emptyset\}$. Note that $\mathcal{S}\mathrm{et}$ is the category of points of the object classifier topos $\mathscr{S}[\mathbb{O}]$. Also, $|| - ||_{-2}$ is the unique functor !, $|| - ||_{-1}$ is the propositional truncation , and $|| - ||_0$ is the ‘underlying set of objects’ functor.

DEFINITION 1.7.5. Suppose $\mathfrak{B}$ is a bicategory. We define the following concepts in $\mathfrak{B}$ representationally: An object A is **bidiscrete** (resp. **biposetal**, resp. **bigroupoidal**) if the representable pseudo functor $\mathfrak{B}(-, A)\colon \mathfrak{B}^{\mathrm{op}} \to \mathfrak{Cat}$ factors, up to an equivalence, through the sub-2-category $\mathcal{S}\mathrm{et}$ (resp. $\mathfrak{Poset}$, resp. $\mathfrak{Grpd}$) of sets (resp. posets, resp. groupoids). The phrase ‘up to equivalence’ means that there is there is a natural equivalence

$$\begin{tikzcd}
 & \mathfrak{K} \arrow[d] \\
\mathfrak{B}^{\mathrm{op}} \arrow[ur, dashed, "\widetilde{\mathfrak{B}(-,A)}"] \arrow[r, "\mathfrak{B}(-,A)"'] \arrow[ur, phantom, "\simeq"', shift right=3] & \mathfrak{Cat}
\end{tikzcd}$$

where $\mathfrak{K}$ is the 2-category $\mathcal{S}\mathrm{et}$ (resp. $\mathfrak{Poset}$, resp. $\mathfrak{Grpd}$).

In more basic terms, A is bigroupoidal iff each 2-morphism

$$\begin{tikzcd}
X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A
\end{tikzcd}$$

is invertible or equivalently, the morphism

$$\mathbf{I} \pitchfork X \to \mathbf{2} \pitchfork X,$$

induced by the categorical embedding $\mathbf{2} \to \mathbb{I}$, is an equivalence in $\mathfrak{B}$. A is biposetal iff there is at most one such 2-morphism between any pair of 1-morphism. Finally A is bidiscrete iff it is both bigroupoidal and biposetal.

REMARK 1.7.6. The analogue of definition above for 2-categories replaces ‘up to equivalence’ by ‘up to isomorphism’.

1.8 Adjunctions, extensions, and liftings

In addition to the definition of equivalence, adjoints, and adjoint equivalences in bicategories, which we have discussed to before, a host of other basic concepts of categories and functors functors can be internalized in bicategories.

PROPOSITION 1.8.1. Every adjunction can be promoted to an adjoint equivalence.

EXAMPLE 1.8.2. Every adjunction in the 2-category $\mathfrak{Grpd}$ is automatically an adjoint equivalence. Also, it is a theorem of formal category theory that every adjunction of categories can be promoted to an adjoint equivalence. This works mutatis mutandis in every bicategory. If we consider groups as one-object groupoids (Example 1.5.2), then an adjunction $\ell\colon G \rightleftarrows H \colon r$ of groups consists of elements $\eta \in G$ and $\varepsilon \in H$ such that $\varepsilon^{-1} = \ell(\eta)$ and $\eta^{-1} = r(\varepsilon)$. So, ε is uniquely determined from η. In fact, both $r \circ \ell$ and $\ell \circ r$ are inner automorphisms, given by conjugation with η and ε, respectively.

DEFINITION 1.8.3. (i) A 1-morphism $i\colon X \to Y$ is **faithful** (resp. **full**) if whiskering with i on the left is a faithful functor (resp. full), i.e. for every $W \in \mathfrak{K}_0$ the induced functor $i_*\colon \mathfrak{K}(W,X) \to \mathfrak{K}(W,Y)$ is faithful (resp. full) in $\mathfrak{Cat}$. We can give a first order reformulation: $i\colon X \to Y$ is full iff for any pair of 1-morphisms $f, g\colon W \rightrightarrows X$, any 2-morphism $\alpha\colon i \circ f \Rightarrow i \circ g$ has a lift $\overline{\alpha}\colon f \Rightarrow g$. Moreover i is fully faithful iff such lifts are unique.

$$\begin{tikzcd}
 & X \arrow[d, "i"] \\
W \arrow[ur, bend left=15, "f"] \arrow[ur, bend right=15, "g"'] \arrow[ur, phantom, "\overline{\alpha}\Downarrow"] \arrow[r, "i \circ f" description] \arrow[r, bend right=60, "i \circ g" description] \arrow[r, phantom, shift right=3, "\alpha\Downarrow"] & Y
\end{tikzcd} \tag{1.11}$$

(ii) A **pseudo-retract** of 1-morphism $f\colon X_0 \to X$ is a 1-morphism $r\colon X \to X_0$ together with an iso-2-morphism $\mathrm{id}_{X_0} \cong r \circ f$. A **pseudo-section** of $p\colon E \to B$ is a 1-morphism $s\colon B \to E$ together with an iso-2-morphism $p \circ s \cong \mathrm{id}_B$.

(iii) Given 1-morphisms $f\colon A \to C$ and $j\colon A \to B$, the 2-morphism $\varphi\colon f \Rightarrow g \circ j \in \mathfrak{K}(A,C)$ exhibits $g \in \mathfrak{K}(B,C)$ as the **left extension** of f along j whenever for any 1-morphism $g' \in \mathfrak{K}(B,C)$ we have the bijection of sets

$$\mathfrak{K}(B,C)(g,g') \cong \mathfrak{K}(A,C)(f,g'j)$$

given, from left to right, by the assignment $\theta \mapsto (\theta \centerdot j) \circ \varphi$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\varphi\Downarrow"] & C \\
B \arrow[ur, dashed, "g"'] &
\end{tikzcd} \tag{1.12}$$

(iv) Given 1-morphisms $f\colon A \to B$ and $p\colon E \to B$, the 2-morphism $\psi\colon f \Rightarrow p \circ g \in \mathfrak{K}(A, B)$ exhibits $g \in \mathfrak{K}(A, E)$ as the **left lifting** of f along p whenever for any 1-morphism $g' \in \mathfrak{K}(A, E)$ we have the bijection of sets

$$\mathfrak{K}(A, E)(g, g') \cong \mathfrak{K}(A, B)(f, g'p)$$

given, from left to right, by the assignment $\theta \mapsto (p \centerdot \theta) \circ \psi$.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\psi"] & B
\end{tikzcd} \tag{1.13}$$

The extension (resp. lifting) is **absolute** if it is preserved by all outgoing (resp. incoming) arrows from C (resp. to B).

REMARK 1.8.4. The left liftings in $\mathfrak{K}$ are the left extensions in $\mathfrak{K}^{\mathrm{op}}$. Also we define the right liftings (resp. right extensions) as the left liftings (left extensions) in $\mathfrak{K}^{\mathrm{co}}$. At times, we shall use the notation lan_j^f for the left extension and ran_j^f for the right extension. If all left (resp. right) extensions of morphisms of the type $A \to C$ along j exist, then we get a left (resp.) adjoint $\mathrm{lan}_j^{(-)} \dashv j^*$ (resp. $j^* \dashv \mathrm{ran}_j^{(-)}$) where $j^* = \mathfrak{K}(j, C)\colon \mathfrak{K}(B, C) \to \mathfrak{K}(A, C)$. Note that in particular the 2-morphism $\phi\colon f \Rightarrow \mathrm{lan}_j^f \circ j$ is the unit of the adjunction above at f. The left extension lan_j^f is absolute iff for any $u\colon C \to C'$, we have $u_*(\phi_f) = \phi_{uf}$.

REMARK 1.8.5. The notions of extension and lifting in a bicategory are direct generalization of left and right closed structures of monoidal category. Consider morphisms A, X, and B in the delooping bicategory $\Sigma\mathcal{V}$ of a closed monoidal category $\mathcal{V}$ (Example 1.2.1). A right lifting of X along A gives the counit $[A, X] \otimes A \to X$ of adjunction $- \otimes A \dashv [A, -]$ and a right extension of X along B gives the counit $B \otimes [B, X] \to X$ of adjunction $B \otimes - \dashv [B, -]$ in $\mathcal{V}$. In a symmetric monoidal category there is no difference between left and right closed structures and this can be seen from the previous remark since $(\Sigma\mathcal{V})^{op} \cong \Sigma\mathcal{V}$

PROPOSITION 1.8.6. In the extension (g, ϕ) of diagram (1.12) φ is an iso-2-morphism iff j is an equivalence.

Proof. We only prove the "if" direction. The "only if" direction is similar. Suppose $j\colon A \to B$ is an equivalence. Then $\zeta := \alpha^{-1}_{j,j^{-1},f} \circ (f\eta) \circ \rho_{1_A,f}$ is an iso-2-morphism between f and $(fj^{-1}) \circ j$. $\square$

REMARK 1.8.7. The representably defined notion of fully faithful 1-morphism can be recasted in terms of left lifting: tautologically, $f\colon A \to B$ is fully faithful iff 1_A is an absolute left lifting of f against itself.

REMARK 1.8.8. The unit η of an adjunction $f \dashv u$ exhibits the left adjoint $f\colon A \to B$ as the absolute left lifting of 1_A along the right adjoint u. For any morphism $f'\colon A \to B$ and any 2-morphism $\alpha\colon 1_A \Rightarrow uf'$, we define $\widetilde{\alpha} := (\epsilon \bullet f') \circ (f \bullet \beta)\colon f \Rightarrow f'$. The left adjunction equation in 1.2 yields the equality of pasting diagrams in below:

$$\begin{tikzcd}
& B \arrow[r, "1"] \arrow[d, "u"'] & B \arrow[d, "u"] & & B \arrow[d, bend right, "u"'] \arrow[d, bend left, "u"] \arrow[d, phantom, "="] \\
A \arrow[r, "1"'] \arrow[ur, "f'"] & A \arrow[r, "1"'] \arrow[ur, "f" description] \arrow[u, phantom, shift left=6, "\epsilon\Uparrow"] & A \arrow[u, phantom, shift right=6, "\eta\Uparrow"] \arrow[r, phantom, "="] & \; & A
\end{tikzcd}$$

REMARK 1.8.9. In a 2-category $\mathfrak{K}$ with a terminal object 1, the colimit and limit of a morphism $f\colon A \to B$ can be intrinsically defined as the left and right extensions of f along the unique (up to iso-2-morphism) morphism $!_A\colon A \to 1$, respectively.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!_A"'] \arrow[dr, phantom, "\varphi\Downarrow" near start] & C & & A \arrow[r, "f"] \arrow[d, "!_A"'] \arrow[dr, phantom, "\Uparrow\varphi" near start] & C \\
1 \arrow[ur, dashed, bend right, "\operatorname{colim} f"'] & & & 1 \arrow[ur, dashed, bend right, "\lim f"'] &
\end{tikzcd}$$

Pasting (lan_f, ϕ) with the comma square in below makes $(gb, (\mathrm{lan}_j^f \cdot \delta) \circ (\phi \cdot d_0))$ into a left extension of fd_0 along $d_1 \colon (j \downarrow b) \to 1$.

$$\begin{tikzcd}
(j \downarrow b) \arrow[r, "d_0"] \arrow[d, "d_1"'] & A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dl, phantom, "\delta\Downarrow"] & C \arrow[dl, phantom, "\varphi\Downarrow"] \\
1 \arrow[r, "b"'] & B \arrow[ur, dashed, "\mathrm{lan}_j^f"'] &
\end{tikzcd} \tag{1.14}$$

By definition, we have

$$\mathrm{lan}_j^f b \cong \mathrm{colim}((j \downarrow b) \xrightarrow{d_0} A \xrightarrow{f} C) \tag{1.15}$$

PROPOSITION 1.8.10. From the remarks above we conclude that

(i) Left (resp. right) adjoints preserve left (resp. right) extensions. In particular, they preserve colimits (resp. limits).

(ii) The left adjoint is fully faithful iff the unit is an iso-2-morphism.

(iii) The right adjoint is fully faithful iff the counit is an iso-2-morphism.

EXAMPLE 1.8.11. In the 2-category $\mathfrak{Cat}$ of categories extensions are known as **Kan extensions** as a tribute to the early work of Daniel Kan on adjoints and extension. It is by now a classical result that in the case when A is small, B is locally small, and C is cocomplete then the left Kan extension of any functor f along any j exists, and is pointwise calculated by the coend $\int^{a \in A} B(ja, b) \otimes fa$ ([ML98, § X.4.1-2]). Of course, the expression of coend uses the set-enrichment structure of categories, so $B(ja, b) \otimes fa$ is basically $B(ja, b)$-indexed coproduct of fa with itself. (See §1.9 for formulation of cotensor as a weighted limit and the expression of left extensions in the more general setting of $\mathcal{V}$-enriched categories.)

Now, the coend expression of the left Kan extension suggests that the condition of local smallness of B can be weakened to the requirement that all $B(ja, b)$ are small (i.e. a set), a condition called "admissibility" of j, by Street and Walters in their ‘Yoneda structures’ A familiar case of this equation in the 2-category $\mathfrak{Cat}$ of categories is the following situation:

$$\begin{tikzcd}
(F^{\mathrm{op}} \downarrow d) \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\delta\Downarrow"] & \mathcal{C}^{\mathrm{op}} \arrow[r, "P"] \arrow[d, "F^{\mathrm{op}}"'] \arrow[dr, phantom, "\varphi\Downarrow"] & \mathcal{S}\mathrm{et} \\
1 \arrow[r, "d"'] & \mathcal{D}^{\mathrm{op}} \arrow[ur, dashed, bend right, "\mathrm{Lan}_{F^{\mathrm{op}}} P"'] &
\end{tikzcd}$$

From the general case, we deduce that

$$\mathrm{Lan}_{F^{\mathrm{op}}} P(d) \cong \mathrm{colim}((d \downarrow F)^{\mathrm{op}} \xrightarrow{\pi} \mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathcal{S}\mathrm{et})$$

This is known as ‘push-forward’ of presheaves. It is, by the universality property of left extensions, the left adjoint to the ‘pullback functor’ $F^* \colon \mathcal{P}\mathrm{Shv}(\mathcal{D}) \to \mathcal{P}\mathrm{Shv}(\mathcal{C})$ obtained from pre-composition with F^{op}. Indeed, $F^*(Q) \cong \mathcal{P}\mathrm{Shv}(\mathcal{D})(y_{\mathcal{D}}F(-), Q)$. Note that by this equation, a natural transformation $\theta \colon F \Rightarrow G$ induces a natural transformation $\theta^* \colon G^* \Rightarrow F^*$, and therefore

When F is left exact, then $(d \downarrow F)$ is filtered and since filtered colimits commute with finite limits (See [MLM94, §VII.6]), it follows that $\mathrm{Lan}_{F^{\mathrm{op}}} \colon \mathcal{P}\mathrm{Shv}(\mathcal{C}) \to \mathcal{P}\mathrm{Shv}(\mathcal{D})$ is left exact, and therefore it is the inverse of geometric morphism $(\mathrm{Lan}_{F^{\mathrm{op}}}, F^*) \colon \mathcal{P}\mathrm{Shv}(\mathcal{D}) \to \mathcal{P}\mathrm{Shv}(\mathcal{C})$. Therefore, we have a 2-functor $\mathcal{P}\mathrm{Shv}(-) \colon {\mathfrak{Cat}_{\mathrm{cart}}}^{\mathrm{op}} \to \mathcal{E}\mathfrak{Top}$. (For more details see [Joh02a, Example 4.1.10].)

EXAMPLE 1.8.12. We saw the connection between left extensions and colimits. But, there is a sense which relates lefts extensions to the object of (path) connected components. Let $1 \colon \mathcal{A} \to \mathcal{S}\mathrm{et}$ be the functor which is constant at the terminal set $1 = \{\star\}$. It is straightforward to see that the left extension of 1 along any functor $K \colon \mathcal{A} \to \mathcal{B}$ computes, at $b \in \mathcal{B}$, the set of connected components of comma category $(K \downarrow b)$, i.e.

$$Lan_K 1(b) \cong \mathrm{colim}((K \downarrow b) \xrightarrow{d_0} \mathcal{A} \xrightarrow{1} \mathcal{S}\mathrm{et}) \cong \Pi_0(K \downarrow b)$$

A special case of this situation involves category of elements of a diagram. Suppose $F\colon \mathcal{B} \to \mathcal{S}\mathrm{et}$ is a functor and $\int_{\mathcal{B}} F$ is the category of elements of F obtained by the following comma object.

$$\begin{tikzcd} \int_{\mathcal{B}} F \arrow[r, "!"] \arrow[d, "\pi_{\mathcal{B}}"'] & \mathbf{1} \arrow[d, "1"] \\ \mathcal{B} \arrow[r, "F"'] \arrow[ur, phantom, "\overset{\delta}{\swarrow}"] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

In fact the 2-morphism δ in the comma square above establishes F as the left extension of constant functor $1\colon \int_{\mathcal{B}} F \to \mathcal{S}\mathrm{et}$, and therefore we have

$$F(b) \cong \mathrm{colim}((\pi_{\mathcal{B}} \downarrow b) \xrightarrow{d_0} \int_{\mathcal{B}} F \xrightarrow{1} \mathcal{S}\mathrm{et}) \cong \Pi_0(\pi_{\mathcal{B}} \downarrow b) \tag{1.16}$$

To see the isomorphism $F(b) \cong (\pi_{\mathcal{B}} \downarrow b)$ more concretely, note that in the comma category $(\pi_{\mathcal{B}} \downarrow b)$, an object is of the form (x, d, σ) where $x \in F(d)$ and $\sigma\colon d \to b$ is in $\mathcal{B}$, and a morphism of $(\pi_{\mathcal{B}} \downarrow b)$ is of the form $g\colon (x, d, \sigma) \to (x', d', \sigma')$ where $g\colon b \to b'$ is a morphism in $\mathcal{B}$ with $g \centerdot x = F(g)(x) = x'$ and $\sigma' \circ g = \sigma$.

$$\begin{tikzcd} x \arrow[dd, mapsto] \arrow[dr] & & \\ & x' \arrow[dd, mapsto] & \\ d \arrow[rr, "\sigma"] \arrow[dr, "g"'] & & b \\ & d' \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

The functor $d_0\colon (\pi_{\mathcal{B}} \downarrow b) \to \int_{\mathcal{B}} F$ forgets the b and σ parts. Now, any two objects in the same connected component of $(\pi_{\mathcal{B}} \downarrow b)$ we associate the same element $\sigma \centerdot x = (\sigma' \circ g) \centerdot x = \sigma \centerdot (g \centerdot x) = \sigma' \centerdot x'$. The mappings $x \mapsto (x, b, \mathrm{id}_b) \in$ and $(x, d, \sigma) \mapsto \sigma \centerdot x$ give the isomorphism $F(b) \cong (\pi_{\mathcal{B}} \downarrow b)$.

1.9 2-Categorical and bicategorical limits

The aim of this section is to introduce a consistent language to talk about and delineate between 2-categorical (co)limits and bicategorical (co)limits. As mentioned before bicategorical (co)limits are the correct notion of (co)limits in various 2-categories of toposes, while the important 2-category $\mathfrak{Con}$ of

AU-contexts for us in Chapter 3 the limits are strict and 2-categorical. This demarcation is summarized in the table of Remark 1.9.6. It should be noted therein that although in general by weakening of structures of cones and representation for (co)limits we obtain various notions of 2-limits and bilimits, in particular cases these various notions could well be equivalent. This is manifested in handful of examples in this section.

We use the elegant machinery of weighted limits ([Kel82], [Joh02a]) for giving the definition of most general 2-limits and bilimits. At the start, we shall motivate the notion of weighted limits from the 1-dimensional case of limits of diagrams in categories.

In Remark 1.9.5, we observe that we can divide the universal properties of 2-limits to the 1-dimensional universal properties and the 2-dimensional universal properties. We will stick to this terminology throughout the whole thesis.

Limits of diagrams in category theory, viewed as a representing objects for appropriate $\mathcal{S}$et-functor, generalizes to the notion of *weighted limits* of a weighted diagrams in 2-category theory, defined as representing objects of certain $\mathfrak{Cat}$-valued 2-functor.

We quickly recall a version of 1-dimensional limit and colimits which can be readily generalized to weighted 2-dimensional limits. Example 1.7.2 of product is one of the simplest instance of products in category theory. As with the product, a limit of a diagram in a category represents the presheaf of cones on that diagram. Suppose $\mathcal{J}$ is a small category and $D\colon \mathcal{J} \to \mathcal{C}$ is a diagram of shape $\mathcal{J}$ in the category $\mathcal{C}$. For an object A in $\mathcal{C}$, the set of cones in $\mathcal{C}$ with apex A is in bijection with the set of natural transformations between the constant functor at $1 = \{*\}$, namely $\Delta(1)\colon \mathcal{J} \to \mathbf{1} \to \mathcal{S}\text{et}$, and functor D. More formally,

$$Cone(A, D) \cong [\mathcal{J}, \mathcal{S}\text{et}](\Delta(1), \mathcal{C}(A, D(-)) \cong [\mathcal{J}, \mathcal{C}](\Delta(A), D) \qquad (1.17)$$

Note that this isomorphism is natural in A, and as such we obtain a presheaf

$Cone(-, D)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. A **limit** of diagram D is a representation $(\lim_{\mathcal{J}} D, \eta)$ for the presheaf $Cone(-, D)$ where $\eta\colon \Delta(A) \Rightarrow D$.

We wrote $\mathcal{C}(A, D(-))$ instead of $\hom_{\mathcal{C}}(A, D(-))$ to emphasize the $\mathcal{S}$et-enrichment structure of the category $\mathcal{C}$. Indeed, it is known since long that the theory of limits and colimits of categories has a robust generalization to the categories enriched in closed monoidal categories and they are known as *enriched weighted (aka indexed) limits* ([BK75], [Kel82]). The enriched theory of limits and colimits generalizes ordinary categorical theory of limits and colimits by choosing $(\mathcal{V}, \otimes, I)$ to be the symmetric closed monoidal category $(\mathcal{S}\mathrm{et}, \times, 1)$. In below, we give a brief outline of this generalization, emphasizing why the notion of *weight* must be introduced in the passage from $\mathcal{S}$et-categories to general $\mathcal{V}$-categories.

NOMENCLATURE. Nowadays, the terminology 'weighted (co)limits' is much more commonly used perhaps for the good reason that the term 'indexed' is already overloaded with various meanings in category theory. There is another reason why we should prefer the terminology 'weighted (co)limits': For a family $\{X_i\}_{i \in I}$ of sets, each X_i with cardinality n_i, the cardinality of $\coprod_{i \in I} X_i$ is $\sum_{i \in I} n_i$, and therefore coproducts *are like* sums. Weighted products *are like* weighted sums $\sum_{i \in I} w_i \times n_i$. This view is vindicated by the coend formula

$$\operatorname*{colim}_{W} D = \int^{j \in J} W(j) \otimes D(j)$$

for weighted colimits. Nonetheless, beware that some of the pioneering papers about weighted limits (e.g. [KS74], [BK75], [Kel89]) use the terminology 'indexed limits'.

First, recall that $\mathcal{V}$-enriched representable functors are defined as $\mathcal{V}$-functors $\mathcal{C}(A, -)\colon \mathcal{C} \to \mathcal{V}$, and the action of this enriched functor[10] on hom-objects is determined by the right adjoint

$$\mathcal{C}(X, Y) \to [\mathcal{C}(A, X), \mathcal{C}(A, Y)]_{\mathcal{V}}$$

---

[10] Note that here $\mathcal{V}$ is considered enriched over itself via its closed structure.

to the composition morphism

$$\mathcal{C}(X,Y) \otimes \mathcal{C}(A,X) \to \mathcal{C}(A,Y)$$

In above we rely on the closed structure[11] of $\mathcal{V}$.

NOTE. If $\mathcal{V}$ is *symmetric* monoidal closed, then we can also define enriched representable presheaves $\mathcal{C}(-,A)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. Understanding $\mathcal{V}$-functors as $\mathcal{C} \to \mathcal{V}$ as $\mathcal{V}$-modules, in the absence of symmetry, we need to distinguish between left and right for the module structure; one is used for limits, the other for colimits.

Second, the category of natural transformation, used in equation 1.17, is generalized to a $\mathcal{V}$-category. If the monoidal category $\mathcal{V}$ is complete then we posses the means to make the collection of $\mathcal{V}$-functors between any two $\mathcal{V}$-categories into an $\mathcal{V}$-category. This is usually expressed by considering the *object* of natural transformations between $\mathcal{V}$-functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ as the end $\int^{c \in \mathcal{C}} \mathcal{D}(F(c), G(c))$.

It seems that we now have all the ingredients to generalize the notion of (co)limits to the enriched setting by replacing 1, the unit of monoidal category $\mathcal{S}$et, with I the unit of $\mathcal{V}$. However, a simple-minded generalization will not yield the correct notion for two reason: first that to establish the first isomorphism in equation 1.17 we fundamentally used the fact that $1 = \{*\}$ is the terminal object of $\mathcal{S}$et. This is not true for many interesting monoidal categories. Furthermore, the category $\mathcal{S}$et is well-pointed and the unit 1 is the separator. Moreover, any set X is entirely determined by its points, i.e. morphisms $1 \to X$, and any function of sets is entirely determined by its action on points. Again, these facts do not generalize to a general monoidal category (by a point of object A of $(\mathcal{V}, \otimes, I)$ we mean a morphism $I \to A$). Therefore, to obtain a nicely behaved notion of enriched (co)limit we have to replace $\Delta(1)$ by a fattened up $\mathcal{V}$-functor $W\colon \mathcal{J} \to \mathcal{V}$.

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[11] Steven Vickers noted that we can do away with this reliance: we can understand a $\mathcal{V}$-functor A from $\mathcal{C}$ to $\mathcal{V}$ as a "$\mathcal{V}$-module" over $\mathcal{C}$, for each object X of $\mathcal{C}$ it has a $\mathcal{V}$-object $A(X)$; and for each pair X, Y there is a "scalar multiplication" $\mathcal{C}(X,Y) \otimes A(X) \to A(Y)$.

Suppose $\mathcal{V}$ is a closed monoidal category and $\mathcal{C}$ is a category enriched in $\mathcal{V}$. A **$\mathcal{V}$-weighted diagram** of shape $\mathcal{J}$ consists of a pair of $\mathcal{V}$-functors D (the diagram) and W (the weight) where $\mathcal{J}$ is a small $\mathcal{V}$-category.

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "D"] \arrow[rd, "W"'] & \mathcal{C} \\
 & \mathcal{V}
\end{tikzcd}$$

A weighted cone with apex A in $\mathcal{C}$ is a $\mathcal{V}$-natural transformation $W \Rightarrow \mathcal{C}(A, D(-))$. Consider the transposed $\mathcal{V}$-functor $\widehat{D} \colon \mathcal{C}^{\mathrm{op}} \to [\mathcal{J}, \mathcal{V}]$; it takes a object X of $\mathcal{C}$ to the $\mathcal{V}$-functor $\mathcal{C}(X, D(-)) \colon \mathcal{J} \to \mathcal{V}$, and is defined on hom-objects by the composition morphism $\mathcal{C}(X, D(j)) \otimes \mathcal{C}(Y, X) \to \mathcal{C}(Y, D(j))$. Note that in the case $\mathcal{J} = \mathbf{1}$, the assignment $D \mapsto \widehat{D}$ is nothing but the enriched Yoneda embedding.

A **limit** over the weighted diagram above is a representation $(\lim\limits_{W} D, \eta)$ for the functor

$$\begin{aligned} Cone_W D \colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{V} \\ X &\mapsto [\mathcal{J}, \mathcal{V}](W, \widehat{D}X) \end{aligned} \tag{1.18}$$

where $\eta \colon W \Rightarrow \widehat{D}X$ is a $\mathcal{V}$-natural transformation, that is

$$\mathcal{C}(X, \lim_{W} D) \cong [\mathcal{J}, \mathcal{V}](W, \widehat{D}X) \tag{1.19}$$

natural in X. Note that η is indeed the unit of this isomorphism, i.e. the image of $I \to \mathcal{C}(\lim\limits_{W} D, \lim\limits_{W} D)$ under the isomorphism above.

Dually, one defines the notion of **weighted colimit** over a **weighted cocone** $(D \colon \mathcal{J} \to \mathcal{V}, W \colon \mathcal{J}^{\mathrm{op}} \to \mathcal{V})$ where $\mathcal{J}^{\mathrm{op}}(j, j') := \mathcal{J}(j', j)$. The cocone diagram can be expressed as the span below:

$$\begin{tikzcd}
\mathcal{J}^{\mathrm{op}} \arrow[r, "D^{\mathrm{op}}"] \arrow[rd, "W"'] & \mathcal{C}^{\mathrm{op}} \\
 & \mathcal{V}
\end{tikzcd}$$

The colimit then is defined by the isomorphisms

$$\mathcal{C}(\operatorname*{colim}_{W} D, Y) \cong [\mathcal{J}^{\mathrm{op}}, \mathcal{V}](W, \widecheck{D}Y) \tag{1.20}$$

natural in Y, where $\widecheck{D} \colon \mathcal{C} \to [\mathcal{J}^{\mathrm{op}}, \mathcal{V}]$ is the $\mathcal{V}$-functor which takes X to $\mathcal{C}(D(-), Y)$. A $\mathcal{V}$-enriched category is **complete** whenever $\widehat{D}$ has a left adjoint for all diagrams D. It is **cocomplete** whenever $\widecheck{D}$ has a left adjoint for all diagrams D.

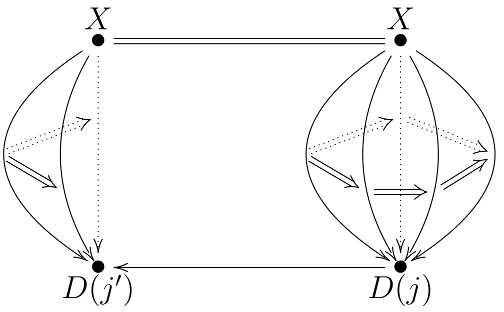
$$\begin{tikzcd}
{[\mathcal{J}, \mathcal{S}\mathrm{et}]} \arrow[r, bend right, "\lim\limits_{(-)} D"'] \arrow[r, phantom, "\top"] & \mathcal{C}^{\mathrm{op}} \arrow[l, bend right, "\widehat{D}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathcal{J}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]} \arrow[r, bend right, "\operatorname*{colim}\limits_{(-)} D"'] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, bend right, "\widecheck{D}"']
\end{tikzcd} \tag{1.21}$$

When $\mathcal{V}$ is the cartesian monoidal category $\mathcal{S}\mathrm{et}$ of sets, as opposed to the general case, then all weighted enriched limits can be expressed by ordinary limits. Nevertheless, weighted limits usually have a simpler diagram functor D as they transfer the complexity of diagrams, over which we take limits and colimits, to the weights. For instance, consider the example of product $\prod\limits_{W} D$, where D is in $\mathcal{C}$ and W is a discrete category, which is the limit of constant diagram $\Delta(D) \colon W \to \mathcal{C}$. It is of course isomorphic to the limit of weighted diagram with weight functor $W \colon \mathbf{1} \to \mathcal{S}\mathrm{et}$ and the diagram $D \colon \mathbf{1} \to \mathcal{C}$. The latter limit is known as *cotensor* (aka *power*) $W \pitchfork D$. In this case we have $W \pitchfork D \cong \prod\limits_{W} D \cong D^{W}$. The limit cone η is given by W-many morphism $W \pitchfork D \to D$, obtained by exponentiating W-many morphism $1 \to W$.

Moreover, even in the case of set-weighted limits, the notion of weighted (co)limit is important on its own merits as it gives a conceptual clarity not offered by ordinary (co)limits. For instance for every complete $\mathcal{V}$-category $\mathcal{C}$, the functor $\operatorname*{colim}\limits_{(-)} D$ in the diagram (1.21) is the left extension of $D \colon \mathcal{J} \to \mathcal{C}$ along the Yoneda embedding.

All of the strict 2-categorical limits can be obtained via weighted limits when we take $\mathcal{V}$ to be the cartesian monoidal category of categories and functors. For the rest of the thesis we will be concerned only with category-weighted limits. We shall give an elementary description of 2-categorical weighted limits. Note that they generalize the enriched limits over the cartesian monoidal category $\mathcal{C}$at of categories and functors in that we can weaken the strict $\mathcal{C}$at-natural transformations, used in definition of category of cones, to pseudo and lax transformations. Also, we can weaken isomorphisms of categories by their equivalence in definition of the limits as representation. But first, it is helpful to contrast picture of category-weighted cones with ordinary cones.

REMARK 1.9.1. In the ordinary case, a cone over a diagram $D\colon \mathcal{J} \to \mathcal{C}$ is given by a apex X of $\mathcal{C}$, and for each j of $\mathfrak{J}$ a *single* morphism $X \to D(j)$ natural with respect to action of morphisms $f\colon j \to j'$ in $\mathcal{J}$. The limit of D is the universal such cone over D. In the case of category-weighted limits, a category-weighted cone over a diagram $D\colon \mathfrak{J} \to \mathfrak{K}$ specifies a category of morphisms $X \to D(j)$, for each object w of the category $W(j)$, and moreover it specifies actions of 1-morphisms and 2-morphisms of $\mathfrak{J}$ as functors and natural transformations between these categories.



DEFINITION 1.9.2. Suppose $\mathfrak{J}$ is a small 2-category and $\mathfrak{K}$ is a 2-category. Moreover, let $D\colon \mathfrak{J} \to \mathfrak{K}$ and $W\colon \mathfrak{J} \to \mathfrak{Cat}$ be strict 2-functors. A **diagram of shape J with weight W** in $\mathfrak{K}$ consists of

$$\begin{tikzcd}
\mathfrak{J} \arrow[r, "D"] \arrow[dr, "W"'] & \mathfrak{K} \\
 & \mathfrak{Cat}
\end{tikzcd}$$

where the 2-functor D is the diagram, and W specifies a weight $W(j)$ for each object $j \in \mathfrak{J}_0$ and a weight transformer $W(f)$ to each morphism $j \xrightarrow{f} j'$ in $\mathfrak{J}$. A **lax weighted cone** over the weighted diagram (D, W) with apex $X \in \mathfrak{K}_0$ is given by the following data:

$(WC1)$ A functor $L(j) : W(j) \to \mathfrak{K}(X, D(j))$ for each $j \in \mathfrak{J}_0$.

($WC2$) A natural transformation $L(f)\colon D(f)_* \circ L(j) \Rightarrow L(j') \circ W(f)$, for each arrow $f\colon j \to j'$ in $\mathfrak{J}$.

$$\begin{tikzcd}
W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "\Downarrow L(f)"] & \mathfrak{K}(X, D(j)) \arrow[d, "D(f)_*"] \\
W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j'))
\end{tikzcd} \tag{1.22}$$

satisfying the coherence condition expressed by equality of pasting diagrams in below:

$$\begin{tikzcd}
W(j) \arrow[r, "L(j)"] \arrow[d, bend right=60, "W(f')"'] \arrow[d, bend left=60, "W(f)"] \arrow[d, phantom, "\overset{W(\alpha)}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow L_f"] & \mathfrak{K}(X, D(j)) \arrow[d, "D(f)_*"] & {} \arrow[d, phantom, "="] & W(j) \arrow[r, "L(j)"] \arrow[d, "W(f')"'] \arrow[dr, phantom, "L_{f'} \Downarrow"] & \mathfrak{K}(X, D(j)) \arrow[d, bend right=60, "D(f')_*"'] \arrow[d, bend left=60, "D(f)_*"] \arrow[d, phantom, "\overset{D(\alpha)_*}{\Leftarrow}"] \\
W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j')) & {} & W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j'))
\end{tikzcd} \tag{1.23}$$

for any 2-morphism $\alpha\colon f \Rightarrow f'\colon j \rightrightarrows j'$ in $\mathfrak{J}$.

Notice that the last condition materializes only when $\mathfrak{J}$ is not a locally discrete 2-category. It appears in the shape diagram of equifier (Example 1.9.28), inverter (Example 1.9.31), and identifier (Example 1.9.32).

CONSTRUCTION 1.9.3. We form the category $\mathcal{L}ax\mathcal{C}one^X_W D$ of lax weighted cones over the weighted diagram (D, W) with apex X. The objects of this category are lax natural transformations $L\colon W \Rightarrow \mathfrak{K}(X, D(-))$ as given in ($WC2$), and a morphism between two such natural transformations L and L' is a modification $m\colon L \Rrightarrow L'$ which specifies for each object j of $\mathfrak{J}$, a natural transformation $m(j)\colon L(j) \Rightarrow L'(j)$ such that

$$L'_f \circ (D(f)_* \bullet m(j)) = (m(j') \bullet W(f)) \circ L_f \tag{1.24}$$

Equation (1.24) expresses commutativity of the obvious diagram of 2-morphisms in diagram (1.25): traversing along the front face and then bottom face yields the same 2-morphism as traversing the top face followed by back face.

$$\begin{tikzcd}
& W(j) \arrow[rr, "L'(j)"] \arrow[dl, equal] \arrow[dd] \arrow[dr, phantom, "m(j)\Nearrow"] & & \mathfrak{K}(X, D(j)) \arrow[dl, equal] \arrow[dd, "D(f)_*"] \\
W(j) \arrow[rr, "L(j)"] \arrow[dd, "W(f)"'] & & \mathfrak{K}(X, D(j)) \arrow[dd] & \\
& W(j') \arrow[rr, "L'(j')"] \arrow[dl, equal] \arrow[dr, phantom, "m(j')\Nearrow"] & & \mathfrak{K}(X, D(j')) \arrow[dl, equal] \\
W(j') \arrow[rr, "L(j')"] & & \mathfrak{K}(X, D(j')) &
\end{tikzcd} \tag{1.25}$$

Consider the 2-functor $\widehat{D}\colon \mathfrak{K}^{\mathrm{op}} \to [\mathfrak{J}, \mathfrak{Cat}]$; it takes a object X of $\mathfrak{K}$ to the functor $\mathfrak{K}(X, D(-))\colon \mathfrak{J} \to \mathfrak{Cat}$, a 1-morphism $f\colon Y \to X$ to the natural transformations of functors $\widehat{D}(f)\colon \widehat{D}(X) \Rightarrow \widehat{D}(Y)$ and a 2-morphism $\alpha\colon f \Rightarrow g$ to a modification $\widehat{D}(\alpha)\colon \widehat{D}(f) \Rrightarrow \widehat{D}(g)$.

The category $\mathcal{L}ax\mathfrak{C}one^{X}_{W}D$ just so constructed is a functor category, that is:

$$\mathcal{L}ax\mathfrak{C}one^{X}_{W}D \cong [\mathfrak{J}, \mathfrak{Cat}]_{lax}(W, \widehat{D}X) \cong [\mathfrak{J}, \mathfrak{Cat}]_{lax}(W, \mathfrak{K}(X, D(-))) \tag{1.26}$$

where the 2-category $[\mathfrak{J}, \mathfrak{Cat}]_{lax}$ consists of strict 2-functors, lax transformations and modifications.

DEFINITION 1.9.4. A **lax weighted limit** over the weighted diagram (D, W) is the representing object $\lim_W D$ of $\mathfrak{K}_0$ for the 2-functor

$$\begin{aligned} \mathcal{L}ax\mathfrak{C}one_{W}D\colon \mathfrak{K}^{\mathrm{op}} &\to \mathfrak{Cat} \\ X &\mapsto \mathcal{L}ax\mathfrak{C}one^{X}_{W}D \end{aligned}$$

This is equivalent to give equivalences

$$\Phi_X\colon \mathfrak{K}(X, \lim_{W} D) \simeq [\mathfrak{J}, \mathfrak{Cat}]_{lax}(W, \widehat{D}X)\colon \Psi_X \tag{1.27}$$

of categories, natural in X. We call $\Phi(1_{lim_W D})$, which gives the structure of limit cone, the **unit** of representation and we denote it by $\eta_{W,D}$.

Dually, a **lax weighted cocone** can be defined by a pair of strict 2-functors $D\colon \mathfrak{J} \to \mathfrak{K}$ and $W\colon \mathfrak{J}^{\mathrm{op}} \to \mathfrak{Cat}$. A lax weighted colimit is an object together with equivalences

$$\Phi^Y : \mathfrak{K}(\operatorname*{colim}_W D, Y) \simeq [\mathfrak{J}^{\mathrm{op}}, \mathfrak{Cat}]_{lax}(W, \breve{D}Y) \colon \Psi^Y \tag{1.28}$$

natural in Y. Thus weighted colimits are the same thing as weighted limits in $\mathfrak{K}^{\mathrm{op}}$.

REMARK 1.9.5. We can break the universal property of limit expressed in (1.27) into two parts:

(i) One-dimensional property which is expressed by the equivalence in (1.27) restricted to the underlying categories:

$$||\mathfrak{K}||_1(X, \lim_W D) \simeq ||[\mathfrak{J}, \mathfrak{Cat}]_{lax}||_1(W, \widehat{D}X) \tag{1.29}$$

where the isomorphism above is a bijection of sets.

(ii) Two-dimensional property which states that for any pair of morphisms $l_0, l_1 \colon X \rightrightarrows \lim_W D$, any modification $\eta l_0 \Rrightarrow \eta l_1$ of cones is equal to $\eta \centerdot \alpha$ for a unique 2-morphism $\alpha\colon l_0 \Rightarrow l_1$.

REMARK 1.9.6. There are several important variations of this definition which provides us with stricter structures. More precisely, the level of strictness of our weighted limits supervenes upon

- the strictness structure of functor 2-category $[\mathfrak{J}, \mathfrak{Cat}]_?$ where $?$ can be filled with lax, psd, or str, and
- the strictness of representation of the limit, that is whether it represents category of cones by isomorphism or equivalence of categories in equation (1.17).

We enumerate some important variations from the most strict to the least.

	Diagram	*Cone*	*Representation*
Strict weighted limits	*strict*	*strict*	$\cong$
Pseudo weighted limit	*strict*	*pseudo*	$\cong$
Lax weighted limit	*strict*	*lax*	$\cong$
Weighted bilimit	*strict*	*pseudo*	$\simeq$
Lax weighted bilimit	*strict*	*lax*	$\simeq$

For instance the paper [PR91] on PIE limits exclusively deals with strict weighted limits but [Joh02a] is mostly concerned with weighted bilimits particuarly in various 2-categories of toposes, although the prefix 'bi' is not used there. We have followed the consensus of Australian category theorists in naming various concepts of 2-categorical weighted limits. For instance See [Kel89]. However, of course not everybody adheres to this convention. Most notably, [Joh02a, §B1.1] takes "lax limit" to mean the limit of a lax diagram D, as opposed to our terminology where we took 'lax' as an attribute of weighted cones. However, the theory of limits of lax diagrams can be reduced to weighted limits with strict diagrams (See [Joh02a, Lemma 1.1.6]).

REMARK 1.9.7. The correct bicategorical notion of weighted limit is that of bilimit (aka weak limits). In bicategories, and also various 2-categories of toposes, we shall only consider bilimits, and we shall explicitly state it when we do. Since isomorphisms of categories are equivalences, any limit is automatically a bilimit, but the converse almost always fails to be true.

REMARK 1.9.8. The theory of weighted limits can be done fibrewise. Here, we only sketch the outline of it. Its details will be the subject of a future study. Suppose diagram D and weight W are given as before. A W-cone L with apex X in $\mathfrak{K}$ is an opfibration map $W \to X \sslash D$ over $\mathfrak{J}$ where $X \sslash D$ is the weak slice constructed as the comma 2-category of $X\colon \mathbf{1} \to \mathfrak{K}$ and $D\colon \mathfrak{J} \to \mathfrak{K}$. By opfibration in above we mean a fibration of 2-categories which will be discussed in chapter 2. The limit $\lim\limits_{W} D$ then is the universal such opfibration $\lim\limits_{W} D \sslash D \to \mathfrak{J}$ with an opfibration map from W over $\mathfrak{J}$.

EXAMPLE 1.9.9. Any weighted limit with weight functor $W = \Delta(\mathbf{1})\colon \mathfrak{J} \to \mathfrak{Cat}$ constant at the terminal category $\mathbf{1}$ is called **conical**. Notice that in this case, an object of $\mathcal{L}ax\mathfrak{C}one^X_W D$ is an ordinary cone over D with apex X in underlying category $||\mathfrak{K}||_1$, and a morphism therein is a modification of such cones. The universal property in (1.27) exhibits something more than just a limit in underlying category $||\mathfrak{K}||_1$. There is also the 2-dimensional universal property. Therefore, every conical limit, such as product, pullback, etc., in a 2-category $\mathfrak{K}$ is an ordinary limit in $||\mathfrak{K}||_1$. However the converse is not true; a binary product in $||\mathfrak{K}||_1$ need not be a conical limit in $\mathfrak{K}$.

EXAMPLE 1.9.10. Consider the weighted diagram where $\mathfrak{J} = \mathbf{1}$ is the terminal 2-category, D is an object of $\mathfrak{K}$ and W is a (small) category. The strict weighted limit $\lim\limits_{W} D$ is known as **cotensor** (aka **power**) of D by W and is denoted by $W \pitchfork D$. Similarly the colimit $\operatorname*{colim}\limits_{W} D$ is known as the **tensor** (aka **copower**) and is usually denoted by $W \otimes D$. Equations 1.27 and 1.28 become specialized to

$$\mathfrak{K}(X, W \pitchfork D) \cong \mathfrak{Cat}(W, \mathfrak{K}(X, D)) \quad \text{and} \quad \mathfrak{K}(W \otimes D, Y) \cong \mathfrak{Cat}(W, \mathfrak{K}(D, Y)) \tag{1.30}$$

In the case of cotensor, the weighted limit cone consist a family $\{d(\phi)\}$ of 2-morphism

$$\begin{tikzcd}
W \pitchfork D \arrow[r, bend left=50, "d(w)", ""{name=U, below}] \arrow[r, bend right=50, "d(w')"', ""{name=L, above}] & D
\arrow[Rightarrow, from=U, to=L, "d(\phi)"]
\end{tikzcd}$$

indexed by morphisms $\phi \colon w \to w'$ in W. The *1-dimensional universal property* states that any other family $\{l(\phi)\}$ factors uniquely through the family $\{d(\phi)\}$. The *2-dimensional universal property* states that for any parallel pair of morphisms $h, k \colon X \rightrightarrows W \pitchfork D$ and a family $\{\beta_w \colon d(w)h \Rightarrow d(w)k\}$ of 2-morphisms in $\mathfrak{K}$ which makes the following diagram of 2-morphisms

$$\begin{tikzcd}
d(w)h \arrow[r, Rightarrow, "\beta_w"] \arrow[d, Rightarrow, "d(\phi)\bullet h"'] & d(w)k \arrow[d, Rightarrow, "d(\phi)\bullet k"] \\
d(w')h \arrow[r, Rightarrow, "g\bullet m_1"'] & d(w')k
\end{tikzcd}$$

commutes, there is a unique 2-morphism $\alpha \colon h \Rightarrow k$ with $d(w) \bullet \alpha = \beta_w$ for each object w of W. The characterization of universal properties of tensor is similar. The tensor and cotensor with the free walking arrow category $\mathbf{2}$ has special status. In fact in the 2-category $\mathfrak{Cat}$, the tensor $\mathbf{2} \otimes \mathcal{C}$ is isomorphic to the product $\mathbf{2} \times \mathcal{C}$ and the cotensor $\mathbf{2} \pitchfork \mathcal{C}$ is the comma category $(\mathcal{C} \downarrow \mathcal{C})$. These two are very different things: for instance $\mathbf{2} \otimes \mathbf{1} \cong \mathbf{2} \not\simeq \mathbf{1} \cong \mathbf{2} \pitchfork \mathbf{1}$.

Of course in the environment of bicategories, and also 2-categories of toposes by tensor and cotensor we really mean the weak version, i.e. a bilimit. In this case, for any 2-morphism $\alpha \colon a_0 \Rightarrow a_1 \colon X \rightrightarrows D$, we have a morphism $\ulcorner \alpha \urcorner \colon X \to \mathbf{2} \pitchfork D$, unique up to a unique iso-2-morphism, together with iso-2-morphisms $\zeta_i \colon a_i \cong d_i \circ \ulcorner \alpha \urcorner$ $(i = 0, 1)$ such that $\zeta_1^{-1} \circ (\delta \centerdot \ulcorner \alpha \urcorner) \circ \zeta_0 = \alpha$. For instance, in the case where $\mathfrak{K} = \mathcal{E}\mathfrak{Top}$, we have a 2-functor $\mathbf{2} \otimes (-) \colon \mathcal{E}\mathfrak{Top} \to \mathcal{E}\mathfrak{Top}$. For a topos $\mathscr{E}$, the underlying category of $\mathbf{2} \otimes \mathscr{E}$ is the comma category $(\mathcal{E} \downarrow \mathcal{E}) = \mathfrak{Cat}(\mathbf{2}, \mathcal{E})$, where $\mathcal{E}$ is the underlying category of $\mathscr{E}$. There are (bounded) inclusions $d_0, d_1 \colon \mathscr{E} \rightrightarrows \mathbf{2} \otimes \mathscr{E}$ whose inverse images are given by domain and codomain functors $(\mathcal{E} \downarrow \mathcal{E}) \rightrightarrows \mathcal{E}$, i.e. $d_0^*(E_0 \xrightarrow{f} E_1) = E_0$ and $d_1^*(E_0 \xrightarrow{f} E_1) = E_1$. The direct images are given by $(d_0)_* E = (E \xrightarrow{!} 1)$ and $(d_1)_* E = (E \xrightarrow{\mathrm{id}} E)$. For the final topos $\mathscr{S}$, we have $\mathbf{2} \otimes \mathscr{S} \simeq \mathcal{S}\mathrm{hv}(\mathbb{S})$. An direct way to see this is to consider sheaves over as discrete opfibration: A sheaf X over $\mathbb{S}$ then is a discrete bundle (opfibration) over points of $\mathbb{S}$, and as such is given by a morphism $X_\perp \to X_\top$ in $\mathscr{S}$. Similarly $\mathbf{2} \otimes \mathcal{S}\mathrm{hv}(X) \simeq \mathcal{S}\mathrm{hv}(\mathbb{S} \times X)$.

PROPOSITION 1.9.11 ([Kel89]). If a 2-category $\mathfrak{K}$ admit strict tensors with $\mathbf{2}$ then all the 2-dimensional universal properties of existing strict weighted limits follows from their respective 1-dimensional universal properties.

Proof. Suppose diagram D and weight W are given as before, and A is an object satisfying strict version (i.e. with isomorphism instead of equivalence) of (1.29) natural in X. Therefore, we have the structure of limit cone of A, and we get functors Φ_X as in 1.27, though not necessarily an isomorphism yet, by whiskering with the structure of limit cone of A. We want to show that Φ_X

is indeed an isomorphism of categories. Consider the commutative diagram of sets in below.

$$\begin{tikzcd}
\|\mathfrak{Cat}\|_1(\mathbf{2}, \mathfrak{K}(X,A)) \arrow[r, "\|\mathfrak{Cat}\|_1(\mathbf{2}, \Phi_X)"] \arrow[d, "\cong"'] & \|\mathfrak{Cat}\|_1(\mathbf{2}, [\mathfrak{J}, \mathfrak{Cat}](W, \widehat{D}(X))) \arrow[d, "\cong"] \\
\|\mathfrak{K}\|_1(\mathbf{2} \otimes X, A) \arrow[r, "\cong"'] & \|[\mathfrak{J}, \mathfrak{Cat}]\|_1(W, \widehat{D}(\mathbf{2} \otimes X))
\end{tikzcd}$$

The left bijection is the expression of the 1-dimensional universal property of tensor $\mathbf{2} \otimes X$, while the bottom row bijection follows from the 1-dimensional universal property of A by our assumption. The right bijection is a combination of currying (with respect to the cartesian monoidal structure of $\|\mathfrak{Cat}\|_1$ and the 1-dimensional universal property of $\mathbf{2} \otimes X$. Now it is an easy exercise to see that $\|\mathfrak{Cat}\|_1(\mathbf{2}, -) \colon \|\mathfrak{Cat}\|_1 \to \mathcal{S}\mathrm{et}$ reflects isomorphisms. Therefore, Φ_X is an isomorphism. $\square$

In such 2-categories, such as $\mathfrak{Cat}$, $\mathcal{E}\mathfrak{Top}$, and $\mathfrak{Con}$, our proofs that a certain object is equivalent to a weighted limit are more economical since we do not need to check the 2-dimensional aspect.

DEFINITION 1.9.12. A 2-category is **complete** (resp. **cocomplete**) if it admits products (resp. coproducts), equalizers (resp. coequalizers), and cotensor products (resp. tensor products). It is **bicomplete** (resp. bicocomplete) if it admits the weak version of these limits. We say that a 2-category is finitely complete (resp. finitely cocomplete) if it admits finite products (resp. coproducts), equalizers (resp. coequalizers), and cotensor (resp. tensor) with $\mathbf{2}$.

PROPOSITION 1.9.13. The following statements hold about strict completeness:

- $\mathfrak{Cat}$ is complete and cocomplete.
- The 2-category $\mathbf{2}\mathfrak{Cat}(\mathfrak{J}, \mathfrak{K})$ is complete (resp. cocomplete) when $\mathfrak{K}$ is so, and the limits (resp. colimits) are calculated pointwise.
- Any full reflective sub-2-category of a complete 2-category is again complete.

EXAMPLE 1.9.14. Suppose $\mathfrak{K}$ be a finitely complete 2-category (or "representable" in terminology of [Str74]). Therefore, $\mathfrak{K}$ has all comma objects (See Remark 1.9.27). For an object C in $\mathfrak{K}$, the pair $d_0, d_1 \colon \mathbf{2} \pitchfork C \rightrightarrows C$ can be enriched to an internal category (See A.8) in the underlying category $||\mathfrak{K}||_1$. The identity 2-morphism id_{1_C} induces a morphism $i \colon C \to \mathbf{2} \pitchfork C$ with $\delta \centerdot i = \mathrm{id}_{1_C}$. Also, the 2-morphism $(\delta \centerdot (d_0^* d_1)) \circ (\delta \centerdot (d_1^* d_0))$ formed by the pasting diagram

$$\begin{tikzcd}
(\mathbf{2} \pitchfork C)\,{}_{d_1}\!\times_{d_0}(\mathbf{2} \pitchfork C) \arrow[r, "d_0^* d_1"] \arrow[d, "d_1^* d_0"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{2} \pitchfork C \arrow[d, "d_0"', ""{name=A}] \arrow[d, bend left=60, "d_1", ""{name=B, swap}] \arrow[from=A, to=B, phantom, "\overset{\delta}{\Rightarrow}"] \\
\mathbf{2} \pitchfork C \arrow[r, "d_1", ""{name=C, swap}] \arrow[r, bend right=40, "d_0"', ""{name=D}] \arrow[from=D, to=C, phantom, "\delta\Uparrow"] & C
\end{tikzcd}$$

induces a morphism $m \colon (\mathbf{2} \pitchfork C)\,{}_{d_1}\!\times_{d_0}(\mathbf{2} \pitchfork C) \to \mathbf{2} \pitchfork C$ with $\delta \centerdot m = (\delta \centerdot (d_0^* d_1)) \circ (\delta \centerdot (d_1^* d_0))$. Indeed, i and m are respectively unit and composition of category object $\mathbb{C} = (d_0, d_1 \colon \mathbf{2} \pitchfork C \rightrightarrows C)$. A morphism $f \colon C \to D$ in $\mathfrak{K}$ lifts to internal functor $(f, \mathbf{2} \pitchfork f) \colon \mathbb{C} \to \mathbb{D}$ since $f \centerdot \delta_C$ must uniquely factor through δ_D.

$$\begin{tikzcd}
\mathbf{2} \pitchfork C \arrow[r, "\mathbf{2} \pitchfork f"] \arrow[d, bend right=50, "d_0"', ""{name=A}] \arrow[d, bend left=50, "d_1", ""{name=B, swap}] \arrow[from=A, to=B, phantom, "\overset{\delta}{\Rightarrow}"] & \mathbf{2} \pitchfork D \arrow[d, bend right=50, "d_0"', ""{name=C}] \arrow[d, bend left=50, "d_1", ""{name=D, swap}] \arrow[from=C, to=D, phantom, "\overset{\delta}{\Rightarrow}"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

Additionally, any 2-morphism $\alpha \colon f \Rightarrow f' \colon C \rightrightarrows D$ in $\mathfrak{K}$ lifts to an internal natural transformation $\widetilde{\alpha} \colon C \to \mathbf{2} \pitchfork D$ from $(f, \mathbf{2} \pitchfork f)$ to $(f', \mathbf{2} \pitchfork f')$. This induces a fully faithful 2-functor $\mathbf{2} \pitchfork - \colon \mathfrak{K} \to \mathfrak{Cat}(||\mathfrak{K}||_1)$. For instance, in $\mathfrak{K} = \mathfrak{Cat}$, this 2-functor takes to a category $\mathcal{C}$ to the double category of commutative squares of $\mathcal{C}$.

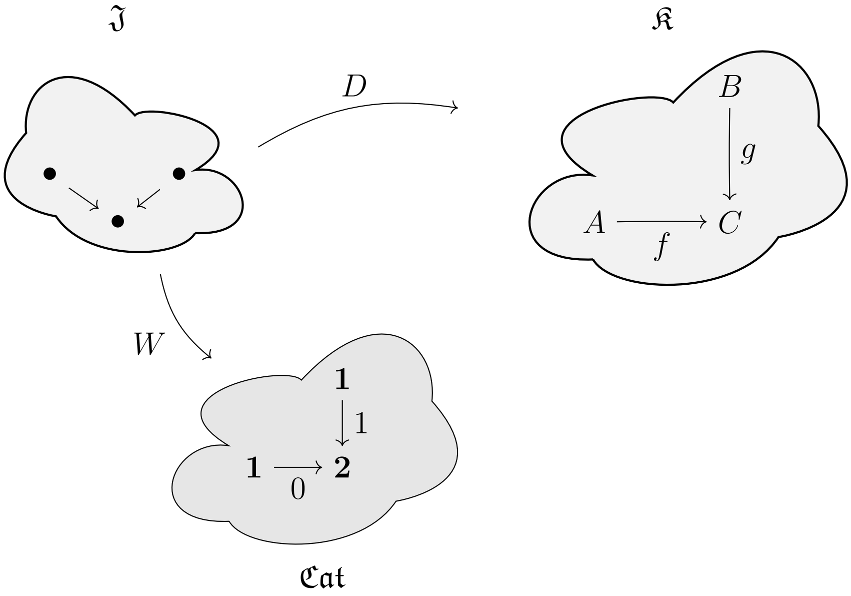
There is a generalization of Yoneda embedding for 2-categories:

CONSTRUCTION 1.9.15. Any small 2-category can be embedded into a complete and cocomplete 2-category: given a small 2-category $\mathfrak{K}$, the Yoneda embedding $\mathbb{Y}\mathrm{on}$ is given as the composite

$$\mathfrak{K} \hookrightarrow \mathfrak{Cat}(||\mathfrak{K}||_1) \hookrightarrow \mathbf{2}\,\mathfrak{Cat}_{str}(||\mathfrak{K}||_1, \mathfrak{Cat})$$

of fully faithful strict 2-functors whereby the second functor is the externalization of an internal category denoted by $\mathbb{F}\mathrm{am}$ (Appendix A.8 A.8.7). Therefore the 2-functor $\mathbb{Y}\mathrm{on}\colon \mathfrak{K} \hookrightarrow \mathbf{2}\,\mathfrak{Cat}_{str}(||\mathfrak{K}||_1{}^{\mathrm{op}}, \mathfrak{Cat})$ takes an object A to $\mathbb{F}\mathrm{am}(\mathbb{A})$. The codomain of $\mathbb{Y}\mathrm{on}$ is equivalent to the 2-category of split normal cloven fibred categories over $||\mathfrak{K}||_1$ (See Chapter 2 2.3). Therefore, we can express the Yoneda embedding of 2-categories by a 2-functor $\mathbb{Y}\mathrm{on}\colon \mathfrak{K} \to \mathrm{splnl}\mathfrak{Fib}(||\mathfrak{K}||_1)$. Note that $\mathbb{Y}\mathrm{on}$ is biconservative in that it reflects equivalences.

EXAMPLE 1.9.16. Consider the weighted diagram



(1.31)

where $\mathbf{2}$ is the category with two objects and a free (walking) arrow between them as its only non-identity morphism. The strict weighted limit of (D, W) is a known as **comma object** of f and g and is usually denoted by $(f \downarrow g)$ (or sometimes $(f \downarrow g)$).

For a object X in $\mathfrak{K}$, a W-cone with apex X over opspan $\langle f, C, g\rangle$ is specified by functors $L(j)\colon W(j) \to \mathfrak{K}(X, D(j))$ satisfying strict naturality condition (with identity for each 2-cell $L(f)$ in the diagram (1.22)).

Therefore, we get two morphisms $l_0\colon X \to A$ and $l_1\colon X \to B$, and also, two morphism $X \rightrightarrows C$ with a 2-morphism λ between them. The strict naturality condition dictates that the source and target of λ must be equal to $f \circ l_0$ and $g \circ l_1$, respectively.

$$\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] \arrow[rd, dotted, bend left=25, ""{name=U, below}, "="{near start, above right}] \arrow[rd, dotted, bend right=25, ""{name=L, above}, "="'{near end, below left}] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\arrow[from=L, to=U, Rightarrow, "\lambda"]
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] \arrow[ru, phantom, "\overset{\lambda}{\Nearrow}"] & C
\end{tikzcd}
\tag{1.32}$$

Now, universal property of $\lim\limits_{W} D = (f \downarrow g)$ says that for any 1-morphism $u\colon X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{K}(Y, (f \downarrow g)) \arrow[r, "\cong"] \arrow[d, "u^*"'] & \mathcal{L}ax\mathcal{C}one^{Y}_{W}D \arrow[d, "\mathcal{L}ax\mathcal{C}one^{u}"] \\
\mathfrak{K}(X, (f \downarrow g)) \arrow[r, "\cong"'] & \mathcal{L}ax\mathcal{C}one^{X}_{W}D
\end{tikzcd}$$

Let the unit $\Phi_{\lim_W D}(1_{\lim_W D})$ be the limit cone $\langle (f \downarrow g), d_0, d_1, \delta\rangle$, where $\delta_{f,g}\colon f d_0 \Rightarrow g d_1$. Then commutativity of the above diagram for object $Y := (f \downarrow g)$ implies that Φ is calculated by whiskering with the limit cone, i.e. $\Phi_X(u) = \langle X, d_0 u, d_1 u, \delta_{f,g} \centerdot u\rangle$ for any 1-morphism $u\colon X \to (f \downarrow g)$.

On the other hand, for any cone $L = \langle X, l_0, l_1, \lambda \rangle$, $u = \Psi_X(L)\colon X \to (f \downarrow g)$ is the unique morphism with $\Phi_X u = \mathrm{id}_{(f\downarrow g)}$. In other words, $d_0 \circ u = l_0$, $d_1 \circ u = l_1$, and $\delta_{f,g} \cdot u = \lambda$.

$$\begin{tikzcd}
X \arrow[rd, dotted, "u"] \arrow[rrd, bend left, "l_1"] \arrow[rdd, bend right, "l_0"'] & & \\
 & (f \downarrow g) \arrow[r, "d_1"] \arrow[d, "d_0"'] & B \arrow[d, "g"] \\
 & A \arrow[r, "f"'] \arrow[ru, Rightarrow, "\delta_{f,g}"] & C
\end{tikzcd} \tag{1.33}$$

Thus the *1-dimensional universal property* of the comma object $(f \downarrow g)$ states that any 2-morphism $\lambda\colon fl_0 \Rightarrow gl_1$ *uniquely* factors through the universal 2-morphism δ up to equality. Now, suppose that $L = \langle X, l_0, l_1, \lambda \rangle$ and $L' = \langle X, l'_0, l'_1, \lambda' \rangle$ are both weighted cones with apex X. A modification $m\colon L \Rrightarrow L'$ consists of 2-morphisms $m_0\colon l_0 \Rightarrow l'_0$ and $m_1\colon l_1 \Rightarrow l'_1$ rendering the diagram below (left) commutative.

$$\begin{tikzcd}
fl_0 \arrow[r, Rightarrow, "f \cdot m_0"] \arrow[d, Rightarrow, "\lambda"'] & fl'_0 \arrow[d, Rightarrow, "\lambda'"] \\
gl_1 \arrow[r, Rightarrow, "g \cdot m_1"'] & gl'_1
\end{tikzcd}
\qquad
\begin{tikzcd}
fd_0u \arrow[r, Rightarrow, "f \cdot \alpha"] \arrow[d, Rightarrow, "\delta \cdot u"'] & fd_0u' \arrow[d, Rightarrow, "\delta \cdot u'"] \\
gd_1u \arrow[r, Rightarrow, "g \cdot \beta"'] & gd_1u'
\end{tikzcd}$$

In such a situation, the unique 2-morphism $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ generates m_0 and m_1 by whiskering with d_0 and d_1 respectively. The *2-dimensional universal property* can be expressed as follows: given morphisms $u, u'\colon X \rightrightarrows (f \downarrow g)$ and 2-morphisms $\alpha\colon d_0u \Rightarrow d_0u'$ and $\beta\colon d_1u \Rightarrow d_1u'$ which make the diagram above (right) commute, there exists a *unique* 2-morphism $\psi\colon u \Rightarrow u'$ with $d_0 \cdot \psi = \alpha$ and $d_1 \cdot \psi = \beta$.

REMARK 1.9.17. Dually, cocomma objects are defined as colimits of spans. In the weighted diagram 1.31, $\mathfrak{J}$ is replaced by its opposite, and the weight functor W takes $\mathfrak{J}$ to

$$\begin{tikzcd}
\mathbf{2} \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \\
\mathbf{1} &
\end{tikzcd}$$

Obviously, cocomma objects in $\mathfrak{K}$ are comma objects in $\mathfrak{K}^{\mathrm{op}}$. This is generally true about all weighted limits.

REMARK 1.9.18. Notice that in the case of weighted diagram (1.31), pseudo weighted limits are equivalent to strict weighted limits: we can construct comma objects as pseudo-weighted limits. Isomorphisms $L(f)$ in (1.22) provide us with two extra iso-2-morphisms $\zeta_0 \colon fl_0 \cong z$ and $\zeta_1 \colon gl_1 \cong z'$ in addition to $\lambda \colon z \Rightarrow z'$. Such a pseudo cone can be strictified to $\langle X, l_0, l_1, \widetilde{\lambda} \rangle$ where $\widetilde{\lambda} := \zeta_1^{-1}\lambda\zeta_0$.

$$\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] \arrow[rd, dotted, bend left, "\cong"] \arrow[rd, dotted, bend right, "\cong"'] & B \arrow[d, "g"] & & X \arrow[r, "l_1"] \arrow[d, "l_0"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C & & A \arrow[r, "f"'] & C
\arrow[from=2-1, to=1-2, phantom, "\overset{\lambda}{\Nearrow}"]
\arrow[from=2-4, to=1-5, phantom, "\overset{\widetilde{\lambda}}{\Nearrow}"]
\arrow[from=1-2, to=2-4, phantom, "\mapsto"]
\end{tikzcd}$$

REMARK 1.9.19. The weighted bilimit over the same diagram as above is the so-called **bicomma object**. We'll use the same notation for bicomma objects, but the context shall indicate whether we use comma or bicomma objects in each instance. The structure of limit cone remains the same but the universal property becomes weaker. First of all, arbitrary cones factor through the limit cone of $(f \downarrow g)$ not necessarily uniquely, but rather the factorization is unique up to a unique iso-2-morphism. Moreover, the equalities $d_0 u = l_0$ and $d_1 u = l_1$ are replaced with cannonical iso-2-morphisms. Nevertheless, the 2-dimensional universal property remains the same. Finally, with this change in the weighted diagram, the weighted bilimit is called the **bipullback** of f and g. We visited them earlier in 1.4.10.

REMARK 1.9.20. Two special well-known cases of comma object $(f \downarrow g)$ are when either f or g is identity morphism or even more specially, both f and g are identity morphisms. In the first case, say when $g = 1_C$, we get, what is known as, the **lax limit of morphism** f, i.e. an object $(f \downarrow C)$ with morphisms $d_0 \colon (f \downarrow C) \to A$ and $d_1 \colon (f \downarrow C) \to C$ and a 2-morphism $\delta \colon fd_0 \Rightarrow d_1$, universal among such data. For instance in $\mathfrak{Cat}$, the coslice category $C/\mathcal{C}$ is obtained as the lax limit of constant functor $X \colon \mathbf{1} \to \mathcal{C}$. In the second case, we have $(1_C \downarrow 1_C) \cong \mathbf{2} \pitchfork C$. Sometimes we denote the latter by $(C \downarrow C)$.

EXAMPLE 1.9.21. If in the structure of weight of diagram (1.31) we replace the category $\mathbf{2}$ with the interval groupoid $\mathbb{I}$ (which is obtained from $\mathbf{2}$ by localizing at the free walking arrow), then the weighted limit is known as **pseudo pullback**. Weighted cones are similar to 1.32 except that λ therein becomes an iso-2-morphism, i.e. an iso-square. It has the same universal properties with respect to iso-squares.

EXAMPLE 1.9.22. Both comma objects and pseudo pullbacks are well-known in the 2-category $\mathfrak{Cat}$ of categories. For functors $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ the **comma category** $(F \downarrow G)$, has as its objects all triples (c, d, δ) where c is an object of $\mathcal{C}$, d is an object of $\mathcal{D}$ and $\delta\colon F(c) \to G(d)$ is a morphism in $\mathcal{E}$. A morphisms between any two such objects is a pair $(f, g)\colon (c, d, \delta) \to (c', d', \delta')$ where $f\colon c \to c'$ is a morphism in $\mathcal{C}$ and $g\colon d \to d'$ is a morphism in $\mathcal{D}$ such that the following square commutes in $\mathcal{E}$.

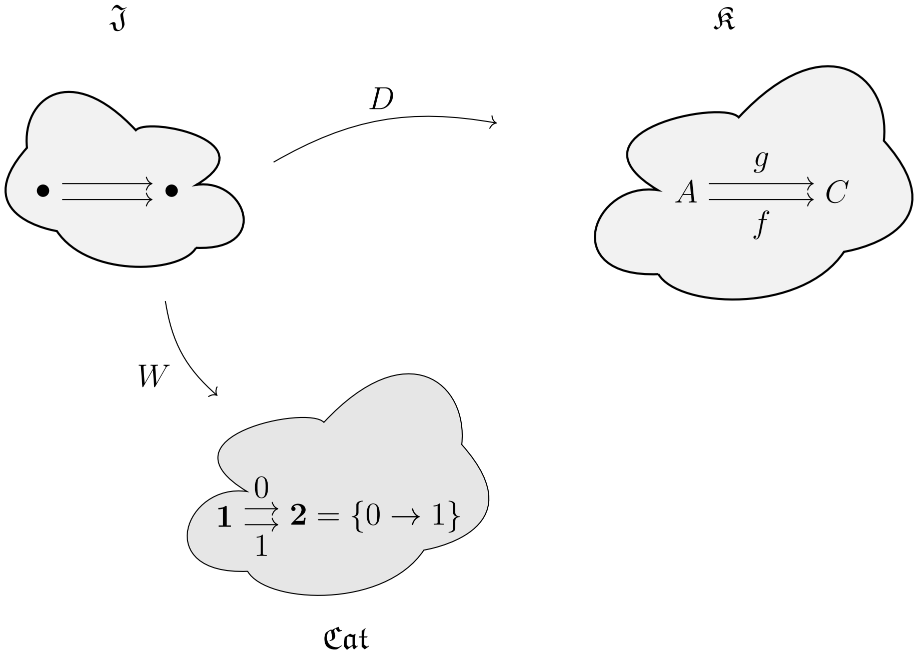
$$\begin{tikzcd}
F(c) \arrow[r, "F(f)"] \arrow[d, "\delta"'] & F(c') \arrow[d, "\delta'"] \\
G(d) \arrow[r, "G(g)"'] & G(d')
\end{tikzcd}$$

The pseudo pullback (aka **iso-comma category**) $\left(F \downarrow_{\cong} G\right)$ can be similarly described but with the difference that the component δ in the object (c, d, δ) is an isomorphism of $\mathcal{E}$. In the 2-category $\mathfrak{Cat}$ of categories, there is no distinction between pseudo pullbacks and bipullbacks. However, strict pullbacks and pseudo pullbacks of functors give inequivalent categories in general. Obviously, the canonical comparison functor

$$\begin{aligned} I\colon \mathcal{C}\,{}_F{\times}_G\,\mathcal{D} &\to \left(F \downarrow_{\cong} G\right) \\ (c, d) &\mapsto (c, \mathrm{id}_{F(c)}, d) \\ (f, g) &\mapsto (f, g) \end{aligned} \tag{1.34}$$

is fully faithful. It is an equivalence if either F or G is an isofibration. The same holds in every bicategory $\mathfrak{B}$. (See [JS93b] for more details.)

EXAMPLE 1.9.23. Consider the weighted diagram in below.



The limit cone is the universal diagram of the from

$$\begin{tikzcd}
I(f,g) \arrow[r, "p"] \arrow[d, "p"'] & A \arrow[d, "g"] \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\phi}{\nearrow\!\!\!\nearrow}"] & C
\end{tikzcd}$$

which is called the **inserter** of f and g. Let us enumerate its universal properties:

$(UP1)$ Given any morphism $q\colon X \to A$ and any 2-morphism $\psi\colon fq \Rightarrow gq$ there exists a unique morphism $u\colon X \to I(f,g)$ such that $pu = q$ and $\phi \centerdot u = \psi$.

$(UP2)$ Given a pair $u, v\colon X \to I(f,g)$ and a 2-morphism $\beta\colon pu \Rightarrow pv$ which makes the diagram

$$\begin{tikzcd}
fpu \arrow[r, Rightarrow, "f\centerdot\beta"] \arrow[d, Rightarrow, "\phi\centerdot u"'] & fpv \arrow[d, Rightarrow, "\phi\centerdot v"] \\
gpu \arrow[r, Rightarrow, "g\centerdot\beta"'] & gpv
\end{tikzcd}$$

commute, there exists a unique 2-morphism $\alpha\colon u \Rightarrow v$ satisfying $p \centerdot \alpha = \beta$.

REMARK 1.9.24. Replacing $\mathbf{2}$ with the groupoid $\mathbb{I} = \{0 \xrightarrow{\cong} 1\}$, we get the **iso-inserter** as the limit. Iso-inserter of f and g agrees with their inserter if the object C is groupoidal. Replacing $\mathbf{2}$ with the terminal category $\mathbf{1}$, we get **equalizer** as the limit.

EXAMPLE 1.9.25. The inserter of functors $F, G \colon \mathcal{A} \to \mathcal{C}$ is the category $\mathcal{I}(F, G)$ whose objects are pairs $(a, \phi \colon Fa \to Ga)$ where a is an object of $\mathcal{A}$ and ϕ is a morphism in $\mathcal{C}$, and whose morphisms are of the form $f \colon (a, \phi) \to (a', \phi')$ where $f \colon a \to a'$ is a morphism in $\mathcal{A}$ with $G(f) \circ \phi = \phi' \circ F(f)$. The category $\mathcal{I}(F, G)$ is a subcategory of $(F \downarrow G)$, however it is not full. The universal properties of inserters in a bicategory (i.e. a weak inserter) can be equivalently formulated by the equivalence

$$\mathfrak{K}(X, I(f, g)) \simeq \mathcal{I}(\mathfrak{K}(X, f), \mathfrak{K}(X, g))$$

of categories, and therefore, it is obvious that the inserter morphism $p \colon I(f, g) \to A$ is both faithful and conservative. It is fully faithful if the object C is posetal. Finally, observe that every inserter is in particular a weak inserter, and any pseudo inserter is equivalent to a strict inserter.

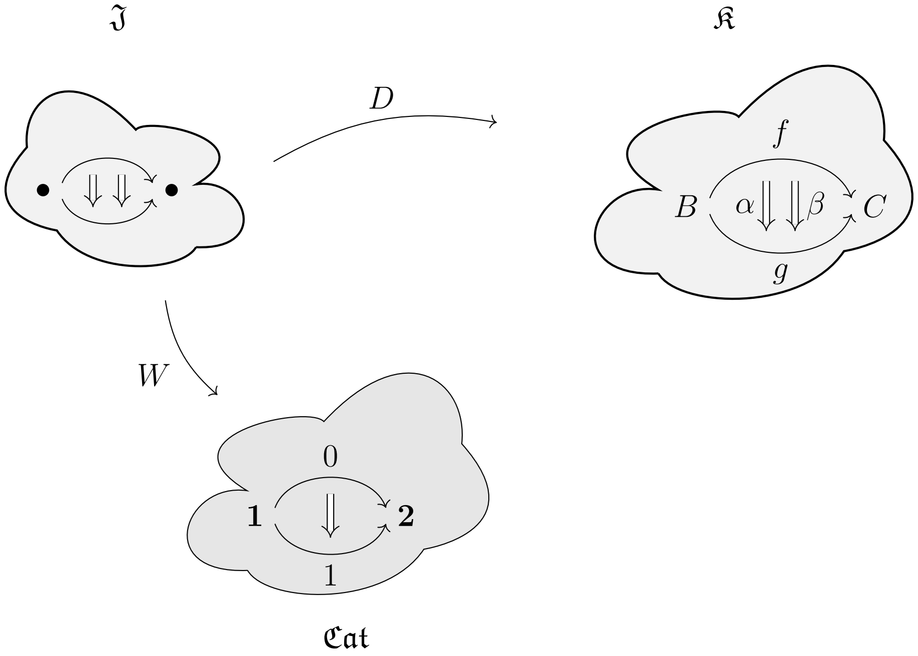
EXAMPLE 1.9.26. The free category $\mathcal{F}(G)$ of a graph $G = (E, V)$, understood as a span $V \xleftarrow{d_0} E \xrightarrow{d_1} V$ where E is the set of edges and V is the set of vertices of the graph, is equivalent to the inserter of the aformentioned span.

REMARK 1.9.27. Inserters and comma objects may be constructed from the products, pullbacks, and cotensor with $\mathbf{2}$.

$$\begin{tikzcd}
I(f,g) \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{2} \pitchfork C \arrow[d, "d_0 \times d_1"] \\
A \arrow[r, "\langle f,g \rangle"'] & C \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
(f \downarrow g) \arrow[r, "\ulcorner \delta_{f,g} \urcorner"] \arrow[d, "d_0 \times d_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{2} \pitchfork C \arrow[d, "d_0 \times d_1"] \\
A \times B \arrow[r, "f \times g"'] & C \times C
\end{tikzcd}
\tag{1.35}$$

Moreover, all comma objects can be obtained from inserters and products, for the comma object $(f \downarrow g)$ can be constructed as the inserter of $f\pi_A, g\pi_B \colon A \times B \rightrightarrows C$, where π_A, π_B are the product projection morphisms.

EXAMPLE 1.9.28. The 2-categorical generalization of equalizers is what is known as **equifier**. It can be constructed as the weighted limit of the weighted diagram below.



(1.36)

Therefore, the strict equifier of α and β is given by an object $Eq(\alpha, \beta)$ and a morphism $e\colon Eq(\alpha, \beta) \to B$ such that $\alpha \centerdot e = \beta \centerdot e$ subject to the following universal properties:

($UP1$) Given any morphism $q\colon X \to B$ with $\alpha \centerdot q = \beta \centerdot q$, there exists a unique morphism $u\colon X \to Eq(\alpha, \beta)$ such that $eu = q$.

($UP2$) Given a pair $u, v\colon X \rightrightarrows Eq(\alpha, \beta)$ and a 2-morphism $\gamma\colon eu \Rightarrow ev$, there exists a unique 2-morphism $\alpha\colon u \Rightarrow v$ satisfying $e \centerdot \alpha = \beta$.

REMARK 1.9.29. The limits reducible to the products, inserters and equifiers are referred to PIE limits and they are characterized in elementary terms and further studied in [PR91] (they are all strict limits). Any pseudo PIE limit is equivalent to a strict PIE limit.

PIE limits are important for us, since the 2-category $\mathfrak{Con}$ of AU-contexts has got all PIE limits ([Vic19]), but not all conical limits (e.g. pullbacks). In 2-categories where we have both products and pullbacks, any strict equifier can be constructed from cotensor with $\mathbf{2}$.

$$\begin{tikzcd}
Eq(\alpha,\beta) \arrow[rr] \arrow[d, "e"'] \arrow[drr, phantom, "\ulcorner" very near start] & & \mathbf{2} \pitchfork C \arrow[d, "\Delta"] \\
B \arrow[rr, "\langle \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle"'] & & (\mathbf{2} \pitchfork C) \times (\mathbf{2} \pitchfork C)
\end{tikzcd} \tag{1.37}$$

In particular the equifier morphism $e\colon Eq(\alpha,\beta) \to B$ is fully faithful.

REMARK 1.9.30. Lax equifiers are defined by a more complicated 2-dimensional universal property. For instance, in the 2-category $\mathfrak{Cat}$, a lax equifier of natural transformations α, β between functors $F, G\colon \mathcal{B} \rightrightarrows \mathcal{C}$ is given by the category $Eq_{\text{lax}}(\alpha,\beta)$ whose objects are quadruples $(b, g, \gamma_0, \gamma_1)$ where b is an object of $\mathcal{B}$, $g\colon c_0 \to c_1$ is a morphism of $\mathcal{C}$, and $\gamma_0\colon F(b) \to c_0$, $\gamma_1\colon G(b) \to c_1$ are morphisms in $\mathcal{C}$ which make both diagrams in below commute.

$$\begin{tikzcd}
F(b) \arrow[r, "\gamma_0"] \arrow[d, "\alpha_b"'] & c_0 \arrow[d, "g"] \\
G(b) \arrow[r, "\gamma_1"'] & c_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(b) \arrow[r, "\gamma_0"] \arrow[d, "\beta_b"'] & c_0 \arrow[d, "g"] \\
G(b) \arrow[r, "\gamma_1"'] & c_1
\end{tikzcd}$$

A morphism $(b, g, \gamma_0, \gamma_1) \to (b', g', \gamma_0', \gamma_1')$ in $Eq_{\text{lax}}(\alpha,\beta)$ is given by a morphism $f\colon b \to b'$ in $\mathcal{B}$ and morphisms $t_i\colon c_i \to c_i'$, for $i = 0, 1$, in $\mathcal{C}$ such that all faces of the cubes below commute.

$$\begin{tikzcd}
F(b) \arrow[rr, "\alpha_b"] \arrow[dr, "\gamma_0"] \arrow[dd, "F(f)"'] & & G(b) \arrow[dr, "\gamma_1"] \arrow[dd, "G(f)" near end] & \\
& c_0 \arrow[rr, "g" near start] \arrow[dd, "t_0" near start] & & c_1 \arrow[dd, "t_1"] \\
F(b') \arrow[rr, "\alpha_{b'}"' near start] \arrow[dr, "\gamma_0'"'] & & G(b') \arrow[dr, "\gamma_1'"] & \\
& c_0' \arrow[rr, "g'"] & & c_1'
\end{tikzcd}
\qquad
\begin{tikzcd}
F(b) \arrow[rr, "\beta_b"] \arrow[dr, "\gamma_0"] \arrow[dd, "F(f)"'] & & G(b) \arrow[dr, "\gamma_1"] \arrow[dd, "G(f)" near end] & \\
& c_0 \arrow[rr, "g" near start] \arrow[dd, "t_0" near start] & & c_1 \arrow[dd, "t_1"] \\
F(b') \arrow[rr, "\beta_{b'}"' near start] \arrow[dr, "\gamma_0'"'] & & G(b') \arrow[dr, "\gamma_1'"] & \\
& c_0' \arrow[rr, "g'"] & & c_1'
\end{tikzcd}$$

In the pseudo case, γ_i $(i = 0, 1)$ are isomorphisms and the objects of $Eq_{\text{str}}(\alpha,\beta)$ have the simpler form of triples (b, γ_0, γ_1) with no extra equations. In the simplest case of strict equifier, γ_i are identity morphisms. Note that the strict equifier $Eq(\alpha,\beta)$ is a full subcategory of $\mathcal{B}$ whose objects are those objects b of $\mathcal{B}$ for which $\alpha_b = \beta_b$. This agrees with the construction of strict equifier as the pullback in (1.37). The fact that

EXAMPLE 1.9.31. Let $\alpha \colon f \Rightarrow g \colon B \rightrightarrows C$ is a 2-morphism in a 2-category $\mathfrak{K}$. The **inverter** of α is the universal morphism $i \colon Inv(\alpha) \to B$ such that the whiskered 2-morphism $\alpha \centerdot i$ is invertible. More precisely, the universal properties state that any morphism $u \colon X \to B$ which is whiskered with α to an invertible morphism factors uniquely through i, and moreover, any 2-morphism $iu \Rightarrow iv \colon X \rightrightarrows B$ is uniquely induced by a 2-morphism $u \Rightarrow v \colon Inv(\alpha) \rightrightarrows B$.

A familiar instance of coinverters is the categories of fractions. See [KLW93] for more details.

EXAMPLE 1.9.32. Let $\alpha \colon f \Rightarrow g \colon B \rightrightarrows C$ is a 2-morphism in a 2-category $\mathfrak{K}$. The **identifier** of α is the universal morphism $i \colon Id(\alpha) \to B$ such that the whiskered 2-morphism $\alpha \centerdot i$ is the identity 2-morphism id_f.

EXAMPLE 1.9.33. Identifiers and coidentifiers are not bicategorical. Consider the cotensor limit cone

$$\begin{tikzcd}
(\mathcal{C} \downarrow \mathcal{C}) \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{C}
\end{tikzcd}$$

in $\mathfrak{Cat}$. The identifier of δ is the globular subcategory of the arrow category $(\mathcal{C} \downarrow \mathcal{C})$ which is isomorphic to $\mathcal{C}$ itself. The coidentifier is the quotient of $\mathcal{C}$ by the equivalence relation of 'being connected by a zig-zag (span) of morphisms' on objects of $\mathcal{C}$. Therefore, the coidentifier is the category of path components of $\mathcal{C}$.

Comma construction preserves adjunctions.

PROPOSITION 1.9.34 ([Str74]). Suppose $\mathfrak{K}$ is a 2-category and $f \colon A \to B$ is a morphism with the right adjoint u, unit η, and counit ϵ. For any morphism $g \colon C \to B$ for which the comma category $(f \downarrow g)$ exists in $\mathfrak{K}$, the filling arrow $v \colon C \to (f \downarrow g)$ obtained by factoring $\epsilon \centerdot g$ through $\delta \colon fd_0 \Rightarrow gd_1$ is the right adjoint to d_1 with identity counit.

The 1-morphism v in the proposition is uniquely determined by equations $d_1 v = 1$, $d_0 v = ug$, and $\delta \centerdot v = \epsilon \centerdot g$. Moreover, the proposition states that we can lift the 2-morphism η in the lower part of the diagram to a 2-morphism $1 \Rightarrow vd_1$ in the upper part.

$$\begin{tikzcd}
& C \arrow[ldd, bend right, "ug"'] \arrow[rd, dashed, "v"] \arrow[rrd, bend left, "1"] & & & \\
& & (f \downarrow g) \arrow[r, "d_1"] \arrow[lld, "d_0"'] \arrow[rr, dotted, bend right, "1"] & C \arrow[r, "v"] \arrow[ld, "g"'] & (f \downarrow g) \arrow[ld, "d_0"] \\
A \arrow[rr, "f"] \arrow[rrr, bend right=50, "1"'] \arrow[rrr, phantom, bend right=25, "\eta\Uparrow"] & & B \arrow[r, "u"] & A &
\end{tikzcd}$$

Proof. We first construct the unit τ_1 of putative adjunction $d_1 \dashv v$. Using the fact $(\epsilon \cdot f) \circ (f \cdot \eta) = 1$, we obtain the equality of pasting diagrams

$$\begin{tikzcd}
(f \downarrow g) \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[rd, phantom, "\Downarrow\delta"] & A \arrow[r, "1"] \arrow[d, "f"'] \arrow[rd, phantom, "\eta\Downarrow"] & A \arrow[d, "1"] & & = & (f \downarrow g) \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[rd, phantom, "\Downarrow\delta"] & A \arrow[d, "f"] \\
C \arrow[r, "g"'] & B \arrow[r, "u"'] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, bend right=25, "\epsilon\Downarrow"] & A \arrow[r, "f"'] & B & & C \arrow[r, "g"'] & B
\end{tikzcd}$$

Therefore,

$$(\delta \cdot vd_1) \circ (f \cdot ((u \cdot \delta) \circ (\eta \cdot d_0)) = (\epsilon \cdot gd_1) \circ (fu \cdot \delta) \circ (f \cdot \eta \cdot d_0) = \delta$$

From the 2-dimensional universal property of the comma object $(f \downarrow g)$, we obtain a unique 2-morphism $\tau_1 \colon 1 \Rightarrow vd_1$ with

$$\begin{aligned} d_0 \cdot \tau_1 &= (u \cdot \delta) \circ (\eta \cdot d_0) \\ d_1 \cdot \tau_1 &= \mathrm{id}_{d_1} \end{aligned} \tag{1.38}$$

One readily verifies that $\mathrm{id} \colon d_1 v = 1_C$ and $\tau_1 \colon 1_{(f\downarrow g)} \Rightarrow vd_1$, d_1 satisfy the triangle equations of adjunction. $\square$

The proposition above has a dual whereby one of the morphisms participating in the construction of comma object has a left adjoint instead.

REMARK 1.9.35. In a 2-category with a terminal object, taking $C = 1$ and $g = b\colon 1 \to B$, the proposition above generalizes the well-known fact of category theory that $(f \downarrow b)$ has a terminal point for every $b\colon 1 \to B$ if f has a right adjoint. Recall that in the 2-category $\mathfrak{Cat}$ the terminal point of $(f \downarrow b)$ is given by the pair $(u(b), \epsilon_b\colon fu(b) \to b)$ and its universality discloses the familiar fact that any morphism $\sigma\colon fa \to b$ lifts along ϵ_b to $f(\widehat{\sigma})$ for a unique $\widehat{\sigma}$. However, $\mathfrak{Cat}$, unlike a general 2-category, is well-pointed, and therefore the fact above holds in the reverse direction as well: if $(f \downarrow b)$ has a terminal object for every $b\colon 1 \to B$ then f has a right adjoint. Dually, if a morphism $u\colon B \to A$ in the 2-category $\mathfrak{K}$ has a left adjoint then the comma object $(a \downarrow u)$ has an initial point, for every $a\colon 1 \to A$.

REMARK 1.9.36. A useful special case of the above proposition is when f and g are both identity morphisms $1\colon E \to E$. In this case $(f \downarrow g) \simeq (E \downarrow E) \simeq \mathbf{2} \pitchfork E$ and $v = i_E\colon E \to (E \downarrow E)$ whiskers with $\delta_E\colon e_0 \Rightarrow e_1$ to give the identity 2-morphism id_{1_E}. The unit $\tau_1\colon 1_{(B \downarrow B)} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$ while the counit is identity. Thus, e_1 is a reflection. similarly, the dual of proposition 1.9.34 yields i_E as the left adjoint of $e_0\colon (E \downarrow E) \to E$. The unit of $i_E \dashv e_0$ is identity, making e_0 a retraction. The counit is given by the unique 2-morphism $\tau_0\colon i_E \circ e_0 \Rightarrow 1_{(E \downarrow E)}$ defined by the equations $e_0 \centerdot \tau_0 = \mathrm{id}_{e_0}$ and $e_1 \centerdot \tau_0 = \delta$. When $\mathfrak{K} = \mathfrak{Cat}$, we have $\tau_0(u) = (\mathrm{id}, u)$, and $\tau_1(u) = (u, \mathrm{id})$ for any $u\colon e_0 \to e_1$ in $(E \downarrow E)$.

$$\begin{tikzcd}
e_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & e_0 \arrow[d, "u"] \\
e_0 \arrow[r, "u"'] & e_1
\end{tikzcd}
\qquad
\begin{tikzcd}
e_0 \arrow[r, "u"] \arrow[d, "u"'] & e_1 \arrow[d, "\mathrm{id}"] \\
e_1 \arrow[r, "\mathrm{id}"'] & e_1
\end{tikzcd}$$

1.10 Notes

The canonical reference for weighted limits and colimits is [Kel82, Chapter 3]. Therein they are known by the name of indexed limits. The origin of the notion itself goes back further than that; see for instance [BK75]. Weighted limits and colimits are studied in areas other than pure category theory and categorical homotopy theory. See their use in study of topological Hochschild homology [MSV97] and in [PRV04] in their study of the Davis-Januszkiewicz spaces.

As we saw in §1.3, the enrichment structure can be realized as a lax functor from an indiscrete (aka chaotic) category to the suspension of a monoidal category. In fact, there are indications which support the view that the theory of enriched categories should be approached as a part of the theory of lax functors ([Bén67] and [Str05]. First steps have been taken in [Bac13] in extending the internal hom of enriched categories to lax functors taking their values in a symmetric monoidal category. More recently, the paper [GH13] introduces a notion of enriched infinity-category analogous to the view of enrichment as a lax functor.

We also saw some serious problems with lax functors, the most severe being that they are not invariant under equivalences. One good solution is to work with double categories instead. Bicategories get 'horizontally' embedded in double categories and the same is true for all bicategorical concepts of this chapter. All examples of 2-categories and bicategories in this chapter have smooth generalization to double categories; the most prominent example being the bicategory of modules and profunctors. In addition, there is a satisfactory notion of lax functors between double categories which *is* invariant under equivalence (See [Shu08]). Lax double functors are laxly functorial on horizontal morphisms, and strictly functorial on the vertical morphism of double categories, whereas the components of the transformations remain vertical and therefore, whiskering preserves naturality. We saw with lax functors of 2-categories we could not do this and that is why the surrogate notion of icon is needed.

Categorical fibrations

2

In this chapter, we review the two styles of internal fibrations in 2-categories, which we shall call the *Chevalley* and *Johnstone* styles. In Chapter 3 we use Chevalley style to define fibrations of AU-contexts and in Chapter 4 we use Johnstone style fibrations as fibrations of toposes. The main theorem of the thesis then connects the fibrations of AU-contexts to the fibrations of toposes.

Our main task in §2.4 is to clarify the 2-categorical structure needed, and the strictness issues, when we apply the Chevalley criterion in $\mathfrak{Con}$.

As an original contribution, we introduce the notion of fibrational object for 2-functors of 2-categories. In §2.6, we prove that Johnstone-style fibrations are in fact fibrational objects of the 2-functor $\mathbf{cod}\colon \mathcal{G}\mathfrak{Top} \to \mathcal{E}\mathfrak{Top}$. This reformulation will be a crucial step in our proof of the main theorem (4.2.2) of the thesis.

2.0 Introduction

The standard notion of categorical fibration, i.e. Grothendieck fibration, expressed as a property of a functor of categories, can be generalized to a property of a 1-morphism in a 2-category, but how this may be done depends on the structure available in that 2-category.

Basically, for a Grothendieck fibration (resp. opfibration) $P\colon \mathcal{E} \to \mathcal{B}$, every morphism $f\colon b \to a$ whose codomain (resp. domain) is in the image of P has a cartesian lift in $\mathcal{E}$. This induces a 'transport' functor from the fibre of P over a to that over b, with a certain universality conditions that express cartesianness. When we generalize from $\mathfrak{Cat}$ to some other 2-category $\mathfrak{K}$, the obvious generalization of Grothendieck fibration may seem to be achieved by replacing $P\colon \mathcal{E} \to \mathcal{B}$ by a 1-morphism $p\colon E \to B$ in $\mathfrak{K}$, a and b with 1-morphisms from the terminal object 1 to B, and with f a 2-morphism between them. Note that Remark 1.4.7 justifies this move for well-pointed 2-categories.

However, in general, even when $\mathfrak{K}$ has a terminal object, there may fail to be enough 1-morphisms from the terminal object 1 to object B to make a satisfactory definition this way. This is generally the case with 2-categories of toposes.

The crude remedy for this is to consider a and b as 1-morphisms from arbitrary objects B' to B in $\mathfrak{K}$, and this underlies Johnstone's definition for $\mathcal{B}\mathfrak{Top}$ in [Joh02a, B4.4]. This definition requires very little structure on $\mathfrak{K}$ other than some – not necessarily all – bipullbacks (Definition 1.4.10), sufficient to have bipullbacks of p along all 1-morphisms to B. We shall call it the *Johnstone* style of definition of fibration. This definition is quite intricate, because it has to deal with several coherence conditions. In § 2.6, we shall give a cogent reformulation of Johnstone-style fibrations in terms of fibrational objects of a certain fibrations of bicategories. The utility of this reformulation is that it repackages lots of coherence data in the definition of Johnstone-style fibrations, arising from bipullbacks involved in the said definition, into universal properties of cartesian morphism of a certain fibration of bicategories.

In the special case whereby $\mathfrak{K}$ has comma objects, corresponding to a generic 2-morphism α between 1-morphisms with codomain B, we get a 1-morphism whose codomain is the cotensor $\mathbf{2} \pitchfork B$ of B with the walking arrow category $\mathbf{2}$, and whose whiskering the free 2-morphism $\lambda_B\colon d_0 \Rightarrow d_1\colon \mathbf{2} \pitchfork B \rightrightarrows B$ is α. In such a 2-category $\mathfrak{K}$, the fibration structure for arbitrary B' and α can be got from generic structure for the generic λ. Therefore, the structure of fibration needs to be given only once, instead of each time for every B'. We shall call this a *Chevalley criterion*. For ordinary fibrations the idea was attributed to Chevalley by Gray ([Gra66]), and subsequently referred to as the Chevalley criterion by Street ([Str74]).

However, unfortunately our 2-categories of interest such as $\mathcal{B}\mathfrak{Top}$ (unlike $\mathcal{B}\mathfrak{Top}/\mathscr{S}$) do not support the structure of comma objects, and as such we can not use the simpler Chevalley criterion to define fibrations inside it.

But, not all hope is lost. The 2-category $\mathfrak{Con}$ of AU-contexts (See chapter 3) has all comma objects and pullbacks we need. Also, $\mathfrak{Con}$ is intimately linked to $\mathcal{B}\mathfrak{Top}$. The strategy which we shall pursue in Chapter 4 is to use Chevalley criterion in $\mathfrak{Con}$ to define fibrations therein and then relate those fibrations to Johnstone style fibrations in $\mathcal{B}\mathfrak{Top}$.

We shall begin this chapter, in §2.1, by a general discussion concerning bundles and fibrations. In the subsequent section (§2.2) we will motivate this discussion by giving examples of 1-categorical fibrations of groupoids and categories from their origin in algebraic topology. For instance the notion of covering spaces in topology gives rise to discrete fibrations of groupoids.

We then pass on from discrete fibrations to Grothendieck fibrations (§2.3). While the fibres of a discrete fibration are discrete categories (i.e. sets), the fibres of a Grothendieck fibration are generally not discrete. As example 2.3.45 shows, non-discrete fibrations are quite important and commonplace in variety of branches of mathematics To state precise definition of Grothendieck (op)fibration we will need to reintroduce the ancillary notion of (op)cartesian morphisms. Readers familiar with the parlance of higher category theory recognize Grothendieck (op)fibration as "(op)cartesian fibrations" as they have 'enough' cartesian lifts (for instance in [Lur09]).

Additionally, we shall review the correspondence between Grothendieck fibrations and indexed categories through the Grothendieck construction, and shall highlight the reasons why it is preferable for us to work with fibrations rather than indexed categories.

The general approach of this chapter is to proceed with the philosophy of seeing constructions on categories as inherently 2-categorical notions, and as such we emphasize the 2-categorical aspects of Grothendieck fibrations. Many of the propositions stated with regard to 1-categorical fibrations are stated in a way that have natural intrinsic 2-categorical formulations. In §2.4, we review the fact that Grothendieck fibrations are Chevalley-style fibrations in $\mathfrak{Cat}$. Chevalley-style fibrations and their characterization in [Str74] as pseudo algebras is summarized in the same section. New calculations concerning the strictness of the counit of Chevalley adjunction are provided. In §2.6, we remark that both Chevalley and Johnstone styles of fibrations are respectively the strict and weak versions of the representational notion of fibration in 2-categories.

In §2.6, using Construction 1.4.12 of display sub-2-category we give a cogent reformulation of Johnstone-style fibration. The utility of this reformulation is that it repackages lots of coherence data in the definition of Johnstone-style fibrations, arising from bipullbacks involved in the said definition, into universal property of cartesian morphism of a certain fibration of bicategories. We shall use this reformulation in obtaining results on fibrations and opfibrations in the 2-category $\mathcal{E}\mathfrak{Top}$ of elementary toposes by taking $\mathfrak{K} = \mathcal{E}\mathfrak{Top}$ and $\mathcal{D}$ as the collection of bounded geometric morphisms in $\mathcal{E}\mathfrak{Top}$.

2.1 Bundles and fibrewise view

In mathematics we do not work only with objects but also with families of objects. In most classical set-based branches of mathematics, influenced by the structuralism of Bourbaki, structures *are* sets determined internally in terms of relations and operations on their elements, and when working with various structures we often introduce definitions and constructions not only on object but also on family of objects exhibiting considered structures.

In ZFC set theory, a cartesian product of I-indexed families $X = \{X_i\}_{i \in I}$ and $Y = \{Y_i\}_{i \in I}$ is an I-indexed family $X \times Y = \{X_i \times Y_i\}_{i \in I}$. Note that a family like X as above can be consider as a functor $X \colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ where I^{d} is considered as the discrete category whose set of objects is I. Given families X and Y a function α between them is defined, according the principle of extensionality, elementwise. Therefore, it can be realized as a natural transformation $\alpha \colon X \Rightarrow Y$.

In category theory we do not have the same language (an admittedly strange language!) as ZFC set theory and we shall not utter such a thing as "an object of a category whose 'elements' are a collection of objects of the same category".

First of all, it is not clear what the word 'element' should mean. If we think along the same lines as Lawvere's ETCS, we may consider an element x of object X of category $\mathcal{S}$ as a morphism $x \colon 1 \to X$. The problem with this approach is that the category $\mathcal{S}$ may not have a terminal object and more seriously, it may not be well-pointed.

So, it is best to change our perspective on families of sets. We can see a family $X \colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ as a bundle $\gamma \colon X \to I$ of sets where the fibre of γ at the element $i \in I$ is $\gamma^{-1}(i) \cong X_i$. In this way, we obtain the equivalence

$$\mathcal{S}\mathrm{et}\,/I \simeq \mathcal{C}\mathrm{at}(I^{\mathrm{d}}, \mathcal{S}\mathrm{et}) \tag{2.1}$$

of categories. Note that I^{d} is the set I considered as a discrete category.

In the language of category theory, the above change of perspective is expressed by stipulating X_i as a pullback of γ along $i \colon 1 \to I$ in $\mathcal{S}$, if such a pullback exists in $\mathcal{S}$. So, for an object I of a category $\mathcal{S}$ an I-indexed family of objects can be simply regarded as a morphism $\gamma \colon X \to I$ in $\mathcal{S}$. One of the first exercises in set theory is that any construction on sets (such as product, union, sum (disjoint union), the set of functions and relations between sets, etc.) can be elementwise carried out for families of sets. Categorically, this means that the slice category $\mathcal{S}\mathrm{et}\,/I$ possesses the same structures as the category $\mathcal{S}\mathrm{et}$. The same holds for any elementary topos and even for any Grothendieck topos and it is known as "the fundamental theorem of topos theory".

In particular, for an elementary topos $\mathscr{S}$, the topos $\mathscr{S}/I$ is cartesian closed since $\mathscr{S}$ is. This means that we get natural isomorphisms

$$\mathscr{S}\big/_I\left(p \times_I q \colon X \times_I Y\big/_I, r \colon Z\big/_I\right) \cong \mathscr{S}\big/_I\left(p \colon X\big/_I, r^q \colon Z^Y\big/_I\right)$$

Unwinding the natural isomorphism of sets above precisely says that for any morphism $f\colon J \to I$ the pullback functor $f^*\colon \mathscr{S}/I \to \mathscr{S}/J$ has a right adjoint Π_f (Note that in addition, f^* has a left adjoint Σ_f given by post-composition with f).

Recall that in a cartesian category $\mathcal{C}$ with an exponentiable object B, the *object of sections* of a morphism $\gamma\colon X \to B$ is obtained by the pullback

$$\begin{tikzcd}
\Pi_B(\gamma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[B,X]} \arrow[d, "B^{\gamma}"] \\
1 \arrow[r, "\widehat{\mathrm{id}_B}"'] & {[B,B]}
\end{tikzcd}$$

where $\widehat{\mathrm{id}_B}$ is the transpose of the isomorphism projection $1 \times B \cong B$. A generalized element of $\Pi_B(\gamma)$ at stage W is equivalent to a morphism $\pi_B^{(W)} \to \gamma$ in the slice category $\mathcal{C}/B$, where $\pi_B^{(W)}\colon W \times B \to B$ is the second product projection. Type theoretically, it can be expressed as a term of type $\prod\limits_{b\colon B}(W \to X_b)$.

For a Grothendieck topos $\mathscr{E}$, and an object (sometimes called a sheaf) I of $\mathscr{E}$, $I^*\colon \mathscr{E} \to \mathscr{E}/I$ is part of an essential geometric morphism where $I^*(X) = I \times X \xrightarrow{\pi_0} I$. In the special situation when $\mathcal{S} = \mathcal{S}\mathrm{et}$, given a set X, we have $I^*(X)$ as a bundle with constant fibre X, and given an I-indexed family $\gamma = \{X_i\}_{i \in I}$, we have $\Pi_I(\gamma) = \Pi_{i \in I} X_i$. Note that the direct image Π_I defined in above, computes the ‘set’ of sections (more precisely, it is the discrete coreflection of the *space* of sections which exists as an internal point-free space). Observe that Π_I uses non-geometric constructions.

If $\mathscr{E}$ is a Grothendieck topos (say over elementary topos $\mathscr{S}$), classifying a theory $\mathbb{T}$, then $\mathscr{E}/I$ classifies the theory of pairs (M, x) where M is a model of $\mathbb{T}$ and x is a global element of $I^*(M)$. The geometric morphism $(I^*, \Pi_I)\colon \mathscr{E}/I \to \mathscr{E}$ then takes the point (M, x) to M.

The crucial observation is that the language of topos theory enables us to compute things such as space of sections of a bundle functorially and synthetically. Indeed, fibrewise topology of bundles (for toposes they are bounded geometric morphisms) shows the advantage of working with point-free topology: the localic bundle theorem of Joyal and Tierney says that point-free spaces internal to a topos $\mathscr{S}$ are equivalent to localic bundles over $\mathscr{S}$.

2.2 Discrete fibrations

We recall from topology that a continuous map $p\colon E \to B$ is said to be a **covering map**, and space E is a **covering space** over B, whenever for every point $x \in B$ there is an open neighbourhood U containing x such that $p^{-1}(U) = \amalg_{i \in I} V_i$, a disjoint union of open sets V_i in E such that $p|_{V_i}\colon V_i \cong U$. A simple example of a covering map is the quotient map $\mathbb{R}^2 \to \mathbb{T}$ where the torus $\mathbb{T}$ is obtained as the quotient space of $\mathbb{R}^2$ by the congruence generated by identifications $(x, y) \sim (x + m, y + n)$ for every $m, n \in \mathbb{Z}$.

Another well-known examples is the helix-shaped real line over 1-sphere. More generally, some of covering spaces are built out of locally constant sheaves. We recall that a sheaf P on a topological space X is *locally constant* if there exists an open cover of X such that the restriction of P to each open set in the cover is a constant sheaf. If the topological space X is locally connected, a locally constant sheaf P on X is, up to an isomorphism, the sheaves of sections of the étale covering $\pi\colon \text{ét}(P) \to X$.

The famous *unique path lifting* property holds for covering maps with connected and locally connected base.

THEOREM 2.2.1. Suppose B is a connected and locally path connected space and $p\colon E \to B$ is a covering map of spaces. Suppose also that $\lambda\colon I \to B$ is a path in B starting at $\lambda(0) = b_0$. Then for each $e \in p^{-1}(b_0)$ there is a unique path $\tilde{\lambda}\colon I \to E$ with $p(\tilde{\lambda}) = \lambda$. Moreover, if there is a homotopy H between two paths λ and γ (with the same starting and ending points) in the base space B, then there is a unique lift $\tilde{H}$ of homotopy H between the lifts $\tilde{\lambda}$ and $\tilde{\gamma}$ (with the same starting and ending points).

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
I \arrow[ur, dashed, "\tilde{\lambda}"] \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

A proof of this theorem can be found in section 3.2. of [May99]. Moreover, covering spaces are 'almost' stable under base change.

REMARK 2.2.2. If $f \colon A \to B$ is a map whereby A is path connected then f^*p, the pullback of p along f, is a covering map. In particular, the fibre E_b is a covering space over a point $b \in B$, and hence E_b must be a discrete space.

$$\begin{tikzcd}
E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\
1 \arrow[r, "b"'] & B
\end{tikzcd}$$

There is a strict 2-functor $\Pi_{\leq 1} \colon \mathcal{T}op_{\leq 2} \to \mathfrak{Grpd}$ which associates to every topological space its fundamental groupoid, to a continuous map of spaces a functor of groupoids, and to a homotopy between maps, an natural isomorphism.

For each groupoid $\mathcal{G}$ and each object c of $\mathcal{G}$, define $\pi(\mathcal{G}, c)$ as the full subgroupoid of $\mathcal{G}$ with only one object namely c. So, $\pi(\mathcal{G}, c)(c, c) = \mathrm{Aut}_{\mathcal{G}}(c)$. Composing this functor with $\Pi_{\leq 1}$, we get the familiar fundamental group at point of a topological space at point c. We can use 2-functor $\Pi_{\leq 1}$ for lifting of paths and homotopies of topological spaces in terms of groupoids and functors: If $p \colon E \to B$ is a covering map of spaces then the functor $e/p \colon e/\Pi_{\leq 1}(E) \to p(e)/\Pi_{\leq 1}(B)$, which sends a homotopy class $[\lambda]$ represented by path $\lambda \colon I \to E$ starting at e in E to homotopy class $[p \circ \lambda]$, is an isomorphism of groupoids for any point $e \in E$.

We now give an algebraic characterization of the notion of covering map of spaces in terms of functors of groupoid:

DEFINITION 2.2.3. A functor $P\colon \mathcal{E} \to \mathcal{B}$ of groupoids is a **covering** functor whenever

(i) P is surjective on objects, and

(ii) $e/P\colon e/\mathcal{E} \to P(e)/\mathcal{B}$ is an isomorphism of categories for every object e in $\mathcal{E}$.

REMARK 2.2.4. For any groupoid $\mathcal{E}$, there is only a unique morphism between any two objects of $e/\mathcal{E}$. So, isomorphism of such co-slice categories means isomorphism of their underlying sets of objects.

THEOREM 2.2.5. (i) For a covering map $p\colon E \to B$ of topological spaces the fundamental groupoid functor $\Pi_{\leq 1}(p)\colon \Pi_{\leq 1}(E) \to \Pi_{\leq 1}(B)$ is a covering functor.

(ii) Covering functors of groupoids are closed under composition.

(iii) Covering functors of groupoids are stable under base change.

REMARK 2.2.6. By the unique path lifting property it is trivial to see that $\Pi_{\leq 1}(E)_b$ does not have no non-identity morphisms and therefore, it is discrete. We note that $\Pi_{\leq 1}(E)_b \simeq \Pi_{\leq 1}(E_b)$ since both are discrete groupoids with the same set of objects.

By the unique path lifting theorem, for any point $b \in \mathcal{B}$, there is a transitive action of fundamental group $\pi(\mathcal{B}, b)$ on the fibre $\mathcal{E}_b$:

$$\phi\colon \pi(\mathcal{B}, b) \times \mathcal{E}_b \to \mathcal{E}_b$$

defined by $\phi(l)(e) = \tilde{l}(1)$, where $\tilde{l}$ is the unique lift of l with $\tilde{l}(0) = e$.

$$\begin{tikzcd}
e_0 \arrow[rr, dashed, bend left, "\tilde{l}"] \arrow[dr, mapsto] & & e_1 \arrow[dl, mapsto] & & \mathcal{E} \arrow[d, "P"] \\
& b \arrow[loop right, "l"] & & & \mathcal{B}
\end{tikzcd}$$

Notice that for any $e, e' \in \mathcal{E}_b$, $P(\pi(\mathcal{E}, e))$ and $P(\pi(\mathcal{E}, e'))$ are conjugate subgroups of $\pi(\mathcal{B}, b)$ and each is isomorphic to isotropy group of the action. Hence

$$\mathcal{E}_b \cong \pi(\mathcal{B}, b)/P(\pi(\mathcal{E}, e))$$

as $\pi(\mathcal{B}, b)$-sets.

DEFINITION 2.2.7. Suppose $\mathcal{B}$ is a connected groupoid. We define $\mathcal{C}ov(\mathcal{B})$ to be the category whose objects are coverings with base $\mathcal{B}$ with morphisms between any two coverings $P \colon \mathcal{E} \to \mathcal{B}$ and $Q \colon \mathcal{F} \to \mathcal{B}$ being functors $G \colon \mathcal{E} \to \mathcal{F}$ such that $Q \circ G = F$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "G"] \arrow[dr, "P"'] & & \mathcal{F} \arrow[dl, "Q"] \\
& \mathcal{B} &
\end{tikzcd}$$

REMARK 2.2.8. Any such morphism G is necessarily a covering itself if $\mathcal{F}$ is connected.

PROPOSITION 2.2.9. For a connected groupoid $\mathcal{B}$, we have the following bijection

$$\mathcal{C}ov(\mathcal{B})\ (\mathcal{E}, \mathcal{F}) \cong \pi(\mathcal{B}, b)\text{-}\ \mathcal{S}\mathrm{et}\ (\mathcal{E}_b, \mathcal{F}_b)$$

where b is any base point in $\mathcal{B}$. This bijection is natural with respect to the choice of b.

See [May99, p.29] for a proof. In fact, we can study covering of spaces entirely by covering of their fundamental groupoids and not lose any information. This is a pretty atypical situation in algebraic topology. Generally, we have the strict hierarchy of subclasses of morphisms of topological spaces:

$$\{\text{homeomorphisms}\} \subset \{\text{homotopy equivalences}\} \subset \{\text{weak homotopy equivalences}\}$$

We can of course generalize the notion of covering functors of groupoid to the functors of categories. Note, however that there is a breaking of symmetry in passing from groupoids to categories. For a groupoid $\mathcal{E}$, we have $e/\mathcal{E} \cong (\mathcal{E}/e)^{\mathrm{op}}$

and we could have instead formulated the notion of covering of groupoids in term of slice groupoids. The breaking of symmetry leads to the *covariant* and *contravariant* notions of covering for categories.

We shall also drop the condition of surjectivity on objects. This omission gives a structure more easily attuned to the setting of categories and internal categories. Note that a functor $P\colon \mathcal{E} \to \mathcal{B}$ of groupoids which satisfies the condition (ii) of 2.2.3 is the same thing as a functor $\mathcal{B} \to \mathcal{C}\mathrm{ore}(\mathcal{S}\mathrm{et})$, where $\mathcal{C}\mathrm{ore}$ is the maximal subgroupoid functor. Therefore, for a groupoid $\mathcal{B}$ we have an equivalence

$$d\mathcal{F}\mathrm{ib}(\mathcal{B}) \simeq \mathfrak{Cat}(\mathcal{B}, \mathcal{C}\mathrm{ore}(\mathcal{S}\mathrm{et})) \tag{2.2}$$

DEFINITION 2.2.10. A functor $P\colon \mathcal{E} \to \mathcal{B}$ of categories is a **discrete fibration** if for every object e of $\mathcal{E}$, every morphism $f\colon b \to P(e)$ in $\mathcal{B}$ has a unique lift $\tilde{f}\colon \tilde{b} \to e$ in $\mathcal{E}$. A functor $F\colon \mathcal{E} \to \mathcal{B}$ is a **discrete opfibration** whenever the functor $F^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a discrete fibration. For a category $\mathcal{B}$, discrete fibrations (resp. opfibrations) over $\mathcal{B}$ form a full subcategory of $\mathcal{C}\mathrm{at}/\mathcal{B}$ which we shall denote by $d\mathcal{F}\mathrm{ib}(\mathcal{B})$ (resp. $do\mathcal{F}\mathrm{ib}(\mathcal{B})$. The category $\mathcal{B}$ is sometimes referred to as the *base category* of fibration.

REMARK 2.2.11. Unwinding the above definition of discrete opfibration, we note that F is a discrete opfibration precisely whenever for every object e of $\mathcal{E}$, every morphism $f\colon Fe \to b$ in $\mathcal{B}$ has a unique lift $\tilde{f}\colon e \to \tilde{b}$ in $\mathcal{E}$.

REMARK 2.2.12. The word 'discrete' refers to the fact that the fibres of functor P form discrete categories. To see why, assume that $\mathcal{E}_b$ is the fibre given by the following pullback of categories:

$$\begin{tikzcd}
\mathcal{E}_b \arrow[r, hook] \arrow[d, "!"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "\mathcal{P}"] \\
\mathbf{1} \arrow[r, "b"'] & \mathcal{B}
\end{tikzcd}$$

over any object b in the base, and take any arrow $u\colon e' \to e$ in $\mathcal{E}_b$. Of course u is a lift of id_b with codomain e. However, id_e is the unique lift of id_b with codomain e and thus $u = \mathrm{id}_e$ and $e' = e$.

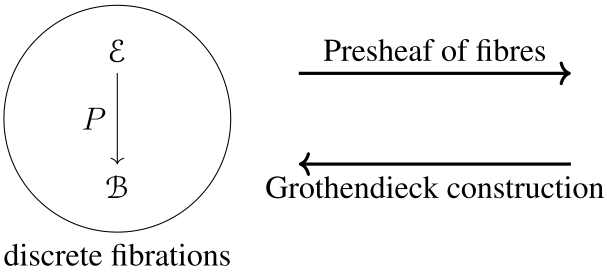
REMARK 2.2.13. Note that for a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$, even if each fibre is discrete, it may not be the case that $\mathcal{E}$ is discrete.

Remark 2.2.14. We can reformulate Definition 2.2.10 so that it can be extended to internal categories in any finitely complete category $\mathcal{S}$. For internal categories[1] $\mathbb{B} = (B_1 \rightrightarrows B_0)$ and $\mathbb{E} = (E_1 \rightrightarrows E_0)$ in $\mathcal{S}$, an internal functor $\mathbb{P} \colon \mathcal{E} \to \mathcal{B}$ is an **internal discrete fibration** if

$$\begin{tikzcd} E_1 \arrow[r, "d_1"] \arrow[d, "P_1"'] \arrow[dr, phantom, very near start, "\ulcorner"] & E_0 \arrow[d, "P_0"] \\ B_1 \arrow[r, "d_1"'] & B_0 \end{tikzcd} \tag{2.3}$$

is a pullback diagram in the category $\mathcal{S}$. The dual notion of *internal discrete opfibration* is defined by replacing d_1 with d_0 in the diagram (2.3).

Construction 2.2.15. The Grothendieck construction for presheaves of sets (i.e. discrete categories) establishes an adjoint equivalence $d\mathcal{F}\mathrm{ib}(\mathcal{B}) \simeq \mathcal{P}\mathrm{Shv}(\mathcal{B})$.



(2.4)

the presheaf $\mathbb{P}$ is defined as follows:

$$\begin{aligned} \mathbb{P} \colon \mathcal{B}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\ b &\longmapsto \mathcal{E}_b \\ (b' \xrightarrow{f} b) &\longmapsto (\mathcal{E}_b \xrightarrow{f^*} \mathcal{E}_{b'}) \end{aligned} \tag{2.5}$$

where f^* maps an object in the fibre of b to $\mathrm{dom}(\tilde{f})$, where $\tilde{f}$ is the unique lift of f. The functoriality of $\mathbb{P}$ precisely follows from the uniqueness of lifts.

---

[1] For an internal category $\mathbb{C} = (C_1 \rightrightarrows C_0)$ we shall call C_0 *the object of objects* and C_1 the *object of morphisms*. Occasionally we shall use the notations $C_0 = \mathrm{Ob}(\mathbb{C})$, and $C_1 = \mathrm{Mor}(\mathbb{C})$. See Appendix A.8.1 for more details.

For instance for an object b in a locally small category $\mathcal{B}$, the functor $\pi_b\colon \mathcal{B}/b \to \mathcal{B}$ formed by the lax pullback

$$\begin{tikzcd}
\mathcal{B}/b \arrow[r, "!"] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\Nearrow"] & \mathbf{1} \arrow[d, "b"] \\
\mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B}
\end{tikzcd} \tag{2.6}$$

is a discrete fibration and the presheaf of fibres is indeed the representable presheaf $y(b) = \mathrm{Hom}_{\mathbb{B}}(-, b)$. We shall refer to π_b as the representable fibration.

Conversely, starting from a presheaf $\mathbb{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the Grothendieck construction yields the so-called category of elements $\mathbb{X} \rtimes \mathcal{B}$ with a forgetful functor $\pi_{\mathbb{X}}\colon \mathbb{X} \rtimes \mathcal{B} \to \mathcal{B}$. In fact, $\pi_{\mathbb{X}}$ can be constructed as the lax pullback of $\star^{\mathrm{op}}$ along $\mathbb{X}^{\mathrm{op}}\colon \mathcal{B} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ whereby $\star\colon \mathbf{1} \to \mathcal{S}\mathrm{et}$ is the unique left exact functor.

$$\begin{tikzcd}
\mathbb{X} \rtimes \mathcal{B} \arrow[r, "!"] \arrow[d, "\pi_{\mathbb{X}}"'] \arrow[dr, phantom, "\Nearrow"] & \mathbf{1} \arrow[d, "\star^{\mathrm{op}}"] \\
\mathcal{B} \arrow[r, "\mathbb{X}^{\mathrm{op}}"'] & \mathcal{S}\mathrm{et}^{\mathrm{op}}
\end{tikzcd} \tag{2.7}$$

We readily observe that $\pi_{\mathbb{X}}$ is a discrete fibration: the fibre $(\mathbb{X} \rtimes \mathcal{B})_b$ is isomorphic to the set $\mathbb{X}(b)$ and this yields the equivalence 2.4. The Grothendieck construction of representable presheaves are slice categories:

$$\mathrm{Hom}(-, b) \rtimes \mathcal{B} \cong \mathcal{B}/b$$

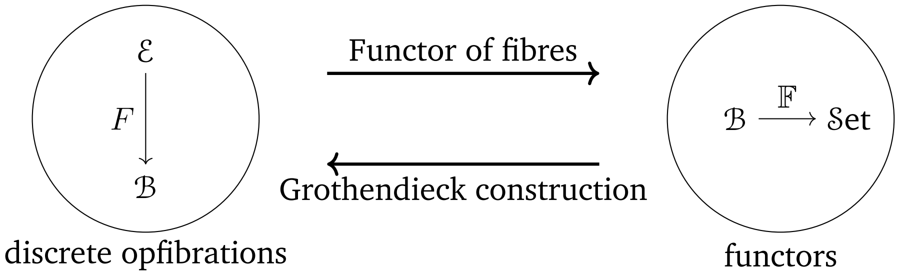
Hence, the equivalence 2.4 restricts to

$$\left\{ \begin{array}{c} \text{Discrete fibrations} \\ \pi_b\colon \mathcal{B}/b \to \mathcal{B} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Representable presheaves} \\ \mathrm{Hom}(-, b)\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \end{array} \right\}$$

Moreover,

$$d\mathcal{F}\mathrm{ib}(\pi_b, P) \cong \mathcal{E}_b \cong \mathbb{P}(b) \cong \mathcal{P}\mathrm{Shv}(\mathrm{Hom}(-, B), \mathbb{P})$$

Similarly, we have the equivalence



(2.8)

Adopting the fibrational viewpoint of presheaves (resp. functors) enables us to internalise them to other categories. Taking an internal presheaf essentially as an internal discrete fibration (See Remark 2.2.14), we define an internal presheaf (resp. internal diagram) as follows.

DEFINITION 2.2.16. For an internal category $\mathbb{C} = (C_1 \rightrightarrows C_0)$ in a finitely complete category $\mathcal{S}$, an **internal presheaf** $\mathbb{X}$ over $\mathbb{C}$ consists of

- an object X of $\mathcal{S}$,
- a *bundle* morphism $\gamma \colon X \to C_0$, and
- an *action* morphism $\alpha \colon X \,{}_{\gamma}\!\times_{d_1} C_1 \to X$

such that the left square in below commutes, i.e. $\gamma \circ \alpha = d_0 \circ \pi_1$ where π_1 is the pullback of γ along d_1.

$$\begin{tikzcd}
X \arrow[d, "\gamma"'] & X \,{}_{\gamma}\!\times_{d_1} C_1 \arrow[l, "\alpha"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\gamma"] \\
C_0 & C_1 \arrow[l, "d_0"] \arrow[r, "d_1"'] & C_0
\end{tikzcd} \tag{2.9}$$

and moreover, α satisfies the unit and associativity axioms for a (right) action, expressed by the commutativities in below:

$$\begin{tikzcd}
& X \,{}_{\gamma}\!\times_{d_1} C_1 \arrow[dr, "\alpha"] & \\
(X \,{}_{\gamma}\!\times_{d_1} C_1) \,{}_{d_0\pi_1}\!\times_{d_1} C_1 \arrow[ur, "\alpha \times \mathrm{id}"] \arrow[d, "\cong"'] & & X \\
X \,{}_{\gamma}\!\times_{d_1 d_2} (C_1 \,{}_{d_0}\!\times_{d_1} C_1) \arrow[dr, "\mathrm{id} \times d_1"'] & & \\
& X \,{}_{\gamma}\!\times_{d_1} C_1 \arrow[uur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\mathrm{id} \times i\gamma"'] \arrow[dr, "\mathrm{id}"] & \\
X \,{}_{\gamma}\!\times_{d_1} C_1 \arrow[r, "\alpha"'] & X
\end{tikzcd}
\tag{2.10}$$

Of course any set-valued presheaf is an internal presheaf in the category $\mathcal{S}\mathrm{et}$.

REMARK 2.2.17. Suppose $P \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a presheaf where $\mathcal{C}$ is a small category. We can view P as an internal presheaf in the category $\mathcal{S}\mathrm{et}$: take $X = \coprod_{c \in \mathcal{C}_0} P(c)$ with the map $\gamma \colon X \to \mathcal{C}_0$ as the first projection, and the action given by $\alpha(c, x \in Pc, f \colon d \to c) = (d, Pf(x))$. We have $X \rtimes \mathcal{C} \simeq \int_{\mathcal{C}} P$ where the latter is the familiar *category of elements* of P.

From Definition 2.2.16, it is easily observed that $\alpha, \pi_1 \colon X \,{}_{\gamma}\!\times_{d_1} C_1 \rightrightarrows X$ form an internal category in $\mathcal{S}$ where α is the domain morphism, π_1 is the codomain morphism, and identity and composition are given by identity and composition in $\mathbb{C}$. We call this internal category the **internal action category**[2] and we denote it by $\mathbb{X} \rtimes \mathbb{C}$. Furthermore, commutativity of diagrams 2.9 and 2.10 are indeed the (internal) functoriality axioms for $\pi_{\mathbb{X}} := \langle \pi_1, \gamma \rangle \colon \mathbb{X} \rtimes \mathbb{C} \to \mathbb{C}$. We note that

$$\begin{tikzcd}
(\mathbb{X} \rtimes \mathbb{C})_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathbb{X} \rtimes \mathbb{C})_0 \arrow[d, "\gamma"] \\
C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}
\tag{2.11}$$

is a pullback diagram in $\mathcal{S}$. By Remark 2.2.14, the forgetful functor $\langle \pi_1, \gamma \rangle$ is an internal discrete fibration. This process describes the internal version of Grothendieck construction earlier described in 2.2.15. It is similar to see that an internal discrete fibration has the structure of an internal presheaf in the sense of Definition 2.2.16.

---

[2] This is the internal version of category of elements.

We would like to conclude this section by discussing the *universal* discrete fibrations and opfibrations of categories.

PROPOSITION 2.2.18. The forgetful functor $U\colon \mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}$, where $\mathcal{S}\mathrm{et}_*$ is the category of pointed sets, is a discrete opfibration of large categories, and the fibre over each set X is isomorphic the set X itself (viewed as a discrete category). We occasionally refer to U as the **tautological discrete bundle**. Moreover, U classifies all discrete opfibrations of small categories: for a small category $\mathcal{B}$, the equivalence $do\mathcal{F}\mathrm{ib}(\mathcal{B}) \simeq \mathcal{F}\mathrm{un}(\mathcal{B}, \mathcal{S}\mathrm{et})$ of Grothendieck construction is achieved by pulling back along $U\colon \mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}$.

More concretely, for any small category $\mathcal{B}$ and every functor $F\colon \mathcal{B} \to \mathcal{S}\mathrm{et}$, the pullback of U along F gives us a discrete opfibration $\pi_F\colon \mathcal{B} \rtimes F \to \mathcal{B}$ with the fibre over $b \in \mathcal{B}$ being the discrete category $F(b)$, as shown in the diagram

$$\begin{tikzcd}
\mathcal{B} \rtimes F \arrow[r, "\pi_1"] \arrow[d, dashed, "\pi_F"'] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{S}\mathrm{et}_* \arrow[d, "U"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{S}\mathrm{et}
\end{tikzcd}$$

where $U(X, x) = X$, and $\pi_1(b, x) = (F(b), x)$. Moreover, any discrete opfibration $P\colon \mathcal{E} \to \mathcal{B}$, is gotten as a pullback of U along a unique (up to isomorphism) functor $F\colon \mathcal{B} \to \mathcal{S}\mathrm{et}$. Of course, by definition $U^{\mathrm{op}}\colon \mathcal{S}\mathrm{et}_*^{\mathrm{op}} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ is the universal discrete fibration of categories. Observe that an immediate consequence of proposition above is that the discrete fibrations and discrete opfibrations are stable under pullback.

The sheaf condition can be expressed fibrewise.

REMARK 2.2.19. Recall that a presheaf P on a site $(\mathcal{C}, \mathscr{J})$ is a sheaf if and only if for any object U of $\mathcal{C}$ and any covering sieve $S \in \mathscr{J}(U)$, any matching family $\chi\colon S \to P$ can be uniquely extended to $\overline{\chi}\colon yU \to P$ in $\mathcal{P}\mathrm{Shv}(\mathcal{C})$ (the diagram on the left). Fibrewise, this is expressed by saying that $\chi \rtimes \mathcal{C}$ has a unique extension to the discrete fibred category $\mathcal{C}/U$ (the diagram on the right).

$$\begin{tikzcd} yU \arrow[r, dashed, "\overline{\chi}"] & P \\ S \arrow[u, hook] \arrow[ur, "\chi"'] & \end{tikzcd} \quad\mapsto\quad \begin{tikzcd} & \mathcal{C}/U \arrow[dd] \arrow[dr, dashed, "\overline{\chi}\rtimes\mathcal{C}"] & \\ S \rtimes \mathcal{C} \arrow[ur, hook] \arrow[rr, "\chi\rtimes\mathcal{C}"] \arrow[dr] & & P \rtimes \mathcal{C} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

2.3 Grothendieck fibrations

In this section we will review the notions of precartesian and cartesian morphisms. They are introduced by Grothendieck which he used to develop the notion of fibration of categories. The standard present-day notions of 'precartesian' morphisms and 'cartesian' morphisms were originally named by Grothendieck 'cartesian' morphisms and 'strongly cartesian' morphisms (See [GR71, Exposé VI], especially its beautiful introduction). For us, as it is the standard nomenclature nowadays, the corresponding notion of functor with enough cartesian (resp. precartesian) lifts will be 'fibration' (resp. 'prefibration').

In learning about fibrations and writing this chapter, I have also benefited from consulting [Vis05, Chapter 3], [Str18], [Joh02a, Part B], and [Jac99, Chapter 1].

2.3.1 Precartesian and cartesian morphism

DEFINITION 2.3.1. Let $P\colon \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $u\colon X \to Y$ in $\mathcal{E}$ is said to be P-**precartesian** whenever for any $\mathcal{E}$-morphism $v\colon Z \to Y$ with $P(u) = P(v)$, there exists a unique $\mathcal{E}$-morphism w such that $u \circ w = v$ and $P(w) = 1_{P(X)}$. Morphism $u\colon X \to Y$ is said to be P-**cartesian** whenever for any $\mathcal{E}$-morphism $v\colon Z \to Y$ and any $h\colon P(Z) \to P(X)$ with $P(u) \circ h = P(v)$, there exists a unique lift w of h such that $u \circ w = v$. The notion of opcartesian morphism is the dual of the notion of cartesian morphism.

NOMENCLATURE. In the diagrams we write $X \mapsto A$, for $X \in \mathcal{E}_0$ and $A \in \mathcal{B}_0$ to indicate that 'X is sitting above A', that is $P(X) = A$. Besides, morphisms in the fibre category $\mathcal{E}_B$, that is all $\mathcal{E}$-morphisms $v \colon X \to Y$ with $P(v) = \mathrm{id}_B$, are called *vertical*. Furthermore, when functor P is obvious from the context, then we simply use the term cartesian instead of P-cartesian.

REMARK 2.3.2. Definition 2.3.1 essentially says u being cartesian means that any lifting of $P(v)$ along $P(u)$ in the base category ($\mathcal{B}$) is uniquely induced from a lifting of v along u in ($\mathcal{E}$).

$$\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "w"'] \arrow[drr, "v"] & & \\
PW \arrow[dr, "h"'] \arrow[drr, "P(v)"] & X \arrow[r, "u"'] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\
& PX \arrow[r, "P(u)"'] & PY
\end{tikzcd} .$$

In the next proposition we list some basic observations about precartesian and cartesian morphisms:

PROPOSITION 2.3.3. Suppose $P \colon \mathcal{E} \to \mathcal{B}$ is a functor.

(i) Any cartesian morphism is precartesian.

(ii) Precartesian lifts, if they exists, are unique up to unique isomorphism.

(iii) An immediate consequence of the remark above is that any precartesian vertical arrow in $\mathcal{E}$ is an isomorphism.

(iv) Any isomorphism is cartesian.

(v) A precartesian morphism with a right inverse is an isomorphism.

LEMMA 2.3.4. An $\mathcal{E}$-morphism $u\colon X \to Y$ is P-cartesian (resp. P-opcartesian) if and only if the left (resp. right) commuting square is a pullback diagram in $\mathcal{S}\mathrm{et}$ for each object W in $\mathcal{E}$:

$$\begin{tikzcd}
\mathcal{E}(W,X) \arrow[r, "u\circ -"] \arrow[d, "P_{W,X}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}(W,Y) \arrow[d, "P_{W,Y}"] & \mathcal{E}(Y,W) \arrow[r, "-\circ u"] \arrow[d, "P_{Y,W}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}(X,W) \arrow[d, "P_{X,W}"] \\
\mathcal{B}(PW,PX) \arrow[r, "P(u)\circ -"'] & \mathcal{B}(PW,PY) & \mathcal{B}(PY,PW) \arrow[r, "-\circ P(u)"'] & \mathcal{B}(PX,PW)
\end{tikzcd}$$

From this lemma and pullback-pasting lemma it follows that

PROPOSITION 2.3.5. The closure properties of cartesian morphisms with respect to composition are:

(i) Cartesian morphisms are stable under composition.

(ii) For a cartesian morphism $u\colon X \to Y$, a morphism $v\colon X' \to X$ is cartesian if and only if $u \circ v\colon X' \to Y$ is cartesian.

(iii) Given a commutative square of $\mathcal{E}$-morphisms

$$\begin{tikzcd}
Y' \arrow[r, "u"] \arrow[d, "v'"'] & Y \arrow[d, "v"] \\
X' \arrow[r, "u'"'] & X
\end{tikzcd}$$

where v, v' are vertical and u' is cartesian we have that u is cartesian iff the square is a pullback diagram.

Note however that these closure properties do not hold for precartesian morphisms. By the proposition above we can associate to every functor $P\colon \mathcal{E} \to \mathcal{B}$ a strict double category $\mathbb{D}(P)$ which has P-vertical morphisms in $\mathcal{E}$ as its vertical morphisms, P-cartesian morphisms as its horizontal morphisms, and commutative squares as 2-morphisms. Evidenlty, $\mathbb{D}(\mathrm{Id}_{\mathcal{E}})$ is the standard double category $\mathbb{D}(\mathcal{E})$ of commutative squares in $\mathcal{E}$.

EXAMPLE 2.3.6. Let's see what precartesian and cartesian morphisms look like in the simplest of cases.

- For any category $\mathcal{B}$, there is a unique functor $\mathcal{B} \to 1$. All morphisms of $\mathcal{B}$ are vertical, a morphisms is cartesian iff it is precartesian iff it is an isomorphisms.

- Let $\mathcal{B}$ be a category with pullbacks. The codomain functor $\mathrm{cod}\colon (\mathcal{B} \downarrow \mathcal{B}) \to \mathcal{B}$ takes an object $\gamma\colon X \to B$ of $(\mathcal{B} \downarrow \mathcal{B})$ to its codomain B, and takes a morphism $\langle g, f\rangle\colon \gamma' \to \gamma$ of $(\mathcal{B} \downarrow \mathcal{B})$, i.e. a commuting square, to f. Interestingly, cod-cartesian morphisms in $(\mathcal{B} \downarrow \mathcal{B})$ are exactly pullback squares of $\mathcal{B}$. Also a morphisms if cod-precartesian iff it is cod-cartesian. (See Appendix for a proof of these facts.)

$$\begin{tikzcd}
\mathcal{B}^{\downarrow} \arrow[dd, "\mathrm{cod}"'] & Y \arrow[r, dotted, "g"] \arrow[d, dotted, "\gamma'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "\gamma"] \\
 & B' \arrow[r, "f"'] & B \\
\mathcal{B} & B' \arrow[r, "f"] & B
\end{tikzcd} \tag{2.12}$$

The fibre $(\mathcal{B} \downarrow \mathcal{B})(B)$ is isomorphic to the slice category $\mathcal{B}/B$. The cartesian vertical morphisms in that fibre form $\mathcal{C}\mathrm{ore}(\mathcal{B}/B)$, that is the maximal subgroupoid of $\mathcal{B}/B$.

2.3.2 Prefibrations and fibrations

DEFINITION 2.3.7. A functor $P\colon \mathcal{E} \to \mathcal{B}$ is said to be a **Grothendieck fibration** (resp. **Grothendieck prefibration**) whenever for each $X \in \mathcal{E}$, every morphism $A \xrightarrow{f} PX$ in $\mathcal{C}$ has a cartesian (resp. precartesian) lift in $\mathcal{E}$. A functor $F\colon \mathcal{E} \to \mathcal{B}$ is a **Grothendieck opfibration** if $F^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a Grothendieck fibration.

Grothendieck fibrations were originally introduced in the classical setting where axiom of choice is valid. In order to not rely on the axiom of choice, a choice of cartesian lifts is often required to be added to the structure of fibrations and this choice for a fibration is called *cleavage*. A fibration equipped with a cleavage is called *cloven*.

DEFINITION 2.3.8. A **cleavage** for a (pre)fibration $P\colon \mathcal{E} \to \mathcal{B}$ is a choice for each X in $\mathcal{E}_0$ and morphism $f\colon B \to PX$ in $\mathcal{B}$, a (pre)cartesian lift $\mathfrak{c}(f,X)\colon \mathfrak{c}_f X \to X$ of f in $\mathcal{E}$. More formally, the data of a cleavage is a term $\mathfrak{c}$ of the following type:

$$\mathfrak{c}\colon \prod_{B,A\colon \mathrm{Ob}(\mathcal{B})} \prod_{f\colon \mathcal{B}(B,A)} \prod_{X\colon \mathcal{X}(A)} \sum_{Y\colon \mathcal{X}(B)} \mathbf{Cart}_{\mathcal{E}}(Y,X)$$

where the type $\mathbf{Cart}_{\mathcal{E}}(Y,X)$ is type of all cartesian morphisms from Y to X. If the fibration P is equipped with a cleavage $\mathfrak{c}$, then $(P,\mathfrak{c})$ is called a *cloven fibration*. The cleavage $\mathfrak{c}$ is said to be **splitting** if for any composable pair of morphisms f, g:

$$\mathfrak{c}(g \circ f, X) = \mathfrak{c}(g, X) \circ \mathfrak{c}(f, \mathfrak{c}_g X)$$

And **normal** whenever for every object X in $\mathcal{E}$:

$$\mathfrak{c}(\mathrm{id}_{PX}, X) = \mathrm{id}_X$$

REMARK 2.3.9. In the presence of axiom of choice, every Grothendieck fibration *is* cloven. But in this chapter we will be quite explicit in working with cloven fibrations, in that we will keep track of the effect of various operations on fibrations (such as pullback, composition, etc.) on the cleavage as well. Nonetheless some fibrations (for instance category of modules fibred over category of rings, see 2.3.45(i)) have a ‘canonical’ choice of a cleavage. However, this is not true in some important examples of fibrations (e.g. as codomain fibration of 2.3.44(ii)), since pullbacks are only defined up to isomorphism. In fact, there the data of cleavage proves us with interesting things (e.g. choice of pullbacks) which we ought to book keep. This is particularly true when one work in strict settings such as semantics of dependent type theory where it is important that semantics of substitution, given by pullbacks, should be strict. We will see in section 2.4.2 a cleavage for a fibration is determined uniquely up to a canonical isomorphism. Thus, a fibration is a ‘non-algebraic’ approach of formulating base change functors (e.g. indexed categories 2.3.3): the operation f^* is characterized by a universal property, and the definition *merely* stipulates that an object with that property exists, rather than selecting a particular such object as part of the structure. In the terminology of [Mak01], they are *virtual* operations (as opposed to *honest* operations of say a bicategory or pseudo functors.).

REMARK 2.3.10. Sometimes when there is no risk of confusion about the cleavage of a (pre)fibration , we usually use the suppressed notation $\tilde{f} \colon \mathfrak{c}_f X \to X$ instead of cartesian lift $\mathfrak{c}(f, X)$ of $f \colon B \to PX$. Further still, when the cleavage $\mathfrak{c}$ is clear from the context, we use the more compact notation $\tilde{f} \colon X_f \to X$

REMARK 2.3.11. A cleavage of a cloven fibration can be modified to make the fibration normal cloven not necessarily splitting normal. The simplest example, given in [Str18], is the delooping $\Sigma(mod_2) \colon \Sigma(\mathbb{Z}) \to \Sigma(\mathbb{Z}_2)$ of the non-trivial group morphism $mod_2 \colon \mathbb{Z} \to \mathbb{Z}_2$. The data of a normal cleavage for $\Sigma(mod_2)$ is just a function $\mathbb{Z}_2 \to \mathbb{Z}$ which takes the identity element 0 to the identity element 0 of $\mathbb{Z}$, and takes 1 to an odd element of $\mathbb{Z}$. But a splitting cleavage for $\Sigma(mod_2)$ is a group homomorphism $s \colon \mathbb{Z}_2 \to \mathbb{Z}$ with $s(1)$ an odd integer. Such s does not exist. Nevertheless, any fibration is equivalent to a split fibration by changing the domain of fibration to an equivalent category. The groupoid $\Sigma(\mathbb{Z})$ (with one object) is equivalent to the groupoid $\mathcal{G}$, generated by two objects $\star_e$ and $\star$, inverse morphisms $\alpha \colon \star_e \rightleftarrows \star \colon \beta$, and an invertible $\delta \colon \star \to \star$, via the equivalence $U \colon \mathcal{G} \to \Sigma(\mathbb{Z})$ which takes α to $+1$, β to -1, and δ to $+1$. By taking α and β in the cleavage, $\Sigma(mod_2) \circ U$ is a splitting fibrations (and opfibration): the lift of $1 \colon \mathbb{Z}_2 \to \mathbb{Z}_2$ with the codomain $\star$ is taken to be α and the lift of $1 \colon \mathbb{Z}_2 \to \mathbb{Z}_2$ with the codomain $\star_e$ is taken to be β. Note that $\alpha \circ \beta = \mathrm{id}_\star$ which is the chosen lift of identity $0 \colon \mathbb{Z}_2 \to \mathbb{Z}_2$.

Assuming the stability of precartesian morphisms under composition, there is no difference between fibrations and prefibrations. The proof of proposition below is given in Appendix A.9

PROPOSITION 2.3.12. A (cloven) prefibration is a (cloven) fibration if and only if precartesian morphisms are closed under composition.

EXAMPLE 2.3.13. We continue Example 2.3.6 by examining the simplest cases of fibrations and opfibrations.

(i) The unique functor $\mathcal{B} \to 1$ is a Grothendieck fibration. The canonical choice of cartesian lift for each $X \in \mathcal{E}$ is id_X, and with this choice the fibration is a normal split fibration.

(ii) For any category $\mathcal{B}$, the codomain functor $\mathrm{cod}\colon (\mathcal{B}\downarrow\mathcal{B}) \to \mathcal{B}$ is always an opfibration, and it is a fibration if and only if $\mathcal{B}$ has all pullbacks. A cloven fibration $(\mathrm{cod}, \mathfrak{c})\colon (\mathcal{B}\downarrow\mathcal{B}) \to \mathcal{B}$ is precisely a category $\mathcal{C}$ with a choice of pullbacks in $\mathcal{B}$. For a morphism $f\colon B' \to B$, the base change functor $f^*\colon (\mathcal{B}\downarrow\mathcal{B})(B) \to (\mathcal{B}\downarrow\mathcal{B})(B')$ are the familiar pullback functor $f^*\colon \mathcal{B}/B \to \mathcal{B}/B'$. Similarly dom is always a Grothendieck fibration and it is a Grothendieck opfibration if and only if $\mathcal{B}$ has all pushouts.

(iii) Any discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration: any morphism in $\mathcal{E}$ is P-cartesian and there are no non-trivial vertical morphisms.

The following proposition is a rewriting of Definition 2.3.7 in terms of adjunction on slice categories. We include the proof in Appendix A.9 for the sake of completeness.

PROPOSITION 2.3.14. $(P, \mathfrak{c})\colon \mathcal{E} \to \mathcal{B}$ is a cloven Grothendieck fibration if and only if for each object $X \in \mathcal{E}$, the induced functor $P_X\colon \mathcal{E}/X \to \mathcal{B}/PX$ has a right adjoint right inverse S_X , that is the counit of adjunction is identity.

The important thing about the proof of this proposition is that S_X is defined by cartesian lifts, and for any $\mathcal{E}$-morphism $u\colon Y \to X$, the unit $\eta(u)$ followed by the cartesian lift $S(Pu)$ in $\mathfrak{c}$ gives the vertical-cartesian factorisation of u:

$$\begin{tikzcd}
Y \arrow[d, dashed, "\eta_X(u)"'] \arrow[dr, "u"] & \\
S_{Pu}(X) \arrow[r, "\widetilde{Pu}"'] & X
\end{tikzcd} \tag{2.13}$$

A similar proof also yields the following proposition.

PROPOSITION 2.3.15. $(P, \mathfrak{c})\colon \mathcal{E} \to \mathcal{B}$ is a cloven Grothendieck fibration if and only if the canonical functor $(\mathcal{E}\downarrow\mathcal{E}) \to \mathcal{B}/P$ has right adjoint right inverse.

The Chevalley fibrations of Section 2.4.2 are generalisation of this formulation of fibration to appropriate 2-categories.

PROPOSITION 2.3.16. (Cloven) Grothendieck fibrations are closed under composition and pullback.

The proof of this classical result is included in Appendix A.9.

We are at a stage to define the 2-category of Grothendieck fibrations:

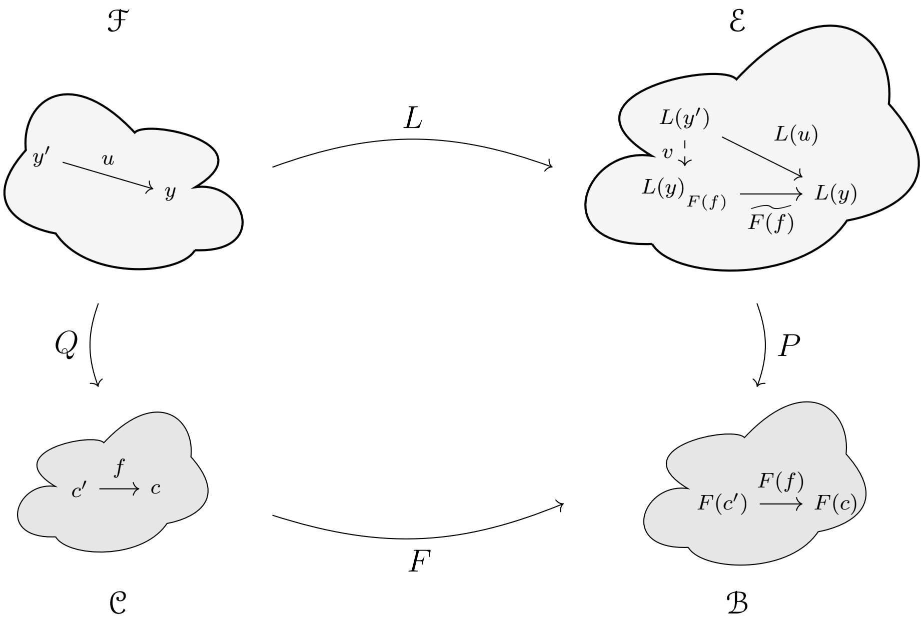
DEFINITION 2.3.17. A **(pre)fibration map** between two (pre)fibrations $Q\colon \mathcal{F} \to \mathcal{C}$ and $P\colon \mathcal{E} \to \mathcal{B}$ consists of two functors $F\colon \mathcal{C} \to \mathcal{B}$ and $L\colon \mathcal{F} \to \mathcal{E}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "L"] \arrow[d, "Q"'] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd} \tag{2.14}$$

commutes, and moreover, L carries Q-cartesian (resp. precartesian) morphisms to P-cartesian (resp. precartesian) morphisms. A **(pre) fibration transformation** is a pair of natural transformations $(\beta\colon L_0 \to L_1, \alpha\colon F_0 \to F_1)$ such that $P \centerdot \beta = \alpha \centerdot Q$. A fibration map of cloven fibrations $(Q, \mathfrak{c}_Q)$ and $(P, \mathfrak{c}_P)$ is similarly defined with the additional requirement that L takes morphisms in the cleavage $\mathfrak{c}_Q$ to $\mathfrak{c}_P$.

To spell out the definition of fibration map $(L, F)\colon Q \to P$ in above, take a morphism $f\colon c' \to c$ in the base category $\mathcal{C}$ and a Q-cartesian morphism $u\colon y' \to y$ over it in $\mathcal{F}$. Apply F to f, and L to u. Commutativity of the diagram

(2.14) says that $L(u)$ lies over $F(f)$. The unique lift of $L(u)$ along the cartesian lift $\widetilde{F(f)}$ in $\mathcal{E}$ is a vertical morphism, say $v\colon L(y') \to L(y)_{F(f)}$.



(2.15)

The fact that L preserves cartesian morphisms makes v an isomorphism. In particular, we have $L(y_f) \cong L(y)_{F(f)}$. We call the fibration map (L, F) **strict** if this isomorphism is indeed an identity.

REMARK 2.3.18. On the surface, we could have defined maps of fibration differently by requiring a natural isomorphism instead of identity in square 2.14. However, Remark 2.3.23 explains why that modification is anyway immaterial as we would obtain a 2-category biequivalent to $\mathfrak{Fib}$.

Fix a category $\mathcal{B}$. In the 2-category $\mathfrak{Fib}(\mathcal{B})$, the discrete objects are exactly discrete fibrations: for any pair of maps of fibrations to a discrete fibration, there is at most one natural transformation between them.

REMARK 2.3.19. Fixing a base $\mathcal{B}$, a fibration map to a discrete fibration in $\mathfrak{Fib}(\mathcal{B})$ is itself a fibration. The assumption that the codomain is discrete is essential. Consider the (non-discrete) fibration $\mathbf{2} \to \mathbf{1}$. A global section of this fibration in $\mathfrak{Fib}(\mathbf{1}) \simeq \mathfrak{Cat}$ exists but it is not a fibration. Moreover, if the domain is a discrete fibration, then the fibration map is too a discrete fibration (For a proof, see [Joh02a, Lemma 1.3.11]).

CONSTRUCTION 2.3.20 (The 2-category of Grothendieck fibrations). Grothendieck (pre)fibrations, (pre)fibration maps, and (pre)fibration transformations form a 2-category $\mathfrak{Fib}$ (resp. $\mathrm{pre}\mathfrak{Fib}$). We also use $\mathfrak{Fib}(\mathcal{B})$ to denote the full sub 2-category of $\mathfrak{Fib}$ which as objects has only categories fibred over $\mathcal{B}$ with 1-morphisms and 2-morphisms only those who sit above $\mathrm{Id}_{\mathcal{B}}$ and $\mathrm{id}_{\mathrm{Id}_{\mathcal{B}}}$. Obviously, $\mathfrak{Fib}(\mathbf{1}) \simeq \mathfrak{Cat}$. Similarly, $\mathrm{clv}\mathfrak{Fib}$ shall stand for the 2-category of cloven Grothendieck fibrations and $\mathrm{clvpre}\mathfrak{Fib}$ shall stand for 2-category of cloven Grothendieck prefibrations. Furthermore, $\mathrm{spl}\mathfrak{Fib}$ (resp. $\mathrm{splnl}\mathfrak{Fib}$) shall stand for the 2-category of cloven splitting (resp. splitting and normal) Grothendieck fibrations. We have the following chains of (forgetful) embedding of 2-categories:

$$\begin{tikzcd}
\mathrm{splnl}\mathfrak{Fib} \arrow[d, hook] & \\
\mathrm{spl}\mathfrak{Fib} \arrow[d, hook] & \\
\mathrm{clv}\mathfrak{Fib} \arrow[r, hook] \arrow[d, hook] & \mathrm{clvpre}\mathfrak{Fib} \\
\mathfrak{Fib} \arrow[r, hook] & \mathrm{pre}\mathfrak{Fib}
\end{tikzcd}$$

REMARK 2.3.21. Note that in diagram (2.14) since F preserves identity morphisms, then L respects vertical morphisms. Hence, L preserves the vertical-cartesian factorization and therefore, we get a morphism of double categories $\mathbb{D}(Q) \to \mathbb{D}(P)$. By the commutativity of diagram (2.14), a fibration map produces a family of functors on fibre categories $(\mathcal{F}_D \to \mathcal{E}_{F(C)} \mid C \in \mathrm{Ob}(\mathcal{C}))$. In fact, this family is the fibre of 1-morphism $L_P \colon Q \to F^*P$ in $\mathfrak{Fib}(\mathcal{C})$ induced by $L \colon Q \to P$ in $\mathfrak{Fib}$.

The result below was proved in [Gra66]. Its proof is not particularly difficult: it can be done componentwise. We state it here to make a connection later with representably-defined notion of fibration internal to 2-categories.

PROPOSITION 2.3.22. A functor $P \colon \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration if and only if $\mathfrak{Cat}(\mathcal{F}, P) \colon \mathfrak{Cat}(\mathcal{F}, \mathcal{E}) \to \mathfrak{Cat}(\mathcal{F}, \mathcal{B})$ is a Grothendieck fibration for any category

$\mathcal{F}$ and for any functor $A\colon \mathcal{F}' \to \mathcal{F}$ the commutative diagram below is a map of fibrations.

$$\begin{tikzcd}
\mathfrak{Cat}(\mathcal{F}, \mathcal{E}) \arrow[r, "A^*(\mathcal{E})"] \arrow[d, "P_*(\mathcal{F})"'] & \mathfrak{Cat}(\mathcal{F}', \mathcal{E}) \arrow[d, "P_*(\mathcal{F}')"] \\
\mathfrak{Cat}(\mathcal{F}, \mathcal{B}) \arrow[r, "A^*(\mathcal{B})"'] & \mathfrak{Cat}(\mathcal{F}', \mathcal{B})
\end{tikzcd} \tag{2.16}$$

The proposition above parallels a similar results about fibrations of spaces (e.g. Kan fibration of simplicial sets). For a fibration $p\colon E \to B$ of spaces, the induced map $p_*\colon \ \mathrm{Map}(X, E) \to \mathrm{Map}(X, B)$ of mapping spaces is again a fibration for every locally compact space X. Also, p induces a fibration $\Omega E \to \Omega B$ of the loop spaces. Since the traditional modelling of spaces uses groupoids and higher groupoids, to model fibrations of spaces categorically, we do not need lift of all morphisms in the base, but rather only isomorphisms. The notion of **isofibration** of categories is a weaker notion than Grothendieck fibration; it only requires a lift of isomorphism (with appropriate codomain) of the base category. This means that $P\colon \mathcal{E} \to \mathcal{B}$ is an isofibration iff the induced functor $\mathcal{C}\text{ore}(P)\colon \ \mathcal{C}\text{ore}(\mathcal{E}) \to \mathcal{C}\text{ore}(B)$ of maximal sub-groupoids is a Grothendieck fibration. Isofibrations relates to the study of spaces up to their first homotopical dimension via their fundamental groupoids. In particular there is a *canonical model structure* $(\mathscr{F}, \mathscr{C}, \mathscr{W})$ on the 1-category $\mathcal{G}rpd$ of groupoids and functors where

- the class $\mathscr{F}$ of fibrations consists of isofibrations.
- the class $\mathscr{W}$ of weak equivalences consists of categorical equivalences.
- the class $\mathscr{C}$ of cofibrations consists of functors which are injections on object parts. All objects are both fibrant and cofibrant and this makes the model category quite simple.

The canonical model structure on $\mathcal{G}rpd$ has nice properties: for instance, it is left proper and cofibrantly generated. Some original ideas go back to work is done in [Bro70], but the model category structure was first presented in

[And78]. An excellent survey of this model structure with its applications can be found [JT08].

REMARK 2.3.23. By Proposition 2.3.3(v), any Grothendieck fibration is an isofibration, and in particular the functor $P_*(\mathcal{F})\colon \mathfrak{Cat}(\mathcal{F}, \mathcal{E}) \to \mathfrak{Cat}(\mathcal{F}, \mathcal{B})$ is an isofibration. This justifies the choice of strict equality instead of natural isomorphism in the definition of a fibration map in the diagram (2.14): any natural isomorphism $FQ \cong PL$ can be lifted to a natural isomorphism $L' \cong L$ with $PL' = FQ$.

CONSTRUCTION 2.3.24. The tautological discrete bundle $U\colon$ $\mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}$ can be constructed as a part of the comma object of the unit $1\colon \mathbf{1} \to \mathcal{S}\mathrm{et}$ and $\mathrm{Id}\colon$ $\mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ in the 2-category $\mathfrak{Cat}$. For this reason, we denote it by $\partial_1(1)$. Similarly, the functor $\partial_1(\mathbf{1})\colon$ $\mathcal{C}\mathrm{at}_* \to \mathcal{C}\mathrm{at}$ obtained from the comma object

$$\begin{tikzcd}
\mathcal{C}\mathrm{at}_* \arrow[r, "\partial_0"] \arrow[d, "\partial_1(\mathbf{1})"'] & \mathbf{1} \arrow[d, "\mathbf{1}"] \arrow[dl, phantom, "\overset{\delta}{\swarrow}"] \\
\mathcal{C}\mathrm{at} \arrow[r, "\mathrm{Id}"'] & \mathcal{C}\mathrm{at}
\end{tikzcd}$$

is indeed a Grothendieck opfibration of large categories. By the construction above, $\mathcal{C}\mathrm{at}_*$ has as its objects pairs $(\mathcal{C}, c)$ where c is an object of $\mathcal{C}$, and as its morphisms pairs $(F, f)\colon (\mathcal{C}, c) \to (\mathcal{D}, d)$ where $f\colon F(c) \to d$ is a morphism in $\mathcal{D}$. The opfibration $\partial_1(\mathbf{1})$ classifies all Grothendieck opfibrations of small categories: Any opfibration $F\colon \mathcal{E} \to \mathcal{B}$ is equivalent to the pullback of $\partial_1(\mathbf{1})$ along the fibre functor $\mathbb{F}\colon \mathcal{B} \to \mathcal{C}\mathrm{at}$.

2.3.3 Fibrations and indexed categories

The equivalences 2.1, 2.2, 2.2.15 and their internal versions suggest a pattern for a bigger picture. As we discussed in the very first section of this chapter a fundamental principle in mathematics is that objects do not exist only in isolation, rather they occur in families. The adjectives “indexed, parameterized, familial" appearing in the title of many fields and concepts in mathematics is a witness to our claim. In category theory, “indexing” is mainly expressed by functors, pseudo functors, ..., ∞-functors, etc. However, as we climb the tower of dimensions, there naturally appears an increasing number of coherence conditions to make sure the indexing is ‘functorial’. Particularly when our higher categories are weak (such bicategories, etc.) to specify and verify the coherence conditions are difficult to track. If we take the bundle view though, these coherence conditions can be repackaged under a single universal property of cartesianness. The process of turning indexed n-categories to fibrations of n-categories is known as Grothendieck construction and we have already seen examples of it for discrete fibrations. In this section we are going to describe Grothendieck construction of indexed categories and indexed 2-categories. By an indexed category we mean a homomorphism of bicategories of the type $\mathcal{C}^{\mathfrak{d}} \to \mathfrak{Cat}$ where $\mathcal{C}$ is a (small) category and $\mathcal{C}^{\mathfrak{d}}$ is the associated discrete bicategory.

An interesting feature of the Grothendieck construction is that it reduces category level as illustrated in the table below[3]:

Indexed families of n-categories	*Fibrations of n-categories*
A set-indexed family of sets $X\colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ in $\mathcal{C}\mathrm{at}$	A bundle of sets $\gamma\colon X \to I$ in $\mathcal{S}\mathrm{et}$
A category-indexed family of sets $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ in $\mathfrak{Cat}$	A discrete bundle of categories $F \rtimes \mathcal{C} \to \mathcal{C}$ in $\mathcal{C}\mathrm{at}$
A category-indexed family of categories $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \boldsymbol{\mathfrak{Cat}}$ in $\mathbf{2}\,\boldsymbol{\mathfrak{Cat}}_{psd}$	A bundle of categories $\mathbb{P} \rtimes \mathcal{B} \to \mathcal{B}$ in $\boldsymbol{\mathfrak{Cat}}$
⋮	⋮

Other than a change in viewpoint it makes a world of difference when we work in higher levels. For instance, an ∞-stack in algebraic geometry can be conceived as a "category fibred in spaces" instead of an ∞-functor to the ∞-category of spaces.

In what follows we shall describe in details how to associate to a normal split cloven Grothendieck fibration the 2-functor of fibres, to a cloven Grothendieck

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[3] Of course there is a dual to this table which relates pseudo functors to opfibrations.

fibration a pseudo functor of fibres, and to a cloven Grothendieck prefibration a lax functor of fibres.

Suppose $(P\colon \mathcal{E} \to \mathcal{B}, \mathfrak{c})$ is a cloven prefibration. We define $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ as follows: For an object A of $\mathcal{B}$, we define $\mathbb{P}(A)$ to be the fibre of P whose objects and morphisms are objects and morphisms of $\mathcal{E}$ which are mapped to A and id_A by P, respectively. Note that for any morphism $f\colon A \to B$, we get a 'change of base' functor $\mathbb{P}(f)\colon \mathbb{P}(B) \to \mathbb{P}(A)$ sending Y to $\mathfrak{c}_f Y$ and $u\colon Y \to Y'$ in $\mathbb{P}(B)$ to $\mathfrak{c}_f(u)$, the unique vertical morphism which makes the following diagram commute.

$$\begin{tikzcd}
 & \mathfrak{c}_f Y \arrow[rr, "\mathfrak{c}(f{,}Y)"] \arrow[dl, dotted, swap, "\mathfrak{c}_f(u)"] \arrow[ddl, mapsto] & & Y \arrow[dl, swap, "u"] \arrow[ddl, mapsto] \\
\mathfrak{c}_f Y' \arrow[rr, "\mathfrak{c}(f{,}Y')"] \arrow[d, mapsto] & & Y' \arrow[d, mapsto] & \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

Now suppose $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathcal{B}$. We have $\mathbb{P}(gf)(Z) = \mathfrak{c}_{gf} Z$ and $\mathbb{P}(f) \circ \mathbb{P}(g)(Z) = \mathfrak{c}_f \mathfrak{c}_g Z$. Notice that since $P(\mathfrak{c}(g, Z) \circ \mathfrak{c}(f, \mathfrak{c}_g Z)) = P(\mathfrak{c}(gf, Z)) = gf$, and precartesian property of morphisms $\mathfrak{c}(gf, Z)$ yields a unique vertical morphism $v\colon \mathfrak{c}_f \mathfrak{c}_g Z \to \mathfrak{c}_{gf} Z$ such that $\mathfrak{c}(gf, Z) \circ v = \mathfrak{c}(g, Z) \circ \mathfrak{c}(f, \mathfrak{c}_g Z)$. (The fact that composition of precartesian morphisms may not be precartesian precludes v from being an isomorphism.) All squares in the diagram below commute and this shows the choice of v is natural.

$$\begin{tikzcd}
 & \mathfrak{c}_f\mathfrak{c}_g Z \arrow[rr, "\mathfrak{c}(f{,}\mathfrak{c}_g Z)"] \arrow[dl, dotted, "\mathfrak{c}_f(\mathfrak{c}_g u)"'] \arrow[dd, "v"] & & \mathfrak{c}_g Z \arrow[rr, "\mathfrak{c}(g{,}Z)"] \arrow[dl, dotted, "\mathfrak{c}_g(u)"'] & & Z \arrow[dl, "u"'] \arrow[dd, equal] \\
\mathfrak{c}_f\mathfrak{c}_g Z' \arrow[rr, "\mathfrak{c}(f{,}\mathfrak{c}_g Z')"] \arrow[dd, "v'"'] & & \mathfrak{c}_g Z' \arrow[rr, "\mathfrak{c}(f{,}Z')"] & & Z' \arrow[dd, equal] & \\
 & \mathfrak{c}_{gf} Z \arrow[rrrr, "\mathfrak{c}(gf{,}Z)"] \arrow[dl, "\mathfrak{c}_{gf}(u)"] & & & & Z \arrow[dl, "u"'] \\
\mathfrak{c}_{gf} Z' \arrow[rrrr, "\mathfrak{c}(gf{,}Z')"] & & & & Z' &
\end{tikzcd}$$

This turns $\mathbb{P}$ into a lax functor. If P was indeed a cloven fibration then v in the diagram above would be an isomorphism and we would get a pseudo functor $\mathbb{P}$ instead. Also, if we have a prefibration map $(F, L)\colon (Q, \mathfrak{c}_Q) \to (P, \mathfrak{c}_P)$ as in Definition 2.3.17, then $L_P\colon Q \to F^*(P)$ in $\mathfrak{Fib}(\mathcal{C})$ induces a pseudo natural transformation $\lambda\colon \mathbb{Q} \Rightarrow \mathbb{P} \circ F^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "L"] \arrow[d, "Q"'] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, bend left, "\mathbb{Q}"] \arrow[rr, phantom, "\Downarrow\lambda"] \arrow[dr, "F"'] & & \mathfrak{Cat} \\
 & \mathcal{B}^{\mathrm{op}} \arrow[ur, "\mathbb{P}"'] &
\end{tikzcd}$$

The pseudo-naturality squares are given, for a morphism $f\colon c' \to c$, by

$$\begin{tikzcd}
\mathbb{Q}(c) \arrow[r, "\lambda_c"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\cong_{\lambda_f}"] & \mathbb{P}(Fc) \arrow[d, "(Ff)^*"] \\
\mathbb{Q}(c') \arrow[r, "\lambda_{c'}"'] & \mathbb{P}(Fc')
\end{tikzcd} \tag{2.17}$$

where the natural isomorphism λ_f at component $y \in \mathbb{Q}(c)$ is exactly the vertical isomorphism v of the diagram (2.15). The fibration map (L, F) is *strict* iff λ is a strict 2-transformation.

What's more, we get a bijection between fibration transformations on the left side and modifications of pseudo transformation of indexed categories on the right side. Indeed, we obtain 2-functors

$$\mathfrak{Fib}(\mathcal{B}) \to \mathbf{2}\,\mathfrak{Cat}_{psd}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \tag{2.18}$$

$$\mathrm{splnl}\mathfrak{Fib}(\mathcal{B}) \to \mathbf{2}\mathfrak{Cat}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \tag{2.19}$$

which are *biequivalence* of 2-categories.

The quasi-inverse is known as the "Grothendieck construction for indexed categories" which we are going to explicate in below. Note that there is no biequivalence for the case of prefibrations since there is no 3-category of 2-categories having lax functors as their morphisms. Suppose $\mathcal{B}$ is a category and $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ is a pseudo functor. We would like to associate a Grothendieck fibration to $\mathbb{P}$ such that fibres are categories equivalent to $\mathbb{P}(U)$ for objects U in $\mathcal{B}$.

CONSTRUCTION 2.3.25. Define the category $\mathbb{P} \rtimes \mathcal{B}$

(i) whose objects are pairs (I, A) where I is an object of $\mathcal{B}$ and A is in an object of category $\mathbb{P}(I)$, and

(ii) whose morphisms are $(f, u)\colon (J, B) \to (I, A)$ where $f\colon J \to I$ is a morphism in $\mathcal{B}$, and $u\colon B \to f^*(A)$ a morphism in $\mathbb{P}(J)$.

Moreover,

- the identity morphism at (J, A) is given by the pair $(\mathrm{id}_J, \tau_J(A))$, and
- the composition of

  $$(K, C) \xrightarrow{(g,v)} (J, B) \xrightarrow{(f,u)} (I, A)$$

  is given by

  $$(K, C) \xrightarrow{(f \circ g, h)} (I, A)$$

  where $h := \phi_{f,g}(A) \circ g^*(u) \circ v$.

In above, $\tau_J \colon \mathrm{Id}_{\mathbb{P}(J)} \Rightarrow \mathbb{P}(\mathrm{id}_J)$ and $\phi_{f,g} \colon \mathbb{P}(g) \circ \mathbb{P}(f) \Rightarrow \mathbb{P}(f \circ g)$ is part of coherence data of $\mathbb{P}$.

The figure below provides us a with a snapshot of the category $\mathbb{P} \rtimes \mathcal{B}$ at moments I, J, K.

$$\begin{tikzcd}
\mathbb{P}(K) & \mathbb{P}(J) & \mathbb{P}(I) \\
C \arrow[d, "v"'] & & \\
g^*(B) \arrow[d, "g^*(u)"'] & & \\
g^*f^*(A) \arrow[d, "\phi_{f,g}(A)"'] & B \arrow[d, "u"] & \\
(gf)^*(A) & f^*(A) & A \\
K \arrow[r, "g"'] & J \arrow[r, "f"'] & I
\end{tikzcd} \tag{2.20}$$

It's plainly clear that $\Pi_{\mathbb{P}} \colon \mathbb{P} \rtimes \mathcal{B} \to \mathcal{B}$ taking object (I, A) to I is a Grothendieck fibration. Moreover, every morphism in $\mathbb{P} \rtimes \mathcal{B}$ factors as vertical morphism followed by a horizontal one:

$$\begin{tikzcd}
(J, B) \arrow[d, "(\mathrm{id}, \tau_J(f^*) \circ u)"'] \arrow[dr, "(f, u)"] & \\
(J, f^*(A)) \arrow[r, "(f, \mathrm{id})"'] & (I, A)
\end{tikzcd}$$

REMARK 2.3.26. The biequivalences in (2.18) sends composition of indexed categories to pullback of fibrations. Given a functor $F\colon \mathcal{C} \to \mathcal{B}$ and an indexed category $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$, we get a pullback of categories

$$\begin{tikzcd}
(\mathbb{P}F) \rtimes \mathcal{C} \arrow[r, "L"] \arrow[d, dashed, "\Pi_{\mathbb{P} \circ F^{op}}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{P} \rtimes \mathcal{B} \arrow[d, "\Pi_{\mathbb{P}}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

where $L((J,B) \xrightarrow{(f,u)} (I,A)) = (F(J),B) \xrightarrow{(F(f),u)} (F(I),A)$.

COROLLARY 2.3.27. Since monads in a 2-category $\mathfrak{Cat}$ are nothing but lax functors $1 \to \mathfrak{Cat}$, we conclude from the above equivalence that monads are indeed the same as prefibred categories over the terminal category.

An application of Grothendieck construction is the formation of homotopy quotients. Suppose G is a group, X is a topological groupoid, and G acts on X. Therefore, X induces a functor $\Sigma G \to \mathcal{G}rpd$. The Grothendieck construction applied to this functor gives the **homotopy quotient** of X by G, denoted by $X//G$. It is isomorphic to the groupoid whose objects are points of X, and whose morphisms from point x to y are given by pairs (g, ϕ) where $\phi\colon g \centerdot x \cong y$ in X. Here's why homotopy quotients are important. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a map of groupoids. The homotopy pullback (i.e. pseudo pullback) $\mathcal{E}_b \to \mathcal{E}$ of an element $b\colon \mathbf{1} \to \mathcal{E}$ is always faithful but not full. The image of $\mathcal{E}_b$ in $\mathcal{E}$ is connected and for b and b' in the same connected component of $\mathcal{B}$, we have $\mathcal{E}_b \simeq \mathcal{E}_{b'}$. Also, the group $\mathrm{Aut}(b) = \mathcal{E}(b,b)$ canonically acts on the homotopy fibre $\mathcal{E}_b$. There is a fully faithful functor $\mathcal{E}_b// \mathrm{Aut}(b) \to \mathcal{E}$. Therefore, we can write one of the most fundamental equations of theory of groupoids, that is

$$\mathcal{E} \cong \sum_{b \in \Pi_0(\mathcal{B})} \mathcal{E}_b // \mathrm{Aut}(b)$$

for any groupoid $\mathcal{E}$.

Another application of Grothendieck construction is the so-called *externalization process* which turns internal categories into fibred categories. The heavy machinery of indexed categories is an essential component of Part B

[Joh02a] and Part C of [Joh02b] to access and define internal constructions in toposes via their externalized indexed categories.[4] For instance one of the key theorem of relativised topos theory is that to any base topos $\mathscr{S}$ and any geometric theory $\mathbb{T}$ one can assocciate an the classifying $\mathscr{S}$-topos $\mathscr{S}[\mathbb{T}]$ which is a Grothendieck topos in the sense that it is equivalent to the category of internal sheaves over internal syntactic site of $\mathbb{T}$.

CONSTRUCTION 2.3.28. Suppose $\mathbb{C}$ is an internal category in $\mathcal{S}$. In Appendix A.8 it is explained how an indexed category $\mathbb{F}\mathrm{am}(\mathbb{C})\colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ can be constructed from an internal category $\mathbb{C}$ in a finitely complete category $\mathcal{S}$. Applying the Grothendieck construction yields a fibration $\Pi\colon \mathbb{F}\mathrm{am}(\mathbb{C}) \rtimes \mathcal{S} \to \mathcal{S}$. The category $\mathbb{F}\mathrm{am}(\mathbb{C}) \rtimes \mathcal{S}$ has

- as its objects (I, X) where I is an object of $\mathcal{S}$ and $X\colon I \to C_0$ is a morphism in $\mathcal{S}$, and

- as its morphisms $(\alpha, f)\colon (J, Y) \to (I, X)$ where $\alpha\colon J \to I$ is a morphism in $\mathcal{S}$ and $f\colon Y \to \alpha^* X$ is given by a morphism $f\colon J \to C_1$ in $\mathcal{S}$ with $d_0 \circ f = Y$ and $d_1 \circ f = X \circ \alpha$ in $\mathcal{S}(J, C_0)$.

The first projection gives a split normal cloven fibration $\Pi_{\mathbb{C}}\colon \mathbb{F}\mathrm{am}(\mathbb{C}) \rtimes \mathcal{S} \to \mathcal{S}$. Note that a morphism (α, f) is cartesian iff f is an isomorphism in $\mathbb{F}\mathrm{am}(\mathbb{C})(J)$. The canonical cleavage assigns to each $\alpha\colon J \to I$ the morphism $(\alpha, \mathrm{id}_{\alpha^* X})$.

EXAMPLE 2.3.29. Let $\mathcal{B}$ be a category. Consider the associated fibration $\mathcal{F}\mathrm{am}(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ of the 2-functor

$$\mathcal{F}\mathrm{un}(-, \mathcal{B})\colon \mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathfrak{Cat}$$

where for an (indexing) set I, $\mathcal{F}\mathrm{un}(I, \mathcal{B})$ is the category of functors from discrete category I to $\mathcal{B}$. The objects of this fibred category are families $\{X_i\}_{i \in I}$ of objects of $\mathcal{B}$ indexed by a set I, and a morphism is a pair (α, f) where $\alpha\colon J \to I$ and f a family of morphisms $\{f_j\colon Y_j \to X_{\alpha(j)}\}_{j \in J}$ in $\mathcal{B}$. In the case where $\mathcal{B}$ is a small category this exactly matches the externalization of category $\mathcal{B}$ (realized as an internal category in $\mathcal{S}$et) in Construction 2.3.28. A morphism $(\alpha, \{f_j\colon Y_j \to X_{\alpha(j)}\}_{j \in J})$ is cartesian iff each f_j is a bijection.

CONSTRUCTION 2.3.30. The Grothendieck construction of an indexed category is a special case of a 2-monad $\mathbb{F}\mathrm{am}_{\mathcal{S}}\colon [\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}] \to [\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}]$ called **indexed family**

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[4] In other places such as [Str18] and [Lur09] a fibrational approach is preferred.

construction. For an $\mathcal{S}$-indexed category $\mathbb{P} \colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ define $\mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})$ to be the $\mathcal{S}$-indexed category of '$\mathcal{S}$-indexed families of objects' of $\mathbb{P}$, i.e. for each object I of $\mathcal{S}$, $\mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})(I)$ is the category whose objects are pairs $(\alpha \colon J \to I, A)$ where A is an object of $\mathbb{P}(J)$, and whose morphisms are of the form $(\beta, f) \colon (\alpha, A) \to (\alpha', A')$ where $\beta \colon \alpha \to \alpha'$ is a morphism in the slice category $\mathcal{S}/I$ (i.e. the left diagram in below commutes) and $f \colon A \to \beta^* A'$ is a morphism in the category $\mathbb{P}(J)$.

$$\begin{tikzcd}
J \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & J' \arrow[dl, "\alpha'"] \\
& I &
\end{tikzcd}
\quad , \qquad A \xrightarrow{f} \beta^* A'$$

Note that if $\mathcal{S}$ has a terminal object 1, then in particular $\mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})(1)$ is equivalent to the total category $\mathbb{P} \rtimes \mathcal{S}$ of Grothendieck fibration of $\mathbb{P}$. The reindexing (aka change of base) functor ϕ^* for a morphism $\phi \colon K \to I$ in $\mathcal{S}$ is given by the pullback functor which takes an object (α, A) to $(\phi^* \alpha, \pi_2^* A)$, and morphism (β, f) to $(\phi^* \beta, \phi^* f \circ \tau)$ where τ is the canonical natural isomorphism $\pi_2^* \beta^* \cong (\phi^* \beta)^* (\pi_2')^*$ as part of the data of indexed category $\mathbb{P}$.

$$\begin{tikzcd}
& (\phi^*\beta)^*(\pi'^*_2 A') & & \beta^* A' \\
& \pi_2^* A \arrow[u, "\phi^* f \circ \tau"] \arrow[d, mapsto] & & A \arrow[u, "f"] \arrow[d, mapsto] \\
& \phi^* J \arrow[rr, "\pi_2"] \arrow[dl, "\phi^*\beta"'] \arrow[ddl] & & J \arrow[dl, "\beta"'] \arrow[ddl, "\alpha"] \\
\phi^* J' \arrow[rr, "\pi_2'"] \arrow[d, "\phi^*\alpha'"'] \arrow[dr, phantom, "\ulcorner", very near start] & & J' \arrow[d, "\alpha'"] & \\
K \arrow[rr, "\phi"] & & I &
\end{tikzcd}$$

Now, any reindexing functor ϕ^* has a left adjoint $\Sigma_\phi \colon \mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})(K) \to \mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})(I)$ which takes an object $(\gamma \colon L \to K, B)$, with B an object of $\mathbb{P}(L)$, to $(\phi \circ \gamma, B)$. Moreover, they satisfy Beck-Chevalley condition. Therefore, $\mathbb{F}\mathrm{am}_{\mathcal{S}}(\mathbb{P})$ is the free cocompletion of indexed category $\mathbb{P}$. In fact, the 2-monad $\mathbb{F}\mathrm{am}_{\mathcal{S}}$ is a KZ-monad whose algebras are exactly $\mathcal{S}$-indexed categories with $\mathcal{S}$-indexed coproducts.

2.3.4 Yoneda's lemma for fibred categories

We have an embedding $\mathcal{B}^\partial \to \mathfrak{Fib}(\mathcal{B})$ of 2-categories by taking an object U of $\mathcal{B}$ to the slice fibration $\pi_U \colon \mathcal{B}/U \to \mathcal{B}$, and a morphism $f \colon V \to U$ to the cartesian functor $f_* \colon \mathcal{B}/V \to \mathcal{B}/U$ over $\mathcal{B}$. In section 2.2 we showed that the discrete fibration π_U is representable amongst discrete fibrations, in that we have the equivalence

$$d\mathcal{F}\mathrm{ib}(\mathcal{B})(\pi_U, P) \simeq P(U)$$

for any discrete fibration $P \colon \mathcal{E} \to \mathcal{B}$. However if we are willing to pay the cost of considering π_U in the 2-category $\mathfrak{Fib}(\mathcal{B})$ rather than in the category $d\mathcal{F}\mathrm{ib}(\mathcal{B})$, we then win the prize of having it as a representable fibration.

PROPOSITION 2.3.31. For any object U in $\mathcal{B}$, and any fibred category $(P, \mathfrak{c}) \colon \mathcal{E} \to \mathcal{B}$ over $\mathcal{B}$, we have a family of equivalences of categories

$$\Phi_U \colon \; \mathrm{clv}\mathfrak{Fib}(\mathcal{B})(\pi_U, P) \simeq P(U) \colon \Psi_U$$

natural in U.

Proof. For a fibration map $L \colon \pi_U \to P$, define $\Phi(L) := L(U \xrightarrow{\mathrm{id}} U)$. Also for a vertical natural transformation $\alpha \colon L \Rightarrow L'$, define $\Phi(\alpha) := \alpha(\mathrm{id}_U)$. Φ is a functor. For an object X in $\mathcal{E}$ over $U = P(X)$, we define the fibration map $\Psi(X) \colon \mathcal{B}/U \to \mathcal{E}$ as the following functor: $\Psi(X)(V \xrightarrow{f} U) = \mathfrak{c}_f X$, and for $h \colon f' \to f$ in $\mathcal{B}/U$, $\Psi(X)(f' \xrightarrow{h} f) = \overline{h}$. One easily checks that $\Psi(X)$ is indeed a functor. Moreover, by Proposition 2.3.5 $P \circ \Psi(X) = \pi_U$ and $\Psi(X)$ preserves cartesian morphisms of $\mathcal{B}/U$. (That is every morphism of $\mathcal{B}/U$ since slice fibration is discrete.) Note that $\Psi \circ \Phi(L) \cong L$ for any fibration map L: since L sends each morphism of $\mathcal{B}/U$ to a cartesian one in $\mathcal{E}$, $L(f \colon f \to \mathrm{id}_U)$ is cartesian, and therefore, $\Psi \circ \Phi(L)(f) = \mathfrak{c}_f(L(\mathrm{id}_U)) \cong L(f)$. □

2.3.5 Categories fibred in groupoids

We start by the following observation whose proof is given in Appendix A.9.

PROPOSITION 2.3.32. Suppose $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Grpd}$ is a pseudo functor. Every morphism in $\mathbb{P} \rtimes \mathcal{B}$ is $\Pi_{\mathbb{P}}$-cartesian.

DEFINITION 2.3.33. A Grothendieck fibration $P\colon \mathcal{E} \to \mathcal{B}$, equivalent to $\Pi_{\mathbb{P}}$ for a pseudo functor $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Grpd}$, is said to be **a category fibred in groupoids**.

So, we deduce that

A pseudo functor $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ gives rise to a category fibred in groupoids
if and only if
it factors through the embedding $\mathfrak{Grpd} \hookrightarrow \mathfrak{Cat}$ of $(2,1)$-category of groupoids
into the 2-category of (small) categories.

Categories fibred in groupoids have an easier description than categories fibred in categories. We do not need to concern ourselves with the cartesianness of the lifts, since every lift is automatically cartesian due to Proposition 2.3.32.

THEOREM 2.3.34. $P\colon \mathcal{E} \to \mathcal{B}$ is category fibred in groupoids if and only if

(CFG 1) For every arrow $f\colon V \to U$ in $\mathcal{B}$ and every object X in $\mathcal{E}$ sitting above U, there is an arrow $\widetilde{f}\colon Y \to X$ with $P(\widetilde{f}) = f$.

(CFG 2) Given a commutative triangle in $\mathcal{B}$, and a lift $\widetilde{f}$ of f and a lift $\widetilde{g}$ of g, there is a unique arrow $\overline{h}\colon Y \to Z$ such that $\widetilde{f} \circ \overline{h} = \widetilde{g}$ and $P(\overline{h}) = h$.

$$\begin{tikzcd}
Z \arrow[dr, "\widetilde{g}"] \arrow[dd, dotted, "\exists! \overline{h}"'] & & & W \arrow[dr, "g"] \arrow[dd, "h"'] & \\
& X & \mapsto & & U \\
Y \arrow[ur, "\widetilde{f}"'] & & & V \arrow[ur, "f"'] &
\end{tikzcd}$$

REMARK 2.3.35. By taking nerves we get quasi-categories $N(\mathcal{E})$ and $N(\mathcal{B})$, we can express the two lifting conditions in above as horn-filling conditions below:

$$\begin{tikzcd}
\Lambda^1[1] \arrow[r] \arrow[d, "i"'] & N(\mathcal{E}) \arrow[d, "N(P)"] \\
\Delta[2] \arrow[r] \arrow[ru, dashed, "\exists"] & N(\mathcal{B})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2[2] \arrow[r] \arrow[d, "i"'] & N(\mathcal{E}) \arrow[d, "N(P)"] \\
\Delta[2] \arrow[r] \arrow[ru, dashed, "\exists!"] & N(\mathcal{B})
\end{tikzcd}$$

Because of theorem above categories prefibred in groupoids and categories fibred in groupoids are the same thing, and we only shall talk about the latter.

REMARK 2.3.36. Note that a fibration is discrete iff in the left diagram in above the diagonal filler exists uniquely as well.

A fibration map between two categories fibred in groupoids $Q\colon \mathcal{F} \to \mathcal{C}$ and $P\colon \mathcal{E} \to \mathcal{B}$ is a pair of functor $L\colon \mathcal{F} \to \mathcal{E}$ and $F\colon \mathcal{C} \to \mathcal{B}$ such that $FQ = PL$. We can drop the condition that L preserves cartesian morphisms (Definition 2.3.17) because of Proposition 2.3.32.

PROPOSITION 2.3.37. Categories fibred in groupoids form a full sub-2-category $\mathfrak{CFG}$ of $\mathfrak{Fib}$. $\mathfrak{CFG}$ inherits stability properties of fibrations in Proposition 2.3.16: categories fibred in groupoids are stable under composition and pullback along all functors.

CONSTRUCTION 2.3.38. For a fibration (resp. prefibration) $P\colon \mathcal{E} \to \mathcal{B}$ we associate a category $\mathfrak{Core}(P)\colon \mathcal{E}_{cart} \to \mathcal{B}$ fibred (resp. prefibred) in groupoids. The category $\mathcal{E}_{cart}$ is a subcategory of $\mathcal{E}$ with the same objects but only P-cartesian (resp. P-precartesian) morphisms between them. The functor $\mathfrak{Core}(P)$ is P restricted to the subcategory $\mathcal{E}_{cart}$. It turns out $\mathfrak{Core}(P)$ is a subfibration of P (i.e. a subobject in $||\,\mathfrak{Fib}(B)||_1$) and in fact it is fibred in groupoids: (CFG 1) holds by the fact that P is a fibration and (CFG 2) is true due to Proposition 2.3.3. This construction induces a 2-functor $\mathfrak{Core}\colon \mathfrak{Fib} \to \mathfrak{CFG}$ which is right 2-adjoint to the embedding 2-functor $\mathfrak{CFG} \hookrightarrow \mathfrak{Fib}$ with identity unit. The counit gives the fibration inclusion $\mathfrak{Core}(P) \to P$ in $\mathfrak{Fib}$. Therefore, $\mathfrak{CFG}$ is a coreflective sub-2-category of $\mathfrak{Fib}$.

REMARK 2.3.39. The 2-adjunction $\mathrm{Inc} \dashv \mathfrak{Core}$ induces a family of 2-adjunctions parameterized over the base $\mathcal{B}$.

$$\begin{tikzcd}
\mathfrak{CFG}(\mathcal{B}) \arrow[r, bend right, "\mathfrak{Core}"'] \arrow[r, phantom, "\perp"] & \mathfrak{Fib}(\mathcal{B}) \arrow[l, bend right, "\mathrm{Inc}"']
\end{tikzcd}$$

Note in particular for $\mathcal{B} = \mathbf{1}$, the left adjoint $\mathfrak{Core}$ gives the core groupoid of a category which in turn in a categorification of core group of a monoid (i.e. the maximal subgroup of the monoid).

REMARK 2.3.40. Every category fibred in groupoid $P \colon \mathcal{E} \to \mathcal{B}$ is a groupoidal object in the 2-category $\mathfrak{Fib}(\mathcal{B})$. This simply follows from the fact that every vertical morphism in $\mathcal{E}$ is an isomorphism since it is both vertical and cartesian. Moreover, $\mathfrak{CFG}(\mathcal{B})$ is equivalent to the full sub-2-category of groupoidal objects of $\mathfrak{Fib}(\mathcal{B})$.

2.3.6 Grothendieck fibrations and the principle of equivalence

Grothendieck fibrations are not invariant under equivalences of categories, so they are not a bicategorical notion as they violate the principle of equivalence. (See A.2.) Given a Grothendieck fibration $Q \colon \mathcal{F} \to \mathcal{B}$ and an equivalence $K \colon \mathcal{E} \to \mathcal{F}$ of categories, unfortunately $Q \circ K \colon \mathcal{E} \to \mathcal{B}$ is no longer a fibration. An easy way to see this is to take an indiscrete groupoid $\mathcal{G}$ with more than one objects and notice that $\mathbf{1} \xrightarrow{\simeq} \mathcal{G}$ is not a Grothendieck fibration.

Nevertheless a composite $P \colon \mathcal{E} \to \mathcal{B}$ of an equivalence $K \colon \mathcal{E} \to \mathcal{F}$ followed by a Grothendieck fibration $Q \colon \mathcal{F} \to \mathcal{B}$ has the following property: for any object E of $\mathcal{E}$ and any morphism $f \colon B \to PE$ we have a P-cartesian morphism $\widetilde{f} \colon f^{*}E \to E$ together with an isomorphism $P(\widetilde{f}) \cong f$ in $\mathcal{B}/PE$ and the unit gives the vertical-cartesian factorisation of morphisms in $\mathcal{E}$.

DEFINITION 2.3.41. Any functor $R \colon \mathcal{F} \to \mathcal{B}$ with the above property is called a **weak fibration** (aka *Street*[5] fibration aka *abstract* fibration).

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[5] See [Str81].

Weak fibrations are the correct notion of fibrations in bicategories as they adhere to the principle of equivalence. One can associate to every weak fibration an equivalent Grothendieck fibration, that is, every weak fibration can be factored as an equivalence followed by a Grothendieck fibration.

The Proposition 2.3.14 has a parallel for weak fibrations:

PROPOSITION 2.3.42. A functor $P\colon \mathcal{E} \to \mathcal{B}$ is a weak cloven fibration iff for every object E of $\mathcal{E}$ the induced slice functors $P_E\colon \mathcal{E}/E \to \mathcal{B}/PE$ has a right adjoint S_E which is fully faithful.

The proof is similar to the proof of 2.3.14 except one thing: the counit in this case is an isomorphism instead of identity.

EXAMPLE 2.3.43. Of course every Grothendieck fibration is a weak fibration. In below, we list few examples of weak fibrations which are not Grothendieck fibrations.

(i) For a groupoid $\mathcal{B}$, every functor $P\colon \mathcal{E} \to \mathcal{B}$ is a weak fibration. By Proposition 2.3.42, we need to prove $\mathcal{E}/E \to \mathcal{B}/PE$ has a fully faithful right adjoint. But, this is evident since $\mathcal{B}/PE \simeq \mathbf{1}$ since $\mathcal{B}$ is a groupoid and the unique functor $!\colon \mathcal{E}/E \to \mathbf{1}$ has a fully faithful right adjoint since the slice category $\mathcal{E}/E$ has a terminal object.

(ii) This example appears in [Jan90] in the context of Magid's Galois Theory. Let P be the composite

$$\mathcal{C}\mathrm{Ring}^{\mathrm{op}} \xrightarrow{\mathrm{BA}_{\mathrm{idem}}} \mathcal{B}\mathrm{ool}^{\mathrm{op}} \xrightarrow{\mathrm{Spec}} \mathcal{S}tone$$

The functor P contravariantly takes a commutative ring R to its *Pierce spectrum*, i.e. the Stone space[6] whose points are ultrafilters of the Boolean algebra $\mathrm{BA}_{\mathrm{idem}}(R)$ of idempotents in R, and whose topology is generated by the basic open sets $\mathcal{O}_H = \{F \in \mathrm{Spec}(\mathrm{BA}_{\mathrm{idem}}(R)) \mid H \not\subset F\}$. The functor P is a weak fibration of categories but not a Grothendieck fibration.

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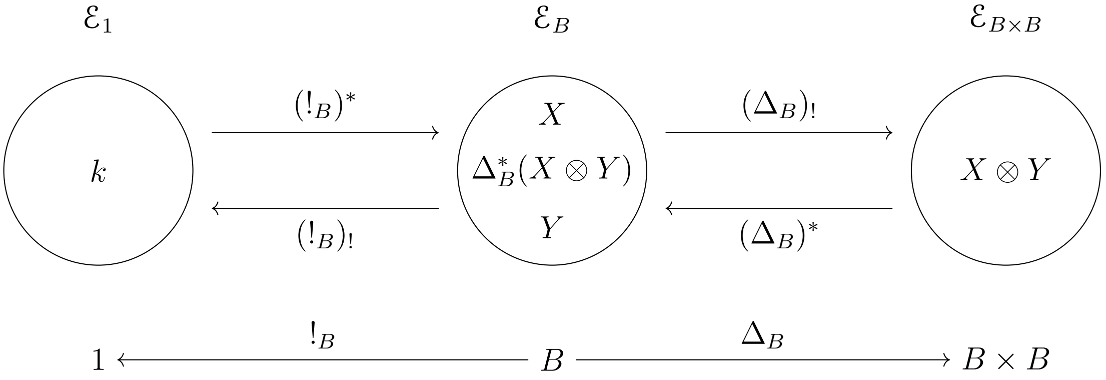
[6] Recall that a *Stone space* is a compact, Hausdorff, and totally disconnected topological space. Any Stone space is homeomorphic to the spectrum of the Boolean algebra of its clopen parts. See [Joh86] for more details about the famous *Stone duality*.

2.3.7 Few examples of categorical fibrations

EXAMPLE 2.3.44. (i) [Shu08] defines a **monoidal fibration** between monoidal categories $(\mathcal{E}, \otimes, k)$ and $(\mathcal{B}, \otimes', k')$ as a Grothendieck fibration $P\colon \mathcal{E} \to \mathcal{B}$ which is also a (strict) monoidal and the tensor product $\otimes$ preserves P-cartesian arrows. The codomain fibration of Example (ii) is a special case where P is a monoidal bifibration and the base category $\mathcal{B}$ is cartesian monoidal. In such cases, in addition to the external monoidal structure of $\mathcal{E}$, given by tensor product $\otimes$ and unit k, there is an internal tensor product on fibres, denoted by $\boxtimes$, which is strictly preserved by base change functors.



In the case of cloven bifibration $(\mathrm{cod}, \mathfrak{c})\colon (\mathcal{B} \downarrow \mathcal{B}) \to \mathcal{B}$ the fibrewise/internal tensor product in $\mathcal{C}/B$ is the fibre product: if $p\colon X \to B$, and $q\colon Y \to B$, then $X \boxtimes Y = X \times_B Y$, and $p \boxtimes q = \Delta^*(p \times q)$ since

$$\begin{tikzcd}
X \times_B Y \arrow[r] \arrow[d, "p \boxtimes q"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[d, "p \times q"] \\
B \arrow[r, "\Delta"'] & B \times B
\end{tikzcd}$$

(ii) A fibration $P\colon \mathcal{E} \to \mathcal{B}$ is called **cartesian** whenever the indexed functor $\mathbb{P}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ factors through the inclusion $\mathfrak{Cat}_{lex} \hookrightarrow \mathfrak{Cat}$ where $\mathfrak{Cat}_{lex}$ is the sub 2-category of finitely complete categories and functors. It turns out the equivalent condition for P to be cartesian is $\mathcal{E}$ has all finite limits and P preserves them. (See [Joh02a, B.1.4.1]) This turns P into a cartesian monoidal fibration. We remark that by Corollary A.9.2 in order to check that P is cartesian we only need to check that the fibre category $\mathbb{P}(I)$ has all finite limits for each object I of $\mathfrak{B}$. Moreover, P is **cartesian closed** whenever $\mathcal{E}$ is cartesian closed and P preserves the exponentials. Again, this condition can equivalent be expressed in term of indexed category $\mathbb{P}$: P is cartesian closed iff each fibre $\mathbb{P}(I)$ is cartesian closed and reindexing along projections $\pi_I \colon I \times J \to I$ has a right adjoint. (This gives dependent products from which exponentials in $\mathcal{E}$ are made.)

(iii) Every discrete (op)fibration is a Grothendieck (op)fibration. This easily follows from Proposition 2.3.14. Note that since in this case we do not have non-trivial vertical morphisms, the unit η_X therein is identity and so is the counit. Therefore, a discrete (op)fibration induces isomorphisms on (co)slices.

(iv) One of the simplest non-discrete fibrations is constructed as follows: consider an I-indexed family $\{G_i\}_{i \in I}$ of groups where I is a set. The groupoid $\coprod_{i \in I} G_i$ is fibred over the discrete category I. Obviously, the fibres are not discrete (set) but groups.

EXAMPLE 2.3.45. Non-discrete fibrations are commonplace in mathematics.

(i) For a suitable monoidal category $(\mathcal{V}, \otimes, I)$, there is a category $\mathcal{M}\mathrm{od}(\mathcal{V})$ of (left) modules (See Appendix A.8), and there are forgetful functors

$$\begin{tikzcd} & \mathcal{M}\mathrm{od}(\mathcal{V}) \arrow[dl] \arrow[dr] & \\ \mathcal{M}on(\mathcal{V}) & & \mathcal{V} \end{tikzcd}$$

Indeed, $\mathcal{M}\mathrm{od}(\mathcal{V})$ is bifibred (both fibred and opfibred) over the category $\mathcal{M}on(\mathcal{V})$ of monoids in $\mathcal{V}$. The most familiar special case of this construction is when $\mathcal{V}$ is the monoidal category $(\mathcal{A}\mathrm{b}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ of abelian groups, $\mathcal{M}on(\mathcal{V})$ is the category of rings, and $\mathcal{M}\mathrm{od}(\mathcal{V})$ is the category of all pairs (R, M) where R is a ring and M is an R-module. First, let us show that for any precartesian morphism $(f, \phi) \colon (R, M) \to (S, N)$ the morphism ϕ of abelian groups must be an isomorphism. Take y in N. Consider the R-module $R\langle y \rangle$ of formal elements $\langle r, y \rangle$ where $r \in R$. Of course, it is an abelian group with the group structure inherited from R. It is also an R-module with the scalar multiplication given by $r'\langle r, y\rangle = \langle r'r, y\rangle$. Moreover, there is a morphism $(f, i)\colon R\langle y\rangle \to N$ in $\mathcal{M}od$ where $i\langle r, y\rangle := f(r)y$. Since (f, ϕ) is precartesian, the morphism (f, i) can be lifted along it. This means there is a unique element x in M such that $\phi(x) = y$. Therefore, ϕ is an isomorphism of abelian groups.

Furthermore, for an S-module N, any ring homomorphism $f\colon R \to S$ has a canonical cartesian lift with the codomain (S, N), namely $(f, \mathrm{id})\colon (R, {}_fN) \to (S, N)$. Note that the R-module ${}_fN$ has the same underlying group as N but different scalar multiplication given by $r \centerdot y := f(r)y$ where $y \in {}_fN$. Also, for an R-module M, any ring homomorphism $f\colon R \to S$ has a canonical opcartesian lift with the domain (R, M), namely $(f, \rho)\colon (R, M) \to (S, S_f \otimes_R M)$, where $\rho(x) = 1_S \otimes x$. Note that S_f is regarded as a left-S-, right-R-bimodule; the left action being the canonical action of S on itself, and the right action being the restriction of scalars action along f.

The bifibrations structure gives the adjunction $f_! \dashv f^*\colon {}_S\mathcal{M}od \to {}_R\mathcal{M}od$ where f^*, given by the formula $f^*(N) = {}_fN$, is known as the *restriction of scalars* functor while $f_!$, given by the formula $f_!(M) = S_f \otimes_R M$, is known as the *extension of scalars* functor. Moreover, f^* has a further right adjoint f_* which is know as the *coextension of scalars*.

$$\begin{tikzcd} {}_R\mathcal{M}od \arrow[r, bend left=70, "f_!"] \arrow[r, phantom, bend left=35, "\perp"] \arrow[r, phantom, bend right=35, "\perp"] \arrow[r, bend right=70, "f_*"'] & {}_S\mathcal{M}od \arrow[l, "f^*" description] \end{tikzcd}$$

Since $f^*(N) = {}_f \cong {}_fS \otimes_S N \cong f^*(S) \otimes_S N$, natural in any left S-module N, we have $f^* \cong f^*(S) \otimes_S (-)$, and therefore by tensor-Hom adjunction (See A.17), we have $f_* \cong \underline{Hom}_R(f^*(S), -)$. Thus, we have $f_*(M) \cong \underline{Hom}_R(f^*(S), M)$, natural in M. The left action of S on $f_*(M)$ is given by $s \centerdot h\colon s' \mapsto h(s's)$. Curiously, the unit of adjunction $f^* \dashv f_*$ is precisely the structure of scalar multiplication of N as a left S-module. The whole story above holds at the more general level of fibrations $\mathcal{M}\mathrm{od}(\mathcal{V}) \to \mathcal{M}on(\mathcal{V})$, and even more generally within the framed bicategories of [Shu08].

The following example shows how powerful the universal property of cartesian morphisms could be in codifying the substantial amount of coherence data of a symmetric monoidal category.

(ii) Consider the category $\mathcal{F}\mathrm{in}_*$ of pointed finite sets which is constructed as the comma category $(* \downarrow \mathcal{S}\mathrm{et}_{\mathrm{fin}})$ where $*\colon \mathbf{1} \to \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ takes the only object of $\mathbf{1}$ to the terminal set. We present the objects of $\mathcal{F}\mathrm{in}_*$ as $\mathbf{m}_+ := (\{0, \ldots, m\}, 0)$, and morphisms as $\alpha\colon \mathbf{m} \to \mathbf{n}$ where α fixes 0. In particular, define $\mu\colon \mathbf{2}_+ \to \mathbf{1}_+$ by $\mu(1) = \mu(2) = 1$, and $\eta\colon \mathbf{0}_+ \to \mathbf{1}_+$ the unique such morphism. By the Bar construction (A.6.1) a symmetric monoidal category $(\mathcal{V}, \otimes, I)$ can be identified with a pseudo functor $\mathbf{Bar}\colon \mathcal{F}\mathrm{in}_* \to \mathfrak{Cat}$ where $\mathbf{Bar}(\mathbf{n}_+) := \mathcal{V}^{\times n}$ and $\alpha_* = \mathbf{Bar}(\alpha)\colon \mathcal{V}^{\times m} \to \mathcal{V}^{\times n}$ defined by the action

$$(c_1, \ldots, c_m) \mapsto (\bigotimes_{k \in \alpha^{-1}(i)} c_k\,)_{i=1,\ldots,n}$$

In particular, $\mu_*(c_1, c_2) = c_1 \otimes c_2$, and $\nu_* = I$.[7] Applying Grothendieck construction to $\mathbf{Bar}$ yields an opfibred category $\mathcal{V}^{\otimes}$ over $\mathcal{F}\mathrm{in}_*$ which has as its objects (possibly empty) m-tuples $(c_1, \ldots c_m)$ for all non-negative integer m, and as its morphisms pairs $(\alpha, f)\colon (c_1, \ldots c_m) \to (d_1, \ldots d_n)$ where $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ and $f = (f_1, \ldots, f_n)$ where $f_i\colon \bigotimes_{k \in \alpha^{-1}(i)} c_k \to d_i$, for $i = 1, \ldots, n$, are morphisms in $\mathcal{V}$. Let's denote the resulting opfibration by $\pi_{\mathcal{V}}\colon \mathcal{V}^{\otimes} \to \mathcal{F}\mathrm{in}_*$. Note that both morphism $\widetilde{\mu}\colon (c_1, c_2) \to c_1 \otimes c_2$ and $\widetilde{\lambda}\colon (c_1, c_2) \to c_1$, and $\widetilde{\rho}\colon (c_1, c_2) \to c_2$ are respectively opcartesian over μ, λ, and ρ all morphisms from $\mathbf{2}_+$ to $\mathbf{1}_+$ with $\mu^{-1}(1) = \{1, 2\}$, $\lambda^{-1}(1) = \{1\}$, and $\rho^{-1}(1) = \{2\}$. Now, the associator and unitors of monoidal category $\mathcal{V}$ and the coherence equations are all encoded to the uniqueness of opcartesian lifts up to unique isomorphism. For instance, there exists a unique vertical isomorphism $\alpha\colon (c_1 \otimes c_2) \otimes c_3 \to$

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[7] By convention, we take empty tensor product to be the unit I of monoidal category.

$c_1 \otimes (c_2 \otimes c_3)$ which makes the diagram below commute since obviously $\mu \circ (\lambda + \mathrm{id}) = \mu \circ (\mathrm{id} + \rho)$.

$$\begin{tikzcd}
 & (c_1 \otimes c_2, c_3) \arrow[dl, "\widetilde{\mu}"'] & \\
(c_1 \otimes c_2) \otimes c_3 \arrow[dd, dashed, "\cong"'] & & \\
 & & (c_1, c_2, c_3) \arrow[uul, "\widetilde{\lambda + \mathrm{id}}"'] \\
c_1 \otimes (c_2 \otimes c_3) & & \\
 & (c_1, c_2 \otimes c_3) \arrow[ul, "\widetilde{\mu}"] \arrow[uur, "\widetilde{\mathrm{id} + \rho}"'] &
\end{tikzcd}$$

Similarly -but using different opcartesian morphisms- we obtain the left and right unitors and their coherence equations. Where does the symmetry come from? Consider the *switch* endomorphism $\sigma \colon \mathbf{2}_+ \to \mathbf{2}_+$ in $\mathcal{F}\mathrm{in}_*$ which takes 1 to 2 and 2 to 1. Both morphisms $\widetilde{\mu} \circ \widetilde{\sigma}$ and $\widetilde{\mu}$ in $\mathcal{V}^{\otimes}$ lie above μ, since evidently $\mu \circ \sigma = \mu$. Therefore there is a unique vertical isomorphism $\overline{\sigma} \colon c_1 \otimes c_2 \to c_2 \otimes c_1$ such that $\widetilde{\mu} \circ \widetilde{\sigma} = \overline{\sigma} \circ \widetilde{\mu}$. Observe that the opfibration $\pi_{\mathcal{V}}$ is special in the sense that the fibre $\mathcal{V}^{\otimes}_{\mathbf{n}_+}$ is equivalent to the n-fold product of fibre $\mathcal{V}^{\otimes}_{\mathbf{1}_+}$. Therefore, we have comparison equivalences $\mathcal{V}^{\otimes}_{\mathbf{n}_+} \xrightarrow{\simeq} \mathcal{V}^{\times n}$ which are called *Segal maps*. It can be checked that every opfibration $P \colon \mathcal{C} \to \mathcal{F}\mathrm{in}_*$ with the data of Segal maps is equivalent to an opfibration of the form $\pi_{\mathcal{V}}$ for some monoidal category $\mathcal{V}$. For symmetric monoidal categories $\mathcal{V}, \otimes, I$ and $\mathcal{V}', \otimes', I'$ an opfibration map $L \colon \pi_{\mathcal{V}} \to \pi_{\mathcal{V}'}$ over $\mathcal{F}\mathrm{in}_*$ takes opcartesian morphism $\widetilde{\mu} \colon (c_1, c_2) \to c_1 \otimes c_2$ to opcartesian morphism $L(\widetilde{\mu}) \colon (L(c_1), L(c_2)) \to L(c_1 \otimes c_2)$ which lies over μ. Therefore, we have a unique opcartesian isomorphism $\phi \colon L(c_1) \otimes' L(c_2) \to L(c_1 \otimes c_2)$ which makes the diagram below commute.

$$\begin{tikzcd}
(L(c_1), L(c_2)) \arrow[r, "\widetilde{\mu'}"] \arrow[dr, "L(\widetilde{\mu})"'] & L(c_1) \otimes' L(c_2) \arrow[d, dashed, "\phi"] \\
 & L(c_1 \otimes c_2)
\end{tikzcd} \tag{2.21}$$

Similarly, we obtain $\tau\colon I' \to L(I)$ by opcartesianness of $\widetilde{\eta}'$. It is straightforward to verify that ϕ and τ equip L with a structure of a strong monoidal functor. Therefore, we have

$$\left\{ \begin{array}{c} \text{Strong symmetric monoidal functors} \\ \mathcal{V} \to \mathcal{V}' \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Opfibration maps} \\ \pi_{\mathcal{V}} \to \pi_{\mathcal{V}'} \text{ in } \mathfrak{Fib}(\mathcal{F}\mathrm{in}_*) \end{array} \right\}$$

Notice that only invertibility of ϕ in diagram (2.21) relies on the fact that L preserves opcartesian morphisms not its existence. Indeed, we have

$$\left\{ \begin{array}{c} \text{Lax symmetric monoidal} \\ \text{functors } \mathcal{V} \to \mathcal{V}' \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Inert cartesian-preserving} \\ \text{morphisms } \pi_{\mathcal{V}} \to \pi_{\mathcal{V}'} \text{ in } \mathfrak{Cat} \,/\, \mathcal{F}\mathrm{in}_* \end{array} \right\}$$

By $\pi_{\mathcal{V}}$-*inert* morphism in $\mathcal{V}^{\otimes}$ we mean a morphism, say u, which lies over a morphism $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ with the property that $\alpha^{-1}(i)$ is a singleton for any $1 \leq i \leq n$.

(iii) The category of vector bundles over manifolds, the category of topological spaces over sets, and the category of groupoids over sets are all exmaple of fibred categories. The common phenomenon shared among them all is that the base change functor is given by pulling back the given structure. For instance, for the last example, given a groupoid $Y = (Y_1 \rightrightarrows Y_0)$ and a function $f_0\colon X_0 \to Y_0$, we define the lift $f = (f_1, f_0)$ of f_0 by the following pullback of sets:

$$\begin{tikzcd} f^*Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "d_0 \times d_1"] \\ B \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0 \end{tikzcd}$$

(iv) The idea of *stack* is a categorification of sheaves: given an indexed functor $\mathcal{X}\colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ and a covering family $\{U_i \to U | i \in I\}$ in $\mathcal{S}$, we would like to see under what conditions we can glue fibre categories $\mathcal{X}(U_i)$ together to get $\mathcal{X}(U)$ up to an equivalence. This condition is known as *descent condition* and is a generalisation of matching families for presheaves. The fibrational view of stacks is originally due to Grothendieck. See [Joh02a, B1.5] for a precise definition. In connection to non-abelian cohomology see [Moe02]. For a great exposition in connection to the use of stacks in algebraic geometry see [Vis05].

2.4 Chevalley-style fibrations internal to 2-categories

In [Str74] (and later in [Str80]), Ross Street develops an elegant algebraic approach to study fibrations, opfibrations, and two-sided fibrations internal to 2-categories (resp. bicategories).

In the case of (op)fibrations the 2-category is required to be *finitely complete,* with strict finite conical limits[8] and cotensors with the (free) walking arrow category $\mathbb{2}$. Given those, it also has strict comma objects. Then he defined a fibration (opfibration) as a pseudo-algebra of a certain right (resp. left) slicing 2-monad. In the case of bicategories they are defined via "hyperdoctrines" on bicategories.

For (op)fibrations internal to 2-categories, he showed [Str74, Proposition 9] that his definition gave rise to Chevalley criterion for fibrations.

Also, Street weakened the original Chevalley criterion of [Gra66], by allowing the adjunction to have counit an isomorphism. Note that, even when we can use the Chevalley style, there are questions about strictness to which we shall deal with in §2.4.2. Is a certain counit of an adjunction an isomorphism (as in [Str74]) or an identity (as in [Gra66]) and how do they relate to the structure of pseudo-algebra? We will note that the relationship is not a direct correspondence. In chapter 3 working in the 2-category $\mathfrak{Con}$, we shall revert to the original requirement for an identity, and we shall call the involved adjunction the *strict Chevalley adjunction*.

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[8] i.e. weighted limits with set-valued weight functors. They are ordinary limit as opposed to a more general weighted limit.

We do not wish to assume existence of all pullbacks since our main 2-category $\mathfrak{Con}$ in Chapter 3 does not have them. Instead, we assume our 2-categories in this section to have all finite strict PIE-limits [PR91]. All PIE limits exist in $\mathfrak{Con}$. This is enough to guarantee existence of all strict comma objects since for any opspan $A \xrightarrow{f} B \xleftarrow{g} C$ in a 2-category $\mathfrak{K}$ with (strict) finite PIE-limits, the comma object $(f \downarrow g)$ can be constructed as an inserter of $f\pi_A, g\pi_C \colon A \times C \rightrightarrows B$. Pullbacks are not PIE-limits, so sometimes we shall be interested in whether they exist.

For all these reason, in the 2-category $\mathfrak{Con}$, we prefer to mainly work with the Chevalley criterion (See chapter 3). Nonetheless, we will give an overview of Street's characterisation using pseudo algebras. We first describe the Chevalley criterion in the style of [Str74], and then go into details of Street's work which connects Chevalley fibrations to pseudo algebras.

Suppose B is an object of $\mathfrak{K}$, and p is an object in the strict slice 2-category $\mathfrak{K}/B$. By the universal property of (strict) comma object $(B \downarrow p)$, there is a unique 1-morphism $\Gamma_1 \colon (E \downarrow E) \to (B \downarrow p)$ satisfying $\partial_0(p)\Gamma_1 = d_0(p \downarrow p)$, $\pi_2\Gamma_1 = e_1$, and $\delta_p \centerdot \Gamma_1 = p \centerdot \delta_E$.

$$\begin{tikzcd}
(E \downarrow E) \arrow[d, "(p\downarrow p)"'] \arrow[dr, dotted, "\Gamma_1"'] \arrow[drr, bend left, "e_1"] & & \\
(B \downarrow B) \arrow[dr, bend right, "d_0"'] & (B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(p)"'] \arrow[dr, phantom, "\delta_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}$$

(2.22)

Definition 2.4.1 (Chevalley). Consider p as above. We call p a **fibration** if the morphism Γ_1 has a right adjoint Λ_1 with counit ε an identity in the 2-category $\mathfrak{K}/B$. Dually one defines (Chevalley) **opfibrations** as 1-morphisms $p \colon E \to B$ for which the morphism $\Gamma_0 \colon (E \downarrow E) \to (p \downarrow B)$ has a left adjoint Λ_0 with unit η an identity.

Nomenclature. We shall call the adjunctions above **Chevalley adjunctions**.

Gray [Gra66] showed that Chevalley fibrations in the 2-category $\mathfrak{Cat}$ of (small) categories correspond to cloven Grothendieck fibrations. We give an illustrated and elementary discussion of this in below.

In the case where p is carrable, the comma objects $(p \downarrow B)$ and $(B \downarrow p)$ can be expressed as pullbacks along the two projections from $(B \downarrow B)$ to B.

REMARK 2.4.2. A consequence of the counit of the adjunction $\Gamma_1 \dashv \Lambda_1$ being the identity is that the adjunction triangle equations are expressed in simpler forms; we have $\Gamma_1 \centerdot \eta_1 = \mathrm{id}_{\Gamma_1}$ and $\eta_1 \centerdot \Lambda_1 = \mathrm{id}_{\Lambda_1}$.

Using the tools developed in the next section, we shall prove that $\partial_0(f)$ is a (Chevalley-style) fibration for any morphism f in $\mathfrak{K}$ (See 1.9.36). An implication of this result is that any morphism $f \colon A \to B$ in $\mathfrak{K}$ can be approximated by a fibration: the 2-morphism id_f factors through the comma 2-morphism δ_f, and this yields a unique morphism $i(f) \colon f \to \partial_0(f)$ in $\mathfrak{K}/B$ with $\pi_2 \circ i(f) = 1_A$ and $\delta_f \centerdot i(f) = \mathrm{id}_f$

$$\begin{tikzcd}
A \arrow[r, dashed, "i(f)"] \arrow[rr, bend left, "1"] \arrow[dr, bend right, "f"'] & (B \downarrow f) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(f)"'] \arrow[dr, phantom, "\delta_f \Uparrow"] & A \arrow[d, "f"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}$$

Indeed $\pi_2 \dashv i(f)$ with identity counit. In particular, $i(f)$ is fully faithful. If B is groupoidal then $\partial_0(f) \centerdot \tau_1(f) = \delta_f$ and $\partial_1(f) \centerdot \tau_1(f) = \mathrm{id}$ are invertible and therefore $\tau_1(f)$ is invertible. Hence, the adjunction $\pi_2 \dashv i(f)$ is indeed an adjoint equivalence with identity counit. Therefore, any functor with a groupoid codomain is equivalent to a fibration.

EXAMPLE 2.4.3. Let's take $\mathfrak{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $P \colon \mathcal{E} \to \mathcal{B}$, the comma category $(\mathcal{B} \downarrow P)$ is given as a category whose objects are of the form shown in the left diagram and and whose morphisms are of the form of right diagram in below, where $e \mapsto b_1$ indicates that $p(e) = b_1$.

$$\begin{tikzcd}
& e \arrow[d, mapsto, "p"] \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
\qquad
\begin{tikzcd}
& & e \arrow[dd, mapsto, "p"] \arrow[dr, "\tilde{h_1}"] & \\
& & & e' \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, "h_0"'] & & b_1 \arrow[dr, "h_1"] & \\
& c_0 \arrow[rr, "g"'] & & c_1
\end{tikzcd}$$

A functor $F \colon \mathcal{A} \to \mathcal{B}$ is approximated to a fibration $\partial_0(F)$ whereby $i(F) \colon F \to \partial_0(F)$ is given by the functorial assignment $a \mapsto \langle F(a), \mathrm{id}_{Fa}, a \rangle$. The unit of adjunction $i(F) \dashv \pi_2$ is given by component-wise by $\langle b, \alpha \colon b \to F(a), a \rangle \xrightarrow{\langle \alpha, \mathrm{id} \rangle} \langle F(a), \mathrm{id}_{Fa}, a \rangle$.

In the next part we shall overview the construction of fibrations as pseudo algebras of the slicing 2-monad introduced originally in [Str74] with one small difference: since we primarily work with fibrations (instead of opfibrations) we emphasize on co-KZ-monads (instead of KZ-monads).

2.4.1 A swift review of pseudo algebras and KZ 2-monads

In this part by a *2-monad* we mean a strict 2-monad: it consists of a strict 2-functor $T \colon \mathfrak{K} \to \mathfrak{K}$, and strict natural transformations $\mu \colon T^2 \Rightarrow T$ and $\eta \colon \mathrm{Id}_{\mathfrak{K}} \Rightarrow T$ satisfying unit and associativity laws strictly. A strict 2-monad is precisely a $\mathfrak{Cat}$-enriched monad. As with the case with monads, 2-monads provide us with the right tools to discuss 2-dimensional universal algebra. Many examples of 2-monads are concerned with studying 2-categories with additional structures, such as finite limits and colimits.

We saw in Chapter 1 that the theory of 2-categories really goes beyond the theory of $\mathfrak{Cat}$-enriched categories, not merely with respect to the size of 2-categories but more importantly due to the existence of weak morphism of 2-categories (i.e. pseudo and lax) and weak notions of limits and colimits.

Same phenomenon occurs with 2-monads: passing to 2-dimensional monads, we are faced with several choices of algebra morphisms of 2-monads. For instance, the notion of *pseudo algebra* for a 2-monad is a weakening of the notion of algebra for a monad: a pseudo algebra is *weakly* associative and *weakly* unital. For a precise definition of pseudo algebras and their morphisms see Appendix A.10.

As an example consider the *list* (aka *free monoid*) 2-monad on $\mathfrak{Cat}$. It is defined by $List(\mathcal{C}) = \coprod\limits_{n \in \mathbb{N}} \mathcal{C}^{\times n}$, and a functor $F \colon \mathcal{C} \to \mathcal{D}$ induces canonical functors $F^{\times n} \colon \mathcal{C}^{\times n} \to \mathcal{D}^{\times n}$ on components by $F^{\times n}(c_1, \ldots, c_n) = (F(c_1), \ldots, F(c_n))$. With the obvious action on functors and natural transformations, $List$ is a 2-monad on $\mathfrak{Cat}$ with unit $i_{\mathcal{C}}$ being the inclusion of elements of $\mathcal{C}$ as one-element lists in $List(\mathcal{C})$ and the multiplication being the concatenation of lists into a single list. A strict $List$-algebra is precisely a strict monoidal category while a pseudo $List$-algebra is an *unbiased monoidal category*[9]. In both cases, the tensor product is given by the structure map $\otimes \colon List(\mathcal{C}) \to \mathcal{C}$.

Even if we restrict to strict algebras there are still three notions of morphisms between them: strict, pseudo, and lax.

To illuminate this point, we give the world's simplest example of a 2-monad: consider the 2-category $\mathfrak{Cat}$, and let the 2-monad $T \colon \mathfrak{Cat} \to \mathfrak{Cat}$ take a category to its *free completion with a terminal object* (i.e. $T(\mathcal{C})$ is $\mathcal{C}$ together with a freely added terminal object). A strict algebra of T is a category with a marked terminal object, and a strict algebra homomorphism is a functor which preserves the marked terminal object up to *equality*, while a pseudo homomorphism of algebras preserves the marked terminal object only up to a *specified isomorphism*. A colax homomorphism of algebras is simply a functor while any lax homomorphism of algebras is automatically a pseudo homomorphism.

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[9] It includes an n-ary tensor product $c_1 \otimes c_2 \otimes \ldots \otimes c_n$ for all $n \geq 0$ (for $n = 0$, the tensor gives the unit $I = ()$), with associativity isomorphisms $((c_1 \otimes c_2) \otimes () \otimes (c_3)) \cong (c_1 \otimes c_2 \otimes c_3)$, etc. satisfying appropriate axioms. The biased (aka the usual definition of monoidal category) and unbiased are indeed equivalent and the proof of equivalence uses a non-trivial coherence theorem.

In the case of the list 2-monad, a lax homomorphism of pseudo algebras is a lax monoidal functor, and an oplax homomorphism is an oplax monoidal functor. The various notions of algebras and homomorphisms of algebras has been systematically studied in various places, perhaps most notably in the celebrated paper [BKP89]. For instance, it is proved therein that for a finitely complete 2-category $\mathfrak{K}$ and a 2-monad $T\colon \mathfrak{K} \to \mathfrak{K}$, the 2-category $\mathfrak{Alg}_{ps}(T)$ of algebras and pseudo homomorphisms has all PIE-limits as well as inverters and co-tensors. Moreover the forgetful 2-functor $\mathfrak{Alg}_{ps}(T) \to \mathfrak{K}$ creates these limits.

There is a certain symmetry between lax morphisms and colax morphisms of algebras, and, following [Kel74a], this is known as *doctrinal adjunction*. Given an adjunction $f \dashv u$ in a 2-category $\mathfrak{K}$, there is a bijection

$$\left\{ \begin{tikzcd} TA \arrow[r, "Tu"] \arrow[d, "a"'] \arrow[dr, phantom, "\check{u}\Downarrow"] & TX \arrow[d, "x"] \\ A \arrow[r, "u"'] & X \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "x"'] \arrow[dr, phantom, "\hat{f}\Uparrow"] & TA \arrow[d, "a"] \\ X \arrow[r, "f"'] & A \end{tikzcd} \right\}$$

between lax algebra homomorphisms $(u, \check{u})$ from a to x and colax algebra homomorphisms $(f, \hat{f})$ from x to a. This bijection is obtained by the operation of mating (§ A.7) using the counit $\varepsilon\colon fu \Rightarrow 1_A$ of adjunction $f \dashv u$, and the unit $T(\eta)\colon 1_{TX} \Rightarrow T(u)T(f)$ of adjunction $T(f) \dashv T(u)$.

Generally we are more interested in certain structured 2-categories, and we ask ourselves what are the monads whose algebras provide those structures. Usually it is the algebras which we care more about, but finding the 2-monad itself is not always straight-forward.

A good motivation for the following definition is the well-known example of *free cocompletion* (under a certain class of diagrams) 2-monad. consider the 2-monad $T\colon \mathfrak{Cat} \to \mathfrak{Cat}$ whereby $T(\mathcal{C})$ is the free cocompletion of $\mathcal{C}$ under a given class of colimits and the algebras $T(\mathcal{C}) \to \mathcal{C}$ are the categories with chosen colimits of that particular class (for example finite coproducts) and the strict morphisms of algebras are the functors which not only preserve these colimits, but also preserve the chosen colimits. Then the pseudo morphisms of algebras are the functors preserving the colimits in the usual sense. Now, for any diagram D of that particular class in $\mathcal{C}$, we get a unique morphism $\operatorname{colim} T(F(D)) \to F(\operatorname{colim} D)$ by the universal property of colimits. This is the idea behind the lax idempotent monads. Any structure arising from an algebra of such monad is necessarily unique up to unique isomorphism. They are called "property-like structures" [KLW97].

DEFINITION 2.4.4. A 2-monad $T \colon \mathfrak{K} \to \mathfrak{K}$ is said to be **lax idempotent** if given any two (pseudo) T-algebras $\mathfrak{a} \colon TA \to A$, $\mathfrak{b} \colon TB \to B$ and a 1-morphism $f \colon A \to B$, there exists a unique 2-morphism $\check{f} \colon \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering $(f, \check{f})$ a lax morphism of pseudo T-algebras.

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

REMARK 2.4.5. Dually, reverse the direction of $\check{f}$ in Definition 2.4.4, then we get the notion of ***co-lax idempotent*** monad.

Lax idempotency is a *property* of algebras of the 2-monad rather than the 2-monad itself. To see the difference, compare it to the analogous situation of knowing a property of a group G versus a property of the category of G-actions. It turns out (See Theorem 2.4.11) that it can be defined purely in terms of structure of monad itself without appealing to its algebras.

DEFINITION 2.4.6. A 2-monad $T \colon \mathfrak{K} \to \mathfrak{K}$ is said to be a ***KZ- monad***[10] if $m \dashv i \centerdot T$ with identity counit in the 2-category $[\mathfrak{K}, \mathfrak{K}]$.

REMARK 2.4.7. Dual to the definition above, we define a monad T to be a ***co-KZ-monad*** by requiring $i \centerdot T \dashv m$ with identity unit.

In what follows the discussion takes place in the 2-category $[\mathfrak{K}, \mathfrak{K}] = \mathbf{2}\,\mathfrak{Cat}_{str}(\mathfrak{K}, \mathfrak{K})$ and we choose our notations accordingly. Therefore, 2-morphisms are really modifications. Suppose T is a co-KZ-monad. In particular, the identity

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[10] KZ: short for 'Kock-Zöberlein'

$m \circ (i \centerdot T) = 1$ is the unit of this adjunction. Moreover, the identity 2-morphism and its mate $\lambda \colon i \centerdot T \Rightarrow T \centerdot i$

$$\begin{tikzcd}
T \arrow[r, "1"] & T & & T \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & T \arrow[d, "i \centerdot T"] \\
T \arrow[u, "1"] \arrow[r, "T \centerdot i"'] \arrow[ur, phantom, "\mathrm{id}\Downarrow"] & T^2 \arrow[u, "m"'] & & T \arrow[r, "T \centerdot i"'] & T^2
\end{tikzcd} \tag{2.23}$$

satisfy the equations

$$\begin{aligned} m \centerdot \lambda &= \mathrm{id}_{1_T} \\ \lambda \centerdot i &= \mathrm{id}_{(T \centerdot i) \circ i} \end{aligned} \tag{2.24}$$

The first equation follows directly from the left triangle equation of adjunction $i \centerdot T \dashv m$ whereas the second equation in above follows from the right triangle equation of adjunction $i \centerdot T \dashv m$ together with the equation $(i \centerdot T) \circ i = (T \centerdot i) \circ i$ which in turn expresses the naturality of i.

THEOREM 2.4.8. Let T be a KZ-monad, and A an object of $\mathfrak{K}$. There is a one-to-one correspondence between the pseudo T-algebras on A and the left adjoints to unit i_A with invertible counit. Dually, there is a one-to-one correspondence between the pseudo algebras of a co-KZ-monad and the right adjoints to unit of the monad with invertible unit.

Proof. We give the proof of the theorem for the case of co-KZ-monads. We first establish that any pseudo algebra $a \colon TA \to A$ is a right adjoint to i_A:

$$\begin{tikzcd}
TA \arrow[r, bend right, "\mathfrak{a}"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "i_A"']
\end{tikzcd}$$

The (invertible) unit of adjunction above is given by $\zeta\colon 1 \Rightarrow \mathfrak{a} i_A$ (Recall that ζ is part of the data of pseudo algebra (A.20)). Here is the putative counit[11] using the mate λ_A introduced in diagram (2.23).

$$\begin{tikzcd}
 & A \arrow[dr, bend left, "i_A"] & \\
TA \arrow[ur, bend left, "\mathfrak{a}"] \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A\Downarrow"] \arrow[rr, bend right=70, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] & TA \\
 & T\zeta^{-1}\Downarrow &
\end{tikzcd} \tag{2.25}$$

To prove the adjunction triangle equations, we need the following lemma whose proof is given in the Appendix A.10.

LEMMA 2.4.9. Suppose $(\mathfrak{a}, \theta, \zeta)\colon TA \to A$ is a pseudo algebra for a KZ-monad $T\colon \mathfrak{K} \to \mathfrak{K}$. We have

$$\begin{tikzcd}
 & & & & & & & TA \arrow[dr, "\mathfrak{a}"] & \\
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A\Downarrow"] \arrow[rr, bend right=70, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] & TA \arrow[r, "\mathfrak{a}"] & A & = & TA \arrow[r, "\mathfrak{a}"] & A \arrow[ur, "i_A"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\zeta^{-1}\Downarrow"] & & A \\
 & T\zeta^{-1}\Downarrow & & & & & & &
\end{tikzcd} \tag{2.26}$$

We prove the triangle identities of adjunction with the proposed unit and counit:

$$\begin{aligned}
\mathfrak{a} \centerdot (T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \circ (\zeta \centerdot \mathfrak{a}) &= (\zeta^{-1} \centerdot \mathfrak{a}) \circ (\zeta \centerdot \mathfrak{a}) && \{\text{by Lemma 2.4.9 }\} \\
&= \mathrm{id}_{\mathfrak{a}} && \{\text{factoring out } \mathfrak{a}\}
\end{aligned}$$

Also,

$$\begin{aligned}
((T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \centerdot i_A) \circ (i_A \centerdot \zeta) &= (T\zeta^{-1} \centerdot i_A) \circ (i_A \centerdot \zeta) && \{\lambda_A \centerdot i_A = \mathrm{id}\} \\
&= (i_A \centerdot \zeta^{-1}) \circ (i_A \centerdot \zeta) && \{\text{2-naturality of } i\colon 1 \Rightarrow T\} \\
&= \mathrm{id}_{i_A} && \{\text{factoring out } i_A\}
\end{aligned}$$

---

[11] The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of [Str74].

□

REMARK 2.4.10. For a (co-)KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a (co-)KZ-monad is a nicely-behaved 2-monad whose pseudo algebras are 'property-like'.

Indeed, the theorem above ensures that

THEOREM 2.4.11 ([Str74],[Koc95]). Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent).

Proof. Given algebras $\mathfrak{a} \colon TA \to A$ and $\mathfrak{b} \colon TB \to B$ of a (co-)KZ-monad and a morphism $f \colon A \to B$ in $\mathfrak{K}$, the mate of identity 2-morphism $i_B \circ f = Tf \circ i_A$ exhibits f as a (co)lax morphism of algebras. □

In [Str74], we also see a converse of the theorem above.

LEMMA 2.4.12. Suppose $T \colon \mathfrak{K} \to \mathfrak{K}$ is a co-KZ-monad and suppose a object A, a morphism $\mathfrak{a} \colon TA \to A$, and an iso 2-morphism $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i_A$ are given in $\mathfrak{K}$, and furthermore, ζ^{-1} satisfies pasting equality (2.26). Then, we have:

(i) ζ is the unit for an adjunction $i_A \dashv \mathfrak{a}$ whose counit ε is given by $(T\zeta^{-1}) \circ (T\mathfrak{a} \bullet \lambda_A)$ (composite 2-morphism in diagram (2.25)).

(ii) The 2-morphism $\theta \colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, obtained by taking thee double mate of $\lambda_A \bullet i_A = \mathrm{id}$, is an iso 2-morphism.

$$\begin{tikzcd}
T^2A & TA \arrow[l, "Ti_A"'] & & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[u, "i_{TA}"] & A \arrow[l, "i_A"] \arrow[u, "i_A"'] \arrow[ul, phantom, "\mathrm{id}\Uparrow"] & \leftrightsquigarrow & TA \arrow[r, "\mathfrak{a}"'] & A \arrow[ul, phantom, "\theta\Downarrow"]
\end{tikzcd}$$

The double mate is obtained by first using the unit of $i_{TA} \dashv m_A$ and the counit of $i_A \dashv \mathfrak{a}$, and secondly by using the unit of $i_A \dashv \mathfrak{a}$ and the counit of $Ti_A \dashv T\mathfrak{a}$.

(iii) The 2-morphism θ enriches $(A, \mathfrak{a}, \zeta)$ with the structure of a pseudo T-algebra.

2.4.2 Fibrations as pseudo-algebras of slicing co-KZ-monad

Let $\mathfrak{K}$ be a representable 2-category. Recall that $\mathfrak{K}/B$ is the strict slice 2-category over B (See Construction 1.4.13). Consider the strict 2-functor $\partial_0 \colon \mathfrak{K}/B \to \mathfrak{K}/B$ which takes an object (E, p) to its lax pullback $((B \downarrow p), \partial_0(p))$ along the identity morphism 1_B, that is

$$\begin{tikzcd}
(B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(p)"'] \arrow[dr, phantom, "\delta_p \Uparrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B
\end{tikzcd} \tag{2.27}$$

is a comma square in $\mathfrak{K}$.

Remark 2.4.13. If p is carrable then the 2-morphism δ_p can be obtained by the pasting of pullback of p along $d_1 \colon (B \downarrow B) \to B$ and the generic comma square for B.

$$\begin{tikzcd}
(B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(p)"'] \arrow[dr, phantom, "\delta_p \Uparrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
(B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\
(B \downarrow B) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\delta \Uparrow"] & B \arrow[d, "1"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}$$

The action of ∂_0 on morphisms is given as follows: if $f \colon (E', p') \to (E, p)$ is a morphism in $\mathfrak{K}/B$, then define $\partial_0(f)$ to be the unique morphism induced by the universal property of comma object $(B \downarrow p)$. Therefore, $\pi_2 \circ \partial_0(f) = f \circ \pi'_2$ and $\partial_0(p) \circ \partial_0(f) = \partial_0(p')$. Similarly if $\sigma \colon f \Rightarrow g$ is a 2-morphism in $\mathfrak{K}/B$, then we have a unique induced 2-morphism $\partial_0(\sigma) \colon \partial_0(f) \Rightarrow \partial_0(g)$ with $\pi_2 \centerdot \partial_0(\sigma) = \sigma \centerdot \pi'_2$ and $\partial_0 \centerdot \partial_0(\sigma) = \mathrm{id}_{\partial_0(p')}$.

Proposition 2.4.14. The 2-functor $\partial_0 \colon \mathfrak{K}/B \to \mathfrak{K}/B$ is a co-KZ-monad.

Proof. The unit $i \colon \mathrm{id} \Rightarrow \partial_0$ at component (E, p) is given by the unique arrow $i(p) \colon E \to (B \downarrow p)$ with property that $\partial_0(p) \circ i(p) = p$, $\pi_2 \circ i(p) = 1_E$, and moreover $\delta_p \centerdot i(p) = \mathrm{id}_p$, all inferred by the universal property of comma object $(B \downarrow p)$.

$$\begin{tikzcd}
E \arrow[r, dashed, "i(p)"] \arrow[rr, bend left=40, "1"] \arrow[dr, bend right, "p"'] & (B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(p)"'] \arrow[dr, phantom, "\delta_p \Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}$$

It also follows that $\pi_2 \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in Proposition 1.9.34, when $f = 1$ and $g = p$. From there, we also get the unit $\tau_1(p)$ of adjunction with $\partial_0(p) \centerdot \tau_1(p) = \delta_p$.

The multiplication $m\colon \partial_0^2 \Rightarrow \partial_0$ at component (E, p) is given by the unique arrow $m(p)\colon (B \downarrow \partial_0(p)) \to (B \downarrow p)$

$$\begin{tikzcd}
(B \downarrow \partial_0(p)) \arrow[r, "\bar{\pi}_2"] \arrow[r, bend left=40, "m(p)"] \arrow[d, "\partial_0(\partial_0(p))"'] \arrow[dr, phantom, "\delta_{\partial_0(p)} \Nearrow"] & (B \downarrow p) \arrow[r, "\pi_2"] \arrow[d, "\partial_0(p)"'] \arrow[dr, phantom, "\delta_p \Nearrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B \arrow[r, "1"'] & B
\end{tikzcd} \tag{2.28}$$

with the property that $\partial_0(p) \circ m(p) = \partial_0^2(p)$, $\pi_2 \circ m(p) = \pi_2 \circ \bar{\pi}_2$, and moreover, $\delta_p \centerdot m(p) = (\delta_p \centerdot \bar{\pi}_2) \circ \delta_{\partial_0(p)}$, all derived by universal property of comma object $(B \downarrow p)$. Now, it follows that $i \centerdot \partial_0 \dashv m$ with unit being identity. $\square$

EXAMPLE 2.4.15. In this example we shall see examine the special case of above situation for the 2-monad $\partial_0\colon \mathfrak{Cat}/\mathcal{B} \to \mathfrak{Cat}/\mathcal{B}$. First recall from the Example 2.4.3 that for a functor $P\colon \mathcal{E} \to \mathcal{B}$, the objects of $(\mathcal{B} \downarrow P)$ are of the form $\langle f, e \rangle$ where $f\colon b \to Pe$ is a morphism in $\mathcal{B}$. The functor $\partial_0(P)$ takes a pair $\langle f, e \rangle$ to $b_0 = \mathrm{dom}(f)$, and $\pi_2\colon (\mathcal{B} \downarrow P) \to \mathcal{E}$ is simply the second projection; it takes $\langle f, e \rangle$ to e. The unit $i(P)\colon \mathcal{E} \to (\mathcal{B} \downarrow P)$ takes an object e of $\mathcal{E}$ to the object $\langle \mathrm{id}_{P(e)}, e \rangle$ (below, on the left) and $\tau_1(P)\colon 1_{(\mathcal{B}\downarrow P)} \Rightarrow i(P) \circ \pi_2$ induces a functor $(\mathcal{B} \downarrow P) \to \mathbf{2} \pitchfork (\mathcal{B} \downarrow P)$

which takes an object $\langle f, e\rangle$ of $(\mathcal{B} \downarrow P)$ to the morphism depicted in below on the right.

$$\begin{tikzcd}
& e \arrow[d, mapsto, "P"] \\
P(e) \arrow[r, equal] & P(e)
\end{tikzcd}
\qquad
\begin{tikzcd}
& e \arrow[d, mapsto, "P"] \arrow[dr, equal] & \\
b_0 \arrow[r, "f"] \arrow[dr, "f"'] & b_1 \arrow[dr, equal] & e \arrow[d, mapsto, "P"] \\
& b_1 \arrow[r, equal] & b_1
\end{tikzcd}$$

Also, the functors $\bar{\pi}_2$ and the multiplication $m(P)$ are given by the following actions:

$$\begin{tikzcd}
& & e \arrow[d, mapsto] & & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] & & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_1 \arrow[r, "g"'] & b_2 & , & b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_0 \arrow[r, "g\circ f"'] & b_2
\end{tikzcd}$$

Finally, Observe that functors $\partial_0(i(P))\colon (\mathcal{B} \downarrow P) \to (B \downarrow \partial_0(P))$ (on the left) and $i(\partial_0(P))\colon (\mathcal{B} \downarrow p) \to (B \downarrow \partial_0(P))$ (on the right) are given as follows:

$$\begin{tikzcd}
& e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] & & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 & \mapsto & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1 & , & b_0 \arrow[r, "f"'] & b_1 & \mapsto & b_0 \arrow[r, equal] & b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}$$

The counit of $i\partial_0(P) \dashv m$ is illustrated on the left hand side in below, and the mate 2-morphism λ appears as a natural transformations where $\lambda_P\colon i(\partial_0(P)) \Rightarrow \partial_0(i(P))$, which is the whiskering of this counit with $\partial_0(i(P))$, is illustrated on the right hand side.

$$\begin{tikzcd}
& & e \arrow[d, mapsto, "P"] \arrow[dr, equal] & & & & & e \arrow[d, mapsto, "P"] \arrow[dr, equal] & \\
b_0 \arrow[r, equal] \arrow[dr, equal] & b_0 \arrow[r, "gf"] \arrow[dr, "f"] & b_2 \arrow[dr, equal] & e \arrow[d, mapsto, "P"] & \overset{g=1}{\rightsquigarrow} & b_0 \arrow[r, equal] \arrow[dr, equal] & b_0 \arrow[r, "f"] \arrow[dr, "f"] & b_1 \arrow[dr, equal] & e \arrow[d, mapsto, "P"] \\
& b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_1 & & & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}$$

Now going back to the case of a general 2-category $\mathfrak{K}$, we would like to see what a pseudo algebra $\mathfrak{a}\colon \partial_0(p) \to p$ in $\mathfrak{K}/B$ looks like. The fact that $\mathfrak{a}$ is a morphism in $\mathfrak{K}/B$ provides us with a morphism $\mathfrak{a}$ which makes the diagram

$$\begin{tikzcd}
(B \downarrow p) \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\partial_0(p)"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd} \tag{2.29}$$

commute. Moreover, being a co-KZ-monad, ∂_0 generates an adjunction $i(p) \dashv \mathfrak{a}$ whose unit is the invertible 2-morphism $\zeta\colon 1 \Rightarrow \mathfrak{a} \circ i(p)$ by remark 2.4.10. The counit ε of this adjunction is given by $\partial_0(\zeta^{-1}) \circ (\partial_0\mathfrak{a} \centerdot \lambda_p)$. Whiskering with π_2 yields a 2-morphism $\pi_2 \centerdot \varepsilon\colon \mathfrak{a} \Rightarrow \pi_2$ Observe that $p \centerdot (\pi_2 \centerdot \varepsilon) = \delta_p$ and $p \centerdot \zeta = \mathrm{id}_p$.

$$\begin{tikzcd}
E \arrow[r, "i(p)"] \arrow[rr, bend left=60, "1"] \arrow[rr, bend left=60, phantom, "\Downarrow\zeta"'] \arrow[dr, bend right, "p"'] & (B \downarrow p) \arrow[r, bend left, "\mathfrak{a}"] \arrow[r, bend right, "\pi_2"'] \arrow[r, phantom, "\Downarrow\pi_2 \centerdot \varepsilon"] \arrow[d, "\partial_0(p)"] & E \arrow[dl, bend left, "p"] \\
& B &
\end{tikzcd} \tag{2.30}$$

The example below shows that a pseudo algebra of $\partial_0\colon \mathfrak{Cat}/\mathcal{B} \to \mathfrak{Cat}/\mathcal{B}$ is exactly a cloven Grothendieck fibrations.

EXAMPLE 2.4.16. Let $\mathfrak{a}\colon \partial_0(P) \to P$ be a pseudo algebra for the 2-monad ∂_0. By commutativity of diagram (2.29) we know that $P(\mathfrak{a}\langle f, e\rangle) = \mathrm{dom}(f)$ (below, the left diagram). As observed in above, we get an invertible lift $\zeta(e)$ of identity $\mathrm{id}_{P(e)}$ (below, the right diagram).

$$\begin{tikzcd}
\mathfrak{a}\langle f, e\rangle \arrow[d, mapsto, "P"'] & \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "\zeta(e)"] \arrow[d, mapsto, "P"'] & \mathfrak{a}\langle \mathrm{id}_{P(e)}, e\rangle \arrow[d, mapsto, "P"] \\
p(e) \arrow[r, equal] & P(e)
\end{tikzcd}$$

In addition, the invertible natural transformation $\theta(P)\colon \mathfrak{a} \circ \partial_0(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(P)$ provides us with an isomorphism $\mathfrak{a}\langle f, \mathfrak{a}\langle g, e\rangle\rangle \xrightarrow{\cong} \mathfrak{a}\langle gf, e\rangle$, for any pair of composable morphisms $f\colon b_0 \to b_1$ and $g\colon b_1 \to b_2$ in $\mathcal{B}$, and any e in $\mathcal{E}$ over b_2. Notice that

$\partial_0(\mathfrak{a}) \circ \partial_0(i(P))\langle f, e\rangle = \langle f, \mathfrak{a}\langle \mathrm{id}_{b_1}, e\rangle\rangle$, and $\partial_0(\zeta)\langle f, e\rangle$ may be illustrated as in below.

$$\begin{tikzcd}
 & & e \arrow[dr, "\zeta(e)"] \arrow[dd, mapsto, "P"] & \\
 & & & \mathfrak{a}\langle \mathrm{id}_{b_1}, e\rangle \arrow[dd, mapsto, "P"] \\
b_0 \arrow[rr, "f"] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd} \tag{2.31}$$

Now, the coherence conditions of weak unicity and weak associativity of A.10.1, translated to the special situation of this example, are expressed by the commutativity of diagrams of morphisms in $\mathcal{E}$.

$$\begin{tikzcd}
\mathfrak{a}\langle f, e\rangle \arrow[r, "\mathfrak{a}\partial_0(\zeta)"] \arrow[d, "\zeta_{\mathfrak{a}}"'] \arrow[dr, "\mathrm{id}"] & \mathfrak{a}\langle f, \mathfrak{a}\langle \mathrm{id}_{b_1}, e\rangle\rangle \arrow[d, "\theta_{\partial_0(i(P))}"] \\
\mathfrak{a}\langle \mathrm{id}_{b_0}, \mathfrak{a}\langle f, e\rangle\rangle \arrow[r, "\theta_{i(\partial_0(P))}"'] & \mathfrak{a}\langle f, e\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{a}\langle f, \mathfrak{a}\langle g, \mathfrak{a}\langle h, e\rangle\rangle\rangle \arrow[r, "\theta_{\partial_0^2(\mathfrak{a})}"] \arrow[d, "\mathfrak{a}_{\theta_{\mathfrak{a}}}"'] & \mathfrak{a}\langle gf, \mathfrak{a}\langle h, e\rangle\rangle \arrow[d, "\theta_{m(\partial_0(P))}"] \\
\mathfrak{a}\langle f, \mathfrak{a}\langle hg, e\rangle\rangle \arrow[r, "\theta_{\partial_0(m(P))}"'] & \mathfrak{a}\langle hgf, e\rangle
\end{tikzcd}$$

More specifically, the above commutativities occur in the fibre $\mathcal{E}_{b_0}$. Finally, we are interested in calculating the counit of adjunction $i(P) \dashv \mathfrak{a}$. The counit, computed in the diagram (2.25), gives us the lift $\widetilde{f} = \pi_2 \centerdot \varepsilon = \pi_2 \centerdot (\partial_0\zeta^{-1} \circ (\partial_0\mathfrak{a} \centerdot \lambda_P))$ of f. The picture below illustrates the counit $\varepsilon \colon i(P) \circ \mathfrak{a} \Rightarrow \mathrm{Id}_{(\mathcal{B}\downarrow P)}$ at the component $\langle f, e\rangle$.

$$\begin{tikzcd}
 & & \mathfrak{a}\langle f, e\rangle \arrow[dr, "\pi_2(\partial_0\mathfrak{a}(\lambda))"] \arrow[dd, mapsto, "P"] & & \\
 & & & \mathfrak{a}\langle \mathrm{id}_{b_1}, e\rangle \arrow[dr, "\pi_2(\partial_0\zeta^{-1})"] \arrow[dd, mapsto, "P"] & \\
b_0 \arrow[rr, equal] \arrow[dr, equal] & & b_0 \arrow[dr, "f"] & & e \arrow[dd, mapsto, "P"] \\
 & b_0 \arrow[rr, "f"'] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}$$

It remains to prove that $\widetilde{f}$ is P-cartesian. One couldan try to prove this directly. However, we prove this in a more general setting in Example 2.4.21.

REMARK 2.4.17. Instead of notation $\mathfrak{a}\langle \mathrm{id}_{P(e)}, e\rangle$, which has certain redundant data, we shall from now on use the notation $\mathfrak{a}\langle e\rangle$.

2.4.3 Chevalley criterion

Suppose p is a object in $\mathfrak{K}/B$. Recall the situation in Definition 2.4.1: we have a unique morphism $\Gamma_1 \colon (E \downarrow E) \to (B \downarrow p)$ satisfying $\partial_0(p)\Gamma_1 = d_0 \circ (p \downarrow p)$, $\pi_2\Gamma_1 = e_1$, and $\delta_p \cdot \Gamma_1 = p \cdot \delta_E$.

The lemma below will be crucial in certain calculations of 2-morphisms in the proof of proposition 2.4.19. Recall that $\tau_0 \colon i_E \circ e_0 \Rightarrow 1_{(E\downarrow E)}$ is the counit of adjunction $i_E \dashv e_0$, and $\tau_1 \colon 1_{(E\downarrow E)} \Rightarrow i_E \circ e_1$ is the unit of adjunction $e_1 \dashv i_E$ (Remark 1.9.36). Also, $\tau_1(p)$ is the unit of $\pi_2 \dashv i(p)$ (Proposition 2.4.14). Furthermore, by the triangle equations of adjunction, we have $e_0 \cdot \tau_0 = \mathrm{id}_{e_0}$, $e_1 \cdot \tau_1 = \mathrm{id}_{e_1}$, and $\pi_2\tau_1(p) = \mathrm{id}_{\pi_2}$.

In $\mathfrak{K} = \mathfrak{Cat}$, we have $\tau_0(u) = \langle \mathrm{id}, u \rangle \colon \mathrm{id}_{e_0} \to u$, $\tau_1(u) = \langle u, \mathrm{id} \rangle \colon u \to \mathrm{id}_{e_1}$, and $\tau_1(p)\langle f, e \rangle = \langle f, \mathrm{id}_e \rangle$.

LEMMA 2.4.18. In the situation above, we have

(i) $\Gamma_1 i_E = i(p)$

(ii) $\pi_2\Gamma_1 \cdot \tau_0 = \delta_E$

(iii) $\partial_0(p)\Gamma_1 \cdot \tau_0 = \mathrm{id}_{\partial_0(p)\Gamma_1}$

(iv) $\tau_1(p) \cdot \Gamma_1 = \Gamma_1 \cdot \tau_1$, which is best expressed diagrammatically:

$$\begin{tikzcd}
 & & E \arrow[dr, "i(p)"] & & & & E \arrow[dr, "i_E"] & & \\
(E\downarrow E) \arrow[r, "\Gamma_1"] & (B\downarrow p) \arrow[ur, "\pi_2"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\tau_1(p)\Uparrow"] & & (B\downarrow p) & = & (E\downarrow E) \arrow[ur, "e_1"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\tau_1\Uparrow"] & & (E\downarrow E) \arrow[r, "\Gamma_1"] & (B\downarrow p)
\end{tikzcd}$$

(v) $(\tau_1(p) \cdot \Gamma_1) \circ (\Gamma_1 \cdot \tau_0) = i(p) \cdot \delta_E$

Proof. The first of these equations holds due to the facts that $\pi_2\Gamma_1 i_E = e_1 i_E = \mathrm{id} = \pi_2 i(p)$, $\partial_0(p)\Gamma_1 i_E = p e_0 i_E = p = \partial_0 i(p)$, and the 2-dimensional universal property of comma cone $(B \downarrow p)$. The second equation holds since $e_1 \centerdot \tau_0 = \delta_E$. For the third one observe that $\partial_0(p)\Gamma_1 \centerdot \tau_0 = pe_0 \centerdot \tau_0 = \mathrm{id}_{pe_0} = \mathrm{id}_{\partial_0(p)\Gamma_1}$. Using the equations $\pi_2\Gamma_1 \centerdot \tau_1 = \mathrm{id}_{e_1}$ and $\partial_0(p)\Gamma_1 \centerdot \tau_1 = p \centerdot \delta_E$, we get the following equations.

$$\pi_2 \centerdot \tau_1(p) \centerdot \Gamma_1 = \mathrm{id}_{\pi_2\Gamma_1} = \mathrm{id}_{e_1} = \pi_2 \centerdot \Gamma_1 \centerdot \tau_1$$
$$\partial_0(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \delta_p \centerdot \Gamma_1 = p \centerdot \delta_E = \partial_0(p)\Gamma_1 \centerdot \tau_1$$

Hence, by the 2-dimensional universal property of $(B \downarrow p)$ we obtain $\tau_1(p) \centerdot \Gamma_1 = \Gamma_1 \centerdot \tau_1$. The last equation follows from the penultimate one and the first one:

$$(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0) = \Gamma_1 \centerdot (\tau_1 \circ \tau_0) = \Gamma_1 \centerdot i_E \centerdot \delta_E = i(p) \centerdot \delta_E$$

□

PROPOSITION 2.4.19. Given morphism $\Gamma_1 \colon (E \downarrow E) \to (B \downarrow p)$ as defined before, we have a bijection

$$\left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } \partial_0 \text{ at } p \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Chevalley adjunctions} \\ \Gamma_1 \dashv \Lambda_1 \end{array} \right\}$$

Moreover, the pseudo algebra is normal (i.e. ζ is identity.) if and only if the counit $\varepsilon \colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1_{(B\downarrow p)}$ is the identity 2-morphism.

A major part of the proof we are about to give is present in [Str74] in a much denser form. However the last statement of the proposition and its proof is new.

Proof. Given a pseudo algebra $\langle \mathfrak{a} \colon \partial_0(p) \to p, \theta, \zeta \rangle$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Note that the unit $\tau_1(p)$ of adjunction $\pi_2 \dashv i(p)$ defines a unique morphism $k \colon (B \downarrow p) \to$

$\mathbf{2} \pitchfork (B \downarrow p)$ with $d'_0 k = 1_{(B\downarrow p)}$ and $d'_1 k = i(p)\pi_2$, and $\delta' \centerdot k = \tau_1(p)$. Define $\Lambda_1 \colon = (\mathfrak{a} \downarrow \mathfrak{a}) \circ k$.

$$\begin{tikzcd}
 & \mathbf{2} \pitchfork (B \downarrow p) \arrow[dd, dotted, "(\mathfrak{a}\downarrow\mathfrak{a})"] \arrow[dr, dotted, "d'_i"] \arrow[ddr, dotted] & \\
(B \downarrow p) \arrow[ur, "k"] \arrow[rr, "\pi_2"] \arrow[dr, "\Lambda_1"'] \arrow[dd, "\Gamma_1\Lambda_1"'] & & E \arrow[d, "i(p)"] \\
 & (E \downarrow E) \arrow[dl, "\Gamma_1"'] \arrow[dr, "e_i"'] & (B \downarrow p) \arrow[d, "\mathfrak{a}"] \\
(B \downarrow p) \arrow[rr, "\pi_2"'] & & E
\end{tikzcd} \qquad (i = 1, 2)$$

We note that

$$\begin{aligned}
e_0\Lambda_1 &= e_0(\mathfrak{a} \downarrow \mathfrak{a})k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a} d'_0 k && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\
&= \mathfrak{a} && \{\text{definition of } k\}
\end{aligned} \tag{2.32}$$

This establishes that Λ_1 is indeed a morphism in $\mathfrak{K}/B$ from pe_0 to $\partial_0(p)$, since $pe_0\Lambda_1 = p\mathfrak{a} = \partial_0(p)$. Also, a diagram chase shows that the front square in the diagram above commutes:

$$\begin{aligned}
\pi_2\Gamma_1\Lambda_1 &= e_1\Lambda_1 && \{\text{definition of } \Gamma_1\} \\
&= e_1(\mathfrak{a} \downarrow \mathfrak{a})k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a} d'_1 k && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\
&= \mathfrak{a} i(p)\pi_2 && \{\text{definition of } k\}
\end{aligned} \tag{2.33}$$

We also note that

$$\begin{gathered}
\partial_0(p)\Gamma_1\Lambda_1 = d_0(p \downarrow p)\Lambda_1 = pe_0\Lambda_1 = p\mathfrak{a} = \partial_0(p) \\
\delta_p \centerdot (\Gamma_1\Lambda_1) = p \centerdot \delta_E \centerdot \Lambda_1 = p\mathfrak{a} \centerdot \delta'_{(B\downarrow p)} \centerdot k = p\mathfrak{a} \centerdot \tau_1(p) = \partial_0(p) \centerdot \tau_1(p) = \delta_p
\end{gathered} \tag{2.34}$$

Equations (2.33) and (2.34), and the definition of $\partial_0(\mathfrak{a} i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = \partial_0(\mathfrak{a} i(p)) = \partial_0(\mathfrak{a}) \circ \partial_0(i(p))$$

and we propose the counit $\varepsilon\colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ to be given by $\partial_0(\zeta^{-1})$ which is invertible.[12] This guarantees that the counit lives in $\mathfrak{K}/B$ since $p\pi_2 \centerdot \varepsilon = p\pi_2 \centerdot \partial_0(\zeta^{-1}) = p \centerdot \zeta^{-1} \centerdot \pi_2 = \mathrm{id}_{p\pi_2}$, and $\partial_0(p) \centerdot \varepsilon = \partial_0(p) \centerdot \partial_0(\zeta^{-1}) = \mathrm{id}_{\partial_0(p)}$. Moreover, the definition of $\partial_0(\zeta)$ implies that $\delta_p \centerdot \varepsilon = \delta_p$. Now, we propose the unit; define the 2-morphism $\eta\colon 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-morphism with

$$\begin{gathered} e_0 \centerdot \eta = (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) \\ e_1 \centerdot \eta = \zeta \centerdot e_1 \end{gathered} \tag{2.35}$$

Note that the vertical composition of 2-morphisms in (2.35) is possible since $\mathfrak{a}i(p)e_0 = \mathfrak{a}\Gamma_1 i_E e_0$ which holds in virtue of Lemma 2.4.18. Of course in order for equations above to define the a 2-morphism η at all, $e_0 \centerdot \eta$ and $e_1 \centerdot \eta$ must be compatible. The compatibility is checked in below.

$$\begin{aligned}
(\delta_E \centerdot \Lambda_1\Gamma_1) \circ (e_0 \centerdot \eta) &= (\delta_E \centerdot (\mathfrak{a} \downarrow \mathfrak{a})k\Gamma_1) \circ (e_0 \centerdot \eta) && \{\text{definition of } \Lambda_1\} \\
&= (\mathfrak{a}\delta'_{(B\downarrow p)} \centerdot k\Gamma_1) \circ (e_0 \centerdot \eta) && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (e_0\eta) && \{\text{definition of } k\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) && \{\text{substituting } e_0 \centerdot \eta\} \\
&= \mathfrak{a}((\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)) \circ (\zeta \centerdot e_0) && \{\text{factoring out } \mathfrak{a}\} \\
&= (\mathfrak{a}i(p) \centerdot \delta_E) \circ (\zeta \centerdot e_0) && \{\text{Lemma 2.4.18}\} \\
&= (\zeta \centerdot e_1) \circ \delta_E && \{\text{exchange rule}\} \\
&= (e_1 \centerdot \eta) \circ \delta_E && \{\text{substituting } e_1 \centerdot \eta\}
\end{aligned}$$

Perhaps, it is illuminating to see what the unit η, constructed in above, looks like in the case of $\mathfrak{K} = \mathfrak{Cat}$. Indeed, for a morphism $f\colon e_0 \to e_1$ in $(E \downarrow E)$, $\eta(f)$ is given as follows:

$$\begin{tikzcd}
e_0 \arrow[r, "\zeta e_0(f)"] \arrow[d, "f"'] & \mathfrak{a}\langle e_0\rangle \arrow[r, "\mathfrak{a}\Gamma_1\tau_0(f)"] & \mathfrak{a}\langle p(f), e_1\rangle \arrow[d, "\Lambda_1\Gamma_1(f)"] \\
e_1 \arrow[rr, "\zeta e_1(f)"'] & & \mathfrak{a}\langle e_1\rangle
\end{tikzcd}$$

---

[12] When $\mathfrak{K} = \mathfrak{Cat}$, $\partial_0(\zeta)$ is illustrated in diagram (2.31).

Here is a proof that the unit η and counit ε satisfy triangle equations of adjunction. We first show that $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1 = \partial_0(\mathfrak{a}) \centerdot \lambda_p$, expressed diagrammatically as

$$\begin{tikzcd}
 & & E \arrow[dr, "i_E"] & & & & & & \\
(B \downarrow p) \arrow[r, "\Lambda_1"] & (E \downarrow E) \arrow[ur, "e_0"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\tau_0\Downarrow"] & & (E \downarrow E) \arrow[r, "\Gamma_1"] & (B \downarrow p) & = & \partial_0(p) \arrow[r, bend left, "i_{\partial_0(p)}"] \arrow[r, bend right, "\partial_0(i(p))"'] \arrow[r, phantom, "\lambda_p\Downarrow"] & \partial_0^2(p) \arrow[r, "\partial_0(\mathfrak{a})"] & \partial_0(p)
\end{tikzcd}$$

First we verify that the domain and codomain of the involved 2-morphisms match. Indeed, $\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = \partial_0(\mathfrak{a}) i(\partial_0(p))$, and as we observed earlier $\Gamma_1 \circ \Lambda_1 = \partial_0(\mathfrak{a}) \partial_0(i(p))$. Now, using Lemma 2.4.18, observe that

$$\pi_2 \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = \delta_E \centerdot \Lambda_1 = \mathfrak{a}\tau_1(p) = \mathfrak{a}\bar{\pi}_2 \centerdot \lambda_p = \pi_2 \centerdot \partial_0(\mathfrak{a}) \centerdot \lambda_p$$

$$\partial_0(p) \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = \mathrm{id}_{pe_0} \centerdot \Lambda_1 = \mathrm{id} = \partial_0^2(p) \centerdot \lambda_p = \partial_0(p)\partial_0(\mathfrak{a}) \centerdot \lambda_p$$

To prove the first identity, we notice that

$$\partial_0(p) \centerdot [(\varepsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = [\partial_0(p) \centerdot (\varepsilon \centerdot \Gamma_1)] \circ [\partial_0(p) \centerdot (\Gamma_1 \centerdot \eta)] = (\mathrm{id}_{\partial_0(p)} \Gamma_1) \circ (pe_0 \centerdot \eta) = \mathrm{id}_{\partial_0(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0 \centerdot \eta = \mathrm{id}_{pe_0} = \mathrm{id}_{R(p)\Gamma_1}$. Similarly, we have

$$\pi_2 \centerdot [(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = (\zeta^{-1} \centerdot \pi_2 \Gamma_1) \circ (e_1 \centerdot \eta) = (\zeta^{-1} \centerdot e_1) \circ (\zeta \centerdot e_1) = \mathrm{id}_{\pi_2 \Gamma_1}$$

Therefore, $(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta) = \mathrm{id}_{\Gamma_1}$. To prove the second identity, $(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1) = \mathrm{id}_{\Lambda_1}$, we first prove the following lemma: Using lemma above we have,

$$\begin{aligned}
e_0 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] &= (\mathfrak{a} \centerdot \epsilon) \circ ((\mathfrak{a}\Gamma_1\tau_0) \circ (\zeta e_0)) \centerdot \Lambda_1 \\
&= (\mathfrak{a} \centerdot R(\zeta^{-1})) \circ (\mathfrak{a}R(\mathfrak{a}) \centerdot \lambda_p) \circ (\zeta\mathfrak{a}) \\
&= (\zeta^{-1}\mathfrak{a}) \circ (\zeta\mathfrak{a}) \\
&= \mathrm{id}_{e_0\Lambda_1}
\end{aligned}$$

The penultimate equality comes from equality of pasting diagrams 2.26. Similarly, using the fact that $e_1\Lambda_1 = ai(p)\hat{d}_1$, we get

$$e_1 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] = (\mathfrak{a}i(p)\hat{d}_1 \centerdot \epsilon) \circ (\zeta \centerdot e_1\Lambda_1) = (\mathfrak{a}i(p)\zeta^{-1}\hat{d}_1) \circ (\zeta \centerdot \mathfrak{a}i(p)\hat{d}_1) = \mathrm{id}_{e_1\Lambda_1}$$

The last identity is by the exchange law of horizontal-vertical composition of 2-morphisms. From these two equations we deduce the second adjunction identity.

Conversely, suppose we are given a Chevalley adjunction, that is to say a right adjunction Λ_1 of Γ_1 over B:

$$\begin{tikzcd}
(E \downarrow E) \arrow[loop above, "\eta"] \arrow[rr, bend left, "\Gamma_1"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pe_0"'] & & (B \downarrow p) \arrow[loop above, "\varepsilon"] \arrow[ll, bend left, "\Lambda_1"] \arrow[dl, "\partial_0(p)"] \\
& B &
\end{tikzcd} \tag{2.36}$$

such that the counit ε is an isomorphism, $\partial_0(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = \partial_0(p)$, $\partial_0(p) \centerdot \varepsilon = \mathrm{id}_{\partial_0(p)}$, and $pe_0 \centerdot \eta = \mathrm{id}_{pe_0}$. We define the pseudo-algebra $\mathfrak{a} \colon (B \downarrow p) \to E$ as the composite $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = \partial_0(p)\Gamma_1\Lambda_1 = \partial_0(p)$. We propose $e_1 \centerdot \eta \centerdot i_E$ for $\zeta \colon 1 \Rightarrow ai(p)$. First we prove that $\eta \centerdot i_E$ is invertible and thence ζ is invertible. We have the following pasting equality[13]:

$$\begin{tikzcd}
(E \downarrow E) \arrow[r, "1"] \arrow[dr, "\Gamma_1"'] \arrow[r, phantom, shift right=4ex, "\eta\Downarrow"] & (E \downarrow E) \arrow[rr, "1"] \arrow[dr, "\Gamma_1"] \arrow[drr, "e_1"] \arrow[rr, phantom, shift right=4ex, "\tau_1\Downarrow"] & & (E \downarrow E) \\
E \arrow[u, "i_E"] \arrow[r, "i(p)"'] & (B \downarrow p) \arrow[u, "\Lambda_1"] \arrow[r, "1"'] \arrow[r, phantom, shift left=3ex, "\varepsilon\Downarrow"] & (B \downarrow p) \arrow[r, "\pi_2"'] & E \arrow[u, "i_E"']
\end{tikzcd}
=
\begin{tikzcd}
(E \downarrow E) \arrow[r, "1"] & (E \downarrow E) \\
E \arrow[u, "i_E"] \arrow[r, "1"'] \arrow[r, phantom, shift left=4ex, "\mathrm{id}\Downarrow"] & E \arrow[u, "i_E"']
\end{tikzcd}$$

---

[13] This equality in fact lies over B. Also, all of triangles and squares without a designated 2-morphism commute.

$$\begin{tikzcd}
E \arrow[r, "\Lambda_1\Gamma_1 i_E"] \arrow[d, "i(p)"'] & (E\downarrow E) \arrow[r, bend left, "1"] \arrow[r, bend right, "i_E e_1"'] \arrow[r, phantom, "\Downarrow\tau_1"] \arrow[d, "\Gamma_1"] & (E\downarrow E) \arrow[d, "\Gamma_1"] \arrow[dr, "1"] \arrow[d, phantom, shift left=4ex, "\swarrow\eta"] & \\
(B\downarrow p) \arrow[r, "1"'] \arrow[ur, "\Lambda_1"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\varepsilon"] & (B\downarrow p) \arrow[r, dotted, bend left, "1"] \arrow[r, bend right, "i(p)\pi_2"'] \arrow[r, phantom, "\Downarrow\tau(p)"] & (B\downarrow p) \arrow[r, "\Lambda_1"'] & (E\downarrow E)
\end{tikzcd}
=
\begin{tikzcd}
E \arrow[d, bend right, "\Lambda_1 i(p)"'] \arrow[d, bend left, "\Lambda_1 i(p)"] \arrow[d, phantom, "="] \\
(E\downarrow E)
\end{tikzcd}$$

The first pasting equality is deduced from the adjunction triangle equalities and the second one is deduced from the Lemma 2.4.18. Therefore,

$$(i_E\pi_2\varepsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E) \circ (\eta \centerdot i_E) = \mathrm{id}_{i_E}$$

$$(\eta \centerdot i_E) \circ (i_E\pi_2 \centerdot \varepsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E) = \mathrm{id}_{\Gamma_1 i(p)}$$

This proves that $\eta \centerdot i_E$ is indeed an iso 2-morphism. To be more explicit, whiskering with e_1 unveils the inverse of ζ:

$$\zeta^{-1} = (e_1 i_E\pi_2 \centerdot \varepsilon \centerdot i(p)) \circ (e_1 \centerdot \tau_1 \centerdot \Lambda_1\Gamma_1 i_E) = \pi_2 \centerdot \varepsilon \centerdot i(p)$$

Indeed, ζ^{-1} is the counit of composite adjunction in below:

$$\begin{tikzcd}
E \arrow[r, bend left, "i_E"] \arrow[r, phantom, "\perp"] & (E\downarrow E) \arrow[l, bend left, "e_0"] \arrow[r, bend left, "\Gamma_1"] \arrow[r, phantom, "\perp"] & (B\downarrow p) \arrow[l, bend left, "\Lambda_1"] \arrow[r, bend left, "\pi_2"] \arrow[r, phantom, "\perp"] & E \arrow[l, bend left, "i(p)"]
\end{tikzcd}$$

It is straightforward to show that ζ^{-1} satisfies the pasting equality of diagram (2.26). So, Lemma 2.4.12 completes the proof. □

REMARK 2.4.20. Notice that we have proved that $\zeta = e_1 \centerdot \eta \centerdot i_E$ is invertible regardless of invertibility of ε.

EXAMPLE 2.4.21. We now return to prove our promise at the end of Example 2.4.16. We would like to show that $\tilde{f}$, obtained by whiskering π_2 with counit of $i(P) \dashv \mathfrak{a}$, is indeed cartesian. Here, we appeal to the bijection

$$\mathrm{Hom}_{(\mathcal{B}\downarrow P)}(\Gamma_1(g), \langle f, e_1\rangle) \cong \mathrm{Hom}_{(\mathcal{E}\downarrow\mathcal{E})}(g, \Lambda_1\langle f, e_1\rangle)$$

natural in $g\colon d_0 \to d_1$ in $(\mathcal{E} \downarrow \mathcal{E})$ and $\langle e_1, f\rangle$ in $(\mathcal{B} \downarrow P)$. This bijection states that any diagram of the form on the left hand side, where the square in base commutes and u_1 lies above h_1, can be extended to the diagram on the right hand side via a unique morphism $\widetilde{h_0}$.

$$\begin{tikzcd}
 & & d_1 \arrow[dr, "u_1"] \arrow[dd, mapsto, "P"] & & & d_0 \arrow[rr, "g"] \arrow[dr, dashed, "\widetilde{h_0}"'] \arrow[dd, mapsto] & & d_1 \arrow[dr] \arrow[drr, "u_1"] \arrow[dd, dotted, mapsto] & & \\
 & & & e_1 \arrow[dd, mapsto, "P"] & & & \mathfrak{a}\langle e_1, f\rangle \arrow[rr] \arrow[dd, mapsto] & & \mathfrak{a}\langle e_1\rangle \arrow[r, "\zeta_{e_1}^{-1}"'] \arrow[dd, mapsto] & e_1 \arrow[dd, mapsto] \\
P(d_0) \arrow[rr, "P(g)"] \arrow[dr, "h_0"'] & & P(d_1) \arrow[dr, "h_1"] & & \rightsquigarrow & P(d_0) \arrow[rr, dotted] \arrow[dr, "h_0"'] & & P(d_1) \arrow[dr] \arrow[drr, dotted] & & \\
 & b_0 \arrow[rr, "f"] & & b_1 & & & b_0 \arrow[rr, "f"] & & b_1 \arrow[r, equal] & b_1
\end{tikzcd}$$

Taking g to be identity we obtain the usual condition which expresses cartesian property of lift $\tilde{f}$. Also, one can easily show that unique morphism $\widetilde{h_0}$ over h_0 is calculated by the expression $(e_0\Lambda_1\langle h_0, h_1, k\rangle) \circ (\mathfrak{a}\Gamma_1\tau_0(g)) \circ (\zeta e_0(g))$.

We have the following bijections:

$$\left\{\begin{array}{c}\text{cleavages}\\ \text{of } p\end{array}\right\} \cong \left\{\begin{array}{c}\text{pseudo algebras}\\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p\end{array}\right\} \cong \left\{\begin{array}{c}\text{right adjoints of } \Gamma_1\\ \text{with isomorphism counit}\end{array}\right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

CONSTRUCTION 2.4.22. The situation in $\mathfrak{Cat}$ can be encapsulated as follows: The forgetful 2-functor $U\colon \mathrm{clv}\mathfrak{Fib}(\mathcal{B}) \to \mathfrak{Cat}/\mathcal{B}$ is 2-monadic: the **free fibration** of a functor $P\colon \mathcal{E} \to \mathcal{B}$ is the fibration $\partial_0(p)\colon (B \downarrow p) \to B$. In general, a cleavage (aka fibration structure) on P is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad $\partial_0 = UF$. Strict algebra structures of ∂_0 correspond to normal splitting fibration structures on P.

$$\begin{tikzcd}
\mathrm{clv}\mathfrak{Fib}(\mathcal{B}) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
\mathfrak{Cat}/\mathcal{B} \arrow[u, bend left, "F"] \arrow[loop right, "\partial_0"]
\end{tikzcd}$$

We also note that for a category $\mathcal{B}$ the domain functor $\mathrm{dom}\colon (\mathcal{B} \downarrow \mathcal{B}) \to \mathcal{B}$ is the free Grothendieck fibration on identity functor $1\colon \mathcal{B} \to \mathcal{B}$, that is $\mathrm{dom} =$

$\partial_0(\mathrm{Id})$. The situation above generalizes 1-categorical case where for an $\mathcal{S}$-internal category $\mathbb{C} = (C_1 \rightrightarrows C_0)$, the forgetful functor $d\mathcal{F}\mathrm{ib}(\mathbb{C}) \to \mathcal{S}/C_0$ taking $(\gamma\colon X \to C_0, \alpha\colon X\,{}_\gamma{\times}_{d_1}\, C_1 \to X)$ to γ is monadic and the category of discrete fibrations is the category of algebras for the corresponding monad.

2.5 Fibrational objects for 2-functors

Our discussion of the Johnstone criterion in §2.6 will involve a use of cartesian morphisms and 2-morphisms for a 2-functor, and the present section discusses those. It is important to note that, although our applications are for 2-functors between 2-categories, the definitions we use are the ones appropriate to bicategories.

[Her99] generalizes the notion of fibration to strict 2-functors between strict 2-categories. His archetypal example of strict 2-fibration is the 2-category $\mathfrak{Fib}$ of Grothendieck fibrations, fibred over the 2-category $\mathfrak{Cat}$ of categories via the codomain functor $\mathbf{cod}\colon \mathfrak{Fib} \to \mathfrak{Cat}$. This result can be generalized to a 2-fibration $\mathbf{cod}\colon \mathfrak{Fib}(\mathfrak{K}) \to \mathfrak{K}$ where $\mathfrak{K}$ is a 2-category and $\mathfrak{Fib}(\mathfrak{K})$ is the 2-category of internal Chevalley-style fibrations in $\mathfrak{K}$. Later [Bak12] in his talk, and [Buc14] in his paper developed these ideas to define fibration of bicategories. Bakovic even defined a notion of fibration internal to general tricategories and proved that fibrations of bicategories are the internal fibrations in the tricategory $\mathbf{Hom}$.

Borrowing the notions of cartesian 1-morphisms and 2-morphisms from their work, we reformulate *Johnstone (op)fibrations* in terms of existence of cartesian lifts of 1-morphisms and 2-morphisms with respect to the codomain 2-functor. This reformulation will be essential in giving a concise proof of our main result in Chapter 4. Johnstone's definition is quite involved and this reformulation effectively organizes the data of certain iso 2-morphisms as part of structure of 1-morphisms in the 2-category $\mathcal{G}\mathfrak{Top}$ of "Grothendieck toposes over varying base". This approach simultaneously makes it fairly painless to mix bounded and unbounded geometric morphisms. It uses the 2-functor $\mathbf{cod}$ to $\mathcal{E}\mathfrak{Top}$ (§1.6), so that the fibre $\mathcal{G}\mathfrak{Top}(\mathscr{S})$ is equivalent to ${}^{\mathcal{B}\mathfrak{Top}}/_{\mathscr{S}}$. Our formulation uses the cartesian 1-morphisms and 2-morphisms for this 2-functor, and we review the theory of those, in its bicategorical form.

We introduced the display 2-category $\mathfrak{K}_{\mathfrak{D}}$ and its ‘upstairs-downstairs’ notation. In this chapter we shall denote a chosen bipullback of a bicarrable morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ by

$$\begin{tikzcd}
\underline{f}^*\overline{x} \arrow[r, "\overline{f}"] \arrow[d, "\underline{f}^*x"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

where the 2-morphism $\overset{\blacktriangledown}{f}$ is an iso 2-morphism.

DEFINITION 2.5.1. Suppose $P\colon \mathfrak{X} \to \mathfrak{B}$ is a 2-functor.

(i) A 1-morphism $f\colon y \to x$ in $\mathfrak{X}$ is **cartesian** with respect to P whenever for each object w in $\mathfrak{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{tikzcd}
\mathfrak{X}(w,y) \arrow[r, "f_*"] \arrow[d, "P_{w,y}"'] \arrow[dr, phantom, "\cong"] & \mathfrak{X}(w,x) \arrow[d, "P_{w,x}"] \\
\mathfrak{B}(Pw,Py) \arrow[r, "P(f)_*"'] & \mathfrak{B}(Pw,Px)
\end{tikzcd}$$

This amounts to requiring that, for every object w, the functor

$$\langle P_{w,y}, f_* \rangle : \mathfrak{X}(w,y) \to P(f)_* \downarrow_{\cong} P_{w,x}$$

should be an equivalence of categories, where the category on the right is the isocomma. (Note that the image of $\mathfrak{X}(w,y)$ has identities in the squares, not isos.)

(ii) A 2-morphism $\alpha\colon f \Rightarrow g\colon y \to x$ in $\mathfrak{X}$ is **cartesian** if it is cartesian as a 1-morphism with respect to the functor $P_{yx}\colon \mathfrak{X}(y,x) \to \mathfrak{B}(Py,Px)$.

The following lemma, which proves certain immediate results about cartesian 1-morphisms and 2-morphisms, will be handy in the proof of Proposition 2.6.10. The statements are similar to the case of 1-categorical cartesian morphisms (e.g. in the definition of Grothendieck fibrations) with the appropriate weakening of equalities by isomorphisms and isomorphisms by equivalences. They follow straightforwardly from the definition above, however for more details see [Buc14]. In what follows, in keeping with the nomenclature of 2.3.1, we regard *vertical* 1-morphisms (resp. vertical 2-morphisms) as those 1-morphisms (resp. 2-morphisms) in $\mathfrak{X}$ which are mapped to identity 1-morphisms (resp. 2-morphisms) in $\mathfrak{B}$ under P.

LEMMA 2.5.2. Suppose $P\colon \mathfrak{X} \to \mathfrak{B}$ is a 2-functor between 2-categories.

(i) Cartesian 1-morphisms (with respect to P) are closed under composition and cartesian 2-morphisms are closed under vertical composition.

(ii) Suppose $k\colon w \to y$ and $f\colon y \to x$ are 1-morphisms in $\mathfrak{X}$. If f and fk are cartesian then k is cartesian. The same is true with 2-morphisms and their vertical composition.

(iii) Identity 1-morphisms and identity 2-morphisms are cartesian.

(iv) Any equivalence 1-morphism is cartesian.

(v) Any iso 2-morphism is cartesian.

(vi) Any vertical cartesian 2-morphism is an iso 2-morphism.

(vii) Cartesian 1-morphisms are closed under isomorphisms: if $f \cong g$ then f is cartesian if and only if g is cartesian.

REMARK 2.5.3. We unwind the essential surjectivity and fully faithfulness conditions on the functor $\langle P_{w,y}, f_* \rangle$ in the definition above to give a more explicit and elementary description of cartesian 1-morphisms. A 1-morphism $f\colon y \to x$ in $\mathfrak{X}$ is P-cartesian if and only if the following conditions hold.

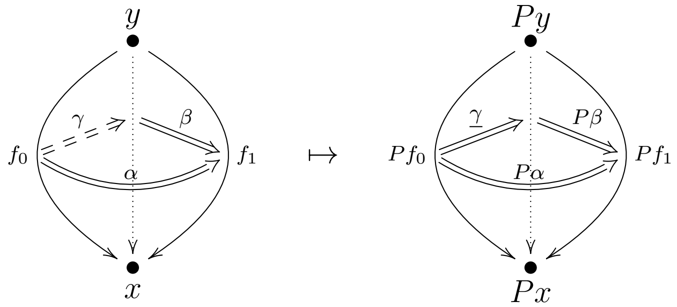
(i) For any 1-morphisms $g\colon w \to x$ and $\underline{h}\colon P(w) \to P(y)$ and any iso-2-morphism $\underline{\alpha}\colon Pf \circ \underline{h} \Rightarrow Pg$, there exist a 1-morphism h and iso-2-morphisms $\underline{\beta}\colon P(h) \Rightarrow \underline{h}$ and $\alpha\colon fh \Rightarrow g$ such that $P(\alpha) = \underline{\alpha} \circ (P(f) \mathbin{\centerdot} \underline{\beta})$. In this situation we call $(h, \underline{\beta})$ a **weak lift** of $\underline{h}$. If $\underline{\beta}$ is the identity 2-morphism then we simply call h a **lift** of $\underline{h}$.

(ii) Given 1-morphisms $h, h'\colon w \rightrightarrows y$, and 2-morphisms $\underline{\delta}\colon P(h) \Rightarrow P(h')$ and $\sigma\colon fh \Rightarrow fh'$ satisfying $P(f) \mathbin{\centerdot} \underline{\delta} = P(\sigma)$, there exists a unique 2-morphism $\delta\colon h \Rightarrow h'$ such that $f \mathbin{\centerdot} \delta = \sigma$ and $P(\delta) = \underline{\delta}$.

$$\begin{tikzcd}
w \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "g"] & \arrow[d, phantom, "\Uparrow \alpha"] & \\
& y \arrow[r, "f"'] \arrow[dd, mapsto, "P"'] & x \\
& & \\
Pw \arrow[dr, "\underline{h}"] \arrow[dr, bend right=60, "Ph"'] \arrow[drr, bend left, "Pg"] & \arrow[d, phantom, "\Uparrow \underline{\alpha}"] & \\
& Py \arrow[r, "Pf"'] & Px
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[dr, bend right, "h"'] \arrow[dr, bend left, "h'"] \arrow[drr, bend left, "fh"] \arrow[drr, bend left=70, "fh'"] \arrow[dr, phantom, "\delta \Rightarrow"] & \arrow[r, phantom, "\Uparrow \sigma \quad \Uparrow \alpha'"] & \\
& y \arrow[r, "f"'] \arrow[dd, mapsto, "P"'] & x \\
& & \\
Pw \arrow[dr, bend right, "P(h)"'] \arrow[dr, bend left, "P(h')"] \arrow[drr, bend left, "P(fh)"] \arrow[drr, bend left=70, "P(fh')"] \arrow[dr, phantom, "\underline{\delta} \Rightarrow"] & \arrow[r, phantom, "\Uparrow P\sigma"] & \\
& Py \arrow[r, "Pf"'] & Px
\end{tikzcd}$$

(2.37)

Also, in elementary terms, a 2-morphism $\alpha\colon f_0 \Rightarrow f_1\colon y \rightrightarrows x$ is cartesian iff for any given 1-morphism $e\colon y \to x$ and any 2-morphisms $\beta\colon e \Rightarrow f_1$ and $\underline{\gamma}\colon P(f_0) \Rightarrow P(e)$ with $P(\alpha) = P(\beta) \circ \underline{\gamma}$, there exists a unique 2-morphism γ over $\underline{\gamma}$ such that $\alpha = \beta \circ \gamma$.



REMARK 2.5.4. Definition 2.5.1 may at first sight seem a bit daunting. Nonetheless the idea behind it is simple; We often think of $\mathfrak{X}$ as bicategory over $\mathfrak{B}$ with richer structures (in practice often as a fibred bicategory). In this situation, $f\colon y \to x$ being cartesian in means that we can reduce the problem of lifting of any 1-morphism g (with same codomain as f) along f (up to an iso 2-morphism) to the problem of lifting of $P(g)$ along $P(f)$ in $\mathfrak{B}$ (up to an iso 2-morphism). The latter is easier to verify since $\mathfrak{B}$ is a poorer category than $\mathfrak{X}$. The second part of definition says that we also have the lifting of 2-morphisms along f and the lifted 2-morphisms are coherent with iso-2-morphisms of lifting structure. This implies the solution to the lifting problem is unique up to a (unique) coherent iso 2-morphism.

REMARK 2.5.5. Note that $f\colon y \to x$ being P-cartesian for a 2-functor P does not imply f is cartesian with respect to the underlying functor $||P||_1$ of P, since the lifts in the 2-category $\mathfrak{X}$ exists only up to an iso 2-morphisms. However f is cartesian in the classifying category of $\mathfrak{X}$ (Construction 1.4.4).

DEFINITION 2.5.6. Let $P\colon \mathfrak{X} \to \mathfrak{B}$ be a 2-functor. We define an object e of $\mathfrak{X}$ to be **fibrational** iff

$(B1)$ every $f\colon b' \to b = P(e)$ has a cartesian lift,

$(B2)$ for every object e' in $\mathfrak{X}$, the functor

$$P_{e',e}\colon \mathfrak{X}(e', e) \to \mathfrak{B}(P(e'), P(e))$$

is a Grothendieck fibration of categories, and

$(B3)$ cartesian 2-morphisms in $\mathfrak{X}$ between morphisms with common codomain e are closed under whiskering on the left with any morphism.

P is a **2-fibration** if every object of $\mathfrak{X}$ is fibrational. It is also noteworthy that conditions $(B2)$ and $(B3)$ together make the 2-functor $P_{-,e} \colon \mathfrak{X}^{\mathrm{op}} \to (\mathfrak{Cat} \downarrow \mathfrak{Cat})$ lift to $P_{-,e} \colon \mathfrak{X}^{\mathrm{op}} \to \mathfrak{Fib}$ for every object e of $\mathfrak{X}$.

REMARK 2.5.7. Our definition of fibration of bicategories differs from [Buc14, Definition 3.1.5] in only one criterion: the latter requires the whiskering *on both sides* to preserve cartesian 2-morphisms. The main motivation behind this is to achieve Grothendieck construction on bicategories. Since in this chapter and the rest of this thesis we have no use of such construction we only suffice to the weaker version of our definition. Incidentally, our weaker condition also appears in [Her99] which is arguably the first time a definition for the concept of 2-fibration[14] was ever proposed.

PROPOSITION 2.5.8. A morphism in $\mathfrak{K}_{\mathfrak{D}}$ is $\mathbf{cod}$-cartesian if and only if it is a bipullback square in $\mathfrak{K}$.

$$\begin{tikzcd}
\overline{y} \arrow[rr, dashed, "\overline{f}"] \arrow[d, dashed, "y"'] \arrow[drr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] \arrow[dr, phantom, very near start, "\lrcorner"] & & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x} \\
& {} \arrow[d, mapsto, "\mathbf{cod}"'] & \\
& {} & \\
\underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x}
\end{tikzcd} \tag{2.38}$$

Before giving the proof there is one step we take to simplify the proof.

LEMMA 2.5.9. Suppose $\underline{h} \colon \underline{w} \to \underline{y}$ is a morphism in $\mathfrak{K}$. Any weak lift $(h_0, \underline{\beta})$ of $\underline{h}$ w.r.t. $\mathbf{cod}$ can be replaced by a lift h in which $\underline{\beta}$ is replaced by the identity 2-morphism. Therefore, conditions (i) and (ii) in Remark 2.5.3 can be rephrased to simpler conditions in which $\underline{\beta}$ is the identity 2-morphism.

---

[14] Although a strict definition unlike our case!

Proof. Define $\overline{h} = \overline{h}_0$, and $\overset{\blacktriangledown}{h} = (\underline{\beta} \mathbin{\scriptscriptstyle\blacksquare} w) \circ \overset{\blacktriangledown}{h}_0$:

$$\begin{tikzcd}
\overline{w} \arrow[r, "\overline{h}_0"] \arrow[d, "w"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}_0 \Downarrow"] & \overline{y} \arrow[d, "y"] \\
\underline{w} \arrow[r, bend left, "\underline{h}_0"] \arrow[r, bend right=60, "\underline{h}"'] \arrow[r, phantom, yshift=-3ex, "\underline{\beta}\Downarrow"] & \underline{y}
\end{tikzcd}$$

Then $h = \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is indeed a lift of $\underline{h}$. Moreover, if α_0 is a lift of 2-morphism $\underline{\alpha} \colon \underline{f} \circ \underline{h} \Rightarrow \underline{g}$ as in part (i) of Remark 2.5.3, then obviously $\underline{\alpha}_0 = \underline{\alpha} \circ (\underline{f} \mathbin{\scriptscriptstyle\blacksquare} \underline{\beta})$, and it follows that $\alpha = (\overline{\alpha}, \underline{\alpha})$ is a 2-morphism in $\mathfrak{K}_{\mathfrak{D}}$ from $f \circ h$ to g which lies over $\underline{\alpha}$. ☐

Proof of Proposition 2.5.8. We first prove the 'only if' part. Suppose that $f \colon y \to x$ is a cartesian 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$. For each object c of $\mathfrak{K}$, let us write $\mathrm{WCone}(c; x, \underline{f})$ for the category of weighted cones (in the pseudo- sense) from c to the opspan $(x, \underline{f})$, in other words pairs of 1-morphisms $\overline{g} \colon c \to \overline{x}$ and $\underline{h} \colon c \to \underline{y}$ as in diagram below, and equipped with an iso 2-morphism $\overset{\blacktriangledown}{g} \colon x \circ \overline{g} \Rightarrow \underline{f} \circ \underline{h}$. We have chosen the notation so that if we define $\underline{g} = \underline{f} \circ \underline{h}$, and if we allow c also to denote the identity on c as object in $\mathfrak{K}_{\mathfrak{D}}$, then $g \colon c \to x$ is a 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$.

Then for each c we have a functor $F_c \colon \mathfrak{K}(c, \overline{y}) \to \mathrm{WCone}(c; x, \underline{f})$, given by $\overline{h} \mapsto (\overline{f} \circ \overline{h}, y \circ \overline{h})$, with the iso 2-morphism got by whiskering $\overset{\blacktriangledown}{f}$, and we must show that each F_c is an equivalence of categories.

First we deal with essential surjectivity. Since f is cartesian we can lift $\underline{h}$ and the identity 2-morphism $\underline{f} \circ \underline{h} = \underline{g}$ to a 1-morphism $h \colon c \to y$ in $\mathfrak{K}_{\mathfrak{D}}$ with isomorphism $\iota = (\overline{\iota}, \mathrm{id})\colon f \circ h \Rightarrow g$, where we have used Lemma 2.5.9 to obtain h as a lift rather than a weak lift.

$$\begin{tikzcd}
c \arrow[r, dashed, "\overline{h}"] \arrow[rr, bend left=60, "\overline{g}", "\overline{\iota}\Uparrow"'] \arrow[d, equal] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"] \arrow[dr, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & \overline{x} \arrow[d, "x"] \\
c \arrow[r, "\underline{h}"'] & \underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

To prove that F_c is full and faithful, take any 1-morphisms $\overline{h}$ and $\overline{h}'$ in $\mathfrak{K}$. In the diagram above we can define $\underline{h} = y \circ \overline{h}$ and $\overset{\blacktriangledown}{h}$ the identity 2-morphism on $\underline{h}$ to get a 1-morphism $h\colon c \to y$ in $\mathfrak{K}_{\mathfrak{D}}$, and similarly $h'\colon c \to y$.

Now suppose we have 2-morphisms $\underline{\delta}\colon y\overline{h} \Rightarrow y\overline{h}'$ and $\overline{\sigma}\colon \overline{fh} \Rightarrow \overline{fh}'$ such that they form a weighted cone over $\underline{f}$ and x, i.e. they satisfy compatibility equation

$$(\underline{f} \mathbin{\centerdot} \underline{\delta}) \circ (\overset{\blacktriangledown}{f} \mathbin{\centerdot} \overline{h}) = (\overset{\blacktriangledown}{f} \mathbin{\centerdot} \overline{h}') \circ (x \mathbin{\centerdot} \overline{\sigma}).$$

If we define $\underline{\sigma} = \underline{f} \mathbin{\centerdot} \underline{\delta}$, then that equation tells us that $\sigma = (\overline{\sigma}, \underline{\sigma})$ is a 2-morphism from fh to fh' in $\mathfrak{K}_{\mathfrak{D}}$. Now the cartesian property tells us that there is a unique $\delta\colon h \to h'$ over $\underline{\delta}$ such that $f \mathbin{\centerdot} \delta = \sigma$, and this gives us the unique $\overline{\delta}\colon \overline{h} \Rightarrow \overline{h}'$ that we require for F_c to be full and faithful.

Conversely, suppose that $\overline{f}$ and y exhibit $\overline{y}$ as the bipullback of $\underline{f}$ and x as illustrated in diagram (2.38). We show that $f\colon y \to x$ is a cartesian 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$, in other words that, for every w, the functor $G_w = \langle P_{w,y}, f_* \rangle$ in Definition 2.5.1 is an equivalence.

To prove essential surjectivity, assume that a 1-morphism $g\colon w \to x$ in $\mathfrak{K}_{\mathfrak{D}}$ is given together with a 1-morphism $\underline{h}\colon \underline{w} \to \underline{y}$ and an iso 2-morphism $\underline{\alpha}\colon \underline{f}\underline{h} \Rightarrow \underline{g}$ in $\mathfrak{K}$.

$$\begin{tikzcd}
\overline{w} \arrow[r, dashed, "\overline{h}"] \arrow[d, "w"'] \arrow[rr, bend left=50, "\overline{g}"] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{w} \arrow[r, "\underline{h}"'] \arrow[rr, bend right=50, "\underline{g}"'] \arrow[rr, phantom, bend right=25, "\underline{\alpha}\Downarrow"] & \underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

The iso 2-morphism $\gamma := (\underline{\alpha}^{-1} \centerdot w) \circ \overset{\blacktriangledown}{g} : x\overline{g} \Rightarrow \underline{g}w \Rightarrow \underline{f}\underline{h}w$ factors through the bipullback 2-morphism with apex $\overline{y}$, and therefore it yields a 1-morphism $\overline{h}\colon \overline{w} \to \overline{y}$ and iso 2-morphisms $\overset{\blacktriangledown}{h} : y \circ \overline{h} \Rightarrow \underline{h} \circ w$ (making a 1-morphism $h\colon w \to y$ in $\mathfrak{K}_{\mathfrak{D}}$) and $\overline{\alpha}\colon \overline{f} \circ \overline{h} \Rightarrow \overline{g}$ such that $\overset{\blacktriangledown}{f}$ and $\overset{\blacktriangledown}{h}$ paste to give $\gamma \circ (x \centerdot \overline{\alpha})$. From this we observe that $h := \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is a lift of $\underline{h}$ and $\alpha := (\overline{\alpha}, \underline{\alpha})$ is an iso 2-morphism from fh to g over $\underline{\alpha}$ as required for cartesianness.

To show that G_w is full and faithful, suppose we have 1-morphisms $h, h'\colon w \to y$. If $\underline{\delta}\colon \underline{h} \Rightarrow \underline{h}'$ and $\sigma\colon fh \Rightarrow fh'$ with $\underline{f} \centerdot \underline{\delta} = \underline{\sigma}$, we must show that there is a unique $\delta\colon h \Rightarrow h'$ over $\underline{\delta}$ with $f \centerdot \delta = \sigma$.

We have 2-morphisms $\overline{\sigma}\colon \overline{fh} \Rightarrow \overline{fh}'$

$$\mu = (\overset{\blacktriangledown}{h}{}'^{-1})(\underline{\delta} \centerdot w)(\overset{\blacktriangledown}{h}) \colon y\overline{h} \Rightarrow \underline{h}w \Rightarrow \underline{h}'w \Rightarrow y\overline{h},$$

and moreover

$$(\overset{\blacktriangledown}{f} \centerdot \overline{h}')(x \centerdot \overline{\sigma}) = (\underline{f} \centerdot \overset{\blacktriangledown}{h}{}'^{-1})(fh')^{\blacktriangledown}(x \centerdot \overline{\sigma}) = (\underline{f} \centerdot \overset{\blacktriangledown}{h}{}'^{-1})(\underline{\sigma} \centerdot x)(fh)^{\blacktriangledown} = (\underline{f} \centerdot \mu)(\overset{\blacktriangledown}{f} \centerdot \overline{h}).$$

It then follows from the bipullback property that we have a unique $\overline{\delta}\colon \overline{h} \Rightarrow \overline{h}'$ such that $y \centerdot \overline{\delta} = \underline{\delta} \centerdot w$ (so we have a 2-morphism $\delta\colon h \Rightarrow h'$ over $\underline{\delta}$) and $\overline{f} \centerdot \overline{\delta} = \overline{\sigma}$, so $f \centerdot \delta = \sigma$ as required. $\square$

2.6 Johnstone-style fibrations refashioned

Another definition of (op)fibration first appeared in [Joh93]; see also [Joh02a, B4.4.1] for more discussion. Johnstone's definition does not require strictness of the 2-category nor the existence of comma objects. Indeed, it is most suitable for weak 2-categories such as various 2-categories of toposes where we do not expect diagrams of 1-morphisms to commute strictly. Moreover, although this definition assumes the existence of bipullbacks, in fact we only need bipullbacks of the class of 1-morphisms one would like to define as (op)fibrations. This enables us to generalize some of Johnstone's results from $\mathcal{B}\mathfrak{Top}$ (where all bipullbacks exist) to $\mathcal{E}\mathfrak{Top}$ (where bounded 1-morphisms are bicarrable).

We have adjusted axiom (i) (lift of identity) in Johnstone's definition so that the (op)fibrations we get have the apposite weak properties. That is to say, unlike Johnstone's definition, we require lift of identity to be isomorphic, rather than equal to identity.

Johnstone's definition is rather complicated, as it has to deal with coherence issues. We have found a somewhat simpler formulation, so we shall first look at that. It is simpler notationally, in that it uses single symbols to describe two levels of structure, "downstairs" and "upstairs" (See Construction 1.4.12). More significantly, it is also simpler structurally in that it doesn't assume canonical bipullbacks and then describe the coherences between them. Instead it borrows from the techniques and results of last section on use of cartesian liftings as bipullbacks. This enables us to show (Proposition 2.6.10) that the Johnstone criterion is equivalent to the fibrational property of Definition 2.5.6.

DEFINITION 2.6.1. Suppose $\mathfrak{K}$ is a 2-category. A 1-morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ is a **Johnstone-style fibration** if the following two conditions hold.

(i) x is bicarrable.

(ii) Any 2-morphism $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}\colon \underline{y} \rightrightarrows \underline{x}$ has a lifting 1-morphism $\overline{r_\alpha}\colon \overline{x}_g \to \overline{x}_f$, and a lifting 2-morphism $\overline{\alpha}\colon \overline{f} \circ \overline{r_\alpha} \Rightarrow \overline{g}$, together with an invertible 2-

morphism $\overset{\blacktriangledown}{r}_\alpha : x_f \circ \overline{r_\alpha} \Rightarrow x_g$, where $x_f \colon \overline{x}_f \to \underline{x}_f = \underline{y}$ and $x_g \colon \overline{x}_g \to \underline{x}_g = \underline{y}$ are respectively bipullbacks of x along $\underline{f}$ and $\underline{g}$.

$$\begin{tikzcd}
& \overline{x}_g \arrow[rr, "\overline{g}"] \arrow[dl, "\overline{r_\alpha}"'] \arrow[dd, "x_g" near start] \arrow[dr, phantom, "\overline{\alpha}\,\Nearrow"] & & \overline{x} \arrow[dl, equal] \arrow[dd, "x"] \\
\overline{x}_f \arrow[rr, "\overline{f}" near end] \arrow[dd, "x_f"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r}_\alpha \Downarrow"] & & \overline{x} \arrow[dd, "x" near start] & \\
& \underline{y} \arrow[rr, "\underline{g}" near start] \arrow[dl, equal] \arrow[dr, phantom, "\underline{\alpha}\,\Nearrow"] & & \underline{x} \arrow[dl, equal] \\
\underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x} &
\end{tikzcd} \qquad (2.39)$$

To proceed further in completing the definition, we first simplify this by taking $\mathcal{D}$ to be the class of all bicarrable 1-morphisms in $\mathfrak{K}$ and working in $\mathfrak{K}_\mathcal{D}$. (We could equally well work with $\mathcal{D}$ any class of display 1-morphisms in $\mathfrak{K}$, as in Construction 1.4.12.) Thus we have cartesian 1-morphisms $f \colon x_f \to x$ and $g \colon x_g \to x$, and a *vertical* 1-morphism $r_\alpha \colon x_g \to x_f$ ($\underline{x}_g = \underline{y} = \underline{x}_f$, and $\underline{r}_\alpha$ is the identity).

The data is subject to the following axioms:

($J1$) $\alpha = (\overline{\alpha}, \underline{\alpha})$ make a 2-morphism in $\mathfrak{K}_\mathcal{D}$ of the form where r_α is vertical and f and g are cartesian.

$$\begin{tikzcd}
& x_g \arrow[rr, "g"] \arrow[dl, "r_\alpha"'] \arrow[dr, phantom, "\alpha\,\Nearrow"] & & x \arrow[dl, equal] \\
x_f \arrow[rr, "f"'] & & x &
\end{tikzcd} \qquad (2.40)$$

($J2$) Suppose we have two composable 2-morphisms $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta} \colon \underline{g} \Rightarrow \underline{h}$ in $\mathfrak{K}$ where $\underline{f}, \underline{g}, \underline{h} \colon \underline{y} \to \underline{x}$; we write $\underline{\gamma} := \underline{\beta} \circ \underline{\alpha}$. Let $\alpha, \beta, \gamma, r_\alpha, r_\beta, r_\gamma$ be as above.

Then there exists a vertical iso 2-morphism $\tau_{\alpha,\beta}\colon r_\alpha \circ r_\beta \cong r_\gamma$ in $\mathfrak{K}_{\mathfrak{D}}$ such that $\beta \circ (\alpha \centerdot r_\beta) = \gamma \circ (f \centerdot \tau_{\alpha,\beta})$.

$$\begin{tikzcd}
 & x_h \arrow[d, "r_\beta"'] \arrow[dd, bend right=70, "r_\gamma"'] \arrow[ddr, bend left=40, "h"] & \\
 \arrow[r, phantom, "\overset{\tau_{\alpha,\beta}}{\Longleftarrow}"] & x_g \arrow[d, "r_\alpha"'] \arrow[dr, "g"] \arrow[r, phantom, "\Nearrow\beta"] & {} \\
 & x_f \arrow[r, "f"'] \arrow[ur, phantom, "\underset{\alpha}{\Nearrow}"] & x
\end{tikzcd}$$

We can phrase this condition by saying that τ provides a vertical iso 2-morphism between the composition of lifts and the lift of composition.

($J3$) For any 1-morphism $\underline{f}\colon \underline{y} \to \underline{x}$ the lift of the identity 2-morphism on $\underline{f}$ is canonically isomorphic to the identity 2-morphism on the lift f via a vertical iso 2-morphism $\tau_f\colon 1_{x_f} \Rightarrow r_{\mathrm{id}_{\underline{f}}}$ in $\mathfrak{K}_{\mathfrak{D}}$ such that $f \centerdot \tau_f^{-1}$ is the lift of identity 2-morphism $\mathrm{id}_{\underline{f}}$.

$$\begin{tikzcd}
 & x_f \arrow[d, "1"] \arrow[d, bend right=70, "r_{\mathrm{id}}"'] \arrow[dr, "f"] & \\
 \arrow[r, phantom, "\overset{\tau_f}{\Longleftarrow}"] & x_f \arrow[r, "f"'] \arrow[ur, phantom, "/\!/"] & x
\end{tikzcd}$$

($J4$) The lift of the whiskering of any 2-morphism $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}\colon \underline{y} \rightrightarrows \underline{x}$ with any 1-morphism $k\colon \underline{z} \to \underline{y}$ is isomorphic, via vertical iso 2-morphisms, to the whiskering of the lifts.

In the following diagram, the right hand square is as usual, f' and g' are cartesian lifts of $\underline{fk}$ and $\underline{gk}$, and the 1-morphisms k_f and k_g are over $\underline{k}$ and the vertical iso 2-morphisms ρ and π are got from cartesianness of f and g. Then the condition is that there should be a vertical iso 2-morphism (over $\underline{k}$) in the left hand square, which pastes with the others to give the lift $\alpha'\colon f'r_{\alpha'} \Rightarrow g'$ of $\underline{\alpha} \centerdot \underline{k}$.

$$\begin{tikzcd}
x_{g'} \arrow[r, "k_g"] \arrow[d, "r_{\alpha'}"'] \arrow[rr, bend left=50, "g'"] \arrow[rr, phantom, bend left=30, "\cong\ \pi"] & x_g \arrow[r, "g"] \arrow[d, "r_\alpha"] \arrow[dr, phantom, "\Uparrow\alpha"] & x \arrow[d, equal] \\
x_{f'} \arrow[r, "k_f"'] \arrow[ur, phantom, "\cong"] \arrow[rr, bend right=50, "f'"'] \arrow[rr, phantom, bend right=30, "\cong\ \rho"] & x_f \arrow[r, "f"'] & x
\end{tikzcd}$$

($J5$) Given any pair of vertical 1-morphisms $v_0\colon y \to x_f$ and $v_1\colon y \to x_g$, any 2-morphism $\alpha_0\colon f \circ v_0 \Rightarrow g \circ v_1$ over $\underline{\alpha}$ factors through α uniquely, that is there exists a unique vertical 2-morphism $\mu\colon v_0 \Rightarrow r_\alpha v_1$ such that the following pasting diagrams are equal.

$$\begin{tikzcd}
y \arrow[r, "v_0"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\Downarrow\alpha_0"] & x_f \arrow[d, "f"] \arrow[r, phantom, "="] & y \arrow[r, "v_0"] \arrow[d, "v_1"'] \arrow[r, phantom, shift right=3, "\mu\Downarrow"] & x_f \arrow[d, "f"] \\
x_g \arrow[r, "g"'] & x & x_g \arrow[r, "g"'] \arrow[ur, "r_\alpha" description] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] & x
\end{tikzcd}$$

REMARK 2.6.2. Dually, **opfibrations** are defined by changing the direction of r_α. For each $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$, we require a 1-morphism $\ell_\alpha\colon x_f \to x_g$ and a 2-morphism $\alpha\colon f \Rightarrow g\ell_\alpha$ with the axioms modified accordingly. The letters ℓ and r used here correspond to Street's 2-monads ∂_1 and ∂_0 in §2.4.2 (In Street's notation they are L and R).

PROPOSITION 2.6.3. A fibration $p\colon E \to B$ is also an opfibration precisely when every 2-morphism $\underline{\alpha}$ induces an adjunction $\ell_\alpha \dashv r_\alpha$. In this situation we call p a **bifibration**.

Proof. The unit and counit of adjunction are respectively obtained by choosing $(1_{x_f}, \ell_\alpha, \alpha)$ and $(r_\alpha, 1_{x_g}, \alpha)$ for (v_0, v_1, α_0) in axiom ($J5$) above. Conversely, given the left adjoints ℓ_α, the opfibration structure of p is exhibited by the composition of 2-morphism $\alpha \bullet \ell_\alpha \colon fr_\alpha\ell_\alpha \Rightarrow g\ell_\alpha$ and $f \bullet \eta_\alpha \colon f \Rightarrow fr_\alpha\ell_\alpha$ for each 2-morphism $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$. □

Both Chevalley-style fibrations and Johnstone-style fibrations can be considered as two flavours of the notion of *representable fibration*. For a morphism $p \colon E \to B$ in a 2-category $\mathfrak{K}$, consider the 2-natural transformation of category-valued representable presheaves $\mathfrak{K}(-,p) \colon \mathfrak{K}(-,E) \Rightarrow \mathfrak{K}(-,B)$. Then we have:

- If $\mathfrak{K}$ has comma objects, then p is a Chevalley-style fibration in $\mathfrak{K}$ iff $\mathfrak{K}(-,p)$ is a Grothendieck fibration (in the sense of Definition 2.3.7), i.e. for any object X of $\mathfrak{K}$, $\mathfrak{K}(X,P)$ is a Grothendieck fibration of categories, and $\mathfrak{K}(X,p) \to$.

- If p is bicarrable in $\mathfrak{K}$, then p is a Johnstone-style fibration in $\mathfrak{K}$ iff $\mathfrak{K}(-,p)$ is a weak fibration (in the sense of Definition 2.3.41).

Now, we describe how Johnstone-style (op)fibrations can be obtained from Chevalley-style (op)fibrations. If $\mathfrak{K}$ has pullbacks of p, then these can be considered the fibres of p. Suppose we have $\alpha \colon g \to f$ between B' and B. Then by the representable definition $\alpha \bullet f^*p$ has a cartesian lift $\alpha' \colon g' \to p^*f$:

$$\begin{tikzcd}
f^*E \arrow[r, bend left=60, "g'"{name=A}] \arrow[r, "p^*f"'{name=B}] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\
B' \arrow[r, bend left=60, "g"{name=C}] \arrow[r, "f"'{name=D}] & B
\arrow[Rightarrow, from=A, to=B, "\alpha'"]
\arrow[Rightarrow, from=C, to=D, "\alpha"]
\end{tikzcd} \tag{2.41}$$

g' now gives us a morphism from f^*E to g^*E, in other words a morphism between the fibres over f and g but in the opposite direction to that of α. This brings us closer to the "indexed category" view of fibrations, with 2-morphisms between base points (f and g) lifting to maps between the fibres (f^*E and g^*E).

CONSTRUCTION 2.6.4. In Propositions 2.3.14 and 2.3.42 we characterized the structures of Grothendieck fibration and Street fibration of categories respectively as the right inverse right adjoint and the fully faithful right adjoint to the induced functors on slice categories. In the construction below, originally due to Johnstone in [Joh93], we obtain the structure of Johnstone fibration $x \colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ as *the* unit semi-oplax right 2-adjoint $\bar{x}^*$ of the 2-functor $\Sigma_x \colon \mathfrak{K} \swarrow \overline{x} \to \mathfrak{K} \swarrow \underline{x}$ (See Construction 1.4.13). The basic idea here is that we consider the 2-morphism $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ of $\mathfrak{K}$ as morphism $\langle 1, \underline{\alpha} \rangle \colon \underline{g} \to \underline{f}$ in the lax slice 2-category $\mathfrak{K} \swarrow \underline{x}$ and $\overline{\alpha} \colon \overline{f} \Rightarrow \overline{g}$ of $\mathfrak{K}$ as morphism $\langle \overline{r_\alpha}, \overline{\alpha} \rangle \colon \overline{g} \to \overline{f}$ in the lax slice 2-category $\mathfrak{K} \swarrow \overline{x}$. Out of the structure of fibration of x we construct a pseudo functor $\bar{x}^* \colon \mathfrak{K} \swarrow \underline{x} \to \mathfrak{K} \swarrow \overline{x}$ which takes object $\underline{f} \colon \underline{a} \to \underline{x}$ to its bipullback along x, i.e. $\overline{f} \colon \overline{x}_f \to \overline{x}$. Moreover, it takes the vertical morphism $\langle 1, \underline{\alpha} \rangle$ to $\langle \overline{r_\alpha}, \overline{\alpha} \rangle$. Using the fact that in lax slices we have factorization of 1-morphisms into vertical followed by strict morphisms, we define $\bar{x}^*$ on general morphisms of $\mathfrak{K} \swarrow \underline{x}$ by the action below:

$$\begin{tikzcd}
\underline{b} \arrow[rr, "\underline{h}"] \arrow[dr, "\underline{g}"'] & \arrow[d, phantom, "\overset{\underline{\alpha}}{\Leftarrow}"] & \underline{a} \arrow[dl, "\underline{f}"] \\
& \underline{x} &
\end{tikzcd}
=
\begin{tikzcd}
\underline{b} \arrow[r, "1"] \arrow[dr, "\underline{g}"'] & \underline{b} \arrow[r, "\underline{h}"] \arrow[d, "\underline{f}\underline{h}"] \arrow[dl, phantom, "\overset{\underline{\alpha}}{\Leftarrow}"] \arrow[dr, phantom, "="] & \underline{a} \arrow[dl, "p"] \\
& \underline{x} &
\end{tikzcd}
\mapsto
\begin{tikzcd}
\overline{x}_g \arrow[r, "\overline{r_\alpha}"] \arrow[dr, "\overline{g}"'] & \overline{x}_{fh} \arrow[r, "\overline{h}"] \arrow[d, "\overline{fh}"] \arrow[dl, phantom, "\overset{\overline{\alpha}}{\Leftarrow}"] \arrow[dr, phantom, "\underset{\overline{\rho}}{\overset{\sim}{\Leftarrow}}"] & \overline{x}_f \arrow[dl, "\overline{f}"] \\
& \overline{x} &
\end{tikzcd}$$

Therefore, $\bar{x}^* \langle \underline{h}, \underline{\alpha} \rangle = \langle \overline{h} \circ \overline{r_\alpha}, \overline{\alpha} \circ (\rho_{\underline{f}, \underline{h}} \centerdot \overline{r_\alpha}) \rangle$. The action of $\bar{x}^*$ on 2-morphisms is slightly more involved: given a 2-morphism $\underline{\beta} \colon \langle \underline{h}, \underline{\alpha} \rangle \Rightarrow \langle \underline{h}', \underline{\alpha}' \rangle$ in the 2-category $\mathfrak{K} \swarrow \underline{x}$, we obtain the following 1,2-morphisms by the fibration property of x.

$$\begin{tikzcd}
& x_{fh'} \arrow[d, "r(f \centerdot \underline{\beta})"] \arrow[dr, "h'"] \arrow[drr, bend left, "fh'"] & \arrow[d, phantom, "\cong"] & \\
x_g \arrow[ur, "r(\alpha')"] \arrow[dr, "r(\alpha)"'] \arrow[r, phantom, "\overset{\tau^{-1}}{\Uparrow}"] & x_{fh} \arrow[r, phantom, "\overset{\widehat{f \centerdot \beta}}{\Uparrow}"] & x_f \arrow[r, "f"] & x \\
& x_{fh} \arrow[ur, "h"'] \arrow[urr, bend right, "fh"'] & \arrow[u, phantom, "\cong"] &
\end{tikzcd}
\overset{\mathbf{cod}}{\longmapsto}
\begin{tikzcd}
& \underline{b} \arrow[d, "1"] \arrow[dr, "\underline{h}'"] \arrow[drr, bend left, "\underline{f}\underline{h}'"] & \arrow[d, phantom, "="] & \\
\underline{b} \arrow[ur, "1"] \arrow[dr, "1"'] \arrow[r, phantom, "="] & \underline{b} \arrow[r, phantom, "\overset{\beta}{\Uparrow}"] & \underline{a} \arrow[r, "\underline{f}"] & \underline{x} \\
& \underline{b} \arrow[ur, "\underline{h}"'] \arrow[urr, bend right, "\underline{f}\underline{h}"'] & \arrow[u, phantom, "="] &
\end{tikzcd}$$

By pasting 2-morphisms $f \centerdot \beta$ and $\tau^{-1}_{\underline{\alpha}', f \centerdot \underline{\beta}}$ we get a 2-morphism $fhr(\alpha) \Rightarrow fh'r(\alpha')$, namely $(f \centerdot \beta) \circ (fh \centerdot \tau^{-1}_{\underline{\alpha}', f \centerdot \underline{\beta}})$, and moreover, by cartesian property of morphism f, this 2-morphism uniquely factors through f to a 2-morphism $\widehat{f \centerdot \beta} \colon hr(f \centerdot \beta) \Rightarrow h'$, shown in the diagram above. Pasting $\tau^{-1}_{\underline{\alpha}', f \centerdot \underline{\beta}}$ and $\widehat{f \centerdot \beta}$ yields the desired 2-morphism $\bar{x}^*(\underline{\beta})$. Alternatively, by the 2-dimensional universal property of bipullback of $\overline{x}_f$, the 2-morphism $\bar{x}^*(\underline{\beta})$ is uniquely determined by a pair of 2-morphisms $\bar{x}^*(\underline{\beta})_0$ and $\bar{x}^*(\underline{\beta})_1$ depicted as

$$\begin{tikzcd}
\overline{x}_g \arrow[rr, bend left, "\overline{f}\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle"] \arrow[rr, bend right, "\overline{f}\bar{x}^*\langle\underline{h},\underline{\alpha}\rangle"'] \arrow[dd, bend right, "x_f\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle"'] \arrow[dd, bend left, "x_f\bar{x}^*\langle\underline{h},\underline{\alpha}\rangle"] & \Uparrow \bar{x}^*(\underline{\beta})_1 & \overline{x} \arrow[dd, "x"] \\
\overset{\bar{x}^*(\underline{\beta})_0}{\Leftarrow} & & \\
\underline{a} \arrow[rr, "\underline{f}"'] & & \underline{x}
\end{tikzcd}$$

which furthermore satisfy the compatibility condition expressed by the commutativity of the diagram of 2-morphisms in $\mathfrak{K}$ in below.

$$\begin{tikzcd}
\underline{f}x_f\bar{x}^*\langle\underline{h},\underline{\alpha}\rangle \arrow[r, Rightarrow, "\underline{f}\blacksquare\bar{x}^*(\underline{\beta})_0"] \arrow[d, Rightarrow, "\overset{\blacktriangledown}{f}\blacksquare\bar{x}^*\langle\underline{h},\underline{\alpha}\rangle"'] \arrow[dr, phantom, "="] & \underline{f}x_f\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle \arrow[d, Rightarrow, "\overset{\blacktriangledown}{f}\blacksquare\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle"] \\
x\overline{f}\bar{x}^*\langle\underline{h},\underline{\alpha}\rangle \arrow[r, Rightarrow, "x\blacksquare\bar{x}^*(\underline{\beta})_1"'] & x\overline{f}\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle
\end{tikzcd} \tag{2.42}$$

We propose $\bar{x}^*(\underline{\beta})_0$ and $\bar{x}^*(\underline{\beta})_1$ to be the dashed 2-morphisms which make the diagrams below commute.

$$\begin{tikzcd}
x_fx^*\langle\underline{h},\underline{\alpha}\rangle \arrow[r, Rightarrow, dashed, "\bar{x}^*(\underline{\beta})_0"] \arrow[d, Rightarrow, "\overset{\blacktriangledown}{r}_\alpha\boxbar\overset{\blacktriangledown}{h}"'] \arrow[dr, phantom, "="] & x_fx^*\langle\underline{h}',\underline{\alpha}'\rangle \arrow[d, Rightarrow, "\overset{\blacktriangledown}{r}_\alpha\boxbar\overset{\blacktriangledown}{h}'"] \\
\underline{h}x_g \arrow[r, Rightarrow, "\underline{\beta}\blacksquare x_g"'] & \underline{h}'x_g
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{fh}\circ\overline{r(f\blacksquare\beta)}\circ\overline{r(\alpha')} \arrow[r, Rightarrow, "(\overline{f\blacksquare\underline{\beta}})\blacksquare\overline{r(\underline{\alpha}')}"] \arrow[d, Rightarrow, "\overline{fh}\blacksquare\tau_{f\blacksquare\underline{\beta},\underline{\alpha}'}"'] \arrow[dr, phantom, "="] & \overline{fh'}\circ\overline{r(\alpha')} \arrow[d, equal] \\
\overline{fh}\circ\overline{r(\alpha)} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\cong"'] \arrow[dr, phantom, "="] & \overline{fh'}\circ\overline{r(\alpha')} \arrow[d, Rightarrow, "\cong"] \\
\overline{f}\circ x^*\langle\underline{h},\underline{\alpha}\rangle \arrow[r, Rightarrow, dashed, "x^*(\underline{\beta})_1"'] & \overline{f}\circ\bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle
\end{tikzcd}$$

Note that $\overset{\blacktriangledown}{r}_\alpha \boxbar \overset{\blacktriangledown}{h}$ and $\overset{\blacktriangledown}{r}_\alpha \boxbar \overset{\blacktriangledown}{h}'$ are invertible.[15] It can be readily checked that $\bar{x}^*(\underline{\beta})_0$ and $\bar{x}^*(\underline{\beta})_1$ satisfy the compatibility condition of diagram (2.42). Therefore, they constitute a unique 2-morphism $\bar{x}^*(\underline{\beta})\colon \bar{x}^*\langle\underline{h},\underline{\alpha}\rangle \Rightarrow \bar{x}^*\langle\underline{h}',\underline{\alpha}'\rangle$ with $x_f \blacksquare \bar{x}^*(\underline{\beta}) = \bar{x}^*(\underline{\beta})_0$ and $\overline{f} \blacksquare \bar{x}^*(\underline{\beta}) = \bar{x}^*(\underline{\beta})_1$.

---

[15] The notation $\boxbar$ is introduced in Construction 1.4.12.

We now show the pseudo functoriality of $\bar{x}^*$. Consider morphisms $\langle \underline{h}, \underline{\alpha} \rangle \colon \underline{g} \Rightarrow \underline{f}$ and $\langle \underline{k}, \underline{\beta} \rangle \colon \underline{f} \Rightarrow \underline{e}$ in $\mathfrak{K} \swarrow \underline{x}$. The fibrational property of x gives us the following morphisms and vertical iso 2-morphisms.

$$\begin{tikzcd}
 & & \overline{x}_g \arrow[dl, "\bar{x}^*\langle \underline{h}, \underline{\alpha} \rangle"'] \arrow[d, "\overline{r_\alpha}"] \arrow[ddll, bend right=50, "\bar{x}^*\langle \underline{kh}, \underline{\alpha} \circ (\underline{\beta} \centerdot \underline{h}) \rangle"'] \arrow[dd, bend left=60, "\overline{r(\underline{\alpha} \circ (\underline{\beta} \centerdot \underline{h}))}"] \\
 & \overline{x}_f \arrow[dl, "\bar{x}^*\langle \underline{k}, \underline{\beta} \rangle"'] \arrow[d, "\overline{r_\beta}"'] \arrow[ur, phantom, "="] & \overline{x}_{fh} \arrow[l, "\overline{h}_f"] \arrow[d, "r_{\underline{\beta} \centerdot \underline{h}}"] \arrow[dl, phantom, "\cong"] \arrow[r, phantom, "\cong"] & {} \\
\overline{x}_e \arrow[ur, phantom, "="] & \overline{x}_{ek} \arrow[l, "\overline{k}_e"] \arrow[d, phantom, "\cong"] & \overline{x}_{ekh} \arrow[l, "\overline{h}_{ek}"] \arrow[ll, bend left=50, "(\overline{kh})_e"] \\
 & {} &
\end{tikzcd}$$

Since by definition, $\bar{x}^*(\langle \underline{k}, \underline{\beta} \rangle \circ \langle \underline{h}, \underline{\alpha} \rangle) = (\overline{kh})_e \circ \overline{r(\underline{\alpha} \circ (\underline{\beta} \centerdot \underline{h}))}$, therefore, we have

$$\bar{x}^*(\langle \underline{k}, \underline{\beta} \rangle \circ \langle \underline{h}, \underline{\alpha} \rangle) \cong \bar{x}^*\langle \underline{k}, \underline{\beta} \rangle \circ \bar{x}^*\langle \underline{h}, \underline{\alpha} \rangle$$

We have $\Sigma \circ \bar{x}^*(\underline{f} \colon \underline{y} \to \underline{x}) = x \circ \overline{f}$, and the counit ε of the 2-adjunction $\Sigma_x \dashv \bar{x}^*$ is given at the component $\underline{f}$ by $\langle x_f, \overset{\blacktriangledown}{f}\,^{-1} \rangle$. For a morphism $\langle \underline{h}, \underline{\alpha} \rangle \underline{g} \to \underline{f}$ in $\mathfrak{K} \swarrow \underline{x}$, we have a iso-square, on the left hand side below, in $\mathfrak{K} \swarrow \underline{x}$, and the corresponding diagram in $\mathfrak{K}$ is drawn on the right hand side, where $\hat{\alpha}$ is $\overline{\alpha} \circ (\overline{\rho}_{\overline{h},\overline{f}} \centerdot \overline{r_\alpha})$ and $\overline{\rho}_{\overline{h},\overline{f}}$ is the canonical iso 2-morphism between cartesian **cod**-morphisms.

$$\begin{tikzcd}
\Sigma_x \bar{x}^*(\underline{g}) \arrow[r, "\langle x_g{,}\overset{\blacktriangledown}{g}^{-1} \rangle"] \arrow[d, "\langle \overline{h} \circ \overline{r_\alpha}{,}\hat{\alpha} \rangle"'] & \underline{g} \arrow[d, "\langle \underline{h}{,}\underline{\alpha} \rangle"] \arrow[dl, phantom, "{\overset{\blacktriangledown}{(r_\alpha} \boxbar \overset{\blacktriangledown}{h})^{-1}} \Swarrow"] \\
\Sigma_x \bar{x}^*(\underline{f}) \arrow[r, "\langle x_f{,}\overset{\blacktriangledown}{f}^{-1} \rangle"'] & \underline{f}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underline{y'} \arrow[dr, bend left, "\underline{h}"] \arrow[dd, dotted, "\underline{g}"'] & \\
\overline{x}_g \arrow[ur, bend left, "x_g"] \arrow[r, dashed, "\overline{h}\overline{r_\alpha}"] \arrow[dr, bend right, "x\overline{g}"'] & \overline{x}_f \arrow[r, dashed, "x_f"] \arrow[d, "x\overline{f}"] \arrow[u, phantom, "{(\overset{\blacktriangledown}{r_\alpha} \boxbar \overset{\blacktriangledown}{h})} \Nearrow"] & \underline{y} \arrow[dl, bend left, "\underline{f}"] \arrow[dl, phantom, "\overset{\blacktriangledown}{f} \Swarrow"] \\
 & \underline{x} \arrow[ul, phantom, "x \centerdot \hat{\alpha} \Swarrow"] &
\end{tikzcd}$$

This proves the pseudo naturality of the counit ε. The unit, however, is only lax natural.

Remark 2.6.5. We presented the construction above in a manner that it is now straightforward to see that the right adjoint pseudo functor $\bar{x}^*$ indeed factors through $\mathfrak{K}_{\mathcal{D}} \swarrow x$ where $\mathcal{D}$ is the chosen class of display morphisms.

$$\begin{tikzcd}
\mathfrak{K} \swarrow \underline{x} \arrow[rrr, bend right=15, "\bar{x}^*"'] \arrow[rrr, phantom, "\perp"] \arrow[dr, "x^*"'] & & & \mathfrak{K} \swarrow \overline{x} \arrow[lll, bend right=15, "\Sigma_x"'] \\
& \mathfrak{K}_{\mathcal{D}} \swarrow x \arrow[r, hook] & \mathbf{cyl}_{\cong}(\mathfrak{K}) \swarrow x \arrow[ur, "\mathbf{dom}"'] &
\end{tikzcd}$$

Example 2.6.6. Let's take $\mathfrak{Cat}$ to be the 2-category of (small) categories, functors and natural transformations. Here we show that a Johnstone fibration in $\mathfrak{Cat}$ is indeed a weak fibration of categories (See §2.3.6). Let $P\colon \mathcal{E} \to \mathcal{B}$ be a Johnstone fibration in $\mathfrak{Cat}$. Let $\mathbf{1}$ be the terminal category, $e \in E$ and $\underline{\alpha}\colon b \to Pe$ a morphism in $\mathcal{B}$. The latter can be viewed as a natural transformation $\underline{\alpha}\colon b \Rightarrow Pe$. The bipullback $\mathcal{E}_b$ has as objects all pairs $\langle x \in \mathcal{E}, \sigma\colon Px \cong b\rangle$, and as morphisms all morphisms $h\colon x \to x'$ in $\mathcal{E}$ making the triangle

$$\begin{tikzcd}
Px \arrow[rr, "Ph"] \arrow[dr, "\sigma"'] & & Px' \arrow[dl, "\sigma'"] \\
& b &
\end{tikzcd}$$

commute. Similarly, the bipullback category $\mathcal{E}_{Pe}$ can be described. Notice that $\langle e, \mathrm{id}_{Pe}\rangle$ is an object of $\mathcal{E}_{Pe}$. Applying r_α to it yields an object x in $\mathcal{E}$ with an isomorphism $\sigma\colon Px \cong b$. Axiom $(J1)$ implies $P(\overline{\alpha}) = \underline{\alpha} \circ \sigma$. The 2-morphism $\overline{\alpha}$ is the lift of $\underline{\alpha}$ and the axioms $(J4)$ and $(J5)$ state that this lift is cartesian. Axioms $(J2)$ and $(J3)$ give coherence equations of lifts for identity and composition.

Example 2.6.7. Let $\mathfrak{Poset}$ be the 2-category of posets and monotone maps with specialization order as 2-morphisms. There is (at most one) 2-morphism between (monotone) maps $F, G\colon E \rightrightarrows B$ whenever $F(e) \leq G(e)$ in B for every $e \in E$. A map $P\colon E \to B$ of posets is a Johnstone-style fibration iff

(i) for all pairs $a, b \in B$ with $a \leq b$ and every $e \in E$ with $P(e) = b$ there is a canonical element $e_a \in E$ with $P(e_a) = a$ and $e_a \leq e$,

(ii) e_a is the largest element with property (i), and

(iii) for all elements $c \leq b \leq a$ in B, and any element e with $F(e) = a$, we have $\left(e_b\right)_c = e_c$.

EXAMPLE 2.6.8. Suppose $B \overset{f}{\rightarrow} D \overset{g}{\leftarrow} C$ is an opspan in a 2-category $\mathfrak{K}$ equipped with bicomma objects and bipullbacks. We prove that first projection morphism $p \colon \left(\underline{f} \downarrow \underline{g}\right) \to B$ of comma object is a fibration in $\mathfrak{K}$. We note that by taking $\underline{f}$ to be identity morphism we obtain a bicategorical analogue of free fibration in 2-categories (See 2.4.22). To see why, take arbitrary 1-morphisms $\underline{h}, \underline{k} \colon A \rightrightarrows B$ and a 2-morphism $\underline{\alpha} \colon \underline{h} \Rightarrow \underline{k}$. First, we construct 1-morphism $\overline{r_\alpha}$ and 2-morphism $\overline{\alpha}$ as shown in diagram below.

$$\begin{tikzcd}
& \underline{k}^*\left(\underline{f} \downarrow \underline{g}\right) \arrow[dl, dotted, "\overline{r_\alpha}"'] \arrow[dr, "\overline{k}"] \arrow[dd, "p_k"'] & & \\
\underline{h}^*\left(\underline{f} \downarrow \underline{g}\right) \arrow[rr, bend right, "\overline{h}"'] \arrow[dd, "p_h"'] & \arrow[r, phantom, "\overline{\alpha}\,\Nearrow"] & \left(\underline{f} \downarrow \underline{g}\right) \arrow[r, "q"] \arrow[dd, "p"] & C \arrow[dd, "\underline{g}"] \\
\arrow[r, phantom, "\Nearrow\,\cong"] & A \arrow[dr, "\underline{k}"] & \arrow[r, phantom, "\theta\,\Nearrow"] & \\
A \arrow[ur, equal] \arrow[rr, bend right, "\underline{h}"'] & \arrow[r, phantom, "\underline{\alpha}\,\Nearrow"] & B \arrow[r, "\underline{f}"'] & D
\end{tikzcd}$$

Bipullbacks $\underline{h}^*\left(\underline{f} \downarrow \underline{g}\right)$ and $\underline{k}^*\left(\underline{f} \downarrow \underline{g}\right)$ may be identified with comma objects $(\underline{f}\underline{h}) \downarrow \underline{g}$ and $(\underline{f}\underline{k}) \downarrow \underline{g}$, respectively. We define 2-morphism $\zeta \colon \underline{f}\underline{h}p_k \Rightarrow \underline{g}q\overline{k}$ to be the following composite of 2-morphisms:

$$\underline{f}\underline{h}p_k \xRightarrow{\underline{f}.\underline{\alpha}.p_k} f\underline{k}p_k \xRightarrow{\underline{f}\centerdot(\overset{\blacktriangledown}{k})^{-1}} \underline{f}p\overline{k} \xRightarrow{\theta\centerdot\overline{k}} \underline{g}q\overline{k}$$

We invoke the universal property of comma object $\left(\underline{f} \downarrow \underline{g}\right)$ to obtain a morphism $\overline{m} \colon \underline{k}^*\left(\underline{f} \downarrow \underline{g}\right) \to \left(\underline{f} \downarrow \underline{g}\right)$ corresponding to 2-morphism ζ, and iso 2-morphisms

$\nu_0 \colon \underline{h}p_k \cong p\overline{m}$ and $\nu_1 \colon q\overline{m} \cong q\overline{k}$ in such a way that they make the following pasting diagrams equal.

$$\begin{tikzcd}
\underline{k}^*\big(\underline{f}\downarrow\underline{g}\big) \arrow[dr, dotted, "\overline{m}"] \arrow[drr, bend left, "q\overline{k}"] \arrow[ddr, bend right, "\underline{h}p_k"'] & & \\
& f\downarrow g \arrow[r, "q"] \arrow[d, "p"'] \arrow[u, phantom, "\nu_1\Uparrow"] \arrow[dl, phantom, "\nu_0 \Swarrow"] & C \arrow[d, "\underline{g}"] \\
& B \arrow[r, "\underline{f}"'] \arrow[ur, phantom, "\theta\Nearrow"] & D
\end{tikzcd}
=
\begin{tikzcd}
\underline{k}^*\big(\underline{f}\downarrow\underline{g}\big) \arrow[drr, bend left, "q\overline{k}"] \arrow[ddr, bend right, "\underline{h}p_k"'] & & \\
& \zeta\Nearrow & C \arrow[d, "\underline{g}"] \\
& B \arrow[r, "\underline{f}"'] & D
\end{tikzcd}$$

Therefore we have $\zeta = (\underline{g} \mathbin{\scriptscriptstyle\blacksquare} \nu_1) \circ (\theta \mathbin{\scriptscriptstyle\blacksquare} \overline{m}) \circ (\underline{f} \mathbin{\scriptscriptstyle\blacksquare} \nu_0^{-1})$. We can now use $\overline{m}$ and ν_0^{-1} and universality of pullback $\underline{h}^*\big(\underline{f}\downarrow\underline{g}\big)$ to get our desired morphism $\overline{r_\alpha} \colon \underline{k}^*\big(\underline{f}\downarrow\underline{g}\big) \to \underline{h}^*\big(\underline{f}\downarrow\underline{g}\big)$ together with an iso 2-morphism $\overset{\blacktriangledown}{r}_\alpha \; \colon p_h \circ \overline{r_\alpha} \cong p_k$. Additionally, we obtain an iso 2-morphism $\overline{\sigma} \colon \overline{h} \circ \overline{r_\alpha} \cong \overline{m}$.

$$\begin{tikzcd}
\underline{k}^*\big(\underline{f}\downarrow\underline{g}\big) \arrow[dr, dotted, "\overline{r_\alpha}"'] \arrow[drr, bend left, "\overline{m}"] \arrow[ddr, bend right, "p_k"'] & & \\
& \underline{h}^*(\underline{f}\downarrow\underline{g}) \arrow[r, "\overline{h}"] \arrow[d, "p_h"'] \arrow[ur, phantom, "\overline{\sigma}\Uparrow"] \arrow[dl, phantom, "\overset{\blacktriangledown}{r}_\alpha\Downarrow"] & f\downarrow g \arrow[d, "p"] \\
& A \arrow[r, "\underline{h}"'] \arrow[ur, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & B
\end{tikzcd}$$

Now, each of $\overline{m}$ and $\overline{k}$, when composed with p and q, yield a comma cone over span $\langle \underline{f}, D, \underline{g} \rangle$, and moreover the resulting comma cones are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd}
\underline{f}p\overline{m} \arrow[r, Rightarrow, "\underline{f}.\gamma"] \arrow[d, Rightarrow, "\theta.\overline{m}"'] & \underline{f}p\overline{k} \arrow[d, Rightarrow, "\theta.\overline{k}"] \\
\underline{g}q\overline{m} \arrow[r, Rightarrow, "\underline{g}.\nu_1"'] & \underline{g}q\overline{k}
\end{tikzcd}$$

where $\gamma := (\overset{\blacktriangledown}{k}\ )^{-1} \circ (\underline{\alpha} \centerdot p_k) \circ \nu_0$. Observe that $(\theta \centerdot \overline{k}) \circ (f \centerdot \gamma) = \zeta \circ (\underline{f} \centerdot \nu_0) = (\underline{g} \centerdot \nu_1) \circ (\theta \centerdot \overline{m})$. So there must be a unique 2-morphism $\overline{\rho}\colon \overline{m} \Rightarrow \overline{k}$ such that $p \centerdot \overline{\rho} = \gamma$ and $q \centerdot \overline{\rho} = \nu_1$. $\overline{\alpha} := \overline{\rho} \circ \overline{\sigma}$ is indeed a lift of $\underline{\alpha}$ which completes the ingredients of fibration p.

Our goal now (Proposition 2.6.10) is to show that, for the 2-functor $\mathbf{cod}\colon \mathfrak{K}_{\mathfrak{D}} \to \mathfrak{K}$, a 1-morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ is a Johnstone-style fibration iff it is a fibrational object in $\mathfrak{K}_{\mathfrak{D}}$ in the sense of Definition 2.5.6.

LEMMA 2.6.9. Suppose x in $\mathfrak{K}_{\mathfrak{D}}$ is a Johnstone-style fibration in the sense of Definition 2.6.1. Let $\underline{f}$, $\underline{g}$ and $\underline{\alpha}$ be as in the definition, giving rise to $f\colon x_f \to x$, $g\colon x_g \to x$ and $\alpha\colon f r_\alpha \Rightarrow g$, and let $u\colon z \to x_g$ be any 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$. Then the whiskering $\alpha \centerdot u\colon f r_\alpha u \Rightarrow g u$ is cartesian.

Proof. First, we deal with the case where u is vertical. Note that this also shows that α itself is cartesian.

Suppose $\gamma_0\colon e_0 \Rightarrow gu$ is a 2-morphism in $\mathfrak{K}_{\mathfrak{D}}$ such that $\mathbf{cod}(\gamma_0) = \underline{\gamma}_0 = \underline{\alpha} \circ \underline{\beta}$ in $\mathfrak{K}$. We seek a unique 2-morphism $\beta_0\colon e_0 \Rightarrow f r_\alpha u$ over $\underline{\beta}$ such that $(\alpha \centerdot u) \circ \beta_0 = \gamma_0$.

Let $e\colon x_e \to x$ be a cartesian lift of $\underline{e}_0$, obtained as a bipullback. Then we can factor e_0, up to a vertical iso 2-morphism, as ev where v is a vertical 1-morphism. We can neglect the iso 2-morphism and assume $e_0 = ev$. Also, let $\beta\colon e \circ r_\beta \Rightarrow f$ and $\gamma\colon e \circ r_\gamma \Rightarrow g$ be lifts of $\underline{\beta}\colon \underline{e} = \underline{e}_0 \Rightarrow \underline{f}$ and $\underline{\gamma} := \underline{\gamma}_0\colon \underline{e} \Rightarrow \underline{g}$ obtained from the fibration structure of x.

From axiom $(J2)$ we get an iso 2-morphism $\tau_{\beta,\alpha}\colon r_\beta \circ r_\alpha \Rightarrow r_\gamma$.

$$\begin{tikzcd}
x_e \arrow[rrd, phantom, ""] & {} & x_g \arrow[ll, bend right=20, "r_\gamma"'] \arrow[ld, "r_\alpha" description] \arrow[ld, bend left=30, "g" description, to=3-2] & z \arrow[l, "u"'] \arrow[lll, bend right=50, "v"'] \\
{} & x_f \arrow[lu, "r_\beta" description] \arrow[d, "f" description] \arrow[u, phantom, "\cong"] \arrow[ld, phantom, "\underset{\beta}{\Rightarrow}"] \arrow[rd, phantom, "\underset{\alpha}{\Rightarrow}"] & {} & {} \\
{} & x & {} & {}
\arrow[from=1-1, to=3-2, bend right=30, "e" description]
\end{tikzcd}$$

Using axiom ($J5$), the unique $\beta_0\colon ev \Rightarrow fr_\alpha u$ that we seek amounts to a unique vertical $\mu_0\colon v \Rightarrow r_\beta \circ r_\alpha \circ u$ such that the diagram on the left below pastes with $\alpha \centerdot u$ to give $\gamma_0\colon ev \Rightarrow gu$. Bringing in τ_β, α, this amounts to finding a unique vertical $\mu_1\colon v \Rightarrow r_\gamma \circ u$ such that the equation on the right holds, and this is immediate from axiom ($J5$).

$$\begin{tikzcd} z \arrow[r, "v"] \arrow[d, "r_\alpha u"'] \arrow[dr, phantom, "\Downarrow\mu_0" near start] & x_e \arrow[d, "e"] \\ x_f \arrow[r, "f"'] \arrow[ur, "r_\beta"] \arrow[ur, phantom, "\Downarrow\beta" near end, shift right=4] & x \end{tikzcd} = \begin{tikzcd} z \arrow[r, "v"] \arrow[d, "r_\alpha u"'] \arrow[dr, phantom, "\Downarrow\beta_0"] & x_e \arrow[d, "e"] \\ x_f \arrow[r, "f"'] & x \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\mu_1" near start] & x_e \arrow[d, "e"] \\ x_g \arrow[r, "g"'] \arrow[ur, "r_\gamma"] \arrow[ur, phantom, "\Downarrow\gamma" near end, shift right=4] & x \end{tikzcd} = \begin{tikzcd} z \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & x_e \arrow[d, "e"] \\ x_g \arrow[r, "g"'] & x \end{tikzcd}$$

Now we prove the result for general u. We can factor u up to an iso 2-morphism as kv, where v is vertical and k is cartesian. Because of Lemma 2.5.2 (i),(v) we might as well assume that $u = kv$. Axiom ($J4$) implies that, up to an iso 2-morphism, $\alpha \centerdot k$ can be obtained as the lift of $\underline{\alpha} \centerdot \underline{k}$. We can thus apply the vertical case, already proved, to see that $(\alpha \centerdot k) \centerdot v$ is cartesian. □

PROPOSITION 2.6.10. A morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{D}$ is a Johnstone-style fibration (in the sense of Definition 2.6.1) iff it is a fibrational object in $\mathfrak{K}_{\mathfrak{D}}$.

Proof. By Proposition 2.5.8, we know that condition ($B1$) is equivalent to bicarrability of x. Now suppose x is a Johnstone-style fibration.

To show ($B2$), assume that $g_0\colon y \to x$ and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}_0\colon \underline{y} \rightrightarrows \underline{x}$ is a 2-morphism in $\mathfrak{K}$. We aim to find a cartesian lift of $\underline{\alpha}$.

Let $f\colon x_f \to x$ and $g\colon x_g \to x$ be cartesian lifts of $\underline{f}$ and $\underline{g}_0$, so $\underline{g} = \underline{g}_0$, and suppose the Johnstone criterion gives them structure $\alpha\colon fr_\alpha \Rightarrow g$. Then we factor g_0 through g and obtain a lift v of $1_{\underline{y}}$ and an iso 2-morphism $\mu\colon gv \Rightarrow g_0$

in $\mathfrak{K}_{\mathfrak{D}}$. Pasting μ and α together we get a 2-morphism $\alpha_0 := \mu \circ (\alpha \centerdot v)$, lying over $\underline{\alpha}$, from $f_0 := f r_\alpha v$ to g_0 in $\mathfrak{K}_{\mathfrak{D}}$.

$$\begin{tikzcd}
y \arrow[d, "v"'] \arrow[ddr, bend left, "g_0"] & \\
x_g \arrow[d, "r_\alpha"'] \arrow[dr, "g"] \arrow[ur, phantom, "\Uparrow \mu"] & \\
x_f \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow \alpha"] & x
\end{tikzcd}$$

Note that α_0 is indeed cartesian. This is because μ is a an iso 2-morphism, and therefore it is cartesian by Lemma 2.5.2(v), $\alpha \centerdot v$ is cartesian according to Lemma 2.6.9, and also vertical composition of cartesian 2-morphisms is cartesian.

For $(B3)$, let $\alpha_0 \colon f_0 \Rightarrow g_0 \colon y \to x$ be any cartesian 2-morphism in $\mathfrak{K}_{\mathfrak{D}}$, and let $k \colon z \to y$ any 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$. We will show that the whiskered 2-morphism $\alpha_0 \centerdot k$ is again cartesian. First, let $f \colon x_f \to x$ and $g \colon x_g \to x$ be cartesian lifts of $\underline{f}_0$ and $\underline{g}_0$, and let $\alpha \colon f r_\alpha \Rightarrow g$ be got from $\underline{\alpha}_0$ in the usual way. Then we factor f_0 and g_0 up to vertical iso 2-morphisms as $\rho \colon f_0 \cong f \circ u$ and $\pi \colon g_0 \cong g \circ v$, where u, v are vertical. Define $\alpha_0' = \pi \circ \alpha_0 \circ \rho^{-1}$. Obviously, α_0' is cartesian and $\alpha_0 \centerdot k$ is cartesian if and only if $\alpha_0' \centerdot k$ is cartesian. By axiom $(J5)$ of fibration, we get a (unique) vertical 2-morphism μ such that $(\alpha \centerdot v) \circ (f \centerdot \mu) = \alpha_0'$. By Lemma 2.6.9 $\alpha \centerdot v$ is cartesian and it follows that $f \centerdot \mu$ is cartesian since α_0' is cartesian. Now the 2-morphism $f \centerdot \mu$ is both vertical and cartesian and thus it is an iso 2-morphism, according to Lemma 2.5.2(vi). So, our task reduces to proving that $(\alpha \centerdot v) \centerdot k$ is a cartesian 2-morphism, and this we know from Lemma 2.6.9.

Conversely, suppose $x \colon \overline{x} \to \underline{x}$ is a fibrational object in $\mathfrak{K}_{\mathfrak{D}}$. We want to extract the structure of Johnstone-style fibration for x out of this data. First of all according to $(B1)$, x is bicarrable. Suppose $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ is any 2-morphism in $\mathfrak{K}$. Let g be a cartesian lift of $\underline{g}$ obtained as a bipullback of $\underline{g}$ along x in $\mathfrak{K}$. By $(B2)$

$\underline{\alpha}$ has a cartesian lift $\alpha'\colon f' \Rightarrow g$. Factor f', up to an iso 2-morphism γ, as $f \circ r_\alpha$ where r_α is vertical and $f\colon x_f \to x$ is cartesian. From α' and γ we obtain a cartesian 2-morphism $\alpha\colon f \circ r_\alpha \Rightarrow g$ which satisfies axiom $(J1)$.

$$\begin{tikzcd}
x_g \arrow[rr, bend left=40, "g"{name=G}] \arrow[rr, "f'"{name=F, description}] \arrow[rd, "r_\alpha"'] & & x \\
& x_f \arrow[ru, "f"'] \arrow[u, phantom, "\cong \gamma"] &
\arrow[from=F, to=G, phantom, "\Uparrow \alpha'"]
\end{tikzcd} \tag{2.43}$$

To show $(J2)$, take a pair of composable 2-morphisms $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta}\colon \underline{g} \Rightarrow \underline{h}$. Carrying out the same procedure as we did in diagram (2.43), we obtain cartesian 2-morphisms $\alpha\colon f \circ r_\alpha \Rightarrow g$ and $\beta\colon g \circ r_\beta \Rightarrow h$, and also $\gamma\colon f \circ r_\gamma \Rightarrow h$ lifting $\underline{\gamma} = \underline{\beta}\underline{\alpha}$. By $(B3)$, the 2-morphism $\beta \circ (\alpha \centerdot r_\beta)\colon f r_\alpha r_\beta \Rightarrow h$ is cartesian. Therefore, there exists a unique vertical iso 2-morphism $\sigma\colon f r_\alpha r_\beta \Rightarrow f r_\gamma$ such that $\gamma \circ \sigma = \beta \circ (\alpha \centerdot r_\beta)$.

$$\begin{tikzcd}
x_h \arrow[rrd, bend left=70, "f r_\gamma"{name=A}] \arrow[rrd, dotted, "f r_\alpha r_\beta"{name=B, description}] \arrow[rd, bend right=40, "r_\alpha r_\beta"'] \arrow[rd, bend left=20, "r_\gamma"] & & \\
& x_f \arrow[r, "f"'] & x
\arrow[from=B, to=A, phantom, "\Uparrow \sigma"]
\end{tikzcd}$$

Since f is cartesian, Remark 2.5.3 (ii) yields a unique vertical iso 2-morphism $\tau_{\alpha,\beta}\colon r_\alpha r_\beta \Rightarrow r_\gamma$ such that $f \centerdot \tau_{\alpha,\beta} = \sigma$. Thus, $(\beta\alpha) \circ (f \centerdot \tau_{\alpha,\beta}) = \beta \circ (\alpha \centerdot r_\beta)$.

For condition $(J3)$, if $\underline{\alpha} = \mathrm{id}$, then α is both cartesian and vertical, and hence an isomorphism. Now we can use Remark 2.5.3(ii) with α^{-1} for σ and an identity for $\underline{\delta}$ to get $\delta\colon 1_{x_f} \Rightarrow r_\alpha$ as well as an inverse for it. It has the property required in $(J3)$.

Now we prove condition $(J4)$, using the notation there, and we wish to define the isomorphism in the left hand square. We find we have two cartesian lifts of $\underline{\alpha}' \centerdot \underline{k}$ to gk_g. The first is the pasting

$$\pi^{-1}\alpha'(\rho \centerdot r_{\alpha'})\colon fk_f r_{\alpha'} \Rightarrow gk_g.$$

This is cartesian by Lemma 2.5.2(i),(v), being composed of isomorphisms and the cartesian α'. The second is $\alpha \centerdot k_g$, cartesian because α is cartesian and, according to $(B3)$, its whiskering with any 1-morphism is cartesian. These two cartesian lifts must be isomorphic, so we get a unique iso 2-morphism between $fk_f r_{\alpha'}$ and $fr_\alpha k_g$, over $\underline{f}\,\mathrm{id}_{\underline{k}}$, that pastes with α, $\mathfrak{c}$ and π to give α'. Now we use Remark 2.5.3(ii) to get a unique isomorphism in the left hand square of the diagram with the required properties.

Finally, we shall prove $(J5)$, which is similar to $(J4)$. Assume vertical 1-morphisms v_0 and v_1 and a 2-morphism α_0 over $\underline{\alpha}$ as in the hypothesis of axiom $(J5)$. We use the cartesian property of the 2-morphism $\alpha \centerdot v_1$ to get a unique vertical 2-morphism $\lambda\colon fv_0 \Rightarrow fr_\alpha v_1$ such that $(\alpha \centerdot v_1) \circ \lambda = \alpha_0$. By the cartesian structure of the 1-morphism f, we can factor λ as $f \centerdot \mu$ for a unique vertical 2-morphism μ with $f \centerdot \mu = \lambda$. Hence, $(\alpha \centerdot v_1) \circ (f \centerdot \mu) = \alpha_0$. □

REMARK 2.6.11. The proof above is rather long and technical. So, we thought our reader may appreciate a summary of various dependencies of B and J conditions. The following table shows how the various structures in a Johnstone fibration relates to structures $(B1)$-$(B3)$. That is which B's we need to prove each J.

Definition 2.6.1	Definition 2.6.10
x is carrable	$(B1)$
Axiom $(J1)$	$(B1)$, $(B2)$
Axiom $(J2)$	$(B1)$, $(B2)$
Axiom $(J3)$	$(B1)$, $(B2)$, $(B3)$
Axiom $(J4)$	$(B1)$, $(B2)$, $(B3)$
Axiom $(J5)$	$(B1)$, $(B2)$, $(B3)$

On the other hand, the table below shows that what J's we need to prove each B:

Definition 2.6.1	Definition 2.6.10
$(B1)$	x is carrable
$(B2)$	$(J1)$, $(J3)$, $(J5)$
$(B3)$	$(J1)$, $(J3)$, $(J4)$, $(J5)$

Theories and contexts

3

In this chapter, we present the third model of generalized spaces, that is the 2-category $\mathfrak{Con}$ of AU-contexts (§3.3) and study its features. We quickly review the main aspects of the theory of AU-contexts, our AU analogue of geometric theories in which the need for infinitary disjunctions in many situations has been satisfied by a type-theoretic style of sort constructions that include list objects (and an nno). The contexts are "sketches for arithmetic universes" [Vic19], and we review the principal syntactic constructions on them that are used for continuous maps and 2-morphisms.

We also introduce the notion of fibration of contexts (§3.4) and in the next chapter we prove that they beget fibrations of toposes.

This accomplishes first steps in fulfilling the bigger goal to see to what extent AUs can replace Grothendieck toposes as models of spaces. In this approach, geometric theories are replaced by AU-contexts, thought of as a kind of *types* of *type theory* of AUs, presented by sketches ([Vic19]), and geometric morphisms are replaced by AU-functors, corresponding to the inverse image functors.

AU-contexts are presented by sketches in [Vic19]. We start by an overview of first order geometric theories and their link to sketches for AUs which is followed by a selective overview of AU-sketches.

The main references for this chapter are [AR94], [Joh02b], [Vic19], [Vic17], and [HV19].

3.0 Introduction

Arithmetic Universes (AUs) were introduced by André Joyal with the insight to provide a categorified proof of the celebrated Gödel Incompleteness theorem. This insight was communicated in his lectures ([Joy73b], [Joy73a]). What initially remained of this insight and the alleged proof in written form was a set of notes taken by Gavin Wraith. Although this significant insight of Joyal never appeared in a published format, it undoubtedly triggered attention and research into Arithmetic Universes (See [Mai99], [Mai00], [Mai03], [Mai05b], [Mai10a], [Mai10b], [MV12], [Vic19]).

What is the idea behind the notion of AU? A philosophical view of the Gödel Incompleteness theorem is that it is a self-reflective ability of a formal system based on its expressiveness and its proof involves the famous arithmetization argument. Joyal proposed an AU to be a structured category whose structure is *expressive enough* to allow the 'internal type theory of the category' to build a replica of the original AU inside itself, analogous to Gödel's arithmetization. The rest of the argument then should use the machinery of internal language to give a categorical incarnation of the Gödel sentence constructed from the AU and its replica.

The 'enough structure' in above has been proposed to be formalized as the structure of a **list-arithmetic pretopos**: a category with finite limits, stable disjoint coproducts, stable effective quotients by monic equivalence relations and parameterized list-objects.

Equivalently an AU is a finitely extensive Barr-exact category with parametric list objects. Note that an AU has all coequalizers, not just the quotients of equivalence relations. This is because the list object allows one to construct the transitive closure of any relations [Mai10a].

The theory of AUs is local, i.e. slices of AUs are AUs. Comma objects of AUs are constructed as comma categories. Therefore, the comma construction is created by the forgetful functor $\mathcal{A}\mathrm{U} \to \mathcal{C}\mathrm{at}$ [MV12].

The above definition of AU parallels (relativized) Giraud's characterization of relative Grothendieck toposes, except that AUs have only a finitary fragment of geometric logic, and instead of infinitary disjunctions being supplied extrinsically by a base topos (e.g. the structure of small-indexed coproducts), they have sort-constructors for **parametrized list object** that allow some infinities intrinsically: e.g. point-free continuum. AUs are presented via sketches in [Vic19].

Sketches (French *esquisses*) were introduced by differential geometer Charles Ehresmann, a student of Cartan, and forerunner of the Bourbaki seminar. He later became a leading proponent of categorical methods and by 1957 he founded the mathematical journal Cahiers de Topologie et Géométrie Différentielle. Collectively, the development of sketches together with contemporary work of Bill Lawvere and earlier work[1] of Halmos (e.g. Halmos's polyadic algebras), Tarski (e.g. his work on cylindric algebras) and Birkhoff has come to be understood under the umbrella term 'categorical logic'.

The simplest kind of sketch is a directed multigraph possibly with loops. Sketches can be underlying graphs of categories but in general they do not have to. The point is that in sketches we do not have the structure of compositions of arrows. Note that models of such sketches in $\mathcal{S}$et cannot accommodate for any nullary, binary, or higher arity operation nor any equations. A remedy is to add more structure to the sketch such as finite products. To express equations, we add commutativities in some extension of our sketch. Starting with a sketch $\mathbb{T}$, we can specify a composition of two composable arrows by adding a third arrow and a commutativity.

Also, to add higher arity operations one works with limit sketches. To still add more structures such as those of regular theories one can work with sketches with cocones. For the purpose of expressing structure of arithmetic universes one has to work with sketches whose models can accommodate for

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[1] These earlier work, sometimes refereed to as algebraic logic, arose from the effort of formulating logical notions and theorems in terms of universal algebraic. It has been argued in [MR11] that categorical logic is logic in an algebraic dressing.

all operations that a generic arithmetic universe allows. Sketches for arithmetic universes are dealt with in [Vic19].

3.1 A swift overview of (geometric) first order theories

In the first part we begin by recalling the notion of syntactic category of a first order theory. The idea here is that we would like to organise the data of $\mathbb{T}$ into a category so that the models of $\mathbb{T}$ in a category $\mathcal{S}$ correspond to the $\mathcal{S}$-valued functors from the *syntactic category* $\mathcal{S}yn(\mathbb{T})$ and the *elementary embeddings* of models correspond to natural transformations between corresponding functors. As we will see, the syntactic category $\mathcal{S}yn(\mathbb{T})$ comes equipped with a *generic* model $M_{\mathbb{T}}$ inside it, in such a way that a formula ϕ is provable in $\mathbb{T}$ (as it is customary we write $\mathbb{T} \vdash \phi$ for the provability relation) if and only if its interpretation in $\mathcal{S}yn(\mathbb{T})$ is satisfied by the model $M_{\mathbb{T}}$ (as it is customary we write $M_{\mathbb{T}} \models \phi$ for the satisfaction relation).

We follow the approach of [Joh02b, p. D1.1], in fact as we shall see in the next part that is necessary in order to deal correctly with geometric logic. We warn the reader that there are some differences from traditional logic. Two major differences from standard approaches are the use of contexts (which is a natural way to make the logic sound for empty carriers), and that axioms are presented by sequents[2] $\phi \vdash_{\vec{x}} \psi$ in context $\vec{x}$, and are not the same as sentences[3].

Also, it is important to allow the logic, the fragment of first-order logic, to vary. Wherever we feel it is necessary we shall point out these differences in practice. Here is a simple example.

EXAMPLE 3.1.1. The theory of posets has one sort X and a binary relation $R \subset X, X$ (where $R(x, y)$ has the intended meaning $x \leq y$) which satisfies the reflexivity,

---

[2] indicated by turnstile symbol $\vdash$ and annotated with the context in which derivation takes place

[3] i.e. formulae with no free variables

the antisymmetry, and the transitivity axioms; they appear on the left hand side in their traditional form while on the right hand side they appear in contexts.

$$\begin{array}{cc}
(\forall x)R(x,x) & \top \vdash_x R(x,x) \\
(\forall x,y,z)((R(x,y)\wedge R(y,z)) \Rightarrow R(x,z)) & (R(x,y)\wedge R(y,z)) \vdash_{x,y,z} R(x,z) \\
(\forall x,y)((R(x,y)\wedge R(y,x)) \Rightarrow (x=y)) & (R(x,y)\wedge R(y,x)) \vdash_{x,y} (x=y)
\end{array}$$

For instance the axioms above are expressed in the so-called "Horn fragment" of (geometric) first order logic (See Table 3.1). Notice that in geometric logic (and its fragments) we do not have the operation of universal quantification over variables, nor do we have implications of formulae (e.g. such as the transitivity axiom on the RHS[4] of Example 3.1.1). The sequent style derivation comes to our rescue. Also, for first order theories, $(\forall x)\phi(x) \nvdash (\exists x)\phi(x)$, however, we have $(\forall x)\phi(x) \vdash_c (\exists x)\phi(x)$ for some other variable c. Writing down our axioms in sequent-style reifies the importance of the contexts.

Another motivation for introducing contexts comes from the phenomenon of enlarging its scope in the process of passing a variable across a logical connective. For instance, in a single sorted first order theory, one can prove that for formulae ψ and ϕ,

$$(\phi \vee \exists x\psi) \iff \exists x(\phi \vee \psi)$$

where $x\colon X$ is not a free variable of ϕ. Now, in any interpretation where the domain of interpretation (i.e. interpretation of sort X) is empty, the equivalence above fails to satisfy which is bad news from the perspective of soundness. To see this, consider the sentence above with $\phi = \forall y(y=y)$ and $\psi = (x=x)$. In classical model theory of first order theories, the remedy is to require non-emptiness of domain of interpretation. Without the use of contexts, however, in categorical model theory where the the domain of interpretations are objects of categories (possibly other than $\mathcal{S}$et) it is not always clear what 'non-emptiness' of an object means.

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[4] Right Hand Side

Finally, it is possible for a particular language to have sorts with no closed terms. Using variables of this sort carries with itself a tacit existential assumption, and therefore we should record each occurrence of such assumption by bookkeeping the variables in the context in our inferences.

The full derivation rules for sequents-in-context are given in [Joh02b, p. D1.3], and it is important to note that they are sound even for *empty carriers*.

In full first-order logic not every structure homomorphism is natural for all formulae, and therefore, it's interesting to look at the restricted class of those that are: these are the so-called elementary embeddings (aka elementary morphisms). In geometric logic the problem doesn't arise, because structure homomorphisms are natural for all geometric formulae. Since in this thesis we are mostly concerned with geometric logic and its fragments we are not paying as much attention to the elementary embeddings.

Briefly, recall that a first order theory is a pair $\mathbb{T} = (\sigma, \Phi)$ where Σ is a first order signature, and Φ is the set of axioms[5] of $\mathbb{T}$. A first order signature Σ comes with a set σ of sorts and a set $\mathbb{P} = \{P_i\}_{i \in I}$ of predicates such that each predicate has an arity which is just a sequence $(X_1, \ldots, X_n)$ of sorts $X_i \in \sigma$. One usually writes $P \subset X_1, \ldots, X_n$. See [Joh02b, p. D1.1.1] Let's call this the *spartan* version.

One may add bells and whistles to this definition and include, in addition to predicate (aka relation) symbols, function symbols (with arity) as well. Notice that for any cartesian theory[6] $\mathbb{T}$ there is a cartesian theory $\mathbb{T}'$ which is Morita equivalent[7] to $\mathbb{T}$ and does not have any function symbols. (See Example 3.1.2 and [Joh02b, Lemma D.1.4.9].) We take the liberty of using either style of presentation depending on the context of discussion and also as a matter of convenience. So a full presentation of a theory includes

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[5] Traditionally, each axiom is a sentence (meaning a formula without any free variables) which become valid sentences in every model of theory $\mathbb{T}$. For us, axioms are going to be sequents, not formulae in general.

[6] The notion of cartesian theory will be defined in Remark 3.1.4.

[7] i.e. Two theories are Morita equivalent if their respective categories of models are equivalent.

- $P \subset X_1, \ldots, X_n$, for each predicate, and

- $f\colon X_1, \ldots, X_n \to X$, for each function symbol.

Two special cases of proposition and constant symbols are included by considering empty arities in the above:

- $P \subset 1$, for each proposition, and

- $c\colon 1 \to X$ for a constant symbol.

The example below contrasts the spartan and the embellished styles of presentation.

EXAMPLE 3.1.2. One can present the theory of groups (on LHS) with one sort G, a ternary relation symbol $M \subset G, G, G$, where the intended meaning of $M(x, y, z)$ is that z "equals the (binary) multiplication of x any y". It also comes equipped with a constant symbol[8] $e\colon G$. Altogether this structure should satisfy the following axioms:

$$\begin{gathered}
M(x,y,u) \wedge M(y,z,v) \wedge M(u,z,w) \vdash_{x,y,z,u,v,w} M(x,v,w) \\
M(x,y,u) \wedge M(y,z,v) \wedge M(x,v,w) \vdash_{x,y,z,u,v,w} M(u,z,w) \\
\top \vdash_x M(x,e,x) \wedge M(e,x,x) \\
\top \vdash_{x,y} (\exists z)\ M(x,y,z) \\
M(x,y,z) \wedge M(x,y,w) \vdash_{x,y,z,w} (z = w) \\
\top \vdash_x (\exists y\colon G\ \exists z\colon G)\ M(x,y,e) \wedge M(z,x,e)
\end{gathered}$$

The fourth and fifth axioms say that M is a functional relation.

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[8] which can be regarded as a constant unary predicate.

Alternatively, instead of ternary relation symbol M, we could have a function symbol $m\colon G \times G \to G$ satisfying the following axioms:

$$\top \vdash_{x,y,z} m(m(x,y),z) = m(x,m(y,z))$$
$$\top \vdash_{x} m(x,e) = x \wedge m(e,x) = x$$
$$\top \vdash_{x} (\exists y\colon G\, \exists z\colon G)\; m(x,y) = e \wedge m(z,x) = e$$

It is often easier and clearer to use function symbols.

3.1.1 Fragments of first order theories

Before we present examples of some well-known theories, we would like to explain some of the nomenclature pertaining to different fragments of first order theories. The table below illustrates the hierarchy of different fragments of first order theory[9]. Each row shows that the axioms of the corresponding fragment are formed by the marked logical operations; for instance, a theory which has any of its axioms formed using implication is not geometric.

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[9] First order refers to the fact that quantification is over variable individuals rather than over subsets or functions of them.

	binary conj. ($\wedge$)	truth ($\top$)	exist. quant. ($\exists$)	binary disj. ($\vee$)	falsity ($\bot$)	neg. ($\neg$)	impl. ($\Rightarrow$)	univ. quant. ($\forall$)	inf. disj. ($\bigvee$)	inf. conj. ($\bigwedge$)
Horn theories	✓	✓								
Cartesian theories	✓	✓	*cartesian*							
Regular theories	✓	✓	✓							
Coherent theories	✓	✓	✓	✓	✓					
(Full) first order theories	✓	✓	✓	✓	✓	✓	✓	✓		
Geometric theories	✓	✓	✓	✓	✓				✓	
Infinitary first order theories	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Fig. 3.1.: Fragments of first order theory

We give a few examples of theories using context-style axioms. In the next sections, we give a different presentation based on AU-sketches.

EXAMPLE 3.1.3. The theory of *linear orders* is obtained from that of posets by adding the axiom below:

$$\top \vdash_{x,y} (R(x,y) \vee R(y,x))$$

Note that the theory of linear orders, unlike that of posets, is not a Horn theory. It is a coherent theory. We can extend it to the theory of *(strict) linear intervals* by adding two constants `t` and `b` of sort X together with the following axioms:

$$\top \vdash_x R(\mathtt{b}, x) \wedge R(x, \mathtt{t})$$
$$(\mathtt{b} = \mathtt{t}) \vdash \bot$$

REMARK 3.1.4. The word "cartesian" in the above table requires further explication. We give an inductive definition of cartesian formulae first. Suppose $\mathbb{T}$ is (at least) a regular theory. A formula is called **cartesian** if it is either *(i)* atomic[10], or *(ii)* finite conjunction of cartesian formulae, or *(iii)* of the form $\exists y \phi$ where $\phi(\vec{x}, y)$ is cartesian and moreover the sequent

$$(\phi \wedge \phi[z/y]) \vdash_{\vec{x},y,z} (y = z) \tag{3.1}$$

is provable in $\mathbb{T}$. A sequent $\phi \vdash_{\vec{x}} \psi$ is cartesian if both ϕ and ψ are cartesian.

A regular theory $\mathbb{T}$ is **cartesian** if there is a well-founded partial ordering of its axioms such that each axiom is cartesian relative to the subtheory, formed by the axioms which precede it in the ordering. As indicated in the table above cartesian theories lie between Horn and regular theories, but they are really closer to Horn theories rather than to regular theories for the following reason: in models, the interpretation of existential quantifiers corresponds to forming images of projection morphisms. By cartesianness, these morphisms are already monic and hence their images are isomorphic to themselves. What we are doing really is to take images of morphisms which are already known to be unique.

It is worth noting that Palmgren and Vickers ([PV07]) show that cartesian theories are equivalent to *partial* Horn theories, i.e. Horn theories in a logic of partial terms.

EXAMPLE 3.1.5. The theory of "lattices equipped with prime filters" can be presented with one sort L and predicates $P \subset L$, $Glb \subset L, L, L$ and $Lub \subset L, L, L$ together with constants $\mathtt{t} : L$, $\mathtt{b} : L$. The intended meaning of $P(x)$ is "x is an element of the prime filter "P of the lattice L", and we need appropriate axioms expressing L as a lattice and P as a prime filter of L. $Glb(a, b, c)$ exhibits c as the

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[10] Either of the form $\vec{x} = \vec{y}$ or $P(\vec{x})$ for some predicate P.

greatest lower bound of a and b while $Lub(a, b, c)$ exhibits c as the least upper bound of a and b. The constant `t` is the top element and `b` is the bottom element. The lattice axioms are as usual, that is idempotency, commutativity, and associativity laws of meet and join plus the identity laws for `t` and `b` with respect to meet and join, and the absorption laws. The axioms (xii), (xiii) express P as a filter and the axioms (xiv), (xv) say that P is indeed a prime filter.

(i) $\top \vdash_a Glb(a, a, a)$

(ii) $\top \vdash_a Lub(a, a, a)$

(iii) $Glb(a, b, c) \vdash_{a,b,c} Glb(b, a, c)$

(iv) $Lub(a, b, c) \vdash_{a,b,c} Lub(b, a, c)$

(v) $Glb(b, c, d) \wedge Glb(a, b, e) \wedge Glb(a, d, f) \vdash_{a,b,c,d,e,f} Glb(e, c, f)$

(vi) $Lub(b, c, d) \wedge Lub(a, b, e) \wedge Lub(a, d, f) \vdash_{a,b,c,d,e,f} Lub(e, c, f)$

(vii) $\top \vdash_a Glb(a, \mathtt{t}, a)$

(viii) $\top \vdash_a Lub(a, \mathtt{b}, a)$

(ix) $Glb(a, b, c) \vdash_{a,b,c} Lub(a, c, a)$

(x) $Lub(a, b, c) \vdash_{a,b,c} Glb(a, c, a)$

(xi) $Glb(a, b, c) \wedge P(a) \wedge P(b) \vdash_{a,b,c:L} P(c)$

(xii) $Lub(a, b, c) \wedge P(a) \vdash_{a,b,c:L} P(c)$

(xiii) $\top \vdash P(\mathtt{t})$

(xiv) $P(\mathtt{b}) \vdash \bot$

(xv) $Lub(a, b, c) \wedge P(c) \vdash_{a,b,c} P(a) \vee P(b)$

REMARK 3.1.6. The theory of posets and groups are cartesian regular, while the theory of linear orders is not regular. The theory of "lattices equipped with prime filters" is not cartesian. Similarly, the theory of local rings is not cartesian.

3.1.2 Homomorphism of theories

DEFINITION 3.1.7. There is a category of (first-order) geometric theories whose morphisms are known as *theory homomorphisms*. For signatures Σ and Σ', a **signature homomorphism** $F\colon \Sigma \to \Sigma'$ is an assignment to each sort X of Σ a sort $F(X)$ of Σ', to each function symbol $f\colon X_1, \ldots, X_n \to Y$ a function symbol $F(f)\colon F(X_1), \ldots, F(X_n) \to F(Y)$ of Σ', and to each relation symbol $R \subset X_1, \ldots, X_n$ of Σ to a relation symbol $F(R) \subset F(X_1), \ldots, F(X_n)$ of Σ'. Note that the above setup ensures that F takes terms to terms and formulae to formulae while keeping their corresponding contexts fixed.

For theories $\mathbb{T} = (\Sigma, \Phi)$ and $\mathbb{T}' = (\Sigma', \Phi')$, a **theory homomorphism** $F\colon \mathbb{T} \to \mathbb{T}'$ is a signature homomorphism which in addition takes an axiom $\phi \vdash_{\vec{x}} \psi$ of $\mathbb{T}$ to an axiom $F(\phi) \vdash_{\vec{x}} F(\psi)$.

There are many obvious examples of theory homomorphisms: for instance the forgetful homomorphism from the theory of monoids to the theory of groups, or the inclusion of theory of groups in the theory of rings.

3.1.3 Interpretations and models

Interpretation of signature of a language

DEFINITION 3.1.8. Suppose we have a first order signature Σ, and $\mathcal{S}$ is a category equipped with all finite products. A Σ**-structure** (aka **interpretation**[11]) M consists of the data

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[11] This is Tarksi interpretation and should be distinguished from BHK (*Brouwer-Heyting-Kolmogorov*) interpretation where the interpretation of relation symbols is defined differ-

(i) an assignment to each sort $X \in \sigma$ an object $M[X]$ of $\mathcal{S}$,

(ii) an assignment to each sequence $X_1, \ldots, X_n$ of sorts the product $M[X_1] \times \ldots \times M[X_n]$ in $\mathcal{S}$ where the empty sequence [] of sorts is interpreted to be the terminal object of $\mathcal{S}$, i.e. $M[] = 1$,

(iii) an assignment to each function symbol $f\colon X_1, \ldots, X_n \to X$ in Σ a morphism $M[f]\colon M[X_1] \times \ldots \times M[X_n] \to M[X]$ in $\mathcal{S}$, and

(iv) an assignment to each relation symbol $R \subset X_1, \ldots, X_n$ in Σ a subobject $M[R] \rightarrowtail M[X_1] \times \ldots \times M[X_n]$ in $\mathcal{S}$.

DEFINITION 3.1.9. Suppose Σ is a first order signature and M and N are interpretations of Σ in a category $\mathcal{S}$. A **Σ-morphism** from M to N is an assignment to each sort $X \in \sigma$ a morphism $\alpha_X\colon M[X] \to N[X]$ such that for every relation symbol $R \subset X_1, \ldots, X_n$ in Σ, there is a (unique) morphism $\alpha_R\colon M[R] \to N[R]$ which makes the diagram

$$\begin{tikzcd}
M[R] \arrow[r, tail] \arrow[d, dashed, "\alpha_R"'] & M[X_1] \times \ldots \times M[X_n] \arrow[d, "\alpha_{X_1} \times \ldots \times \alpha_{X_n}"] \\
N[R] \arrow[r, tail] & N[X_1] \times \ldots \times N[X_n]
\end{tikzcd} \tag{3.2}$$

commute and moreover, for every function symbol $f\colon X_1, \ldots, X_n \to X$ the diagram

$$\begin{tikzcd}
M[X_1] \times \ldots \times M[X_n] \arrow[r, "M[f]"] \arrow[d, "\alpha_{X_1} \times \ldots \times \alpha_{X_n}"'] & M[X] \arrow[d, "\alpha_X"] \\
N[X_1] \times \ldots \times N[X_n] \arrow[r, "N[f]"'] & N[X]
\end{tikzcd} \tag{3.3}$$

commutes.

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ently [Joh02b, Remark D.1.2.2]. BHK interpretation provides semantics of intuitionistic logic.

Notice that if we interpret our signature in the category of sets, then the above commutativity condition 3.2 states that for every n-tuple $(a_1, \ldots, a_n) \in M[X_1] \times \ldots \times M[X_n]$, we have

$$M \models R(a_1, \ldots, a_n) \Rightarrow N \models R(\alpha_{X_1}(a_1), \ldots, \alpha_{X_n}(a_n)) \tag{3.4}$$

REMARK 3.1.10. The commutativity condition (3.3) is a special case of (3.2) once we describe the function f instead by its graph relation. Recall that in any cartesian category the **graph of morphism** $f \colon X \to Y$ is a subobject $\gamma \colon \mathbb{G}r(f) \rightarrowtail X \times Y$ with the property that $\alpha := \pi_X \circ \gamma$ is an isomorphism and $f = \pi_Y \circ \gamma \circ \alpha^{-1}$.

$$\begin{tikzcd}
 & & Y \\
\mathbb{G}r(f) \arrow[r, tail] \arrow[urr] \arrow[drr, "\cong"'] & X \times Y \arrow[ur, "\pi_Y"'] \arrow[dr, "\pi_X"] & \\
 & & X \arrow[uu, "f"']
\end{tikzcd}$$

Moreover, a square

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "k"'] & Y'
\end{tikzcd}$$

commutes iff there is a morphism $g \colon \mathbb{G}r(f) \to \mathbb{G}r(f')$ such that $\gamma'_0 g = \pi_Y \gamma' g = h\pi_Y \gamma = h\gamma_0$ and $\gamma'_1 g = \pi_X \gamma' g = k\pi_X \gamma = k\gamma_0$.

REMARK 3.1.11. An immediate consequence of the above definition is that $M[R]$ is a subobject of $\alpha_1 \times \ldots \times \alpha_n)^* N[R]$. We will soon see that for a class of special Σ-morphisms (elementary embeddings), $M[R] \cong (\alpha_1 \times \ldots \times \alpha_n)^* N[R]$ as subobjects of $M[X_1] \times \ldots \times M[X_n]$.

CONSTRUCTION 3.1.12. For any category $\mathcal{S}$, and a signature Σ, the Σ-structures and Σ-morphisms form a category Σ-$\mathbf{Str}$ where the identity Σ-morphism and the composition of Σ-morphisms is defined component-wise as identity morphism and composition of morphisms in $\mathcal{S}$.

EXAMPLE 3.1.13. A Σ-morphism $\alpha\colon I \to J$ for the theory of (strict) linear intervals is a function which respects the order (commutativity of diagram (3.2)) and moreover, preserves the top and bottom elements (commutativity of diagram (3.3)).

Interpretation of terms

Terms are interpreted as morphisms while formulae are interpreted as subobjects; given an interpretation M of signature Σ of a language L as above, we can interpret a term t of sort Y in a suitable context $\vec{x} = (x_1, \ldots, x_n)$ as a morphism $[\![\vec{x}.t]\!]_M\colon M[X_1] \times \ldots \times M[X_n] \to M[Y]$, where $x_i\colon X_i$, for $1 \leq i \leq n$. Depending on the construction of term t, we define its interpretations in context $\vec{x}$ inductively as follows:

(i) When t is the unique term $*$ of the unit sort 1, $[\![\vec{x}.t]\!]_M$ is defined to be the unique morphism $M[X_1] \times \ldots \times M[X_n] \to 1$ in $\mathcal{S}$.

(ii) When t is a constant term $a\colon X$, $[\![\vec{x}.t]\!]_M$ is defined to be the composite

$$\begin{tikzcd}
\prod\limits_{1\leq i\leq n} M[X_i] \arrow[rr, "{[\![\vec{x}.t]\!]_M}"] \arrow[dr] & & M[X] \\
& 1 \arrow[ur, "{M[a]}"'] &
\end{tikzcd}$$

(iii) When t is the variable $x_i\colon X_i$, $[\![\vec{x}.t]\!]_M$ is defined to be the ith product projection $\pi\colon M[X_1] \times \ldots \times M[X_n] \to M[X_i]$,

(iv) when t is of the form $f(t_1, \ldots, t_m)$ for some function symbol f and some terms $t_i\colon A_i$ in a suitable context $\vec{x} = (x_1, \ldots, x_n)$, then $[\![\vec{x}.t]\!]_M$ is defined to be the composite

$$\begin{tikzcd}
\prod\limits_{1\leq i\leq n} M[X_i] \arrow[rr, "{[\![\vec{x}.t]\!]_M}"] \arrow[dr, "{\langle [\![\vec{x}.t_1]\!]_M, \ldots, [\![\vec{x}.t_m]\!]_M \rangle}"'] & & M[A] \\
& \prod\limits_{1\leq i\leq m} M[A_i] \arrow[ur, "{M[f]}"'] &
\end{tikzcd}$$

Note that (ii) is just the nullary case of (iv). By an inductive argument on construction of terms, we can easily prove the following important property concerning interpretation of substitution of contexts. For instance the item (ii) is when the context $\vec{y}$ in below is empty.

PROPOSITION 3.1.14. Suppose a term $t\colon A$ in a context $\vec{y} = (y_1\colon Y_1, \dots, y_m\colon Y_m)$ is given, and $\vec{s} = (s_1\colon Y_1, \dots, s_m\colon Y_m)$ is a string of terms, each in the suitable context $\vec{x} = (x_1, \dots, x_n)$. Then $\llbracket\ \vec{x}.t[s_1/y_1, \dots, s_n/y_n]\ \rrbracket_M$ is interpreted as the composite of arrows in below:

$$\begin{tikzcd}
\prod\limits_{1\leq i\leq n} M[X_i] \arrow[rr, "\llbracket\ \vec{x}.t[s_1/y_1,\dots,s_n/y_n]\ \rrbracket_M"] \arrow[dr, swap, "\langle \llbracket \vec{x}.s_1 \rrbracket_M,\dots,\llbracket \vec{x}.s_m \rrbracket_M \rangle"] & & M[A] \\
& \prod\limits_{1\leq i\leq m} M[Y_i] \arrow[ur, swap, "\llbracket \vec{y}.t \rrbracket_M"] &
\end{tikzcd}$$

Interpretation of formulae

For the interpretation of terms in a category $\mathcal{C}$ all we needed was for $\mathcal{C}$ to be finitely complete. However, for the interpretation of some formulae, we need more categorical structures depending on the range of logical operators ($=$,$\perp$, $\exists$, $\forall$, $\Rightarrow$, $\bigvee$, $\bigwedge$). Since we are concerned with the geometric logic, we shall focus on giving the interpretation to terms formed by $=, \top, \exists, \bigvee, \wedge$.

Formulae are interpreted as subobjects; given an interpretation M of signature Σ of a language L, we will interpret a formula ϕ in the context $\vec{x}$ as a subobject $\llbracket \vec{x}.\phi \rrbracket_M \rightarrowtail M[X_1] \times \ldots \times M[X_n]$. We do this by induction on construction of formula ϕ. Note that in the case of interpretation of atomic formulae, we need the category $\mathcal{S}$ of models to have all pullbacks (of monomorphims), equalizers, and in the case of interpretation of existential quantifications to have stable image factorizations. On the whole, regular categories suffice. For infinite joins, we need at least the structure of an infinitary coherent category (aka "geometric category", e.g. in [Joh02b, D2.1]). Grothendieck toposes are infinitary coherent.

(i) When ϕ is an atomic formula of the form $R(t_1, \ldots t_m)$ for a predicate/relation symbol $R \subset X_1, \ldots X_m$ and each t_i is a term of type X_i in context $\vec{y} = (y_1 \colon Y_1, \ldots, y_n \colon Y_n)$, for $1 \leq i \leq m$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the pullback

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]_M} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & M[R] \arrow[d, tail] \\
\prod\limits_{1\leq i\leq n} M[Y_i] \arrow[r, "{\langle [\![\vec{y}.t_1]\!]_M, \ldots, [\![\vec{y}.t_m]\!]_M \rangle}"'] & \prod\limits_{1\leq i\leq m} M[X_i]
\end{tikzcd}$$

(ii) When ϕ is an atomic formula of the form $(s = t)$ for terms s, t of sort A defined in a context $\vec{x}$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the equalizer

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]_M} \arrow[r, tail, dashed, "e"] & \prod\limits_{1\leq i\leq n} M[X_i] \arrow[r, shift left, "{[\![\vec{x}.s]\!]_M}"] \arrow[r, shift right, "{[\![\vec{x}.t]\!]_M}"'] & M[A]
\end{tikzcd}$$

(iii) When ϕ is $\top$, then $[\![\vec{x}.\phi]\!]_M$ is the top element of lattice $\mathcal{S}\mathrm{ub}(M[X_1] \times \ldots \times M[X_n])$.

(iv) When ϕ is $\psi \wedge \chi$, where ψ and χ are defined in the same context $\vec{x}$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the pullback of subobjects $[\![\vec{x}.\psi]\!]_M \rightarrowtail \prod\limits_{1\leq i\leq n} M[X_i]$ and $[\![\vec{x}.\chi]\!]_M \rightarrowtail \prod\limits_{1\leq i\leq n} M[X_i]$.

(v) When ϕ is $\psi \vee \chi$, where ψ and χ are defined in the same context $\vec{x}$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the union of subobjects $[\![\vec{x}.\psi]\!]_M \rightarrowtail \prod\limits_{1\leq i\leq n} M[X_i]$ and $[\![\vec{x}.\chi]\!]_M \rightarrowtail \prod\limits_{1\leq i\leq n} M[X_i]$. In practice we work in situations where $\mathcal{S}$ is a pretopos: then the union of subobjects can be constructed as the image of a morphism from a coproduct.

$$\begin{tikzcd}
& \llbracket \vec{x}.\phi \rrbracket_M \vee \llbracket \vec{x}.\psi \rrbracket_M \arrow[dr, tail] & \\
\llbracket \vec{x}.\phi \rrbracket_M + \llbracket \vec{x}.\psi \rrbracket_M \arrow[ur, two heads] \arrow[rr] & & \prod\limits_{1 \leq i \leq n} M[X_i]
\end{tikzcd}$$

In the case of a Grothendieck topos, this can be extended to infinite disjunctions since infinite set-indexed coproducts exist.

(vi) When ϕ is $(\exists y)\psi$ for some formula ψ in context $\vec{x}$, and variable y of sort Y, then the interpretation of ϕ in context $\vec{x}$ is given by the image of $m_\psi \circ \pi_0$, where m_ψ witnesses $\llbracket \psi \rrbracket_{\vec{x},y}$ as a subobject of the product $\prod\limits_{1 \leq i \leq n} M[X_i] \times M[Y]$.

$$\begin{tikzcd}
\llbracket \vec{x}, y.\psi \rrbracket_M \arrow[r, two heads] \arrow[d, tail, "m_\psi"'] & \llbracket \vec{x}.\phi \rrbracket_M \arrow[d, tail] \\
\prod\limits_{1 \leq i \leq n} M[X_i] \times M[Y] \arrow[r, "\pi_0"'] & \prod\limits_{1 \leq i \leq m} M[X_i]
\end{tikzcd}$$

Indeed, originally due to the great insight of Lawvere, there is a universal property to the content of the existential derivation rules which can be expressed by the adjunction $\exists_{\pi_0} \dashv \pi_0^*$ where the right adjoint is the reindexing functor. For a locally cartesian closed $\mathcal{S}$, we have the triple adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$ (the top row of the following diagram) which induces the corresponding triple adjoints $\exists_f \dashv f^* \dashv \forall_f$ (the bottom row of the following diagram) on the lattices of subobjects.

$$\begin{tikzcd}
\mathcal{C}/X \arrow[rr, bend left, "\Sigma_f"] \arrow[rr, bend right, "\Pi_f"'] & & \mathcal{C}/Y \arrow[ll, "f^*" description] \\
\\
\mathcal{S}\mathrm{ub}(X) \arrow[uu, hook] \arrow[rr, bend left, "\exists_f"] \arrow[rr, bend right, "\forall_f"'] & & \mathcal{S}\mathrm{ub}(Y) \arrow[ll, "f^*" description] \arrow[uu, hook]
\end{tikzcd}$$

Each inclusion functor on the sides has a left adjoint which is defined by the image factorization.

Interpretation of sequents and models of theories

Suppose $\mathbb{T}$ is a first order theory with the signature Σ. For a Σ-structure M, we say that M satisfies a sequent $\phi \vdash_{\vec{x}} \psi$ whenever $[\![\vec{x}.\phi]\!]_M \leq [\![\vec{x}.\psi]\!]_M$ in the lattice $\mathcal{S}\mathrm{ub}(\prod_{1 \leq i \leq n} M[X_i])$. Note that this is more than saying that every global element of $[\![\vec{x}.\phi]\!]_M$ is also a global element of $[\![\vec{x}.\psi]\!]_M$, since there might not be enough global elements: the condition of satisfiability of sequents is equivalent to stating that every generalized element of $[\![\vec{x}.\phi]\!]_M$ is also a generalized element of $[\![\vec{x}.\psi]\!]_M$.

An interpretation M is a model of $\mathbb{T}$ if every axiom sequent in the theory is satisfied by M. The category $\mathbb{T}\text{-}\mathbf{Mod}(\mathcal{S})$ of models of $\mathbb{T}$ in $\mathcal{S}$ is a full subcategory of $\Sigma\text{-}\mathbf{Str}$. For any theory homomorphism $F \colon \mathbb{T}_0 \to \mathbb{T}_1$, we have a functor $F^* \colon \mathcal{S}\text{-}\mathbf{Mod}(\mathbb{T}_1) \to \mathcal{S}\text{-}\mathbf{Mod}(\mathbb{T}_0)$ which is called the *F-reduct functor*: it takes a model M to F^*M where the latter is defined on sorts and formulae by

$$[\![(-)]\!]_{F^*M} := [\![F(-)]\!]_M$$

.

Sometimes the reduct functor has a left adjoint: For instance, if both $\mathbb{T}_0$ and $\mathbb{T}_1$ are cartesian theories, then the reduct functor has a left adjoint. A special case of this occurs when $\mathbb{T}_0$ is the empty theory, and the left adjoint to the reduct functor gives the *initial* $\mathbb{T}_1$-model in the category $\mathcal{S}$.

3.1.4 Model morphisms and elementary embeddings

Let Σ be the signature for the theory of groups. A Σ-morphism between models G and H is a group homomorphism $f\colon G \to H$, because of commutativity of diagram (3.2). However, the commutativity of this diagram does not extend to all first-order formulae. To see this, consider the formula $\phi(x) = (\forall y, z)(R(x, y, z) \iff R(y, x, z))$. For a model G of $\mathbb{T}$, $G[\phi]$ is the centre of G, i.e. all elements of G which commute with every element of G. It is obvious that ϕ is not natural with respect to all group homomorphisms since elements of the centre are not necessarily preserved by group homomorphisms. Here is another example: take the formula $\phi(x) = \neg(\exists y)(x = y + y)$. If a is an element of G which is 'not divisible by 2', then commutativity of (3.2) for ϕ would mean that $f(a)$ could not be divisible by 2 in H. An arbitrary group homomorphism need not have this property: e.g. the homomorphism $i\colon \mathbb{Z}_4 \to \mathbb{Z}_{12}$ of (cyclic) groups with $i(1) = 6$.

Note that in both examples above we have used logical operators ($\Rightarrow, \neg$) which are not geometric. It is worth noting that the commutativity of diagram (3.2) does indeed extend to all formulae in geometric logic (See Proposition 3.1.20). The rest of the commentary of this section is illustrating the extra stuff that is needed if we go beyond the geometric logic.

To ensure naturality of all formulae with respect to model morphisms we can build it into a stronger notion of morphism of structures/models. Perhaps we should elaborate at this stage on significance naturality other than its categorical significance. Consider the following question: Let $\mathbb{T}$ be a (fragment of) first-order theory. Suppose that, for every (set) model M of $\mathbb{T}$, we specify a subset $\widetilde{M} \subset M$. Under what conditions does there exist a formula $\phi(\vec{x})$ in the language of $\mathbb{T}$ such that $\widetilde{M} = M[\phi]$ for every model M?

We note that the existence of such formula gives a uniformity in choosing the subsets $\widetilde{M} \subset M$. Therefore, at the very least, we need to demand that the subsets $\widetilde{M}$ have some relation to one another as the model M "varies". To formulate this notion more precisely, we give the definition of *elementary embedding* of models. It will follow that if the answer to the question above is yes, then for every elementary embedding $f \colon M \to N$, we must have $\widetilde{M} = f^{*}\widetilde{N}$. So, we arrived at a necessary condition for the question above to have an affirmative answer.

DEFINITION 3.1.15. Suppose $\mathbb{T}$ is a first order theory and M and N are models of $\mathbb{T}$ in a cartesian category $\mathcal{C}$. Consider a formula ϕ in the context $(x_1 : X_1, \ldots, x_n : X_n)$ in the language of $\mathbb{T}$. Let $f \colon M \to N$ be a Σ-morphism of models of $\mathbb{T}$. Consider the diagram below:

$$\begin{tikzcd}
\llbracket M \rrbracket_{\vec{x}.\phi} \arrow[r, tail] & M[X_1] \times \ldots \times M[X_n] \arrow[d, "f_1 \times \ldots \times f_n"] \\
\llbracket N \rrbracket_{\vec{x}.\phi} \arrow[r, tail] & N[X_1] \times \ldots \times M'[X_n]
\end{tikzcd} \tag{3.5}$$

The morphism $f \colon M \to N$ is called

(i) **elementary** whenever for every *first-order formula* ϕ, the diagram above can be completed to a commutative diagram. (Notice that any such morphism $M[\phi(\vec{x})] \to N[\phi(\vec{x})]$ that completes the diagram is necessarily unique.)

(ii) **embedding** whenever for every *atomic formula*, the diagram above can be completed to a pullback diagram in $\mathcal{C}$. In this situation, f exhibits M as a **substructure/submodel** of N.

(iii) **elementary embedding** whenever for every *first-order formula* ϕ in the language of $\mathbb{T}$, the diagram above can be completed to a pullback diagram in $\mathcal{C}$.

REMARK 3.1.16. Note that the notion of "elementary" morphism of models is meant to depend on the underlying logic. [Joh02b, p. D1.2.10] defines it only for homomorphisms between structures in Heyting categories, and we take that to mean it is with respect to all first-order formulae. Most logicians would understand "elementary" as conveying the restriction on arbitrary structure homomorphisms that allows naturality for negation, implication, and the universal quantification.

REMARK 3.1.17. It is instructive to write down the conditions above in set notation: (i) says that for every formula ϕ as above and every n-tuple $(a_1, \ldots, a_n) \in M[X_1] \times \ldots \times M[X_n]$, we have

$$M \models \phi(a_1, \ldots, a_n) \implies N \models \phi(f(a_1), \ldots, f(a_n)) \tag{3.6}$$

(iii) says that

$$M \models \phi(a_1, \ldots, a_n) \iff N \models \phi(f(a_1), \ldots, f(a_n)) \tag{3.7}$$

And (ii) says the latter is only valid for atomic formulae.

REMARK 3.1.18. Any embedding and therefore any elementary embedding is a monomorphism.

Proof. Apply definition (3.1.15) to the formula $\phi(x, y) := (x = y)$, where x, y are some variables of a type X. If $\mathbb{T}$ does not have any types (hence, no variables) then existence of elementary embedding f between M and N says that $f = \mathrm{id}$ which is a monomorphism. □

REMARK 3.1.19. For structures/models in a Boolean coherent category every elementary morphism is an elementary embedding.

The examples from the beginning of this section suggest that the requirements in definition of elementary morphism may be too restrictive for morphisms of models. However, if our underlying logic is geometric, it turns out there is no such restrictiveness.

PROPOSITION 3.1.20. Let $\mathcal{C}$ be (at least) a cartesian category. Any Σ-morphism of models in $\mathcal{C}$ of a (at most) geometric theory $\mathbb{T}$ is elementary.

Proof. By induction of formation of geometric formulae and their interpretation. For more details see Lemma D.1.2.9 in [Joh02b]. □

3.2 Overview of sketches

Good expositions on theory of sketches are given in [BW05], [AR94, Chapter 1] and [Joh02b, p. D2]. We start by recalling the concept. We remark that our definition follows that of [Joh02b, D2] more closely and is different than definition of other two sources mentioned above. The technical difference is that we define a sketch by a directed graph and not a category; we needs graphs because finiteness is important, and a finite graph can generate an infinite category. Note that there is a forgetful functor from the category of categories to the category of directed graphs which for a category $\mathcal{C}$, gives its underlying graph $|\mathcal{C}|$. The *free functor*, the left adjoint to the forgetful functor, gives us the free category of a directed graph: it has objects for the vertices of the graph, it has morphisms for each generating edge in the graph together with morphisms for formal compositions of them.

REMARK 3.2.1. Suppose $\mathcal{C}$ is a category which has morphisms $f\colon a \to b$ and $g\colon b \to c$ and $h = g \circ f\colon a \to c$. Suppose $\mathcal{F}(|\mathcal{C}|)$ is the free category over the underlying graph of $\mathcal{C}$. In $\mathcal{F}(|\mathcal{C}|)$, $h \neq g \circ f$.

Before defining sketches, we need to introduce some preliminary concepts:

DEFINITION 3.2.2. Suppose G is a directed graph and $\mathcal{C}$ is a category.

(i) A **diagram** of shape G in $\mathcal{C}$ is a homomorphism $d\colon G \to |\mathcal{C}|$ of graphs.

(ii) A diagram $d\colon G \to |\mathcal{C}|$ is **commutative** whenever for any two paths[12] in G with the same source and same target, the two morphisms obtained in $\mathcal{C}$ by composition along the two paths are equal.

---

[12] i.e. a walk in which all vertices (except possibly the first and last) and all edges are distinct; it is given by a finite strings of edges. This string could well be empty in which case the composition along the corresponding path is assumed to be identity in the category.

(iii) A diagram $d\colon G \to |\mathcal{C}|$ is **finite** whenever G is a finite.

(iv) A diagram $d\colon G \to |\mathcal{C}|$ with an apex $g_0 \in G$ is a **cone** if for every vertex g distinct from g_0 there is a unique edge from g_0 to g and no edge from g to g_0. One can say from the *viewpoint* of apex the diagram commutes. For a cone $(d\colon G \to |\mathcal{C}|, g_0)$ with apex g_0, we call the the diagram formed by deleting g_0 and all outgoing edges from g_0 the *base diagram* of d.

(v) Dually, a diagram $d\colon G \to |\mathcal{C}|$ with an apex $g_0 \in G$ is a **cocone** if for every vertex g distinct from g_0 there is a unique edge from g to g_0 and no edge from g_0 to g. Similar to the above, every cocone has a base diagram.

EXAMPLE 3.2.3. Consider directed graphs G (left) and G' (right) in below.

$$\begin{tikzcd}
\mathsf{a} \arrow[loop above, "\mathsf{i}"] & & \mathsf{a} \arrow[r, "\mathsf{j}"] & \mathsf{b}
\end{tikzcd}$$

Let $\mathcal{C}$ be a non-empty category with at least one non-identity endomorphism, say f. Let $d\colon G \to |\mathcal{C}|$ be the diagram specified by $d(\mathsf{a}) = A$ and $d(\mathsf{i}) = f\colon A \to A$. Observe that d commutes if and only if $f = \mathrm{id}_A$. Now, consider the diagram $d'\colon G' \to |\mathcal{C}|$ with $d(\mathsf{a}) = A$, $d(\mathsf{b}) = A$, and $d(\mathsf{j}) = f$. Observe that d' commutes.

DEFINITION 3.2.4. A **limit sketch** $\mathbb{G}$ is a triple $\mathbb{G} = (G, D, L)$ where G is a directed graph, D is a specification of a set of finite diagrams in G, and L is a specification of a set of cones in G.

DEFINITION 3.2.5. A **model** M of a sketch $\mathbb{G}$ in a category $\mathcal{C}$ is a graph homomorphism $M\colon \mathbb{G} \to |\mathcal{C}|$ such that

(i) For each diagram $d\colon I \to G$ in D, the composite $M \circ d\colon I \to |\mathcal{C}|$ is a commutative diagram.

(ii) For each cone $(\ell\colon I \to G, i_0)$ in L with apex $i_0 \in I$, the image under $M \circ \ell\colon I \to |\mathcal{C}|$ form a limit cone in $\mathcal{C}$ with apex i_0 over the base of ℓ.

Note that if a sketch $\mathbb{G}$ does not have any cones, that is L is an empty specification, then a model M of $\mathbb{G}$ in a category $\mathcal{C}$ is essentially the same thing as a functor $\mathcal{F}(\mathbb{G})\big/\langle D\rangle \to \mathcal{C}$, where $\mathcal{F}(\mathbb{G})$ is the free category over sketch $\mathbb{G}$ and $\langle D\rangle$ is the smallest congruence on $\mathcal{F}(\mathbb{G})$ which is generated by identification of all parallel arrows in $\mathcal{F}(\mathbb{G})$ constructed from edges in D. In the case the sketch has cones, the story is a bit more complicated.

EXAMPLE 3.2.6. In this example we sketch the theory of commutative monoids. We denote the sketch by $\mathbb{CM}$. The graph $G_{\mathbb{CM}}$ is defined by four vertices $\mathsf{a}^0, \mathsf{a}^1, \mathsf{a}^2, \mathsf{a}^3$ and the following edges

$$\begin{tikzcd}
\mathsf{a}^3 \arrow[d, bend right=60, "\circ\times\mathrm{id}"'] \arrow[d, bend right=30, "\mathrm{id}\times\circ"'] \arrow[d, shift right=2, "\mathsf{p}^{1,2}"'] \arrow[d, "\mathsf{p}^{0,2}"] \arrow[d, shift left=2, "\mathsf{p}^{0,1}"] \arrow[dr, shift left=2, "\mathsf{p}^0"] \arrow[dr, "\mathsf{p}^1"] \arrow[dr, shift right=2, "\mathsf{p}^2"'] & & \\
\mathsf{a}^2 \arrow[loop left, "\sigma"] \arrow[r, shift left=2, "\mathsf{p}^0"] \arrow[r, "\circ"] \arrow[r, shift right=2, "\mathsf{p}^1"'] & \mathsf{a}^1 \arrow[l, bend right=30, "\mathrm{id}\times\mathsf{e}"'] \arrow[l, bend left=30, "\mathsf{e}\times\mathrm{id}"] \arrow[loop below, "\mathrm{id}"] & \mathsf{a}^0 \arrow[l, "\mathsf{e}"']
\end{tikzcd}$$

The idea is that p^i and $\mathsf{p}^{i,j}$ are meant to express various projections, $\circ$ is meant to express binary multiplication of monoid, and e the identity element with respect to multiplication. To achieve this we must introduce D and L as specification of diagrams and cones to be interpreted in the models by commutativities and limits cones according to Definition 3.2.5.

Take L to be the set of following cones (with respective apex $\mathsf{a}^0, \mathsf{a}^2, \mathsf{a}^3$ from left to right).

$$\begin{tikzcd}
 & & \mathsf{a}^2 \arrow[dl, "\mathsf{p}^0"'] \arrow[dr, "\mathsf{p}^1"] & & & \mathsf{a}^3 \arrow[dl, "\mathsf{p}^0"'] \arrow[d, "\mathsf{p}^1"] \arrow[dr, "\mathsf{p}^2"] & \\
\mathsf{a}^0 & \mathsf{a}^1 & & \mathsf{a}^1 & \mathsf{a}^1 & \mathsf{a}^1 & \mathsf{a}^1
\end{tikzcd}$$

Thus for any category $\mathcal{C}$ with finite limits, and any model M of this sketch, $M[\mathsf{a}^0]$ must the terminal object of $\mathcal{C}$, and $M[\mathsf{a}^2] \cong M[\mathsf{a}^1] \times M[\mathsf{a}^1]$, and $M[\mathsf{a}^3] \cong M[\mathsf{a}^1] \times M[\mathsf{a}^1] \times M[\mathsf{a}^1]$ and $M[\mathsf{p}^i]$ will be the corresponding product projection morphisms in $\mathcal{C}$. Therefore $M[\mathsf{a}^1] \times M[\mathsf{a}^1] \cong M[\mathsf{a}^2] \xrightarrow{\circ} M[\mathsf{a}^1]$ gives the binary multiplication in $\mathcal{C}$.

The set D of diagrams is comprised of

$$\begin{tikzcd}
a \arrow[loop above, "\mathrm{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& a^3 \arrow[dl, "p^0"'] \arrow[d, "p^{0,1}"] \arrow[dr, "p^1"] & \\
a^1 & a^2 \arrow[l, "p^0"] \arrow[r, "p^1"'] & a^1
\end{tikzcd}
\qquad
\begin{tikzcd}
& a^3 \arrow[dl, "p^1"'] \arrow[d, "p^{1,2}"] \arrow[dr, "p^2"] & \\
a^1 & a^2 \arrow[l, "p^0"] \arrow[r, "p^1"'] & a^1
\end{tikzcd}$$

where the first diagram ensures that id must be interpreted as identity morphism in $\mathcal{C}$ and the two others express that p^i and $p^{i,j}$ are appropriately interpreted as product projections. We also need to add two more diagrams to D in order to express the equations of the unit involving edges $\mathrm{id} \times e, e \times \mathrm{id}$. Additionally,

$$\begin{tikzcd}
a^3 \arrow[rr, "\circ \times \mathrm{id}"] \arrow[dr, "p^2"'] & & a^2 \arrow[dl, "p^1"] \\
& a^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
a^3 \arrow[rr, "\mathrm{id} \times \circ"] \arrow[dr, "p^0"'] & & a^2 \arrow[dl, "p^0"] \\
& a^1 &
\end{tikzcd}$$

$$\begin{tikzcd}
a^3 \arrow[r, "\circ \times \mathrm{id}"] \arrow[d, "p^{0,1}"'] & a^2 \arrow[d, "p^0"] \\
a^2 \arrow[r, "\circ"'] & a^1
\end{tikzcd}
\qquad
\begin{tikzcd}
a^3 \arrow[r, "\mathrm{id} \times \circ"] \arrow[d, "p^{1,2}"'] & a^2 \arrow[d, "p^1"] \\
a^2 \arrow[r, "\circ"'] & a^1
\end{tikzcd}$$

belong to D which express the role of $\mathrm{id} \times \circ$ and $\circ \times \mathrm{id}$, and

$$\begin{tikzcd}
a^3 \arrow[r, "\circ \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \circ"'] & a^2 \arrow[d, "\circ"] \\
a^2 \arrow[r, "\circ"'] & a^1
\end{tikzcd}$$

expresses the associativity of binary product, and

$$\begin{tikzcd}
a^2 \arrow[r, "\sigma"] & a^2 \arrow[r, "\sigma"] & a^2
\end{tikzcd}
\qquad
\begin{tikzcd}
& a^2 \arrow[dl, "p^0"'] \arrow[d, "\sigma"] \arrow[dr, "p^1"] & \\
a^1 & a^2 \arrow[l, "p^1"] \arrow[r, "p^0"'] & a^1
\end{tikzcd}
\qquad
\begin{tikzcd}
a^2 \arrow[rr, "\sigma"] \arrow[dr, "\circ"'] & & a^2 \arrow[dl, "\circ"] \\
& a^1 &
\end{tikzcd}$$

express the role of σ as a switch operator and also the commutativity of the binary product.

REMARK 3.2.7. The sketch above is by no means the unique sketch which presents the theory of commutative monoids; it is in fact the minimal such sketch. We could have as well added edges such as $! \colon \mathsf{a}^1 \to \mathsf{a}^0$, other identity edges $\mathsf{id} \colon \mathsf{a}^2 \to \mathsf{a}^2$ and $\mathsf{id} \colon \mathsf{a}^3 \to \mathsf{a}^3$, etc. We also could have added more equations, by adding to the set D diagrams like

$$\begin{tikzcd}
 & \mathsf{a}^1 \arrow[dl, "\mathsf{id} \times \mathsf{e}"'] \arrow[dr, "\mathsf{e} \times \mathsf{id}"] & \\
\mathsf{a}^2 \arrow[rr, "\sigma"'] & & \mathsf{a}^2
\end{tikzcd}$$

Notwithstanding these additions, a models in any category (with finite limits) would remain the same which is exactly an internal commutative monoid.

3.3 The 2-category $\mathfrak{Con}$ of AU-contexts

In this section we are going to give a brief summary of main aspects of the theory of AU-sketches and AU-contexts as developed in [Vic19]. We give a handful of examples, each illustrating some concept of the theory, but we shall avoid repeating proofs of [Vic19]. The exact references to various results of Vickers' paper are given so that the reader could find proofs of various claims which appear in this section.

The observation underlying [Vic19] is that important geometric theories can be expressed in coherent logic (no infinite disjunctions), provided that new sorts can be constructed in a type-theoretic style that includes free algebra constructions. Models can then be sought in any arithmetic universe (list-arithmetic pretopos), and that includes any elementary topos with nno; moreover, the inverse image functors of geometric morphisms are AU-functors.

If a geometric theory $\mathbb{T}$ can be expressed in an 'arithmetic way', then we can compare its models in AUs and in Grothendieck toposes. One advantage of working with AUs over toposes is, usually when working with toposes, infinities we use (for example for infinite disjunction), are supplied extrinsically by base topos $\mathcal{S}$, however, the infinities in $\mathbf{AU}\langle\mathbb{T}\rangle$ come from the intrinsic structures of arithmetic universes, e.g. parametrized list object which at the least gives us

$\mathbb{N} := \mathsf{List}(1)$, $\mathbb{Q}$, and $\mathbb{R}$. In below, we illustrate some of the differences between the AU approach and the topos approach. To see more details about expressive power of AUs we refer the reader to [MV12].

	Arithmetic Universes	Grothendieck toposes
Classifying category	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathscr{S}[\mathbb{T}]$
$\mathbb{T}_1 \to \mathbb{T}_2$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathscr{S}[\mathbb{T}_1] \to \mathscr{S}[\mathbb{T}_2]$
Base	Base independent	Base $\mathscr{S}$
Infinities	Intrinsic; provided by List e.g. $N = \mathsf{List}(1)$	Extrinsic; got from $\mathscr{S}$ e.g. infinite coproducts
Results	A single result in AUs	A family of results by varying $\mathscr{S}$

The system developed in [Vic19] expresses those geometric theories using *sketches*. They are, first of all, finite-limit-finite-colimit sketches: an AU-sketch is a reflexive graph with designated commutativities, initial and terminal objects, pullbacks and pushouts, and list objects. From these we can easily construct, for example, the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$.

A model of a sketch $\mathbb{T}$ in an AU $\mathcal{A}$ is a graph morphism into the underlying reflexive graph of $\mathcal{A}$ which actualizes the designated universals in the AU. If the AU is equipped with chosen limits, colimits, and list objects, then one can distinguish between 'strict' models and 'models up to isomorphism'.

In general, non-strict models cannot be strictified since the same node can be marked as being part of different universals, which may be isomorphic, but not equal in a given AU.

However, AU-sketches that are generated by successively adjoining universals to the empty sketch (in particular without identifying nodes), do admit strictification, as is shown in [Vic19]. These special sketches are called contexts and they are the objects of a 2-category $\mathfrak{Con}$. In [Vic19] it is shown that $\mathfrak{Con}$ admits

PIE-limits and embeds fully and faithfully into the opposite of the category of AUs and strictly structure preserving functors.

An **AU-sketch** is a formalization of the sketches (discussed in §3.2), but fine-tuned for AUs. Any AU-sketch can be used as a system of generators (the nodes and edges) and relations to present an AU. More precisely, we have various structures for sorts and operations shown in the diagram below.

$$\begin{tikzcd}
\mathrm{U}^{\mathrm{pb}} \arrow[d, shift right, "\Gamma^1"'] \arrow[d, shift left, "\Gamma^2"] & \mathrm{U}^{\mathrm{list}} \arrow[l, "\Lambda_2"'] \arrow[r, "\Lambda_0"] \arrow[d, shift right, "\mathrm{c}"'] \arrow[d, shift left, "\mathrm{e}"] & \mathrm{U}^1 \arrow[d, "\mathrm{tm}"] \\
\mathrm{G}^2 \arrow[r, "\mathrm{d}_i\ (i=0{,}1{,}2)"] & \mathrm{G}^1 \arrow[r, shift left, "\mathrm{d}_i\ (i=0{,}1)"] & \mathrm{G}^0 \arrow[l, shift left, "\mathrm{s}"] \\
\mathrm{U}^{\mathrm{po}} \arrow[u, shift left, "\Gamma_1"] \arrow[u, shift right, "\Gamma_2"'] & & \mathrm{U}^0 \arrow[u, "\mathrm{i}"']
\end{tikzcd} \tag{3.8}$$

Here, the elements of G^0, G^1, and G^2 are respectively called **nodes**, **edges**, and **commutativities**.

In comparing with our presentation of first order theories in 3.1, nodes play the role of the sorts, edges play the role of function symbols, and commutativities enable us to write equations between terms. The operations d_0 and d_1 (of diagram (3.8)) give domains and codomains of edges, respectively, while s introduces the identity edge for each node. From $\mathrm{G}^1, \mathrm{G}^0, \mathrm{d}_0, \mathrm{d}_1, \mathrm{s}$, we get a reflexive graph of nodes and edges. A **triangle** in a sketch is given by edges $\mathsf{u}, \mathsf{v}, \mathsf{w}$ such that $\mathrm{d}_0(\mathsf{u}) = \mathrm{d}_0(\mathsf{w})$, $\mathrm{d}_0(\mathsf{v}) = \mathrm{d}_1(\mathsf{u})$, and $\mathrm{d}_1(\mathsf{v}) = \mathrm{d}_1(\mathsf{w})$. We depict such a triangle as $\begin{tikzcd} \mathsf{X} \arrow[r, "\mathsf{u}"'] \arrow[rr, bend left, "\mathsf{w}"] & \mathsf{Y} \arrow[r, "\mathsf{v}"'] & \mathsf{Z} \end{tikzcd}$. The operations $\mathrm{d}_0, \mathrm{d}_2, \mathrm{d}_1 \colon \mathrm{G}^2 \to \mathrm{G}^1$ stipulate *commutative* triangles $\begin{tikzcd} {} \arrow[r, "\mathrm{d}_0(\omega)"'] \arrow[rr, bend left, "\mathrm{d}_1(\omega)"] & \bullet \arrow[r, "\mathrm{d}_2(\omega)"'] & {} \end{tikzcd}$, for any element $\omega \colon \mathrm{G}^2$. We write $\mathsf{uv} \sim_{\mathsf{XYZ}} \mathsf{w}$ for the mere *existence* of a commutativity with that triangle. By the *unary commutativity* $\mathsf{u} \sim_{\mathsf{XY}} \mathsf{u}'$, we mean a commutativity $\mathrm{s}(\mathsf{X})\mathsf{u} \sim_{\mathsf{XXY}} \mathsf{u}'$.

The elements of the other sorts are *universals*, and specify universal properties of their *subjects*. For example, an element of U^{pb} is a *pullback universal* and corresponds to a limit cone in a finite limit sketch. Its subjects are the pullback node and the three projection edges of the pullback cone. We obtain these by using the triangle projection operators Γ^1, Γ^2 (to get the two halves of the pullback square), and further by node and edge projection operators $\mathrm{d_i}$. Similarly, the operator tm takes $\omega\colon \mathrm{U}^1$ to its subject node. We have a dual situation for pushout and initial universals.

An element of $\mathrm{U}^{\mathrm{list}}$ is a *list universal*. Its subjects are the list object and the two structure maps ε and cons. It will also have indirect subjects, since it needs terminal and pullback universals to express the domains of the structure maps. More, precisely, for an element $\omega \in \mathrm{U}^{\mathrm{list}}$ (aka a list universal), the terms $\mathrm{e}(\omega)$ and $\mathrm{c}(\omega)$ are the primary structure morphisms ε and cons for $\mathsf{List}(A(\omega))$, where $A(\omega) = \mathrm{d}_1(\mathrm{d}_0(\Gamma^1(\Lambda_2(\omega))))$. The domains of the structure morphisms (1 and $A(\omega) \times \mathsf{List}(A(\omega))$) are limits, and Λ_0, Λ_2 supply universals to stipulate them.Note that the terminal needed for a product is taken to be the special case of pullback.

We commonly write the subjects, and those of the dependent limit universals, e.g. in a diagram of the form

$$\begin{tikzcd}
\mathsf{T} \arrow[r, "\varepsilon"] & \mathsf{L} & \mathsf{P} \arrow[l, shift right, "\mathsf{cons}"'] \arrow[l, shift left, "\mathsf{p}_1"] \arrow[r, "\mathsf{p}_2"'] & \mathsf{A}\,.
\end{tikzcd} \tag{3.9}$$

where the node T is terminal universal, edges p_1 and p_2 are a product cone making P a product (special from of pullback universal) $\mathsf{A} \times \mathsf{L}$, and ε and cons are the structure morphisms to make L a list object for A.

A **homomorphism** of AU-sketches preserves all structures: it is given by a family of carriers for each sort that also preserves operators, and it maps nodes to nodes, edges to edges, commutativities to commutativities and universals to universals.

We shall need to restrict the sketches to **AU-contexts**. These are built up as **extensions** of the empty sketch $\mathbb{1}$, each extension a finite sequence of **simple extension** steps of the following types: adding a new primitive node, adding a new edge, adding a commutativity, adding a terminal, adding an initial, adding a pullback universal, adding a pushout, and adding a list object. From now on, we shall refer to an AU-context simply as a *context*.

REMARK 3.3.1. An important point about sorts of a context is equality between them: it is an equality that refers to strictness. Any sort is equal to itself. Starting from equal data, the derived sorts constructed in the same way from that data are equal. For example, if $\mathsf{X} = \mathsf{Y}$ then $\mathsf{List}(\mathsf{X}) = \mathsf{List}(\mathsf{Y})$.

For nodes, equality is witnessed by certain edges between them that, in any strict model, will have to be interpreted as identity morphisms between equal objects. The base case is *identity edges* of the form $\mathsf{s}(\mathsf{X})$ (for some node X) in the sketch. Inductively we also have the fillins for limits/colimits/list nodes defined over data for which we already have such edges (e.g. consider extending by a pullback universal over two opspans whose corresponding sorts are equal). Vickers ([Vic19]) proves that these edges are *unique*, when they exist, and gives an equivalence relation on nodes. The uniqueness here is up to *edge equality*: for two edges, equality is witnessed by a commutative square (i.e. two commutativities) with the two given edges and two identity edges. Existence of the equalities is decidable. If two nodes are introduced in different ways then they are not objectively equal; otherwise by recursion through the data from which they are constructed we can prove their equality.

Note that some of these simple extensions does not have any effect on (strict) models since they do add nothing new to the (strict) models of the sketch in arithmetic universes/toposes.

The following is an example of simple extension by adding a pullback universal.

EXAMPLE 3.3.2. Suppose $\mathbb{T}_0$ is a context and X_0 and X_1 are two nodes in it. Consider its equivalent extension $\mathbb{T}_1 = \mathbb{T}_0 + \delta\mathbb{T}_0$ by a terminal node with

$$\begin{aligned}\delta \mathrm{U}^1 &= \{*\}\\ \delta \mathrm{G}^0 &= \{\mathrm{tm}(*)\}\\ \delta \mathrm{G}^1 &= \{\mathrm{s}(\mathrm{tm}(*))\}\end{aligned}$$

Here by $\mathbb{T}_1 = \mathbb{T}_0 + \delta\mathbb{T}_0$, we mean that for every sort Ξ of sketch $\mathbb{T}$, the set $(\mathbb{T}_1)_\Xi$ of elements of sort Ξ can be expressed as a coproduct $\mathbb{T}_\Xi + \delta\Xi$, with a coproduct injection $\mathbb{T}_\Xi \to (\mathbb{T}_1)_\Xi$ and that $\delta\Xi$ is a *strongly finite set* (i.e. isomorphic to a finite cardinal $\{1, \dots, n\}$ for some $n \in \mathbb{N}$).

EXAMPLE 3.3.3. Suppose $\mathbb{T}$ is a sketch that already contains data in the form of a opspan of edges: $\xrightarrow{u_1}\xleftarrow{u_2}$. Then we can make a simple extension of $\mathbb{T}$ to $\mathbb{T}'$ by adding a pullback universal for that opspan, a cone in the form

$$\begin{tikzcd} \mathsf{P} \arrow[r, dotted, "\mathsf{p}^2"] \arrow[d, dotted, "\mathsf{p}^1"'] \arrow[dr, dotted, "\mathsf{p}"] \arrow[dr, phantom, "\bullet" near start, shift left=2] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] \arrow[r, phantom, "\bullet", shift left=2] & {} \end{tikzcd}$$

Along with the new universal itself, we also add a new node P, the pullback; four new edges (the projections $\mathsf{p}^1, \mathsf{p}^2, \mathsf{p}$ and the identity for P) and two commutativities $u_1\mathsf{p}^1 \sim \mathsf{p}$ and $u_2\mathsf{p}^2 \sim \mathsf{p}$. So, more precisely, what is added is $\delta\mathbb{T}$:

$$\delta \mathrm{U}^{\mathrm{pb}} = \left\{ \begin{tikzcd} \mathsf{P} \arrow[r, dotted, "\mathsf{p}^2"] \arrow[d, dotted, "\mathsf{p}^1"'] \arrow[dr, dotted, "\mathsf{p}"] \arrow[dr, phantom, "\bullet" near start, shift left=2] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] \arrow[r, phantom, "\bullet", shift left=2] & {} \end{tikzcd} \right\}$$

$$\delta \mathrm{G}^2 = \{\mathsf{p}^1 u_1 \sim \mathsf{p}, \mathsf{p}^2 u_2 \sim \mathsf{p}\}$$

$$\delta \mathrm{G}^1 = \{\mathsf{p}^1, \mathsf{p}, \mathsf{p}^2, \mathrm{s}(\mathsf{P})\}$$

$$\delta \mathrm{G}^0 = \{\mathsf{P}\}$$

where $\sim$ signifies a commutativity.

An important feature of extensions is that the subjects of the universals (for instance, P and the projections in the above example) must be *fresh* – not already in the unextended sketch. This avoids the possibility of giving a single node two different universal properties, and allows the property that every non-strict model has a canonical strict isomorph (e.g. if we were able to impose an equality between two derived sorts such as List(X) and Y × Z it would violate the canonical strict isomorph theorem).

The next fundamental concept is the notion of *equivalence extension*. This is an extension that can be expressed in a sequence of steps for which each introduces structure that must be present, and uniquely, given the structure in the unextended sketch. Unlike an ordinary extension, we cannot arbitrarily add nodes, edges or commutativities – they must be justified. Examples of equivalence extensions are to add composite edges; commutativities that follow from the rules of category theory; pullbacks, fillins and uniqueness of fillins, and similarly for terminals, initials, pushouts and list objects; and inverses of edges that must be isomorphisms by the rules of pretoposes. Thus the presented AUs for the two contexts are isomorphic.

EXAMPLE 3.3.4. In the case of pullback universal, new edges arise as universal structure edges and fillins.

- A simple extension for a pullback universal is also an equivalence extension.

- Suppose we have a pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ where ω is given as

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\mathsf{p}^2"] \arrow[d, "\mathsf{p}^1"'] \arrow[dr, "\mathsf{p}"] \arrow[dr, phantom, "\bullet", shift left=2ex] \arrow[dr, phantom, "\bullet"', shift right=2ex] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] & {} \end{tikzcd}$$

and π_1, π_2 are commutativities

$$\begin{tikzcd} {} \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, "v"] \arrow[dr, phantom, "\bullet", shift left=2ex] \arrow[dr, phantom, "\bullet"', shift right=2ex] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] & {} \end{tikzcd}$$

with equations

$$\begin{aligned} \mathrm{d}_2(\pi_i) &= \mathrm{d}_2(\Gamma^i(\omega)) = u_i \\ \mathrm{d}_1(\pi_1) &= \mathrm{d}_1(\pi_2) = v. \end{aligned}$$

specifying that π_1, π_2 is another cone on the same data. Then our equivalence extension has

$$\begin{aligned} \delta \mathrm{G}^1 &= \{w = \langle v_1, v_2 \rangle_{u_1, u_2}\} \\ \delta \mathrm{G}^2 &= \{w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2\}. \end{aligned}$$

- Suppose we have a pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ as above, and edges v_1, v_2, w, w' with commutativities $w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2, w'\mathsf{p}^1 \sim v_1, w'\mathsf{p}^2 \sim v_2$. Then our equivalence extension has

$$\delta\mathrm{G}^2 = \{w \sim w'\}.$$

EXAMPLE 3.3.5. We construct the **Sierpinski context** $\mathbb{S}$ by adding two nodes I and 1 where 1 is a terminal node and a 'mono' edge $\mathsf{i}\colon \mathsf{I} \rightarrowtail 1$, where being mono is expressed by two commutativities $\mathsf{si} \sim \mathsf{i}$ and $\pi^1_{\mathsf{i},\mathsf{i}}s \sim \pi^2_{\mathsf{i},\mathsf{i}}$ in an equivalence extension $\mathbb{S}'$ of $\mathbb{S}$.[13]

$$\begin{tikzcd}
\mathsf{P}_{\mathsf{i},\mathsf{i}} \arrow[r, dashed, "\pi^2_{\mathsf{i},\mathsf{i}}"] \arrow[d, dashed, "\pi^1_{\mathsf{i},\mathsf{i}}"'] \arrow[dr, phantom, "\bullet", near start] & \mathsf{I} \arrow[d, "\mathsf{i}"] \\
\mathsf{I} \arrow[r, "\mathsf{i}"'] \arrow[ur, dashed, "\mathsf{s}"] \arrow[ur, phantom, "\bullet", near end, shift right=2ex] & 1
\end{tikzcd}$$

where $\mathsf{P}_{\mathsf{i},\mathsf{i}}$ is the subject of a pullback universal of i along itself.

Any sketch homomorphism between contexts gives a model reduction map (in the reverse direction), but those are much too rigidly bound to the syntax to give us a good general notion of model map. We seek something closer to geometric morphisms, and in fact we shall find a notion of *context map* that captures exactly the strict AU-functors between the corresponding arithmetic universes $\mathbf{AU}\langle\mathbb{T}\rangle$. A **context map** $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ is a sketch homomorphism from $\mathbb{T}_1$ to some equivalence extension $\mathbb{T}_0'$ of $\mathbb{T}_0$. In picture, it is given as an opspan:

$$\begin{tikzcd}
\mathbb{T}_0 \arrow[r, "E", "\in"'] & \mathbb{T}_0' & \mathbb{T}_1 \arrow[l, "F"']
\end{tikzcd}$$

where F is a sketch extension morphism and E an sketch equivalence. We think of a context map $\mathbb{T}_0 \to \mathbb{T}_1$ as a translation F from $\mathbb{T}_1$ into a context equivalent to $\mathbb{T}_0$. We can say morphisms $\mathbb{T}_0 \to \mathbb{T}_1$ are models of $\mathbb{T}_2$ in "stuff

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[13] The upper commutativity is being considered here to express $\pi^1_{\mathsf{i},\mathsf{i}} \sim \pi^2_{\mathsf{i},\mathsf{i}}$. The lower commutativity already existed as derived data for $\mathsf{i} \sim \mathsf{i}$.

derivable from $\mathbb{T}_1$". Still put in terms of classifying AUs and strict AU-functors we get an opspan

$$\mathbf{AU}\langle \mathbb{T}_0\rangle \xrightarrow[\cong]{\mathbf{AU}\langle E\rangle} \mathbf{AU}\langle \mathbb{T}_0'\rangle \xleftarrow{\mathbf{AU}\langle F\rangle} \mathbf{AU}\langle \mathbb{T}_1\rangle$$

Since E is an equivalence extension, $\mathbf{AU}\langle E\rangle$ is an isomorphism ([Vic19, Proposition 18]). Each model M of $\mathbb{T}_0$ gives – by the properties of equivalence extensions – a model of $\mathbb{T}_0'$, and then by model reduction along the sketch homomorphism it gives a model $M \centerdot H$ of $\mathbb{T}_1$.

Thus context maps embody a localization by which equivalence extensions become invertible. Of course, every sketch homomorphism is, trivially, a map in the reverse direction. Context extensions are sketch homomorphisms, and the corresponding maps backwards are *context extension maps*. They have some important properties, which we shall see in the next section. We emphasize that context maps (1-morphisms in the 2-category $\mathfrak{Con}$ of AU-contexts) 'go in the geometric direction' rather than the algebraic one, i.e. if $\mathbb{T}$ is obtained from $\mathbb{S}$ by adjoining new structure, then the corresponding extension map goes in direction $\mathbb{T} \to \mathbb{S}$.

At this point let us introduce the important example of the **hom context** $\mathbb{T}^{\to}$ of a context $\mathbb{T}$. We first take two disjoint copies of $\mathbb{T}$ distinguished by subscripts 0 and 1, giving two sketch homomorphisms $i_0, i_1 \colon \mathbb{T} \to \mathbb{T}^{\to}$. Second, for each node X of $\mathbb{T}$, we adjoin an edge $\theta_{\mathsf{X}} \colon \mathsf{X}_0 \to \mathsf{X}_1$. Also, for each edge $\mathsf{u} \colon \mathsf{X} \to \mathsf{Y}$ of $\mathbb{T}$, we adjoin a connecting edge $\theta_{\mathsf{u}} \colon \mathsf{X}_0 \to \mathsf{Y}_1$ together with two commutativities:

$$\begin{tikzcd}
\mathsf{X}_0 \arrow[r, "\theta_{\mathsf{X}}", "\bullet"'] \arrow[d, "\mathsf{u}_0"'] \arrow[dr, "\theta_{\mathsf{u}}"] & \mathsf{X}_1 \arrow[d, "\mathsf{u}_1"] \\
\mathsf{Y}_0 \arrow[r, "\bullet", "\theta_{\mathsf{Y}}"'] & \mathsf{Y}_1
\end{tikzcd}$$

A model of $\mathbb{T}^{\to}$ comprises a pair M_0, M_1 of models of $\mathbb{T}$, together with a homomorphism $\theta \colon M_0 \to M_1$. In particular, a model of $\mathbb{O}^{\to}$ in a topos $\mathscr{A}$ is exactly a morphism in $\mathscr{A}$. We can define diagonal context map $\delta_{\mathbb{T}} \colon \mathbb{T} \to \mathbb{T}^{\to}$ by the opspan (id, F) of sketch morphisms where F sends edges θ_{X} to $\mathsf{s}(\mathsf{X})$, θ_{u}

to u and commutativities to degenerate commutativities of the form $\mathsf{us(X)} \sim \mathsf{u}$ and $\mathsf{s(Y)u} \sim \mathsf{u}$.

We define a **2-morphism** between context maps $H_0, H_1 \colon \mathbb{T}_0 \to \mathbb{T}_1$ to be a map $H \colon \mathbb{T}_0 \to \mathbb{T}_1^{\to}$ which composes with the maps $i_0, i_1 \colon \mathbb{T}_1^{\to} \to \mathbb{T}_1$ to give H_0 and H_1.

Finally, an **objective equality** between context maps H_0 and H_1 is a 2-morphism for which the homomorphism between strict models must always be an identity. This typically arises when a context introduces the same universal construction twice on the same data.

Let us explain the last point in more details: the (intensional) equality between context maps $f, g \colon \mathbb{T}_1 \rightrightarrows \mathbb{T}_2$ is formulated in [Vic19] by using a common refinement of equivalence extensions, and therefore, we can assume that they are both sketch homomorphisms from $\mathbb{T}_2 \to \mathbb{T}_1'$ where $\mathbb{T}_1'$ is an equivalent extension of context $\mathbb{T}_1$. Thus, every sketch ingredient in $\mathbb{T}_2$ is taken to one of the same kind in $\mathbb{T}_1'$.

We define the equality in two stages. First, an "object equality" is for ingredients already in T1 that serve to witness the equality between f and g. After that, "objective equality" is for when those ingredients can be derived, using an equivalence extension of T1.

From these material [Vic19] constructs the 2-category $\mathfrak{Con}$ whose objects are contexts, morphisms are context maps modulo objective equality, and 2-morphisms are 2-morphisms. It has all PIE-limits (limits constructible from products, inserters, equifiers). Although it does not possess all (strict) pullbacks of arbitrary maps, it has all (strict) pullbacks of context extension maps along any other map.

For instance in $\mathfrak{Con}$, the Sierpinski context $\mathbb{S}$ defined in Example 3.3.5 has two global points $\bot, \top \colon \mathbb{1} \rightrightarrows \mathbb{S}$ where the terminal context $\mathbb{1}$ has empty sketch. These global points correspond to the sketch homomorphisms $F, F' \colon \mathbb{S} \rightrightarrows \mathbb{1}'$ where $\mathbb{1}'$ is the extension of the terminal context by an initial node and a terminal node, and F and F' take the node I of the sketch of $\mathbb{S}$ to the initial and terminal node of $\mathbb{1}'$, respectively. It is easily checked that there is indeed a 2-morphism $\bot \Rightarrow \top$ analogous to the specialization order for the Sierpinski space.

We now list some of the most useful examples of AU-contexts. For more examples see [Vic19, §3.2].

EXAMPLE 3.3.6. The context $\mathbb{O}$ has nothing but a single node, X, and an identity edge $s(X)$ on X. A model of $\mathbb{O}$ in an AU (or topos) $\mathscr{A}$ is a "set" in the broad sense of an object of $\mathscr{A}$, and so $\mathbb{O}$ plays the role of the object classifier in topos theory. The classifying topos of $\mathbb{O}$ is $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et}]$ and with the inclusion functor $\mathrm{Inc}\colon\ \mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$ as its generic model. There is also context $\mathbb{O}_\bullet$ which in addition to the generic node X has another node 1 declared as terminal, that is $\mathrm{tm}(*) = 1$, and moreover, it has an edge $x\colon 1 \to X$ (This is the effect of adding a generic point to the context $\mathbb{O}$). Its models are the pointed sets. This time we must distinguish between strict and non-strict models. In a strict model, 1 is interpreted as *the canonical* terminal object.

The classifying topos of $\mathbb{O}_\bullet$ is the slice topos $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et}]/\mathrm{Inc}$. The generic model of $\mathbb{O}_\bullet$ in $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et}]/\mathrm{Inc}$ is the pair $(\mathrm{Inc}, \pi\colon \mathrm{Inc} \to \mathrm{Inc} \times \mathrm{Inc})$ where π is the diagonal transformation which renders the diagram below commutative:

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\pi"] \arrow[dr, "\mathrm{id}"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

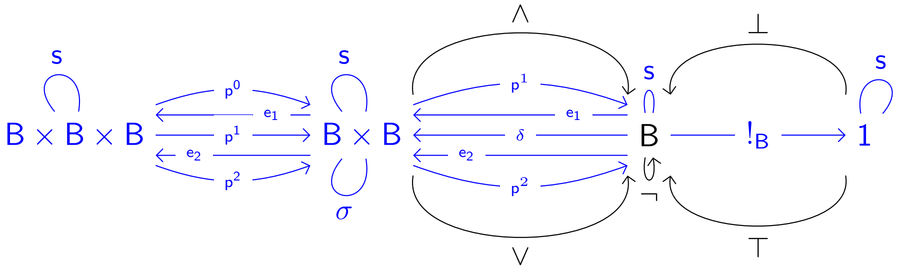
There is a context extension map $U\colon \mathbb{O}_\bullet \to \mathbb{O}$ which corresponds to the sketch inclusion in the opposite direction, sending the generic node in $\mathbb{O}$ to the generic node in $\mathbb{O}_\bullet$. As a model reduction, U simply forgets the point. Note that there is another context map, however not an extension map, $R\colon \mathbb{O}_\bullet \to \mathbb{O}$ corresponding to the sketch map sending the generic node of $\mathbb{O}$ to the terminal node in $\mathbb{O}_\bullet$.

EXAMPLE 3.3.7. The context $\mathbb{O}^{\to}$ comprises two nodes X_0 and X_1 and their identities, and an edge $\theta_{\mathsf{X}}\colon \mathsf{X}_0 \to \mathsf{X}_1$. A model of $\mathbb{O}^{\to}$ in an AU $\mathcal{A}$ is exactly a morphism in $\mathcal{A}$. We define the diagonal context map $\mathbb{O} \to \mathbb{O}^{\to}$ by the opspan (id, F) where the sketch morphism F takes θ_{X} to $\mathsf{s}(\mathsf{X})$, θ_{u} to u and commutativities to degenerate commutativities of the form $\mathsf{us}(\mathsf{X}) \sim \mathsf{u}$ and $\mathsf{s}(\mathsf{Y})\mathsf{u} \sim \mathsf{u}$.

In the next example we give a detailed and somewhat laborious presentation of the **context of Boolean algebra equipped with a prime filter**. We hope this example is complex enough to show the implementation of prior notions of this section in practice. Also, compare the sketch presentation below with the geometric theory presented in Example 3.1.5.

EXAMPLE 3.3.8. Here we present the theory of Boolean algebras with a context $\mathbb{BA}$. A model of $\mathbb{BA}$ in an AU is an internal Boolean algebra. We then construct an extended context $\mathbb{BA}_\bullet$ whose models in a topos are Boolean algebras equipped with a prime filter. Similar to the example of object classifier context, there is an obvious context extension map $\mathbb{BA}_\bullet \to \mathbb{BA}$. which can be thought of as a bundle, for which the fibre over a point of $\mathbb{BA}$ (i.e. a Boolean algebra B) is its spectrum $\mathrm{Spec}(B)$, the Stone space corresponding to B. This allows us to think of the extension map as the “generic Stone space".

The following graph is the sketch corresponding to context $\mathbb{BA}$ of Boolean algebras.



We give a step-by-step construction of it[14]. Our method is very similar to methods of categorical logic, however, as we mentioned before, the technology of sketches is more general than categories and general theory of contexts provides us with a way to keep track of special derived edges we add in our constructions as well as object equalities. Start with the empty context. Add a terminal universal ω with $1 = \mathrm{tm}(\omega)$. Add a fresh node B. Add a pullback universal ω with the node B × B,

---

[14] Some nodes and edges of this diagram are coloured blue to emphasise that they are derived by equivalence extension. The black ones are added freshly.

where $\mathsf{B} \times \mathsf{B} := \mathrm{d}_0\mathrm{d}_0\Gamma^i(\omega) = \mathrm{d}_0\mathrm{d}_1\Gamma^i(\omega)$ for pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ with $\mathrm{d}_1\mathrm{d}_2\Gamma^i(\omega) = \mathrm{tm}(\omega)$ where $i = 1, 2$:

$$\begin{tikzcd}
\mathsf{B} \times \mathsf{B} \arrow[r, dashed, "\mathsf{p}^2"] \arrow[d, dashed, "\mathsf{p}^1"'] \arrow[dr, dashed, "\mathsf{p}"] & \mathsf{B} \arrow[d, "!_{\mathsf{B}}"] \\
\mathsf{B} \arrow[r, "!_{\mathsf{B}}"'] & \mathrm{tm}(\omega)
\end{tikzcd}$$

and $!_{\mathsf{B}} = \mathrm{d}_2\Gamma^1(\omega) = \mathrm{d}_2\Gamma^2(\omega)$, $\mathsf{p}^1 = \mathrm{d}_0\Gamma^1(\omega)$, $\mathsf{p}^2 = \mathrm{d}_0\Gamma^2(\omega)$, and $\mathsf{p} = \mathrm{d}_1\Gamma^i(\omega)$, for $i = 1, 2$. At this stage add pullback fillings $\mathsf{e}_1 = \langle \mathsf{s}_{\mathsf{B}}, \top \circ !_{\mathsf{B}} \rangle_{\langle !_{\mathsf{B}}, !_{\mathsf{B}} \rangle} \colon \mathsf{B} \to \mathsf{B} \times \mathsf{B}$ and $\mathsf{e}_2 = \langle \mathsf{s}_{\mathsf{B}}, \bot \circ !_{\mathsf{B}} \rangle_{\langle !_{\mathsf{B}}, !_{\mathsf{B}} \rangle} \colon \mathsf{B} \to \mathsf{B} \times \mathsf{B}$. Finally add fresh edges $\bot$ and $\top$ for bottom and top elements, $\neg$ for unary negation operator and also, $\wedge$ and $\vee$ for binary meet and join operators. Furthermore, we need to add commutativities to the sketch of our contexts to express Boolean algebra equations.[15] To illustrate this point we formulate a few Boolean algebra equations in terms of commutativities. Obviously we do not attempt at listing all such commutativities as it is quite cumbersome to do so and there is not much new insight one could get from them.

For instance, equations $a \vee \bot = a$ and $a \wedge \top = a$ are expressed by two commutativities

$$\begin{tikzcd}
\mathsf{B} \arrow[r, "\mathsf{e}_2"] \arrow[d, "\mathsf{e}_1"'] \arrow[dr, "\mathsf{s}"] & \mathsf{B} \times \mathsf{B} \arrow[d, "\vee"] \\
\mathsf{B} \times \mathsf{B} \arrow[r, "\wedge"'] & \mathsf{B}
\end{tikzcd}$$

Also, we would like to point out that derived nodes such as $\mathsf{B} \times \mathsf{B} \times \mathsf{B} \times \mathsf{B}$, and derived edges such as $\mathsf{B} \times 1 \to \mathsf{B} \times \mathsf{B}$ do not exist in $\mathbb{BA}$, as presented in the sketch above, but they do exist in some equivalence extension of $\mathbb{BA}$.

Now, we introduce context $\mathbb{BA}_{\bullet}$ which presents the theory of Boolean algebras equipped with a prime filter. To this end, we add finite number of nodes, edges,

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[15] Notice that there are many different order in which we can add nodes, edges, and commutativities to express any context such as the one we just presented. However, these different orders give presentation of isomorphic contexts.

and commutativities to context $\mathbb{BA}$. For start we add a new node F and a 'mono' edge $\mathsf{i}\colon \mathsf{F} \to \mathsf{B}$ (as in Example 3.3.5).

To express that F contains meets of any two of its elements we add the node $\mathsf{F} \times \mathsf{F}$ and introduce an edge $\wedge_{\mathsf{F}}\colon \mathsf{F} \times \mathsf{F} \to \mathsf{F}$

$$\begin{tikzcd}
\mathsf{F} \times \mathsf{F} \arrow[r, "\mathsf{i} \times \mathsf{i}"] \arrow[d, dashed, "\wedge_{\mathsf{F}}"'] \arrow[dr, dashed] \arrow[dr, phantom, "\bullet \quad \bullet"] & \mathsf{B} \times \mathsf{B} \arrow[d, "\wedge"] \\
\mathsf{F} \arrow[r, "\mathsf{i}"'] & \mathsf{B}
\end{tikzcd}$$

Additionally, we want the above square to be a pullback square (implied by upward-closedness of F). Therefore, we require the filling $\langle \wedge_{\mathsf{F}}, \mathsf{i} \times \mathsf{i} \rangle_{\langle \mathsf{p}^1, \mathsf{p}^2 \rangle}$ to be an isomorphism edge[16].

To express the last axiom, we add an edge $\vee_{\mathsf{F}}\colon \mathsf{F} \times \mathsf{B} \to \mathsf{F}$. To say that F does not contain $\bot$ we add an edge $\mathsf{P}_{\mathsf{i},\bot} \to 0$ where 0 is an initial universal. Notice that this edge has to be an isomorphism, due to universality of pullback and initial node as well as stability of initials under pullback. Moreover, we add an edge T_{F} and a commutativity to our sketch to make sure that top element is in our prime filter F:

$$\begin{tikzcd}
0 & \mathsf{P}_{\mathsf{i},\bot} \arrow[l, dashed] \arrow[r, "\pi^2_{\mathsf{i},\bot}"] \arrow[d, dashed, "\pi^1_{\mathsf{i},\bot}"'] \arrow[dr, phantom, "\bullet \quad \bullet"] & 1 \arrow[d, "\bot"] \\
 & \mathsf{I} \arrow[r, "\mathsf{i}"'] \arrow[ur, dashed, "\mathsf{s}"] & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{F} \arrow[d, "\mathsf{i}"] \\
1 \arrow[ur, dashed, "\mathsf{T}_{\mathsf{F}}"] \arrow[r, "\mathsf{T}"'] \arrow[ur, phantom, "\bullet", near end, shift right=3] & \mathsf{B}
\end{tikzcd}$$

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[16] To establish that an edge $\mathsf{u}\colon \mathsf{X} \to \mathsf{Y}$ is an isomorphism we have to supply the data of an edge $\mathsf{v} : \mathsf{Y} \to \mathsf{X}$ together with commutativities

$$\begin{tikzcd}
\mathsf{Y} \arrow[r, "\mathsf{s}"] \arrow[dr, "\mathsf{v}"] \arrow[dr, phantom, "\bullet \quad \bullet"] & \mathsf{Y} \\
\mathsf{X} \arrow[u, "\mathsf{u}"] \arrow[r, "\mathsf{s}"'] & \mathsf{X} \arrow[u, "\mathsf{u}"']
\end{tikzcd}$$

which exhibit that $\mathsf{uv} \sim \mathsf{s}_{\mathsf{Y}}$ and $\mathsf{vu} \sim \mathsf{s}_{\mathsf{X}}$.

Finally, note that $\mathsf{P}_{\mathsf{i},\vee}$ is the pullback universal node which represents all pairs (a, b) such that $a \vee b \in F$. We would like to say any such pair has either its first component or[17] its second component in F. That is achieved by adding an inverse to the edge[18] u in sketch diagram below:

$$\begin{tikzcd}
\mathsf{Im} \arrow[d, dashed, "\mathsf{u}"'] \arrow[drr, dashed, hook, "\mathsf{m}"] \arrow[dr, phantom, "\bullet"] & & \mathsf{F} \times \mathsf{B} + \mathsf{B} \times \mathsf{F} \arrow[ll, dashed, two heads, "\mathsf{e}"'] \arrow[d, "\mathsf{i} \times \mathsf{s} + \mathsf{s} \times \mathsf{i}"] \arrow[dl, phantom, "\bullet"] \\
\mathsf{P}_{\mathsf{i},\vee} \arrow[rr, "\mathsf{p}^2"] \arrow[d, "\mathsf{p}^1"'] \arrow[drr, "\mathsf{p}"] \arrow[dr, phantom, "\bullet"] & & \mathsf{B} \times \mathsf{B} \arrow[d, "\vee"] \arrow[dl, phantom, "\bullet"] \\
\mathsf{F} \arrow[rr, "\mathsf{i}"'] & & \mathsf{B}
\end{tikzcd}$$

Notice that $\mathsf{P}_{\mathsf{i},\vee}$ is the subject of a pullback universal and $\mathsf{F} \times \mathsf{B} + \mathsf{B} \times \mathsf{F}$ is the subject of a coproduct [19].

We outline two more important examples. We do not have space here to give full details as sketches. Rather, our aim is to explain why the known geometric theories can be expressed as contexts.

EXAMPLE 3.3.9. Let $\mathbb{T}_0 = [C\colon \mathrm{Cat}]$ be the theory of categories. It includes nodes C_0 and C_1, primitive nodes introduced for the objects of objects and of morphisms; edges $d_0, d_1\colon C_1 \to C_0$ for domain and codomain and an edge for identity morphisms; another node C_2 for the object of composable pairs and introduced as a pullback; an edge $c\colon C_2 \to C_1$ for composition; and various commutativities for the axioms of category theory. The technique is general and would apply to any finite cartesian theory – this should be clear from the account in [PV07].

Now let us define the extension $\mathbb{T}_1 = [C\colon \mathrm{Cat}][F\colon \mathrm{Tor}(C)]$, where $\mathrm{Tor}(C)$ denotes the theory of torsors (flat presheaves) over C. The presheaf part is expressed by the usual procedure for internal presheaves. We declare a node F_0 with an edge $p\colon F_0 \to C_0$, and let F_1 be the pullback along d_0. Then the morphism part of the

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[17] This ‘or’ is weaker than full intuitionistic one. Although we know that either it is the case that $a \in F$ or it is the case that $b \in F$ but there is necessarily not a way to determine which case occurs.

[18] The existence of u follows from previous assumptions.

[19] constructed as a pushout universal.

presheaf defines xu if $d_0(u) = p(x)$, and this is expressed by an edge from F_1 to F_0 over d_1 satisfying various conditions. In fact this is another cartesian theory.

The flatness conditions are not cartesian, but are still expressible using contexts. First we must say that F_0 is non-empty: the unique morphism $F_0 \to 1$ is epi, in other words the cokernel pair has equal injections. Second, if $x, y \in F_0$ then there are u, v, z such that $x = zu$ and $y = zv$. Third, if $xu = xv$ then there are w, z such that $x = zw$ and $wu = wv$. Again, these can be expressed by saying that certain morphisms are epi.

Now we have a context extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$, which forgets the torsor.

$\mathbb{T}_0$ and $\mathbb{T}_1$, like all contexts, are finite. In §4.1 we shall see how for an infinite category C we can still access the infinite theory $\mathrm{Tor}(C)$ (infinitely many sorts and axioms, infinitary disjunctions) as the "fibre of U over C".

EXAMPLE 3.3.10. Let $\mathbb{T}_0 = [L\colon \mathrm{DL}]$ be the finite algebraic theory of distributive lattices, a context. Now let $\mathbb{T}_1 = [L\colon \mathrm{DL}][F\colon \mathrm{Filt}(L)]$ be the theory of distributive lattices L equipped with prime filters F, and let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be the corresponding extension map. $\mathbb{T}_1$ is built over $\mathbb{T}_0$ by adjoining a node F with a monic edge $F \to L$, and conditions to say that it is a filter (contains top and is closed under meet) and prime (inaccessible by bottom and join). For example, to say that bottom is not in F, we say that the pullback of F along bottom as edge $1 \to L$ is isomorphic to the initial object.

Given a model L of $\mathbb{T}_0$, the fibre of U over L is its spectrum $\mathrm{Spec}(L)$.

One central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion of models is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. Contexts give us good handle over strictness. The following result appears in [Vic17, Proposition 1]:

PROPOSITION 3.3.11. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one got from extending $\mathbb{T}_0$ to $\mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M'_0 of $\mathbb{T}_0$, and an isomorphism $\phi\colon M'_0 \cong M_1 U$.

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[d, "U"'] & M'_1 \arrow[r, dotted, "\widetilde{\phi}", "\cong"'] \arrow[d, dotted, no head] & M_1 \arrow[d, dotted, no head] \\
\mathbb{T}_0 & M'_0 \arrow[r, "\phi", "\cong"'] & M_1 U
\end{tikzcd}$$

Then there is a unique model M'_1 of $\mathbb{T}_1$ and isomorphism $\tilde{\phi}\colon M'_1 \cong M_1$ such that

(i) M'_1 is strict,

(ii) $M'_1 U = M'_0$,

(iii) $\tilde{\phi} U = \phi$, and

(iv) $\tilde{\phi}$ is equality on all the primitive nodes used in extending $\mathbb{T}_0$ to $\mathbb{T}_1$.

We call M'_1 the *canonical strict isomorph* of M_1 along ϕ.

The fact that we can uniquely lift strict models along context extension maps will be crucial in §4.1 and §4.2.

3.4 Fibrations of AU-contexts

In §2.4 we reviewed the notion of Chevalley-style (op)fibration. In this section, we would like to study it more closely in the 2-category $\mathfrak{Con}$. Note that in setting up Chevalley-style (op)fibrations we did not require the existence of pullbacks in the ambient 2-category, and indeed, in $\mathfrak{Con}$ not all pullbacks exists; nonetheless, [Vic19] proves that pullbacks exist along extension maps.

In this section, we assume that our 2-categories are equipped with all finite PIE-limits. Note that for AUs and elementary toposes, we assume that the structure is given *canonically* – this is essential if we are to consider strict models. For our $\mathfrak{K}$ here we do not assume there are canonical PIE-limits or pullbacks. Indeed, in $\mathfrak{Con}$ (so far as we know) they do not exist. morphisms are defined only modulo objective equality, and the construction of those limits depends on the choice of representatives of morphisms.

Recall the notion of Chevalley fibration from Definition 2.4.1. In the case where Chevalley fibration p is carrable, the comma objects $(p \downarrow B)$ and $(B \downarrow p)$ can be expressed as pullbacks along the two projections from $(B \downarrow B)$ to B. Let us at this point reformulate the fibration property using the notation as it will appear in $\mathfrak{Con}$ when p is an extension map $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ – and using the fact that extension maps are carrable.

Let $\mathrm{dom}, \mathrm{cod} \colon \mathbb{T}_0^{\to} \to \mathbb{T}_0$ be the domain and codomain context maps corresponding to sketch homomorphisms $i_0, i_1 \colon \mathbb{T}_0 \to \mathbb{T}_0^{\to}$. We define the context extension maps $\mathrm{dom}^* \, \mathbb{T}_1 \to \mathbb{T}_0^{\to}$ and $\mathrm{cod}^* \, \mathbb{T}_1 \to \mathbb{T}_0^{\to}$ as the pullbacks of U along dom and cod. A model of $\mathrm{dom}^*(\mathbb{T}_1)$ is a pair $(N, f \colon M_0 \to M_1)$ where f is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ such that $N \centerdot U = M_0$. Models of $\mathrm{cod}^*(\mathbb{T}_1)$ are similar, except that $N \centerdot U = M_1$. There are induced context maps $\Gamma_0 \colon \mathbb{T}_1^{\to} \to \mathrm{dom}^*(\mathbb{T}_1)$ and $\Gamma_1 \colon \mathbb{T}_1^{\to} \to \mathrm{cod}^*(\mathbb{T}_1)$. Given a model $f \colon N_0 \to N_1$ of $\mathbb{T}_1^{\to}$, Γ_i sends it to $(N_i, f \centerdot U^{\to} \colon N_0 \centerdot U \to N_1 \centerdot U)$.

$$\begin{tikzcd}
\mathbb{T}_1^{\to} \arrow[rr, bend right=10, "\Gamma_1"'] \arrow[dr, "\Gamma_0"'] \arrow[dddr, bend right, "U^{\to}"'] & & \mathrm{cod}^*(\mathbb{T}_1) \arrow[ll, dotted, bend right=20, "\Lambda_1"'] \arrow[ll, phantom, "\top"] \arrow[dr, "\pi_1"] \arrow[dd] & \\
& \mathrm{dom}^*(\mathbb{T}_1) \arrow[rr, bend right=10, "\pi_0"'] \arrow[dd, "U_0"'] & & \mathbb{T}_1 \arrow[dd, "U"] \\
& & \mathbb{T}_0^{\to} \arrow[dl, equal] \arrow[dr, "\mathrm{cod}"] & \\
& \mathbb{T}_0^{\to} \arrow[rr, bend right=10, "\mathrm{dom}"'] \arrow[rr, phantom, "\overset{\theta_{\mathbb{T}_0}}{\Rightarrow}"] & & \mathbb{T}_0
\end{tikzcd} \tag{3.10}$$

Definition 3.4.1. Consider U as above. We call U a **fibration of contexts** if the morphism Γ_1 has a right adjoint Λ_1 with counit ε an identity.

Similar to Definition 2.4.1, dually one defines **opfibration of contexts** this time Γ_0 has a left adjoint Λ_0 with unit η an identity.

Remark 3.4.2. A consequence of the counit of the adjunction $\Gamma_1 \dashv \Lambda_1$ being the identity is that the adjunction triangle equations are expressed in simpler forms; we have $\Gamma_1 \centerdot \eta_1 = \mathrm{id}_{\Gamma_1}$ and $\eta_1 \centerdot \Lambda_1 = \mathrm{id}_{\Lambda_1}$.

Remark 3.4.3. The composite $\Gamma_0\Lambda_1$ is a morphism from $\mathrm{cod}^*(\mathbb{T}_1)$ to $\mathrm{dom}^*(\mathbb{T}_1)$. Moreover, there is a 2-morphism from $\pi_0\Gamma_0\Lambda_1$ to π_1 constructed as $\pi_0\Gamma_0\Lambda_1 \xRightarrow{\theta_{\mathbb{T}_1}\Lambda_1} \pi_1\Gamma_1\Lambda_1 = \pi_1$. These two, the morphism and the 2-morphism, appear as the central structure needed for the Johnstone-style fibration.

Remark 3.4.4. The (op)fibration results of contexts don't in themselves depend that much on the concrete nature of contexts, more on the 2-categorical structure of $\mathfrak{Con}$.

Example 3.4.5. The context extension $U\colon \mathbb{O}_\bullet \to \mathbb{O}$ (Example 3.3.6) is an opfibration extension.

Proof. First we form the pullbacks of the context extension U along the two context maps dom and cod. U_0 and U_1 are U *reindexed* along dom and cod: the same simple extension steps, but with the data for each transformed by dom or cod.

$$\begin{tikzcd}
\mathrm{dom}^*(\mathbb{O}_\bullet) \arrow[r, "\pi_0"] \arrow[d, "U_0"'] & \mathbb{O}_\bullet \arrow[d, "U"] & \mathrm{cod}^*(\mathbb{O}_\bullet) \arrow[r, "\pi_1"] \arrow[d, "U_1"'] & \mathbb{O}_\bullet \arrow[d, "U"] \\
\mathbb{O}^\to \arrow[r, "\mathrm{dom}"'] & \mathbb{O} & \mathbb{O}^\to \arrow[r, "\mathrm{cod}"'] & \mathbb{O}
\end{tikzcd}$$

$\mathrm{dom}^*(\mathbb{O}_\bullet)$ is a context with three nodes: a terminal 1, primitive nodes X_0 and X_1, and edges $x_0\colon 1 \to X_0$, $\theta_X\colon X_0 \to X_1$, and identities on the three nodes. $\mathrm{cod}^*(\mathbb{O}_\bullet)$ is similar, but with $x_1\colon 1 \to X_1$ instead of x_0.

There is, in addition, the arrow context $\mathbb{O}^{\to}_{\bullet}$ which consists of all the nodes, edges, and two commutativities $\theta_X x_0 \sim \theta_x$, $x_1\theta_1 \sim \theta_x$ (marked by bullet points) as presented in the following diagram plus identity edges.

$$\begin{tikzcd}
1_1 \arrow[r, "x_1", "\bullet"'] & X_1 \\
1_0 \arrow[u, "\theta_1"] \arrow[ur, "\theta_x"] \arrow[r, "\bullet", "x_0"'] & X_0 \arrow[u, "\theta_X"']
\end{tikzcd}$$

There are context maps Γ_0 and Γ_1 which make the following diagram commute:

$$\begin{tikzcd}
 & \mathrm{cod}^* \mathbb{O}_\bullet \arrow[rr, "U_1"] \arrow[dr, "\pi_1"] & & \mathbb{O}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathbb{O}^{\to}_{\bullet} \arrow[ur, "\Gamma_1"] \arrow[dr, "\Gamma_0"'] & & \mathbb{O}_\bullet \arrow[r, "U"] & \mathbb{O} \\
 & \mathrm{dom}^* \mathbb{O}_\bullet \arrow[ul, dotted, "\Lambda_0"'] \arrow[ur, "\pi_0"'] \arrow[rr, "U_0"'] & & \mathbb{O}^{\to} \arrow[u, "\mathrm{dom}"']
\end{tikzcd}$$

Γ_0 is the dual to the sketch morphism $\mathrm{dom}^* \mathbb{O}_\bullet \to \mathbb{O}^{\to}_{\bullet}$ that takes 1 to 1_0 and otherwise preserves notation. Γ_1 is similar.

More interestingly, Γ_0 has a left adjoint $\Lambda_0\colon\ \mathrm{dom}^*(\mathbb{O}_\bullet) \to \mathbb{O}^{\to}_{\bullet}$. For this, X_0, θ_X, X_1 and x_0 in $\mathbb{O}^{\to}_{\bullet}$ are interpreted in $\mathrm{dom}^* \mathbb{O}_\bullet$ by the ingredients with the same name, and 1_0, 1_1 by 1 and θ_1 by the identity on 1. For θ_x and x_1 we need an equivalence extension of $\mathrm{dom}^* \mathbb{O}_\bullet$ got by adjoining the composite $\theta_X x_0$, and a commutativity for one of the unit laws of composition.

It is now obvious that $\Gamma_0\Lambda_0 = \mathrm{id}\colon\ \mathrm{dom}^*(\mathbb{O}_\bullet) \to \mathrm{dom}^*(\mathbb{O}_\bullet)$. Less obvious, but true in this example, is that $\Lambda_0\Gamma_0$ is the identity on $\mathbb{O}^{\to}_{\bullet}$. This follows from the rules for objective equality, and is essentially because in any strict model 1_0 and 1_1 are both interpreted as the canonical terminal object, and θ_1 as the identity on that. ☐

We now outline the argument to show that two further examples should be expected to be (op)fibrations.

EXAMPLE 3.4.6. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be the context extension map of Example 3.3.10, for prime filters of distributive lattices. To show that this is a fibration, consider a distributive lattice homomorphism $f\colon L_0 \to L_1$. The map $\mathrm{Spec}(f)\colon \mathrm{Spec}(L_1) \to \mathrm{Spec}(L_0)$ can be expressed using contexts. It takes a prime filter F_1 of L_1 to its inverse image F_0 under f which is a prime filter of L_0. f restricts (uniquely) to a function from F_1 to F_0, and so we get a $\mathbb{T}_1$-homomorphism $f'\colon (L_1, F_1) \to (L_0, F_0)$. The construction so far can all be expressed using AU-structure, and so gives our $\Lambda_1\colon \mathrm{cod}^*(\mathbb{T}_1) \to \mathbb{T}_1^{\rightarrow}$.

$$\begin{tikzcd}
(L_0, F_0 = f^{-1}(F_1)) \arrow[r, dotted, "f'"] \arrow[d, mapsto, "U"'] & (L_1, F_1) \arrow[d, mapsto, "U"] \\
L_0 \arrow[r, "f"'] & L_1
\end{tikzcd}$$

Aided by the fact that $\Gamma_1\colon \mathbb{T}_1^{\rightarrow} \to \mathrm{cod}^*(\mathbb{T}_1)$ is given by a sketch homomorphism (no equivalence extension of $\mathbb{T}_1^{\rightarrow}$ needed), we find that $\Gamma_1\Lambda_1$ is the identity on $\mathrm{cod}^{-1}(\mathbb{T}_1)$. The unit $\eta\colon \mathrm{id} \Rightarrow \Lambda_1\Gamma_1$ of the adjunction is given as follows. In $\mathbb{T}_1^{\rightarrow}$ we have a generic $f\colon (L_0, F_0) \to (L_1, F_1)$, and clearly f restricted to F_0 factors via $f^{-1}(F_1)$. Taking this with the identity on L_1 gives a $\mathbb{T}_1^{\rightarrow}$-homomorphism from $(L_0, F_0) \to (L_1, F_1)$ to $(L_0, f^{-1}(F_1) \to (L_1, F_1)$, and hence our η. The diagonal equations for the adjunction hold.

EXAMPLE 3.4.7. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be the context extension map of Example 3.3.9, for torsors (flat presheaves) of categories. To show that this is an opfibration, consider a functor $F\colon C \to D$.

If T is a torsor over C, we must define a torsor $T' = \mathrm{Tor}(F)(T)$ over D. In Example 3.3.9 our notation treated the presheaf structure as a right action by C on T. Analogously let us write D as a C-D-bimodule, with a right action by D by composition, and a left action by C by composition after applying F. We define $\mathrm{Tor}(F)(T)$, a D-torsor, as the tensor $T \otimes D$. Its elements are pairs (x, f) with $x \in T$, $f \in D_1$ and $p(x) = d_0(f)$, modulo the equivalence relation generated by $(x, uf) \sim (xu, f)$. This can be defined using AU structure. Let us analyse an equation $(x, f) = (x', f')$ in more detail. It can be expressed as a chain of equations

$$(yu, k) \sim^{-1} (y, uk) = (y, u'k') \sim (yu', k'),$$

each for a quintuple (k, u, y, u', k') with $uk = u'k'$. Hence the overall equation $(x, f) = (x', f')$ derives from sequences (k_i) $(0 \leq i \leq n)$ and $(u_i), (y_i), (u'_i)$ $(0 \leq i < n)$ such that $u_i k_i = u'_i k_{i+1}$, $y_i u'_i = y_{i+1} u_{i+1}$, $f = k_0$, $x = y_0 u_0$, $f' = k_n$ and $x' = y_{n-1} u'_{n-1}$. (We are thinking of k'_i as k_{i+1}.) By flatness of T we can replace the y_is by elements yv_i with $v_i u'_i = v_{i+1} u_{i+1}$, $x = yv_0 u_0$ and $x' = yv_{n-1} u'_{n-1}$.

We outline why $\mathrm{Tor}(F)(T)$ is flat (over D). First, it is non-empty, because T is. If $x \in T$ then $(x, \mathrm{id}_{F(p(x))}) \in \mathrm{Tor}(F)(T)$. Next, suppose $(x, f), (x', f') \in \mathrm{Tor}(F)(T)$. We can find y, u, u' with $x = yu$ and $x' = yu'$, and then $(x, f) = (yu, f) = (y, uf) = (y, \mathrm{id})uf$ and $(x', f') = (\mathrm{id}, y)u'f'$.

Finally, suppose $(x, g)f = (x, g)f'$. We must find h, g', y such that $hf = hf'$ and $(x, g) = (y, g')h$. Composing g' and h, we can instead look for $(y, h) = (x, g)$ such that $hf = hf'$. In fact, we can reduce to the case where $g = \mathrm{id}$. Suppose, then that we have $(x, f) = (x, f')$. By the analysis above, we get y and sequences $(k_i), (u_i), (v_i), (u'_i)$ such that $u_i k_i = u'_i k_{i+1}$, $v_i u'_i = v_{i+1} u_{i+1}$, $f = k_0$, $x = yv_0 u_0$, $f' = k_n$ and $x = yv_{n-1} u'_{n-1}$. Using flatness of T again, we can assume $v_0 u_0 = v_{n-1} u'_{n-1}$. Now put $h := v_0 u_0$, so $(y, h) = (y, v_0 u_0) = (yv_0 u_0, \mathrm{id}) = (x, \mathrm{id})$. Then, as required,

$$hf = v_0 u_0 k_0 = v_0 u'_0 k_1 = v_1 u_1 k_1 = \cdots = v_{n-1} u'_{n-1} k_n = hf'.$$

Although this reasoning is informal, its ingredients – and in particular the reasoning with finite sequences – are all present in AU structure.

Once we have $\mathrm{Tor}(F)(T)$ it is straightforward to define to define the function $T \to \mathrm{Tor}(F)(T)$, $x \mapsto (x, \mathrm{id})$ that makes a homomorphism of $\mathbb{T}_1$-models. Note in particular that the action is preserved: $xu \mapsto (xu, \mathrm{id}) = (x, u) = (x, \mathrm{id})u$. This gives us our Λ_0, and $\Gamma_0 \Lambda_0 = \mathrm{id}$. For the counit of the adjunction, let $(F, \theta)\colon (C, T) \to (D, T')$ be a $\mathbb{T}_1$-homomorphism. Then θ factors via $\mathrm{Tor}(F)(T)$ using $(x, f) \mapsto \theta(x)f$. This respects the equivalence, as $\theta(xu)f = \theta(x)F(u)f$ is a condition of $\mathbb{T}_1$-homomorphisms.

Note that Example 3.4.5 can be got from Example 3.4.7 as a pullback. This is because there is a context map $\mathbb{O} \to [C\colon \mathrm{Cat}]$ taking a set X to the discrete category over it. A torsor over the discrete category is equivalent to an element of X.

We conjecture that further examples can be found as follows, from the basic idea that, given a style of presentation of spaces, homomorphisms between presentations can yield maps between the spaces.

- (Opfibration) Let $\mathbb{T}_0$ be the theory of sets equipped with an idempotent relation, and $\mathbb{T}_1$ extend it with a rounded ideal [Vic93].
- (Opfibration) Let $\mathbb{T}_0$ be the theory of generalized metric spaces, and $\mathbb{T}_1$ extend it with a Cauchy filter (point of the localic completion) [Vic05].
- (Fibration) Let $\mathbb{T}_0$ be the theory of normal distributive lattices, and $\mathbb{T}_1$ extend it with a rounded prime filter [SVW12]. This would generalize Example 3.4.6.
- (Bifibration) Let $\mathbb{T}_0$ be the theory of strongly algebraic information systems, and let $\mathbb{T}_1$ extend it with an ideal [Vic99]. This is a special case of Example 3.4.7 – when the category C is a poset, then a torsor is just an ideal – and hence would be an opfibration. The fibrational nature would come from the fact that a homomorphism between two of these information systems corresponds to an adjunction between the corresponding domains.

3.5 Summary and discussion

AU-contexts form a 2-category $\mathfrak{Con}$ which *gets embedded into* the opposite of the 2-category of AUs and strict AU-functors (which preserve AU-structure on the nose) via the classifying AU-functor $\mathbb{T} \to \mathbf{AU}\langle\mathbb{T}\rangle$. Also, they provide a base-independent model for generalized point-free spaces in the sense that, a result proved for an AU-context $\mathbb{T}$ holds for all toposes $\mathscr{S}[\mathbb{T}]$, for any elementary topos $\mathscr{S}$ (with nno). In this, $\mathbb{T}$ has copies in all the fibre 2-categories $\mathcal{B}\mathfrak{Top}/\mathscr{S} = \mathcal{G}\mathfrak{Top}^{-1}(\mathscr{S})$. The important difference is that AU techniques guarantee (usually) simple proofs of the stronger, predicative, and base-independent results. As a testimony to this claim we shall investigate the case of (op)fibrations in $\mathfrak{Con}$ and $\mathcal{E}\mathfrak{Top}$.

However, of course not all proofs of results about (classifying) toposes can be straightforwardly deduced from AU-contexts proofs; geometricity and predicativity of the constructions involved in the proofs are essential requirements.

Another crucial issue is dealing with strict and non-strict models of AU-contexts which has been mentioned on few occasions in this chapter. One important feature of categorical model theory is that models appear as functors, and the strictness of models of AU-contexts correspond directly to the strictness of AU-functors out of the classifying AU. So, indeed there are two classifying AUs for a context $\mathbb{T}$. The strict classifying AU $\mathbf{AU}_{\text{str}}\langle\mathbb{T}\rangle$ classifies $\mathbb{T}$ by strict AU-functors out of $\mathbf{AU}_{\text{str}}\langle\mathbb{T}\rangle$, and the standard classifying AU $\mathbf{AU}_{\text{str}}\langle\mathbb{T}\rangle$ which classifies $\mathbb{T}$ by functors out of $\mathbf{AU}_{\text{str}}\langle\mathbb{T}\rangle$. In using the methods of universal algebra to construct classifying AUs, that is by using the theory as generators and relations for the classifying AU, we crucially rely on strictness. For AU-structures such as limits, colimits are introduced as syntactic terms, and since the universal characterization of classifying AU works up to equality ([MV12]) we are forced to use strict AU functors to interpret the terms as canonical limits, colimits, etc.

4

Fibrations of toposes from fibrations of AU-contexts

4.0 Introduction

For many special constructions of topological spaces (which for us will be point-free, and generalised in the sense of Grothendieck), a structure-preserving morphism between the presenting structures gives a map between the corresponding spaces. Two very simple examples are: a function $f\colon X \to Y$ between sets already is a map between the corresponding discrete spaces; and a homomorphism $f\colon K \to L$ between two distributive lattices gives a map *in the opposite direction* between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

In topos theory we can relativize this process. A presenting structure in an elementary topos $\mathscr{E}$ gives rise to a bounded geometric morphism $p\colon \mathscr{F} \to \mathscr{E}$, where $\mathscr{F}$ is the topos of sheaves over $\mathscr{E}$ for the space presented by the structure. Then we commonly find that the covariant or contravariant correspondence mentioned above makes every such p an opfibration or fibration in the 2-category of toposes and geometric morphisms.

If toposes are taken as bounded over some fixed base $\mathscr{S}$, as objects in the 2-category $\mathfrak{BTop}/\mathscr{S}$, then there are often easy proofs got by using the Chevalley criterion to show that the *generic* such p, taken over the classifying topos for the relevant presenting structures, is an (op)fibration. See [SVW12] for some simple examples of the idea, though there are still questions of strictness left unanswered there.

However, often there is no natural choice of base topos $\mathscr{S}$. Indeed, Johnstone [Joh02a, B4.4] proves (op)fibrational results in the 2-category $\mathcal{B}\mathfrak{Top}$ where toposes are free to vary over different base toposes. These are harder both to state (the Chevalley criterion is not available) and to prove, but stronger since slicing over a base $\mathscr{S}$ restricts the 2-arrows.

We show how to use to get simple proofs using the Chevalley criterion of the stronger, base-independent (op)fibration results in $\mathcal{E}\mathfrak{Top}$, the 2-category of elementary toposes with nno, and arbitrary geometric morphisms.

Our starting point is the following construction in [Vic17], using the 2-category $\mathfrak{Con}$ of AU-contexts in [Vic19]. Suppose $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map in $\mathfrak{Con}$, and M is a model of $\mathbb{T}_0$ in $\mathscr{S}$, an elementary topos with nno. Then there is a geometric theory $\mathbb{T}_1/M$, of models of $\mathbb{T}_1$ whose $\mathbb{T}_0$-reduct is M, and so we get a classifying topos $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$. Our main result (Theorem 4.2.2) is that –

> *if U is an (op)fibration in $\mathfrak{Con}$, using the Chevalley criterion, then p is an (op)fibration is $\mathcal{E}\mathfrak{Top}$, using the Johnstone criterion.*

Throughout, we assume that *all our elementary toposes are equipped with natural numbers object (nno).* Without an nno the ideas of generalized space do not go far (because it is needed in order to get an object classifier), and AU techniques don't apply.

In §4.1, we review the connection between contexts and toposes as developed in [Vic17], along with some new results. A central construction shows how context extension maps $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ can be treated as *bundles* of generalized spaces: if M is a point of $\mathbb{T}_0$ (a model of $\mathbb{T}_0$ in an elementary topos $\mathscr{S}$), then the fibre of U over M, as a generalized space over $\mathscr{S}$, is a bounded geometric morphism $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ that classifies the models of $\mathbb{T}_1$ whose U-reduct is M. Much of the discussion is about understanding the universal property of such a classifier in the setting of $\mathcal{G}\mathfrak{Top}$.

§4.2 then provides the main result, Theorem 4.2.2. Suppose $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension map, and $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ is a classifier got as in §4.1. Then if U is an (op)fibration, so is p.

4.1 Classifying toposes of contexts in $\mathfrak{GTop}$

In this part, we shall review how [Vic17] exploits the fact that, for any geometric morphism $f\colon \mathscr{E} \to \mathscr{F}$ between elementary toposes with nno, the inverse image functor f^* is an AU-functor. It preserves the finite colimits and finite limits immediately from the definition, and the preservation of list objects follows quickly from their universal property and the adjunction of f.

By straightforwardly applying f^* we transform a model of M of a context $\mathbb{T}$ in $\mathscr{F}$ to a model in $\mathscr{E}$. However, we shall be interested in *strict* models, and f^* is in general non-strict as an AU-functor. For this reason we reserve the notation $f^\star M$ for the canonical strict isomorph of the straightforward application, which we write $f^\star \centerdot M$. By this means, the 1-morphisms of $\mathfrak{ETop}$ act strictly on the categories of strict $\mathbb{T}$-models. This extends to 2-morphisms. If we have $f, g\colon \mathscr{E} \to\to \mathscr{F}$ and $\alpha\colon f \Rightarrow g$, then we get a homomorphism $\alpha^\star M\colon f^\star M \to g^\star M$.

It will later be crucial to know how $(-)^\star$ interacts with transformation of models by context maps. Given a context map $H\colon \mathbb{T}_1 \to \mathbb{T}_0$, the models $f^\star(M\centerdot H)$ and $(f^\star M)\centerdot H$ are isomorphic but not always equal. For instance, take $H\colon \mathbb{O}_\bullet \to \mathbb{O}$ to be the non-extension context map that sends the generic node of $\mathbb{O}$ to the terminal node in $\mathbb{O}_\bullet$, and M a strict model of $\mathbb{O}_\bullet$. However, [Vic17, Lemma 9] demonstrates that if H is an extension map, then they are indeed equal.

One step further is to investigate the action of 1-morphisms and 2-morphisms in $\mathfrak{GTop}$ on strict models of context extensions.

DEFINITION 4.1.1. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be a context extension map and $p\colon \overline{p} \to \underline{p}$ a geometric morphism.

Then a **strict model** of U in p is a pair $M = (\overline{M}, \underline{M})$ where $\underline{M}$ is a strict $\mathbb{T}_0$-model in $\underline{p}$ and $\overline{M}$ is a strict $\mathbb{T}_1$-model in $\overline{p}$ such that $\overline{M} \mathbin{\centerdot} U = p^\star\underline{M}$.

A U**-morphism of models** $\varphi\colon M \to M'$ is a pair $(\overline{\varphi}, \underline{\varphi})$ where $\overline{\varphi}\colon \overline{M} \to \overline{M}'$ and $\underline{\varphi}\colon \underline{M} \to \underline{M}'$ are homomorphisms of $\mathbb{T}_1$- and $\mathbb{T}_0$-models such that $\overline{\varphi} \mathbin{\centerdot} U = p^\star\underline{\varphi}$.

Strict U-models and U-morphisms in p form a category $U\text{-}\mathbf{Mod}\, p$.

CONSTRUCTION 4.1.2. Suppose $f\colon q \to p$ is a 1-morphism in $\mathfrak{GTop}$ and let M be a model of U in p. We define a model $f^\star M$ of U in q, with downstairs part $\underline{f}^\star\underline{M}$, as follows.

$\overset{\blacktriangledown}{f}{}^{\star}\,\underline{M}$ provides us with an isomorphism of $\mathbb{T}_0$-models in $\overline{q}$ and $(\overline{f}^\star\overline{M}) \mathbin{\centerdot} U = \overline{f}^\star(\overline{M} \mathbin{\centerdot} U) = \overline{f}^\star p^\star\underline{M}$. We define the isomorphism $\overset{\blacktriangledown}{f}{}^{\star}\,\overline{M}\colon \overline{f}^\star\overline{M} \to f^\star\overline{M}$ to be the canonical strict isomorph of $\overline{f}^\star\overline{M}$ along $\overset{\blacktriangledown}{f}{}^{\star}\,\underline{M}$, and then $f^\star M := (f^\star\overline{M}, \underline{f}^\star\underline{M})$ is a strict model of U in q.

The construction extends to U-model homomorphisms $\varphi\colon M \to M'$, as in the diagram on the left.

$$\begin{tikzcd}
 & \overline{f}^\star\overline{M}' \arrow[rr, dotted, "\overset{\blacktriangledown}{f}{}^{\star}\,\overline{M}'" description] \arrow[dd, mapsto] & & f^\star\overline{M}' \arrow[dd, mapsto] \\
\overline{f}^\star\overline{M} \arrow[ur, "\overline{f}^\star\overline{\varphi}"] \arrow[rr, dotted, "\overset{\blacktriangledown}{f}{}^{\star}\,\overline{M}" description] \arrow[dd, mapsto] & & f^\star\overline{M} \arrow[ur, dotted, "f^\star\overline{\varphi}"] \arrow[dd, mapsto] & \\
 & f^\star p^\star\underline{M}' \arrow[rr, dashed, "\overset{\blacktriangledown}{f}{}^{\star}\,\underline{M}'" description] & & q^\star\underline{f}^\star\underline{M}' \\
\overline{f}^\star p^\star\underline{M} \arrow[ur, "\overline{f}^\star p^\star\underline{\varphi}"] \arrow[rr, "\overset{\blacktriangledown}{f}{}^{\star}\,\underline{M}" description] & & q^\star\underline{f}^\star\underline{M} \arrow[ur, "q^\star\underline{f}^\star\underline{\varphi}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{g}^\star\overline{M} \arrow[rr, "\overset{\blacktriangledown}{g}{}^{\star}\,\overline{M}" description] \arrow[dd, mapsto] & & g^\star\overline{M} \arrow[dd, mapsto] \\
\overline{f}^\star\overline{M} \arrow[ur, "\overline{\alpha}^\star\overline{M}"] \arrow[rr, "\overset{\blacktriangledown}{f}{}^{\star}\,\overline{M}" description] \arrow[dd, mapsto] & & f^\star\overline{M} \arrow[ur, dotted, "\alpha^\star\overline{M}"] \arrow[dd, mapsto] & \\
 & g^\star p^\star\underline{M} \arrow[rr, dashed, "\overset{\blacktriangledown}{g}{}^{\star}\,\underline{M}" description] & & q^\star\underline{g}^\star\underline{M} \\
\overline{f}^\star p^\star\underline{M} \arrow[ur, "\overline{\alpha}^\star p^\star\underline{M}"] \arrow[rr, "\overset{\blacktriangledown}{f}{}^{\star}\,\underline{M}" description] & & q^\star\underline{f}^\star\underline{M} \arrow[ur, "q^\star\underline{\alpha}^\star\underline{M}"'] &
\end{tikzcd}$$

This can be encapsulated in the functor

$$U\text{-}\mathbf{Mod}\,(f)\colon U\text{-}\mathbf{Mod}\,(p) \to U\text{-}\mathbf{Mod}\,(q),\ M \mapsto f^\star M.$$

By the properties of the canonical strict isomorph, it is strictly functorial with respect to f. Furthermore, if $\alpha\colon f \Rightarrow g$ is a 2-morphism in $\mathfrak{GTop}$, then the bottom square in the above right-hand diagram commutes and we define $\alpha^\star\overline{M}$ to be the unique $\mathbb{T}_1$-model morphism which completes the top face to a commutative square. We may also write $f^\star\underline{M}$ and $\alpha^\star\underline{M}$ for $\underline{f}^\star\underline{M}$ and $\underline{\alpha}^\star\underline{M}$.

The upshot is that each 2-morphism $\alpha\colon f \Rightarrow g$ in $\mathfrak{GTop}$ gives rise to a natural transformation $U\text{-}\mathbf{Mod}\,(\alpha)$ between functors $U\text{-}\mathbf{Mod}\,(f)$ and $U\text{-}\mathbf{Mod}\,(g)$ and $U\text{-}\mathbf{Mod}\,(\alpha)(M) = \alpha^\star M$.

PROPOSITION 4.1.3. $U\text{-}\mathbf{Mod}\,()\colon \mathfrak{GTop}^{\mathrm{op}} \to \mathfrak{Cat}$ is a strict 2-functor.

A main purpose of [Vic17] is to explain how a context extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ may be thought of as a *bundle*, each point of the base giving rise to a space, its fibre. In terms of toposes, a point of the base $\mathbb{T}_0$ is a model M of $\mathbb{T}_0$ in some elementary topos $\mathscr{S}$. Then the space is a Grothendieck topos over $\mathscr{S}$, in other words a bounded geometric morphism. It should be the classifying topos for a theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ that reduce to M.

[Vic17] describes $\mathbb{T}_1/M$ using the approach it calls “elephant theories”, namely that set out in [Joh02a, B4.2.1]. An elephant theory over $\mathscr{S}$ specifies the category of models of the theory in every bounded $\mathscr{S}$-topos $q\colon \mathscr{E} \to \mathscr{S}$, together with the reindexing along geometric morphisms. Then $\mathbb{T}_1/M$ is defined by letting $\mathbb{T}_1/M\text{-}\mathbf{Mod}\,(\mathscr{E})$ be the category of strict models of $\mathbb{T}_1$ in $\mathscr{E}$ that reduce by U to $q^\star M$.

The extension by which $\mathbb{T}_1$ was built out of $\mathbb{T}_0$ shows that the elephant theory $\mathbb{T}_1/M$, while not itself a context, is geometric over $\mathscr{S}$ in the sense of [Joh02a, B4.2.7], and hence has a classifying topos $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$, with generic model G, say. Its classifying property is that for each bounded $\mathscr{S}$-topos $\mathscr{E}$ we have an equivalence of categories

$$\Phi\colon \mathfrak{BTop}/_{\mathscr{S}}\ (\mathscr{E}, \mathscr{S}[\mathbb{T}_1/M]) \simeq \mathbb{T}_1/M\text{-}\mathbf{Mod}\,(\mathscr{E})$$

defined as $\Phi(f) := f^\star G$.

EXAMPLE 4.1.4. Consider the (unique) context map ! from $\mathbb{O}$ to the empty context $\mathbb{1}$. In any elementary topos $\mathscr{S}$ there is a unique model ! of $\mathbb{1}$, and the classifier for $\mathbb{O}/!$ is the object classifier over $\mathscr{S}$, the geometric morphism $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathscr{S}] \to \mathscr{S}$ where $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ here denotes the category of finite sets as an internal category in $\mathscr{S}$, its is object of objects being the nno N. The generic model of $\mathbb{O}$ in $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathscr{S}]$ is the inclusion functor $\mathrm{Inc}\colon\ \mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$. As an internal diagram it is given by the second projection of the order $<$ on N, since $\{m \mid m < n\}$ has cardinality n. Given an object M of $\mathscr{S}$, The classifying topos for $\mathbb{O}_\bullet/M$ is the slice topos $\mathscr{S}/M$. Hence the classifying topos of $\mathbb{O}_\bullet$ is the slice topos $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathscr{S}]/\mathrm{Inc}$. The generic model of $\mathbb{O}_\bullet$ in $[\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathscr{S}]/\mathrm{Inc}$ is the pair $(\mathrm{Inc}, \pi\colon \mathrm{Inc} \to \mathrm{Inc} \times \mathrm{Inc})$ where Δ is the diagonal transformation which renders the diagram below commutative:

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{id}"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

So far the discussion of p as classifier has been firmly anchored to $\mathscr{S}$ and M, but notice that (G, M) is a model of U in p. We now turn to discussing how it fits in more generally with U-**Mod** by spelling out the properties of p as a classifying topos that are shown in [Vic17]. The main result there, Theorem 31, says that P is "locally representable" over Q in the following fibration tower.

$$(\mathfrak{GTop}_{\cong}\text{-}U)^{\mathrm{co}} \xrightarrow{P} (\mathfrak{GTop}_{\cong}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{\mathrm{co}} \xrightarrow{Q} (\mathfrak{ETop}_{\cong}\text{-}\mathbb{T}_0)^{\mathrm{co}}$$

There is a slight change of notation from [Vic17]. $\mathfrak{GTop}$ there, unlike ours, restricts the 2-morphisms to be isomorphisms downstairs. This is needed to make P and Q 2-fibrations. To emphasize the distinction we have written $\mathfrak{GTop}_{\cong}$ above.

The objects of $\mathfrak{GTop}_{\cong}$-U are pairs (q, N) where $q\colon \overline{q} \to \underline{q}$ is a bounded geometric morphism and $N = (\overline{N}, \underline{N})$ is a model of U in q. A 1-morphism from (q_0, N_0) to (q_1, N_1) is a triple (f, f^-, f_-) such that $f\colon q_0 \to q_1$ in $\mathfrak{GTop}$, $(f^-, f_-)\colon N_0 \to f^\star N_1$ is a homomorphism of U-models, *and* f_- *is an isomorphism*. It is P-

cartesian iff f^- too is an isomorphism. A 2-morphism is a 2-morphism $\alpha\colon f \Rightarrow g$ in $\mathcal{G}\mathfrak{Top}_{\cong}$ ($\underline{\alpha}$ an iso) such that $\alpha^\star N_1 \circ (f^-, f_-) = (g^-, g_-)$. $\mathcal{G}\mathfrak{Top}_{\cong}$-($\mathbb{T}_0 \subset \mathbb{T}_0$) is similar, but without the $\overline{N}$s and f^-s.

Let us now unravel the local representability. It says that for each $(\mathscr{S}, M)$ in $\mathcal{E}\mathfrak{Top}_{\cong}$-$\mathbb{T}_0$ there is a classifier $(p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$, where G is the generic model of $\mathbb{T}_1/M$.

PROPOSITION 4.1.5. [Vic17, Proposition 19] The properties that characterize p as classifier are equivalent to the following.

(i) For every object (q, N) of $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$, 1-morphism $\underline{f}\colon \underline{q} \to \underline{p}$ in $\mathcal{E}\mathfrak{Top}$ and isomorphism $f_-\colon \underline{N} \to \underline{f}^\star M$, there is a P-cartesian 1-morphism $(f, f^-, f_-)\colon (q, N) \to (p, (G, M))$ over $(\underline{f}, f_-)$. In other words, there is f over $\underline{f}$ and an isomorphism (P-cartesianness) $f^-\colon \overline{N} \cong f^\star G$ over f_-.

(ii) Suppose $(f, f^-, f_-), (g, g^-, g_-)\colon (q, N) \to (p, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$, with (g, g^-, g_-) being P-cartesian (g^- is an iso). Suppose also we have $\underline{\alpha}\colon \underline{g} \Rightarrow \underline{f}$ so that $\underline{\alpha}^\star M$ commutes with f_- and g_-. (Note the reversal of 2-morphisms compared with [Vic17, Proposition 19]. This is because the fibration tower uses the 2-morphism duals ($\mathcal{G}\mathfrak{Top}_{\cong}$-$U)^{\mathrm{co}}$ etc.) Then $\underline{\alpha}$ has a unique lift $\alpha\colon g \Rightarrow f$ such that $(\alpha^\star G)g^- = f^-$.

In the case where we have identity 1-morphisms and 2-morphisms downstairs, it can be seen that this matches the usual characterization of classifier for $\mathbb{T}_1/M$ in $\mathcal{B}\mathfrak{Top}/\mathscr{S}$.

Although the properties described above insist on the 2-morphisms $\underline{\alpha}$ and model homomorphisms f_- downstairs being isomorphisms, we shall generalize this in a new result, Proposition 4.1.7.

We first remark on the construction of finite lax colimits in the 2-category $\mathcal{E}\mathfrak{Top}$ and more specifically cocomma objects which will be used in our proof. There is a forgetful 2-functor $\mathfrak{U}$ from $\mathcal{E}\mathfrak{Top}^{\mathrm{op}}$ to the 2-category of categories which sends a topos $\mathscr{E}$ to its underlying category $\mathcal{E}$, a geometric morphism $f\colon \mathscr{E} \to \mathscr{F}$

to its inverse image part $f^*\colon \mathcal{F} \to \mathcal{E}$ and a geometric transformation $\theta\colon f \Rightarrow g$ to the natural transformation $\theta^*\colon f^* \Rightarrow g^*$.

The 2-functor $\mathcal{U}$ transforms colimits in $\mathcal{E}\mathfrak{Top}$ to limits in $\mathfrak{Cat}$. This in particular means that the underlying category of a coproduct of toposes, for instance, is the product of their underlying categories. The same is true for cocomma objects. More specifically, for any topos $\mathscr{E}$, with cocomma topos $(\mathrm{id}_{\mathscr{E}} \uparrow \mathrm{id}_{\mathscr{E}})$ equipped with geometric morphisms $i_0, i_1\colon \mathscr{E} \rightrightarrows (\mathrm{id}_{\mathscr{E}} \uparrow \mathrm{id}_{\mathscr{E}})$ and 2-morphism θ between them, the data $\langle i_0^\star, i_1^\star, \theta^\star \rangle$ specifies the corresponding comma category $\left(\mathrm{id}_{\mathcal{U}(\mathscr{E})} \downarrow \mathrm{id}_{\mathcal{U}(\mathscr{E})}\right)$. For more details on the construction of cocomma toposes see [Joh02a, B3.4.2]. Another useful remark is about the relation of topos models of $\mathbb{T}^{\to}$ and models of $\mathbb{T}$.

LEMMA 4.1.6. Models of $\mathbb{T}^{\to}$ in a topos $\mathscr{E}$ are equivalent to models of $\mathbb{T}$ in the cocomma topos $(\mathrm{id}_{\mathscr{E}} \uparrow \mathrm{id}_{\mathscr{E}})$.

PROPOSITION 4.1.7. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension maps of contexts, M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathscr{S}$, and $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ the corresponding classifying topos with generic model G.

Let $q\colon \overline{q} \to \underline{q}$ be a bounded geometric morphism, and let $(f_i, f_i^-, f_{i-})\colon (q, N_i) \to (p, (G, M))$ $(i = 0, 1)$ be two P-cartesian 1-morphisms in $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$.

Suppose $\varphi\colon N_0 \to N_1$ is a homomorphism of U-models and $\underline{\alpha}\colon \underline{f}_0 \Rightarrow \underline{f}_1$ is such that the left hand diagram in below commutes. Then there exists a unique 2-morphism $\alpha\colon f_0 \Rightarrow f_1$ over $\underline{\alpha}$ such that the right hand diagram commutes.

$$\begin{tikzcd}
\underline{N}_0 \arrow[r, "f_{0-}"] \arrow[d, "\underline{\varphi}"] & \underline{f}_0{}^\star M \arrow[d, "\underline{\alpha}^\star M"] & \overline{N}_0 \arrow[r, "f_0^-"] \arrow[d, "\overline{\varphi}"] & f_0{}^\star G \arrow[d, dotted, "\alpha^\star G"] \\
\underline{N}_1 \arrow[r, "f_{1-}"] & \underline{f}_1{}^\star M & \overline{N}_1 \arrow[r, "f_1^-"] & f_1{}^\star G
\end{tikzcd}$$

Proof. Note that we do not assume that $\underline{\alpha}$ and $\underline{\varphi}$ are isomorphisms, so φ need not be a 1-morphism in $\mathcal{G}\mathfrak{Top}_{\cong}$. To get round this, we use cocomma toposes.

Let $\underline{q}' = \underline{q} \uparrow \underline{q}$ and $\overline{q}' = \overline{q} \uparrow \overline{q}$ be the two cocomma toposes, with bounded geometric morphism $q'\colon \overline{q}' \to \underline{q}'$. We now have two 1-morphisms $i_0, i_1\colon q \to q'$

in $\mathcal{G}\mathfrak{Top}$, equipped with identities for $\overset{\blacktriangledown}{i}_0$ and $\overset{\blacktriangledown}{i}_1$, and a 2-morphism $\theta\colon i_0 \Rightarrow i_1$. The pair $\varphi = (\overline{\varphi}, \underline{\varphi})$ is a model of U in q'.

The geometric transformation $\underline{\alpha}$ gives us a geometric morphism $\underline{a}\colon \underline{q}' \to \mathscr{S}$, with an isomorphism $a_-\colon \underline{\varphi} \cong \underline{a}^\star M$, so a 1-morphism in $\mathcal{E}\mathfrak{Top}_{\cong}$-$\mathbb{T}_0$. This lifts to a P-cartesian 1-morphism $(a, a^-, a_-)\colon (q', \varphi) \to (p, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$. We now have the following diagrams in $\mathcal{G}\mathfrak{Top}$ and $\mathcal{G}\mathfrak{Top}_{\cong}$-$U$.

$$\begin{tikzcd}
 & \Downarrow\mu_1 & \\
q \arrow[r, bend left, "i_1"] \arrow[r, bend right, "i_0"'] \arrow[r, phantom, "\Uparrow\theta"] \arrow[rr, bend left=60, "f_1"] \arrow[rr, bend right=60, "f_0"'] & q' \arrow[r, "a"] & p \\
 & \Uparrow\mu_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
(q, N_1) \arrow[dr, "{(i_1,\mathrm{id},\mathrm{id})}"'] \arrow[drr, bend left, "{(f_1,f_1^-,f_{1-})}"] & & \\
 & (q', \varphi) \arrow[r, "{(a,a^-,a_-)}"'] & (p, (G, M)) \\
(q, N_0) \arrow[ur, "{(i_0,\mathrm{id},\mathrm{id})}"] \arrow[urr, bend right, "{(f_0,f_0^-,f_{0-})}"'] & &
\end{tikzcd}$$

In the right hand diagram all the 1-morphisms are P-cartesian, and it follows there are unique iso-2-morphisms $\mu_i\colon (f_1, f_i^-, f_{i-}) \Rightarrow (a, a^-, a_-)(i_1, \mathrm{id}, \mathrm{id})$ lifting the identity 2-morphisms downstairs. Now by composing μ_0, $a \mathbin{\scriptstyle\bullet} \theta$ and μ_1^{-1} we get the required α.

To show uniqueness of the geometric transformation α, suppose we have another, β, with the same properties. In other words, $\underline{\alpha} = \underline{\beta}$ and $\alpha^\star(G, M) = \beta^\star(G, M)$. We thus get two 1-morphisms $a, b\colon q' \rightrightarrows p$, $a = (f_0, \alpha, f_1)$ and $b = (f_0, \beta, f_1)$. We have $\underline{a} = \underline{b}$ and $a^\star(G, M) = b^\star(G, M)$ and it follows, by Proposition 4.1.5 part (ii), that there is a unique vertical 2-morphism $\iota\colon a \Rightarrow b$ such that $\iota^\star(G, M)$ is the identity.

By composing horizontally with θ, we can analyse ι as a pair of 2-morphisms $\iota_\lambda\colon f_\lambda \Rightarrow f_\lambda$ ($\lambda = 0, 1$) such that the following diagram commutes.

$$\begin{tikzcd}
f_0 \arrow[r, "\iota_0"] \arrow[d, "\alpha"'] & f_0 \arrow[d, "\beta"] \\
f_1 \arrow[r, "\iota_1"] & f_1
\end{tikzcd}$$

Now we see that each ι_λ is the the unique vertical 2-morphism such that ${\iota_\lambda}^\star(G, M)$ is the identity, so ι_λ is the identity on f_λ and $\alpha = \beta$. $\square$

4.2 Fibrations of toposes from fibrations of contexts

We are now at a stage that we can state our main theorem. Notice how our reformulation of the Johnstone criterion assists our proof. We do not have to deal with so many bipullback toposes, and there is a single elementary topos $\overline{q}$ where we examine models of the various contexts.

LEMMA 4.2.1. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map of contexts with the fibration property in the Chevalley style (Definition 2.4.1), let M be a model of $\mathbb{T}_0$ in an elementary topos $\mathscr{S}$, and let $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ be the classifier for $\mathbb{T}_1/M$ with generic model G. Suppose $f, g\colon q \rightrightarrows p$ are two 1-morphisms in $\mathcal{G}\mathfrak{Top}$ and $\alpha\colon f \Rightarrow g$ a 2-morphism. We write $\varphi := \alpha^\star(G, M)$, so that $\overline{\varphi} = \alpha^\star G$ is a model of $\mathbb{T}_1^{\to}$ in $\overline{q}$. Then α is a cartesian 2-morphism (in $\mathcal{G}\mathfrak{Top}$ over $\mathcal{E}\mathfrak{Top}$) iff $\eta_{\overline{\varphi}}$ is an isomorphism, where $(\eta_{\overline{\varphi}}, \mathrm{id})$ is the unit for $\Gamma_1\text{-}\mathbf{Mod}\,(\overline{q}) \dashv \Lambda_1\text{-}\mathbf{Mod}\,(\overline{q})$.

Proof. ($\Rightarrow$): Let $\overline{N}$ be the domain of $\overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1$, and let $\underline{N} := \underline{f}^\star M$. Then (see diagram (3.10))

$$\begin{aligned}\overline{N} \centerdot U &= \overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1 \centerdot \Gamma_0 \centerdot \pi_0 \centerdot U = \overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1 \centerdot U^{\to} \centerdot \mathrm{dom} = \overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1 \centerdot \Gamma_1 \centerdot U_1 \centerdot \mathrm{dom}\\ &= \overline{\varphi} \centerdot \Gamma_1 \centerdot U_1 \centerdot \mathrm{dom} = \overline{\varphi} \centerdot U^{\to} \centerdot \mathrm{dom} = \overline{\varphi} \centerdot \mathrm{dom} \centerdot U = (f^\star G) \centerdot U = \underline{N},\end{aligned}$$

and so $N := (\overline{N}, \underline{N})$ is a model of U in q.

$$\begin{tikzcd}
f^\star G \arrow[r, "\overline{\varphi}:=\alpha^\star G"] \arrow[d, "\eta_{\overline{\varphi}}"] \arrow[dd, dotted, bend right=50, "\beta'^\star G"'] & g^\star G \arrow[d, equal] \\
\overline{N} \arrow[r, "\overline{\varphi}\centerdot\Gamma_1\centerdot\Lambda_1"] \arrow[d, dotted, "e^-"] & g^\star G \arrow[d, equal] \\
e^\star G \arrow[r, dotted, "\gamma^\star G"'] \arrow[uu, dotted, bend left=80, "\beta^\star G"] & g^\star G
\end{tikzcd}$$

By the classifier property of p (Proposition 4.1.5), and taking $\underline{e} := \underline{f}$ and $e_- := \mathrm{id}\colon \underline{N} = \underline{f}^\star M$, we obtain $e\colon q \to p$ and $(e^-, \mathrm{id})\colon N \cong e^\star(G, M)$. Now by Proposition 4.1.7 we get a unique $\gamma\colon e \Rightarrow g$ over $\underline{\gamma} := \underline{\alpha}$ such that $\overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1 = (\gamma^\star G)e^-$. Again by Proposition 4.1.7 we get a unique $\beta'\colon f \Rightarrow e$ over $\mathrm{id}_{\underline{f}}$ such that $e^-\eta_{\overline{\varphi}} = \beta'^\star G$, and since $(\gamma^\star G)(\beta'^\star G) = \alpha^\star G$ it follows that $\gamma\beta' = \alpha$.

By cartesianness of α we also have a unique $\beta\colon e \Rightarrow f$ over $\mathrm{id}_{\underline{f}}$ such that $\gamma = \alpha\beta$, and since $\alpha\beta\beta' = \gamma\beta' = \alpha$ it follows that $\beta\beta' = \mathrm{id}_f$. We deduce that $(\beta^\star G)e^-\eta_{\overline{\varphi}} = \mathrm{id}_{f^\star G}$.

Finally $\eta_{\overline{\varphi}}(\beta^\star G)e^- = \mathrm{id}_{\overline{N}}$ follows from the adjunction $\Gamma_1 \dashv \Lambda_1$, because both sides reduce by Γ_1 to the identity. Hence $\eta_{\overline{\varphi}}$ is an isomorphism, with inverse $(\beta^\star G)e^-$.

($\Leftarrow$): Let $e\colon q \to p$ with $\gamma\colon e \Rightarrow f$ such that $\underline{\gamma} = \underline{\alpha}\underline{\beta}$.

$$\begin{tikzcd}
f^\star G \arrow[r, "\overline{\varphi}:=\alpha^\star G"] \arrow[d, "\eta_{\overline{\varphi}}"] & g^\star G \arrow[d, equal] \\
\overline{N} \arrow[r, "\overline{\varphi}\centerdot\Gamma_1\centerdot\Lambda_1"] & g^\star G \arrow[d, equal] \\
e^\star G \arrow[u, dotted, "\overline{\psi}"'] \arrow[uu, dotted, bend left, "\beta^\star G"] \arrow[r, dotted, "\gamma^\star G"'] & g^\star G
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{f}^\star M \arrow[r, "\underline{\varphi}:=\underline{\alpha}^\star M"] \arrow[d, equal] & \underline{g}^\star M \arrow[d, equal] \\
\underline{N} \arrow[r, "\underline{\varphi}:=\underline{\alpha}^\star M"] & \underline{g}^\star M \arrow[d, equal] \\
\underline{e}^\star M \arrow[u, dotted, "\underline{\beta}^\star M"'] \arrow[uu, dotted, bend left, "\underline{\beta}^\star M"] \arrow[r, dotted, "\underline{\gamma}^\star M"'] & \underline{g}^\star M
\end{tikzcd}$$

By the adjunction $\Gamma_1 \dashv \Lambda_1$ there is a unique $\mathbb{T}_1$-morphism $\overline{\psi}\colon e^\star G \to \overline{N}$ over $\underline{\beta}^\star M$ such that $(\overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1)\overline{\psi} = \gamma^\star G$. Because $\eta_{\overline{\varphi}}$ is an isomorphism this corresponds to a unique $\overline{\psi}'\colon e^\star G \to f^\star G$ over $\underline{\beta}^\star M$ such that $\overline{\varphi}\overline{\psi}' = \gamma^\star G$. By Proposition 4.1.7 this corresponds to a unique $\beta\colon e \Rightarrow f$ over $\underline{\beta}$ such that $(\alpha^\star G)(\beta^\star G) = \gamma^\star G$, i.e. unique such that $\alpha\beta = \gamma$. This proves that α is cartesian. $\square$

THEOREM 4.2.2. If $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is an (op)fibration extension map of AU-contexts (in the sense of Definition 2.4.1), and M a model of $\mathbb{T}_0$ in an elementary topos $\mathscr{S}$, then $p\colon \mathscr{S}[\mathbb{T}_1/M] \to \mathscr{S}$ is an (op)fibration of toposes (in the sense of Definition 2.6.1).

Proof. Here we only prove the theorem for the case of fibrations. A proof for the opfibration case is similarly constructed. According to Proposition 2.6.10, in order to establish that p is a fibration in the 2-category $\mathcal{E}\mathfrak{Top}$, we have to verify that the conditions $(B1)$-$(B3)$ in Definition 2.5.6 hold for $P = \mathbf{cod}\colon \mathfrak{K}_{\mathcal{D}} \to \mathfrak{K}$, where $\mathfrak{K} = \mathcal{E}\mathfrak{Top}$, $\mathcal{D}$ is the class of bounded geometric morphisms, and so $\mathfrak{K}_{\mathcal{D}}$ is $\mathcal{G}\mathfrak{Top}$.

By Proposition 2.5.8, the condition $(B1)$ follows from the fact that p is bicarrable.

To prove condition $(B2)$, let $q\colon \overline{q} \to \underline{q}$ be a bounded geometric morphism, let $g\colon q \to p$ be a 1-morphism in $\mathfrak{K}_{\mathcal{D}}$, let $\underline{f}\colon \underline{q} \to \mathscr{S}$ be geometric morphism and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ a geometric transformation.

$$\begin{tikzcd}
\overline{q} \arrow[rr, bend left, "\overline{g}"] \arrow[dd, "q"'] & & \mathscr{S}[\mathbb{T}_1/M] \arrow[dd, "p"] \\
& \overset{\blacktriangledown}{g} \Downarrow & \\
\underline{q} \arrow[rr, bend left, "\underline{g}"] \arrow[rr, bend right, "\underline{f}"'] \arrow[rr, phantom, "\underline{\alpha} \Uparrow"] & & \mathscr{S}
\end{tikzcd}$$

We seek f over $\underline{f}$ with a cartesian lift $\alpha\colon f \Rightarrow g$ of $\underline{\alpha}$. Notice that for the given model M of $\mathbb{T}_0$ in $\mathscr{S}$, the component M of the natural transformation $\underline{\alpha}$ gives us a morphism $\underline{\alpha}^{\star}M\colon \underline{f}^{\star}M \to \underline{g}^{\star}M$ of $\mathbb{T}_0$-models in $\underline{q}$, hence a $\mathbb{T}_0^{\to}$-model in $\underline{q}$. Let us write it as $\underline{\varphi}\colon \underline{N}_f \to \underline{N}_g$. Then $q^{\star}\underline{\varphi}$ is a model of $\mathbb{T}_0^{\to}$ in $\overline{q}$.

Let G be the generic model of $\mathbb{T}_1/M$ in $\mathscr{S}[\mathbb{T}_1/M]$, so that (G, M) is a model of U in p. Hence we get $(\overline{N}_g, \underline{N}_g) := g^{\star}(G, M)$ a model of U in q, and

$$\mathfrak{g} := (\overline{N}_g, q^{\star}\underline{\varphi}) \in \mathrm{cod}^{*}(\mathbb{T}_1)\text{-}\mathbf{Mod}\,\overline{q}.$$

Then $\mathfrak{g} \mathbin{\bullet} \Lambda_1$ (see diagram (3.10)) is a model $\overline{\varphi}\colon \overline{N}_f \to \overline{N}_g$ of $\mathbb{T}_1^{\to}$ in $\overline{q}$, with $\overline{N}_f = \mathfrak{g} \mathbin{\bullet} (\Lambda_1; \Gamma_0; \pi_0)$. We also see that $\overline{\varphi} \mathbin{\bullet} U^{\to} = \mathfrak{g} \mathbin{\bullet} (\Lambda_1; U^{\to}) = q^\star \underline{\varphi}$, so $\varphi := (\overline{\varphi}, \underline{\varphi})\colon N_f \to N_g$ is a homomorphism of U-models in q.

$$\begin{tikzcd}
\overline{N}_f \arrow[r, "\overline{\varphi}"] \arrow[d, mapsto] & \overline{N}_g \arrow[d, mapsto] \\
q^\star \underline{N}_f \arrow[r, "q^\star \underline{\varphi}"] & q^\star \underline{N}_g
\end{tikzcd}$$

Thus we get two objects (q, N_f) and (q, N_g) of P together with φ as in Proposition 4.1.7. In addition we have $(p, (G, M))$, and a P-cartesian 1-morphism

$$(g, (\mathrm{id}\colon \overline{N}_g = g^\star G, \mathrm{id}\colon \underline{N} = \underline{g}^\star M))\colon (q, N_g) \to (p, (G, M)).$$

By the classifier property we can also find a P-cartesian 1-morphism

$$(f, (f^-, f_-))\colon (q, N_f) \to (p, (G, M)).$$

We can now apply Proposition 4.1.7 to find a 2-morphism $\alpha\colon f \Rightarrow g$ over $\underline{\alpha}$ that gives us $\overline{\varphi}$.

Since $\overline{\varphi}$ is defined to be of the form $\mathfrak{g} \mathbin{\bullet} \Lambda_1$, so $\overline{\varphi} \mathbin{\bullet} \Gamma_1 \mathbin{\bullet} \Lambda_1 = \overline{\varphi}$, we find that $\eta_{\overline{\varphi}}$ is the identity and $\eta_{\alpha^\star G}$ is an isomorphism. It follows from Lemma 4.2.1 that α is cartesian.

For proving ($B3$), suppose we have $f, g\colon q \rightrightarrows p$ and a cartesian 2-morphism $\alpha\colon f \Rightarrow g$. By Lemma 4.2.1, $\eta_{\alpha^\star G}$ is an isomorphism. Take any 1-morphism $k\colon q' \to q$ in $\mathfrak{GTop}$ where $q'\colon \overline{q}' \to \underline{q}'$. Relative to the isomorphism of models $k^\star(\mathfrak{g} \mathbin{\bullet} \Lambda_1) \cong (k^\star \mathfrak{g}) \mathbin{\bullet} \Lambda_1$, $k^\star$ preserves the unit η, and so $\eta_{k^\star \alpha^\star G}$ is an isomorphism and, by Lemma 4.2.1, $\alpha \mathbin{\bullet} k$ is cartesian. □

The result can now be applied to the examples in §3.4.

(i) The classifiers for Example 3.3.9 are, by Diaconescu's theorem, those bounded geometric morphisms got as $[\mathbb{C}, \mathscr{S}] \to \mathscr{S}$ for $\mathbb{C}$ an internal category in $\mathscr{S}$. Example 3.4.7 now tells us that such geometric morphisms are opfibrations in $\mathcal{E}\mathfrak{Top}$. This is already known, of course, and appears in [Joh02a, B4.4.9]. Note, however, that our calculation to prove the opfibration property in $\mathfrak{Con}$ is elementary in nature. The proof of [Joh02a] verifies that the class of all such geometric morphisms satisfies the "covariant tensor condition", and such a technique cannot work for AUs as it uses the direct image parts of geometric morphisms.

(ii) The classifiers for Example 3.3.6 are the local homeomorphisms. Their opfibrational character follows simply from our results, though note that it can also be deduced as a special case of the torsor result. Let $\mathscr{B}$ be a bounded topos over the base topos $\mathscr{S}$, and $f\colon M \to N$ a morphism in $\mathscr{B}$. The geometric morphism $p_U\colon \mathscr{S}[\mathbb{O}_\bullet] \to \mathscr{S}[\mathbb{O}]$, induced by the context extension map U, is an opfibration and a local homeomorphism, and we get the 1-morphisms and 2-morphisms in the left diagram in below where the inverse image of the geometric morphism ℓ_f is the base change f^* and the direct image is formed by the dependent product Π_f along f.

$$\begin{tikzcd}
 & \mathscr{B}/N \arrow[dr] \arrow[dd] & \\
\mathscr{B}/M \arrow[ur, dashed, bend left, "l_f"] \arrow[rr, bend right] \arrow[dd] & \arrow[r, phantom, "\overset{\widetilde{f}}{\Nearrow}"] & \mathscr{S}[\mathbb{O}_\bullet] \arrow[dd, "p_U"] \\
 & \mathscr{B} \arrow[dl, equal] \arrow[dr, "\ulcorner N \urcorner"] & \\
\mathscr{B} \arrow[rr, bend right, "\ulcorner M \urcorner"'] & \arrow[u, phantom, "\overset{f}{\Nearrow}"] & \mathscr{S}[\mathbb{O}]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{S}'[\mathbb{O}_\bullet/n^*X] \arrow[dr] \arrow[dd] & \\
\mathscr{S}'[\mathbb{O}_\bullet/m^*X] \arrow[ur, dashed, bend left, "l_\alpha"] \arrow[rr, bend right] \arrow[dd] & \arrow[r, phantom, "\overset{\widetilde{\alpha}}{\Nearrow}"] & \mathscr{S}[\mathbb{O}_\bullet/X] \arrow[dd, "p_{U,X}"] \\
 & \mathscr{S}' \arrow[dl, equal] \arrow[dr, "n"] & \\
\mathscr{S}' \arrow[rr, bend right, "m"'] & \arrow[u, phantom, "\overset{\alpha}{\Nearrow}"] & \mathscr{S}
\end{tikzcd}$$

Note however the crucial point that whereas the left diagram exists only for $\mathscr{S}$-toposes $\mathscr{B}$ (i.e. the opfibration property of p_U is limited to the 2-category $\mathcal{B}\mathfrak{Top}/\mathscr{S}$), the right diagram exists for any elementary topos $\mathscr{S}'$, any object X of $\mathscr{S}$, and arbitrary geometric morphisms m and n, and any geometric transformation α between them.

(iii) By Example 3.4.6, the classifiers for Example 3.3.10 are fibrations. Since the spectra of distributive lattices correspond to propositional coherent theories, this fibrational nature is already known from [Joh02a, B4.4.11], which says that any coherent topos is a fibration. It will be interesting to see how far our methods can cover this general result.

Conclusion

5

5.0 Summary and discussion

We have shown that an important and rather extensive class of fibrations/opfibrations in the 2-category $\mathcal{E}\mathfrak{Top}$ of toposes arises from strict fibrations/opfibrations in the 2-category $\mathfrak{Con}$ of contexts.

There are several advantages: first, the structure of strict fibrations/opfibrations in $\mathfrak{Con}$ is much easier to study because of the explicit and combinatorial description of $\mathfrak{Con}$ and in particular due to the existence of comma objects in there.

Second, proofs concerning properties of based-toposes arising from $\mathfrak{Con}$ are very economical since one only needs to work with strict models of contexts. Not only does this approach help us to avoid taking the pain of working with limits and colimits in $\mathcal{E}\mathfrak{Top}/\mathscr{S}$ and bookkeeping of the coherence issues, but it also gives us insights in inner working of 2-categorical aspects of toposes via a more concrete and constructive approach of context building and context extensions.

The AU-approach to generalized spaces has some of the essential traits of well-studied *formal topology* approach to spaces: in our AU-approach we offered a predicative treatment of some of the main aspects of generalized spaces which has the potential of giving computational content to constructions involving infinite objects (e.g. spaces) and connecting this the finite nature of computers and computational processes.

A good handle of strictness as well as an intensional equality of nodes and edges makes the theory of AU-contexts, modulo certain technical issues in formalization of context extensions, well-suited to computer verification and proof checking. There is also an advantage from foundational point of view; for any $\mathscr{S}$-topos $\mathscr{E}$, there are logical properties internal to $\mathscr{E}$ which are determined by internal logic of $\mathscr{S}$. A consequence of this work is that we can reason in 2-category of contexts to get uniform results about toposes independent of their base $\mathscr{S}$. Crucially, the methods of achieving these results are all predicative.

Above all, we argue that our approach is conceptually stronger than [Joh02a]: if we are to prove that a geometric morphism $p\colon \mathscr{E} \to \mathscr{S}$ in $\mathfrak{ETop}$ is a fibration (resp. opfibration) we have to prove the existence of a lifting structure for every geometric morphism from $\mathscr{A}$ to $\mathscr{B}$, and for every geometric transformation between any such two geometric morphisms. However, if p arises from a fibration of AU-contexts $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ (as in Theorem 4.2.2) we only need to check the (strict) lifting structure along the generic codomain (resp. domain) map $\mathbb{T}_0^{\to} \to \mathbb{T}_0$. Also this lifting structure is strict, which solves in practice the problem of verification of tracking coherence data of involved pullbacks.

We hope that in future work we can investigate in a broader context the question to what extent the 2-categorical structure of $\mathfrak{ETop}$ can be presented by contexts, and more importantly whether we find further simpler proofs in $\mathfrak{Con}$ that can be transported to toposes.

5.1 Further work

We propose three lines of research from here: a new approach to exponentiability via bag AU-contexts, a discussion of 'stuff-structure-property' for AU-contexts and their relevance to factorization system on geometric morphisms of toposes, and the prospects of computer formalization of the theory of AU-contexts.

On the way, we will need a few conjectures which seem plausible but whose proofs need further work and will be subject of future research.

5.1.1 Partial products for AUs

Dyckhoff and Tholen in [DT87] prove that for cartesian categories existence of partial products along a morphism $p\colon E \to B$ is equivalent to the exponentiability of p in the slice category over B.

In [Joh93] and [Joh02a, B4.4], Johnstone generalizes partial products to 2-categories with a modicum of colimits. He then shows that in the 2-category $\mathcal{E}\mathfrak{Top}$ a significant class of partial products exist.

Indeed for toposes, Johnstone proves that partial products exist along an (op)fibration which is exponentiable and satisfies the *PCC condition*. As such, (op)fibrations need PCC if they are to work well. At the end of this section we explain our approach for developing a theory of partial products for AUs.

But first, we shall review Johnstone's definition of partial products in the context of bag toposes and after that, we shall review the PCC condition adapted to our setting of upper bar-lower bar of 2-category $\mathcal{G}\mathfrak{Top}$ from Construction 1.10.

Bag toposes and partial products

Bag spaces were originally conceived as *bagdomains* by Vickers [Vic92] in the context of algebraic dcpos (directed complete posets). In that paper, $\mathscr{B}ag(X)$ is for set-indexed families of points of an algebraic dcpo X.

In a series of papers Johnstone ([Joh92], [Joh93], [Joh94]) generalized the construction of bag domain to toposes and gave a universal characterization of a $\mathscr{B}ag(\mathscr{X})$ as a *2-categorical partial product* using the notion of *(op)fibration* internal to the 2-category of toposes. Johnstone also showed the existence of $\mathscr{B}ag(\mathscr{X})$ for any topos $\mathscr{X}$. He showed how to vary the type of indexing object from 'set' to other structures such as a category or spectral space.

Let $\mathscr{B} = \mathscr{S}_0[\mathbb{O}]$ be the object classifier topos, and $\mathscr{E} = \mathscr{S}_0[\mathbb{O}_\bullet]$ the classifying topos of pointed objects with $p\colon \mathscr{E} \to \mathscr{B}$ taking a point $(I, i \in I)$ to the 'set' I.

Suppose $\mathscr{X}$ is another topos. There is a topos $\mathscr{B}ag(\mathscr{X})$ which classifies *'bags of points'* (aka set-indexed families of points) of $\mathscr{X}$ indexed by points of $\mathscr{B}$ with the map $\lambda_{\mathscr{X}}\colon \mathscr{B}ag(\mathscr{X}) \to \mathscr{B}$ which takes a point $(I, \{x_i\}_{i\in I})$ to the index 'set' I.

The bipullback topos $\lambda_X^*(\mathscr{E})$ classifies triples $(I, \{x_i\}_{i\in I}, i \in I)$. There is a map

$$\begin{gathered} x\colon \lambda_X^*(\mathscr{E}) \to \mathscr{X} \\ (I, \{x_i\}_{i\in I}, i \in I) \longmapsto x_i \end{gathered}$$

$$\begin{tikzcd}
\mathscr{X} & \lambda_{\mathscr{X}}^*\mathscr{E} \arrow[l, "x"'] \arrow[r, "p^*f"] \arrow[d, "f^*p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{E} \arrow[d, "p"] \\
& \mathscr{B}ag\mathscr{X} \arrow[r, "\lambda_{\mathscr{X}}"'] & \mathscr{B}
\end{tikzcd}$$

In fact, $(\mathscr{B}ag(\mathscr{X}), \lambda_{\mathscr{X}}, x)$ is the '*universal*' solution to filling in the question marks of the following diagram in the 2-category $\mathcal{E}\mathfrak{Top}$ of elementary toposes.

$$\begin{tikzcd}
\mathscr{X} & \bullet \arrow[l, "?"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{E} \arrow[d, "p"] \\
& ? \arrow[r, "?"'] & \mathscr{B}
\end{tikzcd}$$

In general in any (cartesian) category, if such a 'universal' solution exists, it is known as the **partial product** of p and X, and is denoted by $\mathcal{P}(p, X)$.

What happens when we move on to 2-categories? In particular how do we express the universality?

A universal structure can be defined by a representation of a certain 2-functor

$$\begin{aligned} \mathcal{F}^\bullet \colon \mathfrak{K}^{\mathrm{op}} &\to \mathfrak{Cat} \\ Q &\longmapsto \mathcal{F}^\bullet(Q) \end{aligned}$$

Clearly, the category $\mathcal{F}^\bullet(Q)$ should have, as its objects, pairs $(\underline{f}\colon Q \to B, \overline{x}\colon \underline{f}^*E \to X)$. But what about its morphisms? How should we define a morphism $(\underline{f}\colon Q \to B, \overline{x}\colon f^*E \to X) \to (\underline{g}\colon Q \to B, \overline{y}\colon g^*E \to X)$? Even if we have a 2-morphism $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$, there seems to be no clear choice for a morphism $(\underline{f}, \overline{x}) \to (\underline{g}, \overline{y})$.

However, if p is a *fibration* (or *opfibration*) we get a transport morphism of fibres $r_\alpha\colon \underline{g}^*E \to \underline{f}^*E$ together with a 2-morphism $\overline{\alpha}\colon p^*f \circ r_\alpha \Rightarrow p^*g$. In short, we get a morphism in lax slice $\mathfrak{K} \swarrow E$.

$$\begin{tikzcd}
 & X \arrow[dl, equal] & & \underline{g}^*E \arrow[ll, "\overline{y}"'] \arrow[rr, "p^*\underline{g}"] \arrow[dl, "r(\alpha)"'] \arrow[dd, dashed, "\underline{g}^*p"] \arrow[dr, phantom, "\overset{\overline{\alpha}}{\Nearrow}"] & & E \arrow[dl, equal] \arrow[dd, "p"] \\
X & & \underline{f}^*E \arrow[ll, "\overline{x}"] \arrow[rr, "p^*\underline{f}"] \arrow[dd, "\underline{f}^*p"'] \arrow[dr, phantom, "\cong"] & & E \arrow[dd, "p"] & \\
 & & & A \arrow[rr, "\underline{g}"] \arrow[dl, equal] & & B \arrow[dl, equal] \\
 & & A \arrow[rr, "\underline{f}"'] \arrow[urr, phantom, "\overset{\underline{\alpha}}{\Nearrow}"] & & B &
\end{tikzcd}$$

In this situation a morphism $(\underline{f}, \overline{x}) \to (\underline{g}, \overline{y})$ is defined by a pair $(\underline{\alpha}, \overline{\beta})$ where $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ and $\overline{\beta}\colon \overline{x} \circ r_\alpha \Rightarrow \overline{y}$.

When p is an (op)fibration, a representing object $\mathcal{P}^\bullet(p, X)$ is given by an equivalence of categories

$$\mathbb{K}(Q, \mathcal{P}^\bullet(p, X)) \simeq \mathcal{F}^\bullet Q$$

where one half of the equivalence (from left to right) is given by pulling back the canonical partial product structure $(\mathcal{P}^\bullet(p, X), \lambda_X, \rho_X)$ along morphisms $Q \to \mathcal{P}(p, X)$.

Suppose $\mathfrak{K}$ is a 2-category and an internal fibration $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ and 1-morphisms $\underline{f}, \underline{g}\colon \underline{y} \rightrightarrows \underline{x}$ and 2-morphism $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ are given in $\mathfrak{K}$. Consider the diagram

$$\begin{tikzcd}
\overline{x}_g \arrow[r, "\overline{r_\alpha}"] \arrow[d, "x_g"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r_\alpha} \Downarrow"] & \overline{x}_f \arrow[d, "x_f"] & & \\
\underline{y} \arrow[r, "1"'] & \underline{y} \arrow[r, bend left, "\underline{d}_0"] \arrow[r, bend right, "\underline{d}_1"'] \arrow[r, phantom, "\Downarrow \underline{\delta}_y"] & \underline{y} \otimes \mathbf{2} \arrow[r, "\ulcorner \underline{\alpha} \urcorner"'] & \underline{x}
\end{tikzcd}$$

Recall that the universal property of $\langle \underline{y} \otimes \mathbf{2}, \underline{\delta}_y \rangle$ is expressed by the following equivalence of categories:

$$\mathfrak{K}(\underline{y} \otimes \mathbf{2}, \underline{w}) \simeq \mathfrak{Cat}(\mathbf{2}, \mathfrak{K}(\underline{y}, \underline{w}))$$

natural in objects $\underline{w}$. This makes $\underline{\delta}_y$ the universal 2-morphism making $\underline{y} \otimes \mathbf{2}$ equivalent to the cocomma object $\left(1_{\underline{y}} \uparrow 1_{\underline{y}}\right)$. In particular, $\underline{\alpha}$ factors through $\underline{y} \otimes \mathbf{2}$ via a 1-morphism $\ulcorner \underline{\alpha} \urcorner\colon \underline{y} \otimes \mathbf{2} \to \underline{x}$ and iso-2-morphisms $\underline{\kappa}\colon \ulcorner \underline{\alpha} \urcorner \circ \underline{d}_0 \cong \underline{f}$ and $\underline{\kappa}'\colon \ulcorner \underline{\alpha} \urcorner \circ \underline{d}_1 \cong \underline{g}$.

Now, the horizontal composition of 2-morphisms $\underline{\delta}_y$ and $\overset{\blacktriangledown}{r_\alpha}$ gives us a 2-morphism from $\underline{d}_0 \circ x_f \circ \overline{r_\alpha}$ to $\underline{d}_1 \circ x_g$ which factors through $\left(\overline{r_\alpha} \uparrow 1_{x_g}\right)$:

$$\begin{tikzcd}
& \overline{x}_g \arrow[dl, "\overline{r_\alpha}"'] \arrow[dr, "1"] & \\
\overline{x}_f \arrow[rr, Rightarrow, "\overline{\rho}"] \arrow[dr, "\overline{d}_0"] \arrow[ddd, "x_f"'] & & \overline{x}_g \arrow[dl, "\overline{d}_1"'] \arrow[ddd, "x_g"] \\
& \left(\overline{r_\alpha} \uparrow 1_{\overline{x}_g}\right) \arrow[ddd, dashed, "z"] & \\
\cong & & \cong \\
\underline{y} \arrow[dr, "\underline{d}_0"'] & & \underline{y} \arrow[dl, "\underline{d}_1"] \\
& \underline{y} \otimes \mathbf{2} &
\end{tikzcd} \tag{5.1}$$

Also, the 2-morphism $\overline{\alpha}\colon \overline{f}\circ\overline{r_\alpha} \Rightarrow \overline{g}$ factors through the cocomma object $\left(\overline{r_\alpha}\uparrow 1_{\overline{x}_g}\right)$, and thus we obtain a 1-morphism $\overline{k}\colon \left(\overline{r_\alpha}\uparrow 1_{\overline{x}_g}\right) \to \overline{x}$ and iso-2-morphisms $\overline{\kappa}\colon \overline{k}\circ\overline{d}_0 \cong \overline{f}$ and $\overline{\kappa}'\colon \overline{k}\circ\overline{d}_1 \cong \overline{g}$. Using condition (J1) of fibration x, we can put the data together in 2-category $\mathfrak{K}^{\downarrow}$:

$$\begin{tikzcd}
 & x_g \arrow[dr, "d_1"] \arrow[drr, bend left, "g"] & \cong & \\
x_g \arrow[ur, "1"] \arrow[dr, "r_\alpha"'] & \rho\Uparrow & z \arrow[r, dashed, "k"] & x \\
 & x_f \arrow[ur, "d_0"'] \arrow[urr, bend right, "f"'] & \cong &
\end{tikzcd} \tag{5.2}$$

where $\rho = (\overline{\rho}, \underline{\delta}_y)$, $\underline{k} = \ulcorner\underline{\alpha}\urcorner$, and the iso-2-morphisms in the diagram above are $\kappa = (\overline{\kappa}, \underline{\kappa})$ and $\kappa' = (\overline{\kappa}', \underline{\kappa}')$. So, the above diagram of 2-morphisms is indeed a factorization of 2-morphism $\alpha\colon f\circ r_\alpha \Rightarrow g$ into ρ, κ, and κ'.

REMARK 5.1.1. It is elementary to observe that comma (resp. cocomma) objects in $\mathfrak{K}^{\downarrow}$ correspond directly to comma (resp. cocomma) objects in $\mathfrak{K}$. More precisely, if $f\colon w \to x$ and $g\colon w \to y$ are 1-morphisms in $\mathfrak{K}^{\downarrow}$, then $\underline{(f\uparrow g)} \simeq \left(\underline{f}\uparrow\underline{g}\right)$ and $\overline{(f\uparrow g)} \simeq \left(\overline{f}\uparrow\overline{g}\right)$, and moreover, the universal 2-morphism is a pair consisting of the universal 2-morphisms of downstairs and upstairs parts. In the light of this observation, we can replace 0-morphism z of $\mathfrak{K}^{\downarrow}$ in diagram (5.2) by more meaningful and equivalent $\left(r_\alpha\uparrow 1_{x_g}\right)$. We also will write $k(\underline{\alpha})$ instead of k to show its dependency on $\underline{\alpha}$.

REMARK 5.1.2. In the case where $\mathfrak{K} = \mathcal{E}\mathfrak{Top}$ and $\mathfrak{K}_{\mathfrak{D}} = \mathcal{G}\mathfrak{Top}$, the cocomma geometric morphism $\left(r_\alpha\uparrow 1_{x_g}\right)$ is bounded and it therefore is a 0-morphism in $\mathcal{G}\mathfrak{Top}$. This is true since for any base topos $\mathscr{S}$, the coproduct and coinserter in $\mathcal{B}\mathfrak{Top}/\mathscr{S}$ exists and they are constructed via categorical product and inserter of their inverse image functors, respectively [Joh02a, Remark 3.4.10]. Now, in diagram (5.1) $\underline{d}_i$ (i=0,1) are bounded over geometric morphisms (indeed with bound 1).

DEFINITION 5.1.3. The fibration x is said to satisfy **PCC** ('Pullbacks commute with cocommas') whenever the 1-morphism $k(\underline{\alpha})\colon z \to x$ is cartesian with respect to the 2-functor $\mathbf{cod}\colon \mathfrak{K}^{\downarrow} \to \mathfrak{K}$ for any 2-morphism $\underline{\alpha}$ targeted at $\underline{x}$ in $\mathfrak{K}$.

REMARK 5.1.4. For $\mathfrak{K} = \mathfrak{Cat}$, all (internal) fibrations (i.e. Grothendieck fibrations) satisfy PCC automatically. To see this, first notice that in $\mathfrak{Cat}$, $\mathcal{A}\otimes\mathbf{2} \simeq \mathcal{A}\times\mathbf{2}$, where the latter is the product of categories.[1] Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration, and $\underline{\alpha}\colon \underline{F} \Rightarrow \underline{G}$ is a natural transformation targeted at $\mathcal{B}$. The objects of category $(\overline{r_\alpha} \uparrow 1)$ are of either the forms (A_0, E_0), where A_0 is an object of $\mathcal{A}$ and E_0 is an object of $\mathcal{E}$ with $P(E_0) = F(A_0)$, or (A_1, E_1), where A_1 is an object of $\mathcal{A}$ and E_1 is an object of $\mathcal{E}$ with $P(E_1) = G(A_1)$. The morphisms of $(\overline{r_\alpha} \uparrow 1)$ are generated by the following morphisms:

$$\left\{\begin{array}{c} (A_0 \xrightarrow{f} A_0', E_0 \xrightarrow{u} E_0') \\ \text{with } P(u) = F(f) \\ \end{array}\right\} \coprod \left\{\begin{array}{c} (A_1 \xrightarrow{g} A_1', E_1 \xrightarrow{v} E_1') \\ \text{with } P(v) = G(g) \\ \end{array}\right\} \coprod \left\{\begin{array}{c} (A, \overline{r_\alpha}(E)) \to (A, E) \\ \text{for any } (A, E) \\ \text{with } P(E) = G(A) \end{array}\right\}$$

Fibration P satisfies PCC if and only if the diagram

$$\begin{tikzcd}
(\overline{r_\alpha} \uparrow 1_{\mathcal{E}_G}) \arrow[r, "pr_1"] \arrow[d, "pr_0"'] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{A} \times \mathbf{2} \arrow[r, "\ulcorner \underline{\alpha} \urcorner"'] & \mathcal{B}
\end{tikzcd}$$

is a pullback. This is satisfied by default precisely because any morphism $(f, r)\colon (A_0, E_0) \to (A_1, E_1)$ in $(\mathcal{A} \times \mathbf{2}) \times_{\mathcal{B}} \mathcal{E}$ factors uniquely as $(A_0, E_0) \xrightarrow{(f, \widetilde{r})} (A_1, \overline{r_\alpha}(E_1)) \xrightarrow{(1, \overline{\alpha}_{A_1})} (A_1, E_1)$. This is depicted in the diagram below:

$$\begin{tikzcd}
E_0 \arrow[dr, dashed, "\widetilde{r}"'] \arrow[rrrd, dashed, "r"] \arrow[dd, mapsto] & & & \\
& \overline{r_\alpha}(E_1) \arrow[rr, "\overline{\alpha}_{A_1, E_1}"'] \arrow[dd, mapsto] & & E_1 \arrow[dd, mapsto] \\
F(A_0) \arrow[rr, dotted] \arrow[dr, "F(f)"'] & & G(A_0) \arrow[dr, dotted] & \\
& F(A_1) \arrow[rr, "\underline{\alpha}_{A_1}"'] & & G(A_1)
\end{tikzcd}$$

---

[1] $\mathrm{Ob}(\mathcal{A} \times \mathbf{2}) = \mathrm{Ob}(\mathcal{A}) \coprod \mathrm{Ob}(\mathcal{A})$, and morphisms are generated by $\theta_a\colon a_0 \to a_1$ for all objects $a \in \mathrm{Ob}(\mathcal{A})$, and $f_i\colon a_i \to b_i$ $(i = 0, 1)$, for all morphisms $f\colon a \to b$ subject to the equations $\theta_b \circ f_0 = f_1 \circ \theta_a$.

Our approach for partial products for AUs

As we noted (pp)fibrations need PCC if they are to work well in 2-categories. Also, definition of PCC requires certain cocomma objects to be present in the ambient 2-category.

There are thorny issues with imitating the methods of partial products of toposes for AUs:

- Topos proofs of exponentiability use direct image functors and won't go through for AUs.

- We currently do not have a construction of the cocommas needed for PCC in the 2-category $\mathfrak{Con}$, but we conjecture, based on a similar method in [Vic99], that they can be conveniently constructed using bag spaces.

So we conjecture that the concrete construction of partial products in $\mathfrak{Con}$ can be taken backwards to get cocommas needed for PCC, as well as exponentiability.

5.2 Conjectures concerning the Sierpinski context

Recall the Sierpinski context from Example 3.3.5. Similar to the case of toposes, where for a topos $\mathscr{E}$ we have $\mathscr{E}[\mathbb{S}] \simeq (\mathscr{E} \uparrow \mathscr{E})$, we conjecture the following for AUs. For AUs the delicate part is about strictness, that is the result below is straightforward when the classifier is defined strictly but not for the non-strict classifier. The techniques of [MV12] might be helpful.

CONJECTURE 5.2.1. For an AU $\mathcal{A}$, we have $\mathcal{A}[\mathbb{S}] \simeq (\mathcal{A} \downarrow \mathcal{A})$.

SKETCH OF PROOF. We define the two functors which are quasi-inverse of each other. First we note that, by classifying property of AU $\mathcal{A}[\mathbb{S}]$, any AU-morphism $\mathcal{A}[\mathbb{S}] \to (\mathcal{A} \downarrow \mathcal{A})$ is defined by its action on objects and morphisms of $\mathcal{A}$, and $\mathsf{i} \colon \mathsf{I} \to 1$.

Define AU-morphism $F\colon \mathcal{A}[\mathbb{S}] \to (\mathcal{A} \downarrow \mathcal{A})$ by taking any object A of $\mathcal{A}$ to $A \xrightarrow{id_A} A$, 1 to $1 \xrightarrow{id_1} 1$, I to the unique arrow $0 \to 1$, and i to the following commutative square:

$$\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, "!"'] & 1 \arrow[d, "id"] \\
1 \arrow[r, "id"'] & 1
\end{tikzcd}$$

Now, we construct another AU-morphism G in the other direction quasi-inverse to F. G takes an arbitrary object $X \xrightarrow{f} Y$ of $(\mathcal{A} \downarrow \mathcal{A})$ to the following pushout in $\mathcal{A}[\mathbb{S}]$:

$$\begin{tikzcd}
\mathsf{I} \times X \arrow[r, "\pi_1"] \arrow[d, "\mathsf{I} \times f"'] & X \arrow[d, "\mathsf{inr}"] \\
\mathsf{I} \times Y \arrow[r, "\mathsf{inl}"'] & Z \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd} \tag{5.3}$$

and any morphism in $(\mathcal{A} \downarrow \mathcal{A})$ is mapped to the induced morphism between corresponding pushouts. (If $\mathsf{I} = 0$ or $\mathsf{I} = 1$ then the pushout Z is X or Y, otherwise somewhere in between along f.)

We now show that G and F are quasi-inverses of each other. First we construct a natural isomorphism $\mathrm{Id} \cong GF$. It is enough to define this natural isomorphism on generators of $\mathcal{A}[\mathbb{S}]$. This is achieved by observing that the following diagrams are pushout diagrams:

$$\begin{tikzcd}
\mathsf{I} \times X \arrow[r, "\pi_1"] \arrow[d, "\mathsf{I} \times id"'] & X \arrow[d, "id"] \\
\mathsf{I} \times X \arrow[r, "\pi_1"'] & X \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{I} \times 0 \arrow[r, "\pi_1"] \arrow[d, "\mathsf{I}\times !"'] & 0 \arrow[d, "\mathsf{inr}"] \\
\mathsf{I} \times 1 \arrow[r, "\mathsf{inl}"'] & \mathsf{I} \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Since F preserves the pushout diagram (5.3) it is easy to see that $FG(X \xrightarrow{f} Y) = F(Z)$ is naturally isomorphic to f in $(\mathcal{A} \downarrow \mathcal{A})$.

The gaps that remain in the proof sketch above are: can we prove that the assignment G is indeed functorial and moreover, can we show that G preserves AU-structures?

CONJECTURE 5.2.2. For a context $\mathbb{U}$, the exponential $\mathbb{U}^{\mathbb{S}}$ exists and it is equivalent to $\mathbb{U}^{\rightarrow}$.

SKETCH OF PROOF. We construct the evaluation map $ev\colon \mathbb{U}^{\rightarrow} \times \mathbb{S} \to \mathbb{U}$ and we show that it is universal among all the maps of the form $\mathbb{T} \times \mathbb{S} \to \mathbb{U}$. First, note that the sketch ingredients for $\mathbb{U}^{\rightarrow} \times \mathbb{S}$ can be summarized as a $\mathbb{U}$-model morphism and a subobject of 1. Thus, we look at the action of ev on points: for any $\mathbb{U}$-model morphism $\alpha\colon M_0 \to M_1$ and a subobject $I \hookrightarrow 1$, and for any sort σ of $\mathbb{U}$ define $ev(M, I)_\sigma$ to be the canonical strict $\mathbb{U}$-model isomorphic to the following pushout:

$$\begin{tikzcd}
I \times (M_0)_\sigma \arrow[r, "\pi_1"] \arrow[d, "I \times f"'] & (M_0)_\sigma \arrow[d] \\
I \times (M_1)_\sigma \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & (M_I)_\sigma
\end{tikzcd} \tag{5.4}$$

Note that this has to be extended to all sketch ingredients, not just nodes.

A

Appendix: Mathematical Background

A.1 Bicategories

DEFINITION A.1.1. A **bicategory** $\mathfrak{B}$ (aka **weak 2-category**) consists of

(BICAT 1) a class of objects $\mathfrak{B}_0$,

(BICAT 2) categories $\mathfrak{B}(x, y)$ for all pairs x, y in $\mathfrak{B}_0$, composition and unit functors

$$\mathrm{c} = \mathrm{c}_{x,y,z} \colon \mathfrak{B}(x, y, z) \to \mathfrak{B}(x, z) \qquad 1_X \colon \mathbf{1} \to \mathfrak{B}(x, x)$$

where $\mathfrak{B}(x, y, z)$ stands for the product $\mathfrak{B}(y, z) \times \mathfrak{B}(x, y)$ of categories, and

(BICAT 3) natural isomorphisms (called *associators* and *left and right unitors*)

$$\begin{tikzcd}
\mathfrak{B}(x,y,z,w) \arrow[r, "1\times\mathrm{c}"] \arrow[d, "\mathrm{c}\times 1"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow\ \alpha_{x,y,z,w}"] & \mathfrak{B}(x,z,w) \arrow[d, "\mathrm{c}"] \\
\mathfrak{B}(x,y,w) \arrow[r, "\mathrm{c}"'] & \mathfrak{B}(x,w)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1} \times \mathfrak{B}(x,y) \arrow[d, "1_Y\times 1"'] \arrow[dr, bend left, "\cong"] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow\ \lambda_{x,y}"] & \\
\mathfrak{B}(x,y,y) \arrow[r, "\mathrm{c}"'] & \mathfrak{B}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{B}(x,y) \times \mathbf{1} \arrow[d, "1\times 1_X"'] \arrow[dr, bend left, "\cong"] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow\ \rho_{x,y}"] & \\
\mathfrak{B}(x,x,y) \arrow[r, "\mathrm{c}"'] & \mathfrak{B}(x,y)
\end{tikzcd}$$

expressing the up-to-isomorphism associativity and unitality of composition. We will write α for α_{XYZW}, λ for λ_{XY}, and ρ for ρ_{XY} when the context is clear and there is no risk of confusion. The composition of 2-morphisms within each Hom-category is *vertical composition* and the effect of the composition functors $\mathrm{c}_{x,y,z}$ on 1,2-morphisms is *horizontal composition*. We shall use the notation $g \circ f$ or sometimes gf for composition of 1-morphisms, $\beta \centerdot \alpha$ for the horizontal composition and $\alpha_1 \circ \alpha_0$ for the vertical composition of 2-morphisms. We also write $\beta \centerdot f$ for $\beta \centerdot id_f$.

The data above is subject to the *coherence conditions* expressed, pointwise, by the commutativity of two diagrams of 2-morphisms in below:

$$\begin{tikzcd}
 & (f(gh))k \arrow[rd, "\alpha"] & \\
((fg)h)k \arrow[ru, "\alpha \centerdot k"] \arrow[d, "\alpha"'] & & f((gh)k) \\
(fg)(hk) \arrow[r, "\alpha"'] & f(g(hk)) \arrow[ru, "f \centerdot \alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(g1)f \arrow[rr, "\alpha"] \arrow[rd, "\rho \centerdot f"'] & & g(1f) \arrow[ld, "g \centerdot \lambda"] \\
 & gf &
\end{tikzcd} \tag{A.1}$$

A 2-category is a bicategory whose associators and unitors are all identities. A k-morphism appears with dimension $k \in \{0, 1, 2\}$. Consider the following arrangement of 2-morphisms:

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f_2"] \arrow[r, "f_1" description] \arrow[r, bend right=60, "f_0"'] \arrow[r, phantom, bend left=30, "\Uparrow \alpha_1"] \arrow[r, phantom, bend right=30, "\Uparrow \alpha_0"] & y \arrow[r, bend left=60, "g_2"] \arrow[r, "g_1" description] \arrow[r, bend right=60, "g_0"'] \arrow[r, phantom, bend left=30, "\Uparrow \beta_1"] \arrow[r, phantom, bend right=30, "\Uparrow \beta_0"] & z
\end{tikzcd} \tag{A.2}$$

The well-known **middle-four interchange law (aka Godement law)** says that it does not differ in which order we compose an arrangement of 2-morphisms in above since the possible two ways of composing them have the same result.

$$(\beta_1 \centerdot \alpha_1) \circ (\beta_0 \centerdot \alpha_0) = (\beta_1 \circ \beta_0) \centerdot (\alpha_1 \circ \alpha_0) \tag{A.3}$$

This law an immediate consequence of the functoriality of composition functor c.

For $f \in \mathfrak{B}(x,y)$ and $g \in \mathfrak{B}(y,z)$, the composition functors c restrict to functors

$$g_* \colon \mathfrak{B}(x,y) \to \mathfrak{B}(x,z) \qquad f^* \colon \mathfrak{B}(y,z) \to \mathfrak{B}(x,z)$$

given respectively with action $\delta \mapsto g \centerdot \delta$ and $\theta \mapsto \theta \centerdot f$. These operations are called *whiskering*. We take note that the notion of bicategory could be equivalently formulated by whiskering operations instead of general horizontal composition of 2-morphisms by adding the **exchange law**: that is for any horizontally composable 2-morphisms

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\theta"] & z
\end{tikzcd}$$

we have

$$(\theta \centerdot f') \circ (g \centerdot \delta) = (g' \centerdot \delta) \circ (\theta \centerdot f). \tag{A.4}$$

which enables us to give an unambiguous expression to the horizontal composition of δ and θ from whiskering operations. Note that the exchange law follows from the middle-four interchange law by inserting identity 2-morphisms in the right top 2-cell and left bottom 2-cell in the pasting diagram (A.2).

The proposition below was first observed by Kelly in [Kel64] for monoidal categories. Joyal & Street in [JS93a, Proposition 1.1] gave an explicit proof of it. It works for bicategories mutatis mutandis.

PROPOSITION A.1.2. The left and right unitors are equal on identity 1-morphisms, that is for any object x, we have $\lambda_{x,x}(1_x) = \rho_{x,x}(1_x) \colon 1_x \circ 1_x \cong 1_x$.

A.2 Bicategories and the principle of equivalence

In sets and set based structures, such as groups, the notion of identity (or equivalence) internal to them is that of equality: two elements of a group are identical (or equivalent) if they are equal as the members of the underlying set of the group. However, the notion of structural equivalence between groups themselves is that of isomorphism. Recall that two groups $G = (G_0, m_G, i_G, e_G)$ and $H = (H_0, m_H, i_H, e_H)$ are isomorphic whenever there is a pair of functions $f\colon G_0 \rightleftarrows H_0 \colon f^{-1}$ such that

- for every member $a \in G_0$, $f^{-1} \circ f(a) =_{G_0} a$ and for every member $b \in H_0$, $f \circ f^{-1}(b) =_{H_0} b$, and
- f preserves the multiplication structure m_G, the inverse structure i_G, and the unit structure e_G.

Now, the first condition of isomorphism explicitly requires notions of equality of elements in both underlying sets G_0 and H_0. Therefore, isomorphism of groups is grounded in isomorphisms of sets which in turn is grounded in equality of elements within sets. Any sensible structural property of groups remains invariant under isomorphisms of groups, and as such any two isomorphic groups *are* indiscernible:

$$G \cong H \iff \forall \text{ group theoretic properties } \mathcal{P}.\ (\mathcal{P}(G) \iff \mathcal{P}(H)).$$

Examples of group theoretic properties are: "Group G has exactly 6 elements.", "Group G is cyclic", "Group G is Abelian", etc. An example of a non-group theoretic property is "$1 \in \mathbb{Z}$" where $\mathbb{Z}$ is the group of integers.

We conclude that in the category $\mathcal{G}rp$ of groups the notion of equivalence of objects is that of isomorphism. This is a general principle for any category and is referred to as "Principle of Isomorphism" (PI):

> *(Principle of Isomorphism) all grammatically correct properties of objects of a fixed category are to be invariant under isomorphism.* [Mak98, p. 161]

Accepting this principle, we expect that all meaningful properties of an object in a fixed category to be invariant under isomorphism. Now, going one level higher, passing from set-bases structures to categories, we may ask what is the correct notion of equivalence of categories? Note that it cannot be isomorphism of categories: isomorphism of categories will use strict equality of objects of categories which is antithetical to the principle of isomorphism.

The Principle of isomorphism dictates to us that the correct notion of equivalence of two categories is that of categorical equivalence: an equivalence of categories $\mathcal{C}$ and $\mathcal{D}$ is a full, faithful, and essentially surjective functor $F\colon \mathcal{C} \to \mathcal{D}$. In the presence of Axiom of Choice, this is the same as a pair of functorial assignments $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$, such that the object $GF(C)$ is the same as C evidenced by an isomorphism $\eta_C\colon C \cong GF(C)$ for each object $C \in \mathcal{C}$, and symmetrically, $\varepsilon_D\colon FG(D) \cong D$ for each object $D \in \mathcal{D}$. On top of this, these isomorphisms are natural[1] in C and D. This is usually formulated as a pair of functors together with a pair of invertible natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\varepsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

This leads us to the Principle of Equivalence (PE) of categories and more generally for objects of any bicategory:

> *It is generally recognized, in exact analogy to sets and set-based structures in relation to the notion of isomorphism, that the "right notion" of "equality" for categories, resp. category-based structures is equivalence of categories, resp. equivalence in the corresponding bicategory. This principle acts, again, in two different ways. First, as the constraint on properties of objects in a bicategory, which we may call the Principle of Equivalence, asserting that any (meaningful) property of an object in a bicategory is invariant under equivalence. Secondly, as the experience that usually, especially in "serious" representation theorems, one gets that a given category can be represented in a*

---

[1] The *naturality* condition is in fact the origin of category theory. It imposes a natural wish that the isomorphisms η and ε should be given uniformly in advance for all objects and not separately based on particularities of each object.

certain desired way up to equivalence but not up to isomorphism. *[Mak98, p. 168]*

Therefore, adopting this point of view, we arrive at the definition of equivalence in bicategories:

DEFINITION A.2.1. An **equivalence** $f\colon X \simeq Y : g$ between two objects of a bicategory $\mathfrak{B}$ is a pair of 1-morphism $f \in \mathfrak{B}(X, Y)$ and $g \in \mathfrak{B}(Y, X)$ together with a pair of iso-2-morphisms $\eta\colon \mathrm{id}_X \cong g \circ f$ in $\mathfrak{B}(X, X)$ and $\varepsilon\colon f \circ g \cong \mathrm{id}_Y$ in $\mathfrak{B}(Y, Y)$. In such a scenario, f (resp. g) is called a quasi-inverse of g (resp. f). We frequently say that a morphism $f\colon X \to Y$ *is an equivalence* if it has a quasi-inverse.

REMARK A.2.2. In general the relation of isomorphism on objects of a bicategory is ill-behaved: it is neither reflexive nor transitive. Even identity morphisms are not isomorphisms but are equivalences. Put another way, while in categories every object is isomorphic to itself by the identity morphism on that very object, in a bicategory every object is only equivalent to itself. Moreover, any morphism isomorphic to an identity morphism is an equivalence.

REMARK A.2.3. Similar to the situation in a category where every isomorphism has a unique inverse, in a bicategory every equivalence has a unique (up to an isomorphism) quasi-inverse. If $f\colon X \to Y$ is an equivalence and g, η, ε and g', η', ε' are quasi-inverses of f, then

$$g' \overset{\sim}{\Rightarrow} 1 \circ g' \xRightarrow[\eta \cdot g']{\sim} (g \circ f) \circ g' \overset{\sim}{\Rightarrow} g \circ (f \circ g') \xRightarrow[g \cdot \varepsilon']{\sim} g \circ 1 \overset{\sim}{\Rightarrow} g$$

evidences an isomorphism between g and g'. However, an iso-2-morphism between g and g' is not necessarily unique. If the quasi-inverse was unique up to unique iso-2-morphism, then any equivalence would have no non-trivial automorphisms. An easy counterexample is to consider the delooping category $\Sigma(G)$ of a group G with non-trivial center; the automorphisms of Id_Σ are in bijection with the central elements of G.

Nonetheless, certain equivalence have unique quasi-inverses up to unique iso-2-morphism. For instance, for a contractible groupoid $\mathcal{G}$, any two quasi-inverses of the equivalence $\mathcal{G} \overset{\sim}{\Rightarrow} \mathbf{1}$ are uniquely isomorphic.

An example of a categorical construction which violates PE and is the (strict) pullback of categories. As such it is occasionally regarded as “evil”. Yet, the pullback construction is entirely legitimate from the point of view of cartesian theory of categories ([Joh02b, Part D], [PV07]): it is because the notion of structural identity incorporated between models of any first order theory is that of isomorphisms. The first order theory of categories is unfortunately impervious to the fundamental notion of equivalence of categories.

EXAMPLE A.2.4. A defect with the pullbacks of categories is that they are not invariant under equivalence of categories: the terminal category **1** and the interval groupoid (with two distinct objects and two non-identity arrows) $\mathbb{I}$ are equivalent as categories, however, for nonempty categories $\mathcal{C}$ and $\mathcal{D}$ the pullback of constant functors $0!_{\mathcal{C}} \colon \mathcal{C} \to \mathbb{I}$ and $1!_{\mathcal{D}} \colon \mathcal{D} \to \mathbb{I}$ is the empty category whereas their pullback over the terminal category is not empty. This shows that the notion of pullback is not the correct one in the 2-category $\mathbf{2}\,\mathcal{C}\mathrm{at}$. The correct notion of pullback in 2-categories and bicategories is that of bipullback (See 1.4.10).

Now, we can go one level up again: using the notion of equivalence within bicategories we arrive at the notion of equivalence *between* bicategories.

DEFINITION A.2.5 (External Equivalence). A **biequivalence** of bicategories $\mathfrak{B}$ and $\mathfrak{C}$ consists of a pair of homomorphisms $F \colon \mathfrak{B} \rightleftarrows \mathfrak{C} \colon G$ together with an equivalence $\mathrm{id}_{\mathfrak{B}} \simeq G \circ F$ in the bicategory $\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}(\mathfrak{B}, \mathfrak{B})$ and an equivalence $F \circ G \simeq \mathrm{id}_{\mathfrak{C}}$ in the bicategory $\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}(\mathfrak{C}, \mathfrak{C})$.

Of course, we have not yet said anything about ‘homomorphisms’ of bicategories nor about the bicategorical structure of $\mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at}(\mathfrak{B}, \mathfrak{B})$. This will be done in the next sections. In the presence of Axiom of Choice, an equivalence $F \colon \mathcal{B} \to \mathcal{C}$ of categories is the same thing as a fully faithful and essentially surjective functor, i.e. for any object c of $\mathcal{C}$ there is some object b in $\mathcal{B}$ such that $Fb \cong c$ and also we have a family of bijections $\{F_{b,b'} \colon \mathcal{B}(b, b') \cong \mathcal{C}(Fb, Fb')\}$. The analogue of this result for bicategories says that, assuming Axiom of Choice, a homomorphism $F \colon \mathfrak{B} \to \mathfrak{C}$ of bicategories is a biequivalence iff for any pair of objects x, y in $\mathfrak{B}$, we have equivalence of categories $\{F_{x,y} \colon \mathfrak{B}(x, y) \simeq \mathfrak{C}(Fx, Fy)\}$, and for any object z of $\mathfrak{C}$ there is some object x in $\mathfrak{B}$ such that $Fx \simeq z$.

Remark A.2.6. There is a stronger notion of **equivalence of bicategories** whereby we require $F \circ G$ and $G \circ F$ to be isomorphic to the identity homomorphisms of bicategories. Notice however that in the light of Remark A.2.2 a bicategory is not in general equivalent, but only biequivalent, to itself via the identity homomorphism. Furthermore, the notion of biequivalence of bicategories generalizes many nice facts of equivalence of categories: for instance, any biequivalence of bicategories can be promoted to a biadjoint biequivalence ([Gur11]).

A.3 Morphisms of bicategories

Definition A.3.1. A **pseudofunctor** $F\colon \mathfrak{B} \to \mathfrak{C}$ between bicategories $\mathfrak{B}$ and $\mathfrak{C}$ is given by the following assignments:

(PsdFun 1) To each object x of $\mathfrak{B}$ a object Fx of $\mathfrak{C}$.

(PsdFun 2) To each objects x and y of $\mathfrak{B}$, a functor $F_{x,y}\colon \mathfrak{B}(x,y) \to \mathfrak{C}(Fx,Fy)$.

(PsdFun 3) To each object x of $\mathfrak{B}$, an invertible natural transformation

$$\begin{tikzcd}
1 \arrow[r, "1_x"] \arrow[dr, "1_{Fx}"'] \arrow[dr, phantom, shift left=4, "\Nearrow \iota_x"] & \mathfrak{B}(x,x) \arrow[d, "F_{x,x}"] \\
 & \mathfrak{C}(Fx,Fx)
\end{tikzcd}$$

(PsdFun 4) To each triple of objects x, y, z of $\mathfrak{B}$, an invertible natural transformation

$$\begin{tikzcd}
\mathfrak{B}(y,z) \times \mathfrak{B}(x,y) \arrow[r, "\mathrm{c}_{x,y,z}"] \arrow[d, "F_{y,z} \times F_{x,y}"'] \arrow[dr, phantom, "\Nearrow \phi_{x,y,z}"] & \mathfrak{B}(x,z) \arrow[d, "F_{x,z}"] \\
\mathfrak{C}(Fy,Fz) \times \mathfrak{C}(Fx,Fy) \arrow[r, "\mathrm{c}_{Fx,Fy,Fz}"'] & \mathfrak{C}(Fx,Fz)
\end{tikzcd}$$

subject to the coherence conditions expressed by the equality of the following pasting diagrams:

$$\begin{tikzcd}
& \mathfrak{B}(x,z,w) \arrow[rd, bend left, "\mathrm{c}"] & \\
\mathfrak{B}(x,y,z,w) \arrow[ru, bend left, "id\times\mathrm{c}"] \arrow[rd, bend right, "\mathrm{c}\times id"'] \arrow[ddd, "F_{x,y,z,w}"'] & \alpha\Uparrow & \mathfrak{B}(x,w) \arrow[ddd, "F_{x,w}"] \\
& \mathfrak{B}(x,y,w) \arrow[ru, bend right, "\mathrm{c}"'] \arrow[ddd, "F(x;y;w)" description] & \\
\phi\times id\Uparrow & & \phi\Uparrow \\
\mathfrak{C}(Fx,Fy,Fz,Fw) \arrow[rd, bend right, "\mathrm{c}\times id"'] & & \mathfrak{C}(Fx,Fw) \\
& \mathfrak{C}(Fx,Fy,Fw) \arrow[ru, bend right, "\mathrm{c}"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathfrak{B}(x,z,w) \arrow[rd, bend left, "\mathrm{c}"] \arrow[ddd, dotted, "F_{x,y,w}" description] & \\
\mathfrak{B}(x,y,z,w) \arrow[ru, bend left, "id\times\mathrm{c}"] \arrow[ddd, "F_{x,y,z,w}"'] & & \mathfrak{B}(x,w) \arrow[ddd, "F_{x,w}"] \\
id\times\phi\Uparrow & & \phi\Uparrow \\
& \mathfrak{C}(Fx,Fy,Fw) \arrow[rd, dotted, "\mathrm{c}"] & \\
\mathfrak{C}(Fx,Fy,Fz,Fw) \arrow[ru, dotted, "id\times\mathrm{c}"] \arrow[rd, bend right, "\mathrm{c}\times id"'] & \alpha\Uparrow & \mathfrak{C}(Fx,Fw) \\
& \mathfrak{C}(Fx,Fy,Fw) \arrow[ru, bend right, "\mathrm{c}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{B}(x,y)\times 1 \arrow[rr, bend left, "\pi_0"] \arrow[rd, bend right, "id\times 1_x"'] \arrow[ddd, "F_{x,y}\times id"'] & \rho\Uparrow & \mathfrak{B}(x,y) \arrow[ddd, "F_{x,w}"] \\
& \mathfrak{B}(x,x,y) \arrow[ru, bend right, "\mathrm{c}"'] \arrow[ddd, "F_{x,x,y}" description] & \\
id\times\iota_x\Uparrow & & \phi\Uparrow \\
\mathfrak{C}(Fx,Fy)\times 1 \arrow[rd, bend right, "id\times 1_{Fx}"'] & & \mathfrak{C}(Fx,Fy) \\
& \mathfrak{C}(Fx,Fx,Fy) \arrow[ru, bend right, "\mathrm{c}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{K}(x,y)\times 1 \arrow[rr, bend left, "\pi_0"] \arrow[ddd, "F_{x,y}\times id"'] & & \mathfrak{K}(x,y) \arrow[ddd, "F_{x,w}"] \\
& & \\
& & \\
\mathfrak{L}(Fx,Fy)\times 1 \arrow[rr, dotted, bend left, "\pi_0"] \arrow[rd, bend right, "id\times 1_{Fx}"'] & \rho\Uparrow & \mathfrak{L}(Fx,Fy) \\
& \mathfrak{L}(Fx,Fx,Fy) \arrow[ru, bend right, "\mathrm{c}"'] &
\end{tikzcd}
\tag{A.5}$$

and similarly there is an equality of pasting diagrams involving left unitor λ as part of coherence conditions.

REMARK A.3.2. More concretely, the third part of data of the above definition assigns to every object x a 2-morphism $\iota_x\colon 1_{Fx} \Rightarrow F(1_x)$. Note that by naturality condition $F(1_{1_x}) = 1_{1_{Fx}}$. Also by part (iv), for every pair of composable 1-morphisms $f\colon x \to y$ and $g\colon y \to z$ we have a 2-morphism $\phi_{f,g}\colon F(g)\circ f(f) \Rightarrow F(gf)$, and the naturality of ϕ implies that for any pair of composable 2-morphisms

$$\begin{tikzcd}
x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\delta"] & y \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\theta"] & z
\end{tikzcd}$$

the square of 2-morphisms

$$\begin{tikzcd}
F(g)F(f) \arrow[r, Rightarrow, "\phi_{f,g}"] \arrow[d, Rightarrow, "F(\theta)\cdot F(\delta)"'] & F(gf) \arrow[d, Rightarrow, "F(\theta\cdot\delta)"] \\
F(g')F(f') \arrow[r, Rightarrow, "\phi_{f',g'}"'] & F(g'f')
\end{tikzcd}$$

commutes. Furthermore, the first coherence condition in Definition A.3.1 guarantees the commutativity of the diagram of 2-morphisms in below,

$$\begin{tikzcd}
(F(h)F(g))F(f) \arrow[r, Rightarrow, "\phi_{g,h}\cdot F(f)"] \arrow[d, Rightarrow, "\alpha_{Ff,Fg,Fh}"'] & F(hg)F(f) \arrow[r, Rightarrow, "\phi_{f,hg}"] & F((hg)f) \arrow[d, Rightarrow, "F(\alpha_{f,g,h})"] \\
F(h)(F(g)F(f)) \arrow[r, Rightarrow, "F(h)\cdot\phi_{f,g}"'] & F(h)F(gf) \arrow[r, Rightarrow, "\phi_{gf,h}"'] & F(h(gf))
\end{tikzcd}$$

where $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to w$ are 1-morphisms in $\mathfrak{B}$. Finally, the second and the third coherence conditions guarantee the commutativity of diagrams of 2-morphisms in below

$$\begin{tikzcd}
F(f)\circ 1_{Fx} \arrow[r, Rightarrow, "F(f)\cdot\iota_x"] \arrow[d, Rightarrow, "\rho_{F(f)}"'] & F(f)\circ F(1_x) \arrow[d, Rightarrow, "\phi_{1_x,f}"] & 1_{Fy}\circ F(f) \arrow[r, Rightarrow, "\iota_y\cdot F(f)"] \arrow[d, Rightarrow, "\lambda_{F(f)}"'] & F(1_y)\circ F(f) \arrow[d, Rightarrow, "\phi_{f,1_y}"] \\
F(f) & F(f\circ 1_x) \arrow[l, Rightarrow, "F(\rho_f)"] & F(f) & F(1_y\circ f) \arrow[l, Rightarrow, "F(\lambda_f)"]
\end{tikzcd}$$

A.4 Transformations of pseudo functors

DEFINITION A.4.1. For parallel pseudo-functors $(F,\phi,\iota),(G,\psi,\kappa)\colon \mathfrak{B} \rightrightarrows \mathfrak{C}$ of bicategories, a **pseudo natural transformation** $\theta\colon (F,\phi,\iota) \Rightarrow (G,\psi,\kappa)$ between them consists of

(PSDNAT 1) a 1-morphism $\theta_x\colon Fx \to Gx$ for every object x of $\mathfrak{B}$, (PSDNAT 2) and an invertible 2-morphism

$$\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\theta_f\Downarrow"] & Gx \arrow[d, "Gf"] \\
Fx' \arrow[r, "\theta_{x'}"'] & Gx'
\end{tikzcd} \tag{A.6}$$

natural for every morphism $f\colon x \to x'$ of $\mathfrak{B}$, subject to the expected compatibility conditions with ϕ and ψ which are detailed in [Lei98].

We comment on compatibility conditions. Modulo associators, the compatibility conditions can be expressed using the pasting diagrams below. Of course full compatibility conditions are attained by placing associator α of $\mathfrak{C}$ for any three composable morphisms in sight which fattens up our diagrams.

$$\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, bend right=70, "1_{Fx}"'] \arrow[d, bend left=20, "F1_x"] \arrow[d, phantom, shift right=4ex, "\overset{\iota^{-1}}{\simeq}"] \arrow[dr, phantom, "\Swarrow\theta_{1_x}"] & Gx \arrow[d, "G1_x"] \\
Fx \arrow[r, "\theta_x"'] & Gx
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, "1_{Fx}"'] & Gx \arrow[d, "1_{Gx}"'] \arrow[d, bend left=70, "G1_x"] \arrow[d, phantom, shift left=4ex, "\overset{\kappa}{\simeq}"] \\
Fx \arrow[r, "\theta_x"'] & Gx
\end{tikzcd}$$

$$\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, "Ff"'] \arrow[dd, bend right=70, "F(gf)"'] \arrow[dr, phantom, "\Swarrow\theta_f"] & Gx \arrow[d, "Gf"] \arrow[dd, bend left=70, "G(gf)"] \\
\overset{\phi^{-1}}{\simeq}\; Fx' \arrow[r, "\theta_{x'}"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Swarrow\theta_g"] & Gx' \;\overset{\psi}{\simeq} \arrow[d, "Gg"] \\
Fx'' \arrow[r, "\theta_{x''}"'] & Gx''
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Swarrow\theta_{gf}"] & Gx \arrow[d, "G(gf)"] \\
Fx'' \arrow[r, "\theta_{x''}"'] & Gx''
\end{tikzcd}$$

Also, the naturality condition in A.6 demands that for every 2-cell $\alpha\colon f \Rightarrow f'$ we have

$$\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\Swarrow\theta_{f'}"] & Gx \arrow[d, "Gf'"'] \arrow[d, bend left=80, "Gf"] \arrow[d, phantom, shift left=4ex, "\overset{G\alpha}{\Leftarrow}"] \\
Fx' \arrow[r, "\theta_{x'}"'] & Gx'
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\theta_x"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, "Ff"] \arrow[d, phantom, shift right=4ex, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow\theta_f"] & Gx \arrow[d, "Gf"] \\
Fx' \arrow[r, "\theta_{x'}"'] & Gx'
\end{tikzcd}$$

This last part is the 2-dimensional meat of naturality. Note that the 2-cells θ_f in the diagram (A.6), parameterized by x, x', and f, can be aggregated in diagrams of hom-categories parameterized by x, x'.

$$\begin{tikzcd}
\mathfrak{B}(x,x') \arrow[r, "G_{x,x'}"] \arrow[d, "F_{x,x'}"'] \arrow[dr, phantom, "\theta\Downarrow"] & \mathfrak{C}(Gx,Gx') \arrow[d, "(\theta_x)^*"] \\
\mathfrak{C}(Fx,Fx') \arrow[r, "(\theta_{x'})_*"'] & \mathfrak{C}(Fx,Gx')
\end{tikzcd}$$

The transformation θ is called a **lax natural transformation** whenever for every x, x' in $\mathfrak{B}$, the natural transformation in the above is not required to be invertible. If they are pointed in the opposite direction, we call it an **oplax natural transformation**. For contrast, a pseudo natural transformation is ocassionally referred to as a **strong** natural transformation. It is called a **strict** transformation if the 2-morphism θ is identity for every x, x'.

In the case of strict natural transformations, the 2-dimensional naturality condition has a simpler characterization, expressed by the equation $G(\alpha) \centerdot \theta_x = \theta_{x'} \centerdot F(\alpha)$, and illustrated by commutativity of whiskering in below.

$$\begin{tikzcd}
Fx \arrow[r, "\theta_x"] & Gx \arrow[r, bend left=50, "Gf", ""{name=U, below}] \arrow[r, bend right=50, "Gf'"', ""{name=D, above}] & Gx' \arrow[r, phantom, "="] & Fx \arrow[r, bend left=50, "Ff", ""{name=U2, below}] \arrow[r, bend right=50, "Ff'"', ""{name=D2, above}] & Fx' \arrow[r, "\theta_{x'}"'] & Gx'
\arrow[Rightarrow, from=U, to=D, "G(\alpha)"]
\arrow[Rightarrow, from=U2, to=D2, "F(\alpha)"]
\end{tikzcd} \tag{A.7}$$

DEFINITION A.4.2. If $\sigma, \theta \colon F \Rightarrow G \colon \mathfrak{B} \rightrightarrows \mathfrak{C}$ are two parallel lax transformations of pseudo-functors, a **modification** $m \colon \sigma \Rrightarrow \theta$ between them consists of 2-morphisms $m_x \colon \sigma_x \Rightarrow \theta_x$ of $\mathfrak{C}$ for every object x of $\mathfrak{B}$, such that the square

$$\begin{tikzcd}
G(f) \circ \sigma_x \arrow[r, Rightarrow, "G(f) \centerdot m_x"] \arrow[d, Rightarrow, "\sigma_f"'] & G(f) \circ \theta_x \arrow[d, Rightarrow, "\theta_f"] \\
\sigma_{x'} \circ F(f) \arrow[r, Rightarrow, "m_{x'} \centerdot F(f)"'] & \theta_{x'} \circ G(f)
\end{tikzcd}$$

commutes for every morphism $f \colon x \to x'$ in $\mathfrak{B}$.

Note that if σ and θ are strict transformations, the commutativity condition in diagram above simplifies to the requirement that

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\sigma_x", ""{name=U, below}] \arrow[r, bend right, "\theta_x"', ""{name=D}] & Gx \arrow[r, "Gf"'] & Gx' \arrow[r, phantom, "="] & Fx \arrow[r, "Ff"] & Fx' \arrow[r, bend left, "\sigma_{x'}", ""{name=U2, below}] \arrow[r, bend right, "\theta_{x'}"', ""{name=D2}] & Gx'
\arrow[Rightarrow, from=U, to=D, "m_x"]
\arrow[Rightarrow, from=U2, to=D2, "m_{x'}"]
\end{tikzcd} \tag{A.8}$$

Moreover, the naturality of θ (equation A.7) entails the equality of following horizontal composition of 2-morphisms. This says that the action of a modification is compatible with action of F and G on 2-morphisms.

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\sigma_x", ""{name=U, below}] \arrow[r, bend right, "\theta_x"', ""{name=D}] & Gx \arrow[r, bend left, "Gf", ""{name=U1, below}] \arrow[r, bend right, "Gf'"', ""{name=D1}] & Gx' \arrow[r, phantom, "="] & Fx \arrow[r, bend left, "Ff", ""{name=U2, below}] \arrow[r, bend right, "Ff'"', ""{name=D2}] & Fx' \arrow[r, bend left, "\sigma_{x'}", ""{name=U3, below}] \arrow[r, bend right, "\theta_{x'}"', ""{name=D3}] & Gx'
\arrow[Rightarrow, from=U, to=D, "m_x"]
\arrow[Rightarrow, from=U1, to=D1, "G(\alpha)"]
\arrow[Rightarrow, from=U2, to=D2, "F(\alpha)"']
\arrow[Rightarrow, from=U3, to=D3, "m_{x'}"]
\end{tikzcd} \tag{A.9}$$

Of course, Definition A.4.2 may be extended to modification of (op)lax transformations of lax functors without any change.

A.5 String diagrams for 2-categories

We saw in our pictorial depiction of diagrams in 2-categories and bicategories n-morphisms ($n = 0, 1, 2$) are modeled by n-cells; a 0-morphism (object) is depicted by a 0-cell (•), a 1-morphism by a 1-cell ($\rightarrow$), and a 2-morphism by a 2-cell ($\Rightarrow$). For 2-categories, other than pasting diagrams pictured by cells of various dimensions, there are *string diagrams* which are dual to pasting diagrams. Objects are depicted as regions, 1-morphisms as lines/wires separating regions, and 2-morphisms as nodes (or boxes) separating (or connecting) lines (or wires). String diagrams are planar dual of cellular pasting diagrams.

String diagrams have become prevalent in higher category theory literature. Some of the early references for string diagrams include [Hot65], [Pen71], [JS91], [FY89]. Good expositions to the calculus of string diagrams, and their utility in proving results in category theory can be found in [Sel09], and [Mar14].

Over the course of the last two decades, there has been a great boon in the business of extending the graphical calculus of string diagrams to monoidal functors and monads, double categories, surface diagrams for 3-categories, etc. String diagrams have found immense applications in quantum computation and quantum foundations, in particular in the Oxford group which culminated in the illustrious book [CK17]. For use of string diagrams in proof theory and game semantics see [Mel06] and [Mel12]. A brief summary of these developments is found on the nLab page [nLa19b] of string diagrams. There is even a proof assistant called Globular[BKV16] which lets the user to visualize proofs categorical proofs in finitely-presented n-categories as string diagrams. String diagrams have also generated deep connections between higher category theory, low dimensional topology, and knot theory.

Without going into details of the theory of string diagrams, we shall briefly explain, by the way of examples and illustrations, how 2-categorical equations can be expressed by equations of connecting strings instead of pasting cells. We shall only suffice to a stringy visualization for 2-categories and not bicategories: morphisms-as-strings are composed by juxtaposition of their corresponding regions and this operation is, at least in what is seen, strictly associative.

However, with some technical enhancement, one can also visualize bicategories with string diagrams and even go up to the dimension three and visualize certain tricategories. In particular, see [BMS12] for a detailed and lengthy (with a lot of pictures) development of string diagrams for $\mathcal{G}$ray-categories with potential application to the study of QFTs. The aforementioned paper also gives precise formal definition of string and surface diagrams in terms of PL (Piece-wise Linear) manifolds.

The table below illustrates how we are going to express various 2-categorical operations in terms of strings. It is an interesting and essential property of string diagrams that morphisms-as-wires have no critical points. Also, any string diagram has two projections to the real line: one which forgets the data of regions (of domain and codomain) and the other one which forgets the data of nodes between wires (See Example A.5.2).

We usually read the string diagrams from top to bottom for the direction of nodes, and from left to right for the direction of wires. Some of these directions are indicated in few places but we usually do not bother with indicating the directions for string diagrams, especially for the direction of wires since, as with 1-cells in pasting diagrams, they usually go from left to right.

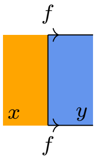
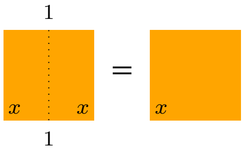
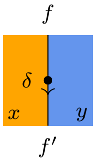
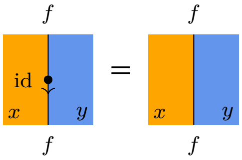
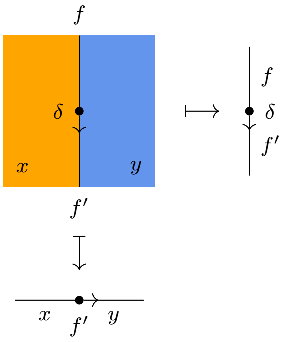
	Pasting diagrams	*String diagrams*
Objects	x	
Morphisms	$x \xrightarrow{f} y$	
Identity morphisms	$x \xrightarrow{1} x$	
2-morphisms	$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \end{tikzcd}$$	
Identity 2-morphisms	$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & y \end{tikzcd}$$	

Fig. A.1.: Pasting vs. string diagrammatic visualization of 2-categories

Below are the 1-dimensional projections of the string diagram of the 2-morphism $\delta\colon f \Rightarrow f'\colon x \rightrightarrows y$.



The projected arrow on the right takes place in the category $\mathfrak{K}(x, y)$ whilst the projected arrow at the bottom lies in the underlying category $||\mathfrak{K}||_1$.

The following table compares various 2-categorical compositions within the two regimes of pasting diagrams and string diagrams.

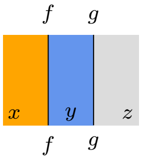
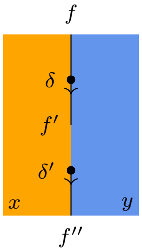
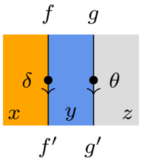
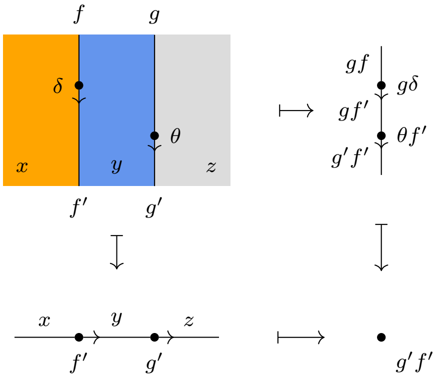
	Pasting diagrams	*String diagrams*
Composition of morphisms	$x \xrightarrow{f} y \xrightarrow{g} z$	
Vertical composition of 2-morphisms	$\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, bend left=30, "\Downarrow\delta"] \arrow[r, phantom, bend right=30, "\Downarrow\delta'"] & y \end{tikzcd}$	
Horizontal composition of 2-morphisms	$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\theta"] & z \end{tikzcd}$	

Fig. A.2.: Compositions in pasting vs. string diagrams

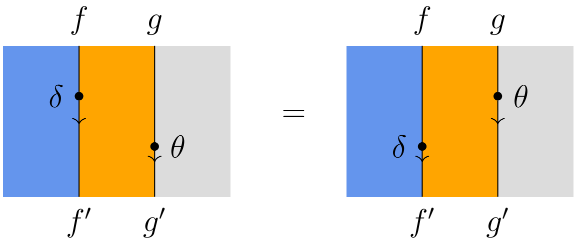
To understand horizontal composition of two string diagrams, we better look at the projections of composite:



String diagrams simplify a considerable amount of complexity in equational bookkeeping of the pasting diagrams of 2-categories. For instance, the exchange law (A.3) in 2-categories which is expressed by the equality of pasting diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \arrow[r, bend left, "g"] & z & & x \arrow[r, bend left, "f"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\theta"] & z \\ & & & = & & & \\ x \arrow[r, bend right, "f'"'] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\theta"] & z & & x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \arrow[r, bend right, "g'"'] & z \end{tikzcd}$$

is represented in its stringy version as the following equality of string diagrams.



This is usually phrased as “we are free to move nodes up and down in so far as there are no obstacles” and it is ocassionally referred to as “the law of elevators”. As a result, either of the string diagrams of the equation above can be identified with the last string diagram of Figure A.2. Once, in using string diagrams, we stop caring about the height of the nodes, in accordance to the law of elevators, we have less equations to track [2] and the 2-categorical proofs become simpler to express and prove.

Recall that

DEFINITION A.5.1. An **adjunction** in $\mathfrak{K}$, often written $\ell \dashv r$, consists of 1-morphisms $\ell \in \mathfrak{K}(x, a)$ and $r \in \mathfrak{K}(a, x)$ together with 2-morphisms (the *unit* and *counit* of adjunction) $\eta \colon 1_x \Rightarrow r \circ \ell$ and $\varepsilon \colon \ell \circ r \Rightarrow 1_a$ satisfying the usual *triangle equalities* (§1.1) $(r \centerdot \varepsilon)(\eta \centerdot r) = \mathrm{id}_r$ and $(\varepsilon \centerdot \ell)(\ell \centerdot \eta) = \mathrm{id}_\ell$. An **adjoint equivalence** is an adjunction where the unit and counit are invertible.

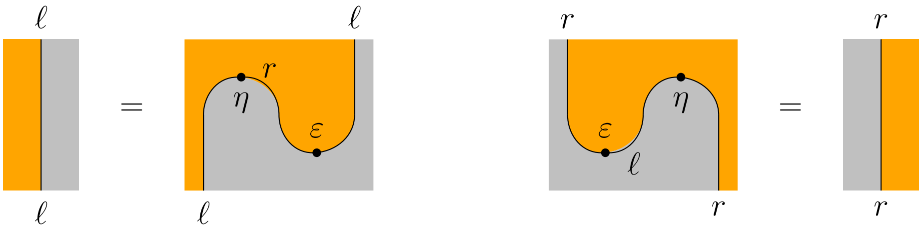
---

[2] resulting, quite possibly, in less headaches!

Any equivalence $(f\colon x \to y, u\colon y \to x, \eta\colon 1_x \Rightarrow uf, \varepsilon\colon fu \Rightarrow 1_y)$ in a 2-category can be promoted to an adjoint equivalence by replacing the invertible 2-morphism ε by $\varepsilon' := \varepsilon \circ (f\eta^{-1}u) \circ (fu\varepsilon^{-1})$. It is a simple algebraic calculation to see that η and ε' satisfy the triangle equations to make f the left adjoint of u. This fact has a nice string diagrammatic proof and we encourage the reader to find such a proof. But before that, let us introduce a stringy visualization of the concept of adjunction.

EXAMPLE A.5.2. For an adjunction $\ell\colon x \leftrightarrows a\colon r$, the unit $\eta\colon 1_x \Rightarrow r \circ \ell$ and the counit $\varepsilon\colon \ell \circ r \Rightarrow 1_a$ of the adjunction are depicted[3] as

and to visualize the two triangular equations of adjunction, we put the string diagram of the right hand side on top of the diagram of the left hand side in a way that the colors match (which reflects the matching of domain and codomain).



(A.10)

This reads as "unit-counit pairs may be straightened by pulling the string".

REMARK A.5.3. The 2-category $\mathfrak{Cat}$ of (small) categories is well-pointed and that means any object of $\mathfrak{Cat}$ is fully determined by its category of points: for a category C in $\mathfrak{Cat}$, an object x of C can be considered as a morphism $x\colon 1 \to C$ in $\mathfrak{Cat}$, and an arrow $f\colon x \to y$ in C can be regarded as a 2-morphism $f\colon x \Rightarrow y\colon 1 \rightrightarrows C$. This move allows us to have a string diagram visualization for the theory of categories. For instance, it is possible to visualize the local characterization of adjoint functors of categories (A.12). See [Mar14] for more details.

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[3] For simplicity and neatness we drop the labelling of regions.

A.6 Strictification

The term “strictification” in the context of higher category theory refers to a host of constructions and results which establish equivalence between certain weak and strict structures in a given dimension. The most well-known of these strictification results occurs in dimension $n = 1$ which is the well-known Mac Lane’s *coherence theorem* for monoidal categories ([ML63], [Kel64]). One way to state it is that every monoidal category is monoidally equivalent to a strict monoidal category. More precisely, the forgetful strict 2-functor $\mathfrak{MonCat}_{str} \to \mathfrak{MonCat}$ has a strict left adjoint and the components of the unit are equivalences in $\mathfrak{MonCat}$.

The simple-minded approach to strictify monoidal structure via the skeleton subcategory does not work. Recall that a category is *skeletal* if any two isomorphic objects are indeed identical, meaning that all isomorphisms are automorphisms. Caution that in general we can not form a skeleton subcategory (e.g. absence of axiom of choice, or lack of nice quotients (to form objects of orbits of action of isomorphisms) in the case of internal categories), and even if we can, a skeleton subcategory Σ of a category $\mathcal{C}$ is *not* comprised of the equivalence classes of $\mathcal{C}$ under the equivalence relation of isomorphism on objects (There are exceptions though, e.g. any thin category (aka preorder)). Rather, a skeleton subcategory Σ of a category $\mathcal{C}$ can be constructed by choosing[4] for every object x of $\mathcal{C}$ a representative object σ_x in the the equivalence class $[x]$ together with an isomorphism $\eta_x \colon x \xrightarrow{\cong} \sigma_x$ in $\mathcal{C}$. Then, Σ is the full subcategory of $\mathcal{C}$ generated by objects σ_x. Indeed, if we define $\sigma(f) = \eta_y \circ f \circ \eta_x^{-1}$ for a morphism $f \colon x \to y$ in $\mathcal{C}$, then σ becomes a left adjoint to the inclusion $\Sigma \hookrightarrow \mathcal{C}$ which makes Σ a reflective full subcategory of $\mathcal{C}$. Notice that the construction of Σ depends on the choice of representative; a different class of representative would yield an equivalence[5] of categories between the generated full subcategories but no canonical one! So, we shall not use the term “the skeleton subcategory”.

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[4] Assuming that we have a mechanism for such a choice!

[5] probably many equivalences!

Now, if $\mathcal{C}$ in addition has a monoidal structure with tensor product $\otimes$, then Σ inherits this monoidal structure in an obvious way: define $\sigma_x \otimes_\sigma \sigma_y := \sigma_{x\otimes y}$. But, note that we get a non-identity isomorphism $\eta\alpha\eta^{-1}$ which becomes a non-identity automorphism in Σ.

$$\begin{tikzcd}
(\sigma_x \otimes_\sigma \sigma_y) \otimes_\sigma \sigma_z \arrow[r, equal] \arrow[d, "\eta\alpha\eta^{-1}"'] & \sigma_{x\otimes y} \otimes_\sigma \sigma_z \arrow[r, equal] & \sigma_{(x\otimes y)\otimes z} \arrow[r, "\eta^{-1}"] & (x \otimes y) \otimes z \arrow[d, "\alpha"] \\
\sigma_x \otimes_\sigma (\sigma_y \otimes_\sigma \sigma_z) \arrow[r, equal] & \sigma_x \otimes_\sigma (\sigma_y \otimes_\sigma \sigma_z) \arrow[r, equal] & \sigma_{x\otimes(y\otimes z)} & x \otimes (y \otimes z) \arrow[l, "\eta"]
\end{tikzcd}$$

We get a similar picture for the unit coherence isomorphism. Therefore we get a monoidal category which is skeletal, but certainly not strict monoidal. Therefore, skeletal construction does nothing to strictify monoidal coherence isomorphisms. For a concrete example, due to Isbell, see the closing remarks in [ML98, Chapter VII, §1]

In a monoidal category with objects x, y, z, w, by tensoring and parenthesizing alone we can make five different objects which are canonically isomorphic. In fact the proof of strictification involved in Mac Lane's coherence theorem does not involve killing off coherence isomorphism by taking a sort of quotients. Rather, the monoidal category $\mathcal{V}$ gets embedded into a strict monoidal category $\mathcal{V}^{str}$.

Similar to the case of monoidal categories there are several important strictification results for bicategories and pseudo functors. These results may be unified as strictification results about pseudo algebras of certain 2-monads.

For instance, there is a 2-monad on the 2-category of $\mathcal{C}$at-enriched graphs whose (strict) algebras are (strict) 2-categories, whose strict/pseudo/lax morphisms are strict/pseudo/lax functors. Moreover, its pseudo algebras are bicategories (See [Lac10a]). A sufficiently general strictification theorem ([BKP89]) states that the pseudo algebras of 2-monads are equivalent, in the category of pseudo algebras, to strict algebras. Therefore, any bicategory is equivalent to a strict 2-category, and any pseudo functor of the form $\mathfrak{K} \to \mathfrak{Cat}$ is equivalent to a strict 2-functor ([Pow89]). Strictification of bicategories and certain pseudo functors can be deduced from this general coherence result. However, [Shu12] shows that not every pseudo algebra is strictifiable using the well-known fact that not every Gray-category is equivalent to a strict 3-category.

We saw earlier that lax functors are less well-behaved in many aspects. However, there are some nice special situations whereby strictifying a lax functor yields quite interesting strict 2-functors. For instance, a monad $T\colon X \to X$ in $\mathfrak{K}$, considered as a lax functor $\mathbf{1} \to \mathfrak{K}$, can be strictified to a strict 2-functor. The most famous example of this, originally due to Lawvere ([Law69]) is explained in below.

EXAMPLE A.6.1. Consider the (strict) monoidal simplex category[6] Δ of finite ordinals where the tensor product is given by the addition bifunctor $+\colon \Delta \times \Delta \to \Delta$ with the action on objects and morphisms defined as

$$\mathbf{n} + \mathbf{m} := \{0, 1, \ldots, n+m-1\}$$

and

$$(f+g)(i) = \begin{cases} f(i) & \text{, if } i = 0, 1, \ldots, n-1 \\ n' + g(i-n) & \text{, otherwise} \end{cases}$$

for $f\colon \mathbf{n} \to \mathbf{n}'$ and $g\colon \mathbf{m} \to \mathbf{m}'$. The unit of Δ is given by the empty ordinal $\mathbf{0}$. Define μ^k to be the unique arrow $\mathbf{k} \to \mathbf{1}$. Set $\mu^0 = \eta$, $\mu^1 = 1 = \mathrm{id}_{\mathbf{1}}$ and $\mu^2 = \mu$. From the uniqueness we get equations such as

$$\mu(\mu + 1) = \mu(1 + \mu) = \mu^3\colon \mathbf{3} \to \mathbf{1},$$

and more generally,

$$\mu^n \left(\mu^{k_1} + \ldots + \mu^{k_n}\right) = \mu^{(k_1 + \ldots + k_n)} \tag{A.11}$$

Note that, in virtue of these equations, the simplex category Δ has a canonical monoid object $\mathbf{1}, \mu\colon \mathbf{1} + \mathbf{1} \to \mathbf{1}, \eta\colon \mathbf{0} \to \mathbf{1}$. The simplex category Δ together

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[6] It includes the empty set as the first ordinal which is the initial object and one-element set as the second ordinal which is the terminal object.

with this monoid object is initial among all strict monoidal categories equipped with a monoid object.

Applying this observation to the cartesian monoidal category $\mathcal{C}$at enables us to identify strict monoidal categories with strict monoidal functors $\Delta \to \mathcal{C}$at. However, hardly any of the monoidal categories in nature are strict. That is why we have to use pseudo functors instead: a monoidal category (i.e. pseudo monoid internal to the 2-category $\mathfrak{Cat}$) can be identified with a monoidal pseudo functor $\Delta \to \mathfrak{Cat}$ where the simplicial identities hold up to invertible natural transformations. This is known known as the *Bar construction*. In Example 2.3.45((ii)), we see how *symmetric monoidal categories* can be considered as Grothendieck fibrations over the category of pointed finite sets, and therefore as pseudo functors to $\mathcal{F}\mathrm{in}_*{}^{\mathrm{op}} \to \mathfrak{Cat}$. A strict 2-functor $T\colon \Sigma\Delta \to \mathfrak{K}$ takes the only object $*$ of $\mathbf{1}$ to an object X of $\mathfrak{K}$, the identity $\mathbf{0}$ to $1_X\colon X \to X$, $\mathbf{1}$ to $T\colon X \to X$ and $\mathbf{2}$ to $T^2 = T \circ T\colon X \to X$. The 2-morphisms η and μ in $\Sigma\Delta$ are mapped to the unit and multiplication of the monad and the equations (A.11) (with lots of redundancies) give the unit and associativity equations of monad T. Therefore, a strict 2-functor from one-object 2-category $\Sigma\Delta$ to $\mathfrak{K}$ is exactly a monad in $\mathfrak{K}$.

A.7 Category theory internal to bicategories

One can generalize enough concepts from category theory to 2-categories and bicategories so that all the fundamental results of category theory hold in 2-categories and bicategories. Alas there is no single reference, akin to the already classic and still excellent [ML98] for theory of categories, treating all fundamental constructions and results for bicategories. However, [KS74] and [Gra74] are great expositions. The latter, while pioneering the study of various weak structures of 2-categories and 3-categories including the treatment of the famous ‘Gray tensor product’, has the disadvantage of using confusing and outdated terminology compared to the standard terminology in theory of 2-categories. Also [RV15] offers a great range of categorical concepts internalized in the homotopy 2-categories of ∞-cosmoi.

In below, we just have a very short glimpse into formal category theory that will be relevant to our further development of categorical and toposical fibrations. We encourage the reader who has not much experience of formal category theory to state and prove the main results categorical adjunctions 2-categorically. We start by reviewing the *calculus of mates*, a very useful tool in calculating pasting of diagrams in 2-categories.

In ordinary category theory, given a pair of adjoint functors $L\colon \mathcal{X} \rightleftarrows \mathcal{A} \colon R$, we have equivalence of sets

$$\mathcal{A}(Lx, a) \cong \mathcal{X}(x, Ra)$$

This is known as the *local bijections* of the adjunction, and in most cases it is the best way of guessing one of the adjoints from the other.

The formulation above has the following extension to 2-categories and bicategories. Consider an adjunction $\ell \dashv r \colon A \to X$ in $\mathfrak{B}$, with unit $\eta\colon 1_X \Rightarrow r\ell$ and counit $\varepsilon\colon \ell r \Rightarrow 1_A$. Then, for any morphisms $a\colon U \to A$, $x\colon U \to X$, $f\colon X \to Z$, and $g\colon A \to Z$ there are natural bijections

$$[\ell_* x, a] \cong [x, r_* a] \qquad \text{and} \qquad [f, \ell^* g] \cong [r^* f, g] \tag{A.12}$$

where $[h, k]$, for general morphisms h and k with the same domain and codomain, is a shorthand notation for $\mathfrak{B}(\operatorname{dom} h, \operatorname{cod} h)(h, k)$. Also $\ell_* x = \ell \circ x$, and $\ell^* g = g \circ l$. The isomorphisms above are given by

$$\begin{tikzcd}
X \arrow[rd, "\ell"] & \\
U \arrow[u, "x"] \arrow[r, "a"'] \arrow[ur, phantom, "\varphi\Downarrow"] & A
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[rd, "\ell"] \arrow[r, phantom, shift left=-12, "\eta\Downarrow"] & X \\
U \arrow[u, "x"] \arrow[r, "a"'] \arrow[ur, phantom, shift right=3, "\varphi\Downarrow"] & A \arrow[u, "r"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rd, "\ell"'] \arrow[rd, phantom, shift left=3, "\phi\Downarrow"] & Z \\
& A \arrow[u, "g"']
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rd, "\ell"] \arrow[r, phantom, shift left=-12, "\phi\Downarrow"] & Z \\
A \arrow[u, "r"] \arrow[r, "1"'] \arrow[ur, phantom, shift right=3, "\varepsilon\Downarrow"] & A \arrow[u, "g"']
\end{tikzcd}$$

Of course we can combine the bijection in (A.12) from both sides:

$$[fr', rg] = [(r')^* f, r_* g] \cong [f, (\ell')^* r_* g] = [f, r_*(\ell')^* g] \cong [\ell_* f, (\ell')^* g] = [\ell f, g\ell'] = [lf, g\ell']$$

DEFINITION A.7.1. Consider adjunctions $\ell' \dashv r' \colon A' \to X'$ and $\ell \dashv r \colon A \to X$. The **mate** ϕ of 2-morphism $\psi \colon fr' \Rightarrow rg$ is given by pasting it on the left with the unit of first adjunction and on the right with the counit of second adjunction.

$$\begin{tikzcd}
X' \arrow[r, "f"] & X \\
A' \arrow[u, "r'"] \arrow[r, "g"'] \arrow[ur, phantom, "\psi\Downarrow"] & A \arrow[u, "r"']
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & X' \arrow[r, "f"] & X \arrow[r, "\ell"] & A \\
X' \arrow[ur, "1"] \arrow[r, "\ell'"'] \arrow[ur, phantom, "\eta'\Downarrow"', shift left=3] & A' \arrow[u, "r'"'] \arrow[r, "g"'] \arrow[ur, phantom, "\psi\Downarrow"] & A \arrow[u, "r"] \arrow[ur, "1"'] \arrow[ur, phantom, "\varepsilon\Downarrow", shift left=3] &
\end{tikzcd}$$

Conversely,

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "\ell'"'] \arrow[dr, phantom, "\varphi\Downarrow"] & X \arrow[d, "\ell"] \\
A' \arrow[r, "g"'] & A
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A' \arrow[r, "r'"] \arrow[dr, "1"'] \arrow[dr, phantom, "\varepsilon'\Downarrow", shift left=3] & X' \arrow[r, "f"] \arrow[d, "\ell'"] \arrow[dr, phantom, "\varphi\Downarrow"] & X \arrow[d, "\ell"'] \arrow[dr, "1"] \arrow[dr, phantom, "\eta\Downarrow"', shift right=3] & \\
 & A' \arrow[r, "g"'] & A \arrow[r, "r"'] & X
\end{tikzcd}$$

In equations, we have

$$\varphi = (\varepsilon \bullet (g\ell')) \circ (\ell \bullet \psi \bullet \ell') \circ ((\ell f) \bullet \eta')$$
$$\psi = ((rg) \bullet \varepsilon') \circ (r \bullet \varphi \bullet r') \circ (\eta \bullet (fr'))$$

EXAMPLE A.7.2. In the delooping bicategory ΣV of a monoidal category $(\mathcal{V}, \otimes, I)$ an adjunction is a given by a pair (L, R) of objects of $\mathcal{V}$ with a unit $\eta \colon I \colon L \otimes R$ and a counit $\varepsilon \colon R \otimes L \to I$ such that the following diagrams commute.

$$\begin{tikzcd}
I \otimes L \arrow[r, "\eta \otimes L"] & (L \otimes R) \otimes L \arrow[r, "\alpha"] & L \otimes (R \otimes L) \arrow[r, "L \otimes \varepsilon"] & L \otimes I \arrow[d, "\lambda_L"] \\
L \arrow[u, "\rho_L^{-1}"] \arrow[rrr, equal] & & & L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes I \arrow[r, "R \otimes \eta"] & R \otimes (L \otimes R) \arrow[r, "\alpha"] & (R \otimes L) \otimes R \arrow[r, "\varepsilon \otimes R"] & I \otimes R \arrow[d, "\rho_R"] \\
R \arrow[u, "\lambda_R^{-1}"] \arrow[rrr, equal] & & & R
\end{tikzcd}$$

The mate-construction in definition A.7.1 is the construction of dual morphisms between dualizable objects. It gives a one-to-one correspondence between morphisms of the form $R' \otimes X \to A \otimes R$ and $X \otimes L \to L' \otimes A$.

REMARK A.7.3. Note that even the mate of identity 2-morphism may not be identity. One way to see this fact is to consider a 2-morphism $\psi \colon f \circ r' \Rightarrow r \circ g$ in the 2-category of groups (Example 1.5.2). Assuming r' and r have respective left adjoints

ℓ' and ℓ, the mate of ψ is given by the element $\varphi = \varepsilon\ell(\psi)\ell(f(\eta'))$. Obviously even if ψ is the unit element then φ does not necessarily equal the unit element. However, mating preserves certain identities; these are the so-called *simple identities* and the process is called *simple mating*. The mate of simple identity 2-morphism $\mathrm{id}_r\colon 1 \circ r \Rightarrow r \circ 1$ is $\mathrm{id}_\ell\colon \ell \circ 1 \Rightarrow 1 \circ \ell$.

The simple identities and simple mates have a special status in the double category $\mathbb{A}\mathbf{dj}(\mathfrak{K})$ of adjunction in the 2-category $\mathfrak{K}$: they are the unit 2-morphisms. Recall that the objects of $\mathbb{A}\mathbf{dj}(\mathfrak{K})$ are objects A of $\mathfrak{K}$, its horizontal morphisms are morphisms $f\colon X \to Y$ of $\mathfrak{K}$, its vertical morphisms are adjoint pairs $\ell \dashv r\colon A \to X$, and its 2-morphisms are $\varphi\colon \ell f \Rightarrow g\ell'$. There is an equivalent double category constructed with the same data except that we take mate of $\psi\colon fr' \Rightarrow rg$ of φ as 2-morphisms.

REMARK A.7.4. Mating commutes with pasting: the mate of pasting

$$\begin{tikzcd}
X'' \arrow[r, "f'"] & X' \arrow[r, "f"] & X \\
A'' \arrow[u, "r''"] \arrow[r, "g'"'] \arrow[ru, phantom, "\psi'\Downarrow"] & A' \arrow[u, "r'"'] \arrow[r, "g"'] \arrow[ru, phantom, "\psi\Downarrow"] & A \arrow[u, "r"']
\end{tikzcd}$$

is equal to pasting

$$\begin{tikzcd}
 & X'' \arrow[r, "f'"] & X' \arrow[r, "1"] \arrow[rd, "\ell'"] & X' \arrow[r, "f"] & X \arrow[r, "\ell"] & A \\
X'' \arrow[ru, "1"] \arrow[r, "\ell'"'] \arrow[ru, phantom, "\eta''\Downarrow"'] & A'' \arrow[u, "r''"'] \arrow[r, "g'"'] \arrow[ru, phantom, "\psi'\Downarrow"] & A' \arrow[u, "r'"'] \arrow[r, "1"'] \arrow[r, phantom, "\varepsilon'\Downarrow", shift left=3] & A' \arrow[u, "r'"'] \arrow[r, "g"'] \arrow[ru, phantom, "\psi\Downarrow"] & A \arrow[u, "r"'] \arrow[ru, "1"'] \arrow[ru, phantom, "\varepsilon\Downarrow"] &
\end{tikzcd}$$

of individual mates of ψ and ψ'. This follows from $(r'\varepsilon') \circ (\eta' r') = \mathrm{id}_{r'}$, and vice versa.

REMARK A.7.5. The process of mating of certain 2-morphisms of $\mathfrak{K}$ extends to 2-morphisms of $\mathbf{cyl}(\mathfrak{K})$ (1.4.12).

$$\begin{tikzcd}[column sep=large, row sep=huge]
X' \arrow[rr, bend left, "f'"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\overline{\alpha}\Uparrow"] \arrow[ddr, phantom, "\Downarrow\psi"] & & X \arrow[ddl, phantom, "\psi'\Downarrow"] \\
\\
A' \arrow[uu, "r'"] \arrow[rr, bend left, "g'"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\underline{\alpha}\Uparrow"] & & A \arrow[uu, "r"']
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=large, row sep=huge]
X' \arrow[dd, "\ell'"'] \arrow[rr, bend left, "f'"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\overline{\alpha}\Uparrow"] \arrow[ddr, phantom, "\Downarrow\varphi"] & & X \arrow[ddl, phantom, "\varphi'\Downarrow"] \\
\\
A' \arrow[rr, bend left, "g'"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\underline{\alpha}\Uparrow"] & & A \arrow[uu, "\ell"']
\end{tikzcd}$$

A.8 The bicategory of internal categories and the bicategory of internal bimodules

DEFINITION A.8.1. Suppose $\mathcal{S}$ is a finitely complete category (e.g. an elementary topos). An **internal category** $\mathbb{C}$ in $\mathcal{S}$ is a diagram

$$\begin{tikzcd}[column sep=large]
C_0 \arrow[r, "i" description] & C_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift right=2, "i_0" description] \arrow[r, shift left=2, "i_1" description] & C_2 \arrow[l, shift right=4, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=4, "d_2"] & C_3 \arrow[l, shift right=4.5, "d_0"'] \arrow[l, shift right=1.5, "d_1" description] \arrow[l, shift left=1.5, "d_2" description] \arrow[l, shift left=4.5, "d_3"]
\end{tikzcd}$$

such that the squares below are pullback squares

$$\begin{tikzcd}
C_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, "\ulcorner" very near start] & C_1 \arrow[d, "d_1"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "d_0"] \arrow[d, "d_3"'] \arrow[dr, phantom, "\ulcorner" very near start] & C_2 \arrow[d, "d_2"] \\
C_2 \arrow[r, "d_0"'] & C_1
\end{tikzcd}$$

and

(IC1) The identity morphism is a common section of domain and codomain morphisms, i_0 is a common section of $d_0, d_1 \colon C_2 \rightrightarrows C_1$, and i_1 is a common section of $d_0, d_1 \colon C_2 \rightrightarrows C_1$. In below, these conditions are expressed by the commutativity of diagrams

$$\begin{tikzcd} C_0 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "\mathrm{id}" description] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "i_0"] \arrow[d, "i_0"'] \arrow[dr, "\mathrm{id}" description] & C_2 \arrow[d, "d_1"] \\ C_2 \arrow[r, "d_0"'] & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "i_1"] \arrow[d, "i_1"'] \arrow[dr, "\mathrm{id}" description] & C_2 \arrow[d, "d_1"] \\ C_2 \arrow[r, "d_0"'] & C_1 \end{tikzcd}$$

in $\mathcal{S}$.

(IC2) $d_j \circ d_k = d_{k-1} \circ d_j$ for all $0 \leq j \leq k \leq 3$

(IC3) $i_0 \circ i = i_1 \circ i$, $d_2 \circ i_0 = i \circ d_1$, and $d_0 \circ i_1 = i \circ d_0$.

We shall call C_0 the object of ‘objects’, C_1 the object of ‘morphisms’, C_2 the object of ‘composable pairs of arrows’, and finally C_2 the object of ‘composable triples of arrows’. Furthermore, we shall call i the ‘identity’ morphism, $d_0 \colon C_1 \to C_0$ the ‘domain’ morphism, $d_1 \colon C_1 \to C_0$ the ‘codomain’ morphism, and finally $d_1 \colon C_2 \to C_1$ the morphism of ‘composition’. Also, we use the term ‘identity arrows’ to refer to the elements of C_1 in the image of $i \colon C_0 \to C_1$. An internal category is **discrete** when its domain and codomain morphisms are equal, and they establish an isomorphism between the object of morphisms and the objects of objects. It is **indiscrete** whenever the object of morphisms is isomorphic to the two-fold product of the object of objects, and the domain and the codomain morphisms are isomorphic to the product projections.

DEFINITION A.8.2. An **internal functor** $F \colon \mathbb{C} \to \mathbb{D}$ consists of three morphisms $F_j \colon C_j \to D_j$, for $j = 0, 1$ such that the following equations of morphisms of $\mathcal{S}$ hold.

(i) $F_0 \circ d_j = d_j \circ F_1$, which expresses that F preserves the domain and codomain of arrows.

(ii) $F_1 \circ i = i \circ F_0$, which expresses that F preserves identity arrows.

(iii) $F_1 \circ d_j = d_j \circ G$, where $G\colon C_2 \to D_2$ is the unique morphism determined entirely solely by F_0 and F. This expresses that F preserves composition of arrows.

DEFINITION A.8.3. Given internal functors $F, G\colon \mathbb{C} \rightrightarrows \mathbb{D}$, an **internal natural transformation** between them is a morphism $\theta\colon C_0 \to D_1$ in $\mathcal{S}$ such that the diagrams below commute in $\mathcal{S}$.

$$\begin{tikzcd} & D_1 \arrow[d, "d_0"] \\ C_0 \arrow[ur, "\theta"] \arrow[r, "F_0"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} & D_1 \arrow[d, "d_1"] \\ C_0 \arrow[ur, "\theta"] \arrow[r, "G_0"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, shift left, "\langle F_1, \theta d_1 \rangle"] \arrow[r, shift right, "\langle \theta d_0, G_1 \rangle"'] & D_2 \arrow[r, "d_1"] & D_1 \end{tikzcd} \tag{A.13}$$

'Whiskering' of natural transformations is given as follows. Given internal functors

$$\mathbb{B} \xrightarrow{S} \mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D} \xrightarrow{S} \mathbb{E}$$

and an internal natural transformation $\theta\colon F \Rightarrow G$, the whiskered natural transformation $S\theta R\colon SFR \Rightarrow SGR$ is defined by the composite $S_1 \circ \theta \circ R_0$ of the 1-morphism $B_0 \xrightarrow{R_0} C_0 \xrightarrow{\theta} D_1 \xrightarrow{S_1} E_1$ in $\mathcal{S}$. Notice that the operation of whiskering is enough to get all horizontal composition of 2-morphisms. The vertical composition of internal natural transformation perhaps has a little bit more interesting construction: suppose we are given natural transformations

$$\begin{tikzcd}[column sep=huge] \mathbb{C} \arrow[r, "G" description] \arrow[r, bend left=70, "H"] \arrow[r, bend right=70, "F"'] \arrow[r, phantom, shift left=4ex, "\Uparrow\lambda"] \arrow[r, phantom, shift right=4ex, "\Uparrow\theta"] & \mathbb{D} \end{tikzcd}$$

where $\theta, \lambda\colon C_0 \rightrightarrows D_1$. We observe that $d_1 \circ \theta = G_0 = d_0 \circ \lambda$. Hence, the cone formed by θ and λ factors through the pullback cone, which defines D_2, via the morphism $\langle \theta, \lambda \rangle$. Now, the vertical composition of θ and λ is defined by the composite $d_1 \circ \langle \theta, \lambda \rangle\colon C_0 \to D_1$. We leave it to the reader to check that horizontal and vertical compositions are unital and associative.

PROPOSITION A.8.4. Internal categories, internal functors, and internal natural transformations in a finitely complete category $\mathcal{S}$ form a finitely complete 2-category which is denoted by $\mathfrak{Cat}(\mathcal{S})$.

Proof. To see that $\mathfrak{Cat}(\mathcal{S})$ is finitely complete, we first note that the underlying category $||\,\mathfrak{Cat}(\mathcal{S})||_1$ is finitely complete since the forgetful functor $||\,\mathfrak{Cat}(\mathcal{S})||_1 \to \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ creates finite limits. To see this result in action, consider an opspan $\mathbb{C} \xrightarrow{F} \mathbb{E} \xleftarrow{G} \mathbb{D}$ of internal functors. The corresponding pullback span $\mathbb{C} \xleftarrow{\pi_0} \mathbb{P} \xrightarrow{\pi_1} \mathbb{D}$ is formed by the pair of spans

$$C_1 \xleftarrow{\pi_0} P_1 \xrightarrow{\pi_1} D_1 \qquad\qquad C_0 \xleftarrow{\pi_0} P_0 \xrightarrow{\pi_1} D_0$$

where P_1 and P_0 are the respective pullbacks of opspans $C_1 \xrightarrow{F_1} E_1 \xleftarrow{G_1} D_1$ and $C_0 \xrightarrow{F_0} E_0 \xleftarrow{G_0} D_0$. It is straightforward to see that with certain induced maps on these pullbacks, $\mathbb{P}$ is indeed an internal category which is also the universal cone on the opspan of F and G. $\square$

Suppose that $\mathbb{C}$ and $\mathbb{D}$ are internal categories in $\mathcal{S}$, and moreover the exponentials $D_0^{C_0}$, $D_0^{C_1}$, $D_1^{C_0}$, and $D_1^{C_1}$ exist in $\mathcal{S}$. Then, the locus of the first condition of Definition A.8.1 can be expressed by the intersection of the subobjects E_0 and E_1 of $D_1^{C_1} \times D_0^{C_0}$ obtained as the following equalizers.

$$\begin{tikzcd}
 & & D_0^{C_1} \times D_0^{C_0} \arrow[dr, "\pi_0"] & \\
E_0 \arrow[r, tail] & D_1^{C_1} \times D_0^{C_0} \arrow[ur, "(d_0)_* \times 1"] \arrow[dr, "1 \times (d_0)^*"'] & & D_0^{C_1} \\
 & & D_1^{C_1} \times D_0^{C_1} \arrow[ur, "\pi_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & D_0^{C_1} \times D_0^{C_0} \arrow[dr, "\pi_0"] & \\
E_1 \arrow[r, tail] & D_1^{C_1} \times D_0^{C_0} \arrow[ur, "(d_1)_* \times 1"] \arrow[dr, "1 \times (d_1)^*"'] & & D_0^{C_1} \\
 & & D_1^{C_1} \times D_0^{C_1} \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Similarly, the locus of second and third conditions in Definition A.8.1 can be expressed as certain subobjects of $D_1^{C_1} \times D_0^{C_0}$. We denote the intersection of the subobjects obtained from conditions (i)-(iii) by $[\mathbb{C}, \mathbb{D}]_0$ and it is to be taken as the object component of the internal category of functors from $\mathbb{C}$ to $\mathbb{D}$.

Furthermore, the object of internal natural transformations between functors from $\mathbb{C}$ to $\mathbb{D}$ is obtained as a subobject of $[\mathbb{C}, \mathbb{D}]_0 \times D_1^{C_0}$.

PROPOSITION A.8.5. If $\mathcal{S}$ is cartesian closed then so is $||\,\mathfrak{Cat}(\mathcal{S})||_1$.

REMARK A.8.6. Every set can be regarded as a discrete category in a canonical way: the objects of category are elements of the set with identity morphisms as the only morphisms. There is an analogue of this construction for internal categories. Any object X of $\mathcal{S}$ is equipped with the structure of internal category $X^{\mathbb{d}} := (X \rightrightarrows X)$ in $\mathcal{S}$ in a trivial way; the domain, codomain, identity, and composition morphisms are all identity morphism id_X. Similarly any set can be made into an indiscrete category by adding exactly one invertible morphism for any pair of its objects. Internally, this is achieved by defining for an object X of $\mathcal{S}$, the internal category $X^{\mathbb{ind}} := (X \times X \rightrightarrows X)$ where the domain and codomain morphisms are product projections $X \times X \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$, the identity morphism is the diagonal $\Delta \colon X \to X \times X$, and the composition morphism is the product projection $\pi_1 \colon X \times X \times X \to X \times X$. These constructions induce the following adjunction triple

$$\begin{tikzcd}
\mathcal{S} \arrow[r, bend left=60, "(-)^{\mathbb{d}}"] \arrow[r, bend right=60, "(-)^{\mathbb{ind}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \lVert \mathfrak{Cat}(\mathcal{S}) \rVert_1 \arrow[l, "\mathrm{Ob}"']
\end{tikzcd} \tag{A.14}$$

where $\lVert \mathfrak{Cat}(\mathcal{S}) \rVert_1$ is the underlying category of $\mathfrak{Cat}(\mathcal{S})$, and Ob is the forgetful object functor. In fact, the adjunction above factors through $\lVert \mathfrak{Grpd} \rVert_1 \hookrightarrow \lVert \mathfrak{Cat} \rVert_1$.

A cartesian functor $F \colon \mathcal{S} \to \mathcal{S}'$ of categories induces a 2-functor $F_* \colon \mathfrak{Cat}(\mathcal{S}) \to \mathfrak{Cat}(\mathcal{S}')$ of 2-categories, and a natural transformation $\alpha \colon F \to F'$ induces a 2-natural transformation $\alpha_* \colon F_* \Rightarrow F'_*$. Therefore, we get a strict meta 2-functor $\mathfrak{Cat}(-) \colon \mathfrak{Cat}_{\text{cart}} \to \mathfrak{CAT}$ which gives the base change.

CONSTRUCTION A.8.7. Any internal category $\mathbb{C}$ can be 'externalized' to a strict 2-functor $\mathbb{F}\mathrm{am}(\mathbb{C}) \colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ where $\mathbb{F}\mathrm{am}(\mathbb{C})(I)$ is a category whose objects are morphisms $X\colon I \to C_0$ in $\mathcal{S}$, and a morphism from X to Y is a morphism $f\colon I \to C_1$ in $\mathcal{S}$ such that the following diagrams commute.

$$\begin{tikzcd}
 & C_1 \arrow[d, "d_0"] \\
I \arrow[ur, "f"] \arrow[r, "X"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D_1 \arrow[d, "d_1"] \\
I \arrow[ur, "f"] \arrow[r, "Y"'] & D_0
\end{tikzcd}$$

The identity morphism on X in $\mathbb{F}\mathrm{am}(\mathbb{C})(I)$ is given by the composite $I \xrightarrow{X} C_0 \xrightarrow{i} C_1$, and the composition of f in $\mathbb{F}\mathrm{am}(\mathbb{C})(I)(X, Y)$ and g in $\mathbb{F}\mathrm{am}(\mathbb{C})(I)(Y, Z)$ is given by the composite $I \xrightarrow{\langle X,Y \rangle} C_2 \xrightarrow{d_1} C_1$ in $\mathcal{S}$. A morphism $\alpha\colon J \to I$ induces a functor $\alpha^*\colon \mathbb{F}\mathrm{am}(\mathbb{C})(I) \to \mathbb{F}\mathrm{am}(\mathbb{C})(J)$ of categories by pre-composition with α (strictness of composition gives the strictness of 2-functor $\mathbb{F}\mathrm{am}(\mathbb{C})$). Therefore, we get a functor $\mathbb{F}\mathrm{am}(\mathbb{C})\colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$. Also, an internal functor $F\colon \mathbb{C} \to \mathbb{D}$ induces a strict 2-natural transformation $\phi\colon \mathbb{F}\mathrm{am}(\mathbb{C}) \to \mathbb{F}\mathrm{am}(\mathbb{D})$ given componentwise by $\phi_I(X) = F_0 \circ X$, and $\phi_I(f) = F_1 \circ f$. In fact, we get a strict 2-functor $\mathbb{F}\mathrm{am}\colon \mathfrak{Cat}(\mathcal{S}) \to \mathbf{2}\,\mathfrak{Cat}_{str}(\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat})$. The previous 2-functor is full and faithful by the Yoneda Lemma. There is even more to this: the functor $||\mathbb{F}\mathrm{am}||_1\colon ||\,\mathfrak{Cat}(\mathcal{S})||_1 \to [\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}]$ is fully faithful on the underlying categories. However, the embedding $\mathfrak{Cat}(\mathcal{S}) \hookrightarrow \mathrm{Hom}(\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat})$ is only fully faithful in the bicategorical sense (Section 1.3). See [Joh02a, B2.3.4] which shows that the latter 2-functor does not reflect isomorphisms, so it cannot be fully faithful at the level of underlying categories.

REMARK A.8.8. The functor $||\mathbb{F}\mathrm{am}||_1$ extends the Yoneda embedding to internal categories.

$$\begin{tikzcd}
\mathfrak{Cat}(\mathcal{S}) \arrow[r, "||\mathbb{F}\mathrm{am}||_1"] & {[\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}]} \\
\mathcal{S} \arrow[u, hook, "(-)^{\mathrm{d}}"] \arrow[r, "y_{\mathcal{S}}"'] & {[\mathcal{S}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]} \arrow[u, hook]
\end{tikzcd}$$

We assume that the reader is familiar with the definition of a monoid object in monoidal categories. Otherwise, we refer the reader to [ML98, §VII.3]. Monoid objects in a monoidal category form the category $\mathcal{M}on(\mathcal{V})$. The category of commutative monoid will be denoted by $\mathcal{CM}on(\mathcal{V})$.

REMARK A.8.9. A monoid object in a cartesian monoidal category $(\mathcal{S}, \times, 1)$ is an internal category whose object of objects is isomorphic to the terminal object 1 of

$\mathcal{S}$. Therefore, the category $\mathcal{M}on(\mathcal{V})$ embeds into the category $||\,\mathfrak{Cat}(\mathcal{S})||_1$ of internal categories in $\mathcal{S}$.

By an $\mathcal{A}$b-like monoidal category, we mean a closed monoidal category with equalizers and coequalizers which are stable under tensoring. Suppose that $(\mathcal{V}, \otimes, I)$ is an $\mathcal{A}$b-like monoidal category. Let (A, μ, ε) be an monoid in $\mathcal{V}$. Define an **internal left A-module** to be the structure (M, m) where M is an object of $\mathcal{V}$ and $m\colon A \otimes M \to M$ is an action morphism in $\mathcal{V}$, in particular, m satisfies the unit and associativity axioms. We form the category $\mathcal{M}\mathrm{od}(\mathcal{V})$ of internal (left) modules in $\mathcal{V}$ in which objects are pairs (A, M), whereby A is a monoid object in $\mathcal{V}$, and M is an A-module. Morphisms are pairs (f, ϕ) whereby $f\colon A \to B$ is a monoid morphism and $\phi\colon M \to N$ in $\mathcal{V}$ is f-equivariant, that is the diagram below commutes:

$$\begin{tikzcd}
A \otimes M \arrow[r, "f\otimes\phi"] \arrow[d, "m"'] & B \otimes N \arrow[d, "n"] \\
M \arrow[r, "\phi"'] & N
\end{tikzcd}$$

Identities and composition in $\mathcal{M}\mathrm{od}(\mathcal{V})$ are respectively given by identities and composition in $\mathcal{V}$. In fact, there is a Grothendieck fibration of categories

$$\begin{tikzcd}
\mathcal{M}\mathrm{od}(\mathcal{V}) \arrow[d] \\
\mathcal{M}on(\mathcal{V})
\end{tikzcd} \tag{A.15}$$

which takes a (left) module (A, M) to its underlying monoid A. The fibre over monoid A is the category A-$\mathcal{M}\mathrm{od}(\mathcal{V})$ of all (left) A-modules. Similarly, one can define notions of internal right module and internal bimodule along the same lines. A motivating example is to consider the symmmetric monoidal category (although not cartesian closed and hence not a topos) $\mathcal{A}$b of Abelian groups. Note that $\mathcal{M}on(\mathcal{A}\mathrm{b})$ is the category of rings, and $\mathcal{CM}on(\mathcal{A}\mathrm{b})$ is the category of commutative rings. Also, $\mathcal{M}\mathrm{od}(\mathcal{A}\mathrm{b})$ is the category of modules over rings and the fibre $\mathbb{Z}$-$\mathcal{M}\mathrm{od}(\mathcal{A}\mathrm{b})$ of fibration above over the ring $\mathbb{Z}$ of integers recovers $\mathcal{V} = \mathcal{A}\mathrm{b}$.

DEFINITION A.8.10. For monoid objects A and B in $\mathcal{V}$, An (A, B)**-bimodule** is given by a monoid object M and an 'action' monoid morphism $m \colon A \otimes M \otimes B \to M$ in $\mathcal{V}$ satisfying the usual unit and associativity axioms of action.

Every such bimodule gives rise to a left A-module and a right B-module which can be seen in the diagram below:

$$\begin{tikzcd}
A \otimes M \otimes I \arrow[r, "A \otimes M \otimes \varepsilon_B"] \arrow[d, "\cong"'] & A \otimes M \otimes B \arrow[d, "m"'] & I \otimes M \otimes B \arrow[l, "\varepsilon_A \otimes M \otimes B"'] \arrow[d, "\cong"] \\
A \otimes M \arrow[r, dashed, "m_A"'] & M & M \otimes B \arrow[l, dashed, "m_B"]
\end{tikzcd}$$

Suppose M is an (A, B)-bimodule and N is a (B, C)-bimodule. We define tensor product of M and N as the following coequalizer:

$$\begin{tikzcd}
M \otimes B \otimes N \arrow[r, shift left, "m_B \otimes 1_N"] \arrow[r, shift right, "1_M \otimes n_B"'] & M \otimes N \arrow[r, dashed, two heads, "q"] & M \otimes_B N
\end{tikzcd} \tag{A.16}$$

The universal property of of q is the familiar universal property of tensor of bi-modules: any bilinear map out of $M \otimes N$ factors via quotient map q to $M \otimes_B N$. We now prove that $M \otimes_B N$ is indeed an (A, C)-bimodule. In the diagram below, notice that the top row is again a coequalizer because $\mathcal{V}$ is $\mathcal{A}$b-like. Since both left squares commute, we obtain a unique map $m_A \otimes_B n_C$ between coequalizers which gives $M \otimes_B N$ the structure of (A, C)-bimodule.

$$\begin{tikzcd}
A \otimes M \otimes B \otimes N \otimes C \arrow[r, shift left, "1_A \otimes m_B \otimes 1_N \otimes 1_C"] \arrow[r, shift right, "1_A \otimes 1_M \otimes n_B \otimes 1_C"'] \arrow[d, "m_A \otimes 1 \otimes n_C"'] & A \otimes M \otimes N \otimes C \arrow[r, two heads, "1_A \otimes q \otimes 1_C"] \arrow[d, "m_A \otimes n_C"'] & A \otimes (M \otimes_B N) \otimes C \arrow[d, dashed, "m_A \otimes_B n_C"] \\
M \otimes B \otimes N \arrow[r, shift left, "m_B \otimes 1_N"] \arrow[r, shift right, "1_M \otimes n_B"'] & M \otimes N \arrow[r, two heads, "q"] & M \otimes_B N
\end{tikzcd}$$

CONSTRUCTION A.8.11. For an $\mathcal{A}$b-like monoidal category $(\mathcal{V}, \otimes, I)$, the bicategory $\mathfrak{BiMod}(\mathcal{V})$ of bimodules is constructed as follows:

- The objects are monoids in $\mathcal{V}$ denoted by A, B, C, etc.

- The 1-morphisms from object A to B are (A, B)-bimodules denoted by $M\colon A \to B$, etc. The composition of 1-morphisms is given by the tensoring of bimodules as in diagram (A.16). For a monoid object A, the identity 1-morphism $1_A\colon A \to A$ is given by the (A, A)-bimodule A whose left and right action morphisms are given by the same monoid multiplication $A \otimes A \to A$.

- The 2-morphisms between 1-morphisms of (A, B)-bimodules M and N are given by (A, B)-bimodule homomorphisms, i.e. the morphisms $f\colon M \to N$ in $\mathcal{V}$ which are equivariant with respect to actions of A and B on M and N. The vertical compositions of 2-morphisms are given simply by compositions in $\mathcal{V}$ and the horizontal compositions are given by the naturality of tensoring in the diagram (A.8)

The crucial observation is that $\mathfrak{BiMod}(\mathcal{V})$ has the structure of a genuine bicategory and not a 2-category as the tensoring of bimodules is weakly unital and weakly associative. The coherence morphisms $\alpha_{M,N,P}\colon (M \otimes_B N) \otimes_C P \cong M \otimes_B (N \otimes_C P)$, $\lambda_M\colon M \otimes_B B \cong M$ and $\rho_M\colon A \otimes_A M \cong M$ are given naturally as the canonical isomorphisms between appropriate coequalizers over the same diagram in $\mathcal{V}$.

EXAMPLE A.8.12. Consider the (symmetric) cartesian closed monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}, \times, \{\star\})$. The category $\mathcal{M}on(\mathcal{V})$ is just the category of monoids and $\mathcal{M}\mathrm{od}(\mathcal{V})$ is the category of monoid actions.

EXAMPLE A.8.13. Consider the (symmetric) monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}^{\mathrm{op}}, \times, \{\star\})$. A monoid object in $\mathcal{V}$ is just a set: the multiplication is given by $\Delta_A\colon A \to A \times A$ and the unit by the unique function $A \to \{\star\}$. The category $\mathcal{M}on(\mathcal{V})$ is just the category $\mathcal{S}\mathrm{et}$ of sets and $\mathcal{M}\mathrm{od}(\mathcal{V})$ is the comma category $(\mathcal{S}\mathrm{et} \downarrow \mathcal{S}\mathrm{et})$ and the fibration A.15 is the codomain fibration. The bicategory $\mathfrak{BiMod}(\mathcal{V})$ is the bicategory $\mathfrak{Span}(\mathcal{S}\mathrm{et})$ of spans. (cf. 1.5.7)

EXAMPLE A.8.14. Consider the (symmetric) monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}, \coprod, \{\emptyset\})$. A monoid object in $\mathcal{V}$ is just a set: the multiplication is given by $\nabla_A\colon A \coprod A \to A$ and the unit by the unique function $\emptyset \to A$. The category $\mathcal{M}on(\mathcal{V})$ is just the category $\mathcal{S}\mathrm{et}$ of sets and $\mathcal{M}\mathrm{od}(\mathcal{V})$ is the comma category $(\mathcal{S}\mathrm{et} \downarrow \mathcal{S}\mathrm{et})$ and the fibration A.15 is the domain fibration. of monoid actions. The bicategory $\mathfrak{BiMod}(\mathcal{V})$ is the bicategory $\mathfrak{opSpan}(\mathcal{S}\mathrm{et})$ of spans. (cf. 1.5.7)

Now, we will generalise the construction of internal hom of bimodules from $\mathcal{A}\mathrm{b}$ to any $\mathcal{A}\mathrm{b}$-like monoidal category. Let M be an (A, B)-bimodule, N a (B, C)-bimodule and P a right C-module. Define internal object of C-linear maps as the following equalizer in $\mathcal{V}$:

$$\begin{tikzcd}
\underline{Hom}_C(N, P) \arrow[r, tail, dashed, "e"] & {[N, P]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & {[N \otimes C, P]}
\end{tikzcd}$$

where ∂_0 and ∂_1 are morphisms in $\mathcal{V}$ whose transpose are given by

$$\begin{tikzcd}
{[N, P] \otimes N \otimes C} \arrow[rr, "\widehat{\partial_0}"] \arrow[dr, "\mathrm{eval} \otimes 1_C"'] & & P \\
& P \otimes C \arrow[ur, "p_C"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
{[N, P] \otimes N \otimes C} \arrow[rr, "\widehat{\partial_1}"] \arrow[dr, "1 \otimes n_C"'] & & P \\
& {[N, P] \otimes N} \arrow[ur, "\mathrm{eval}"'] &
\end{tikzcd}$$

we define a right B-action on $\underline{Hom}_C(N, P)$ which makes it into a right B-module. First observe that $[N, P]$ is a right B-module with action map $\alpha \colon [N, P] \otimes B \to [N, P]$ with $\widehat{\alpha} = \mathrm{eval} \circ (1_{[N,P]} \otimes n_B)$. Similarly, $[N \otimes C, P]$ is a right B-module with action map $\beta \colon [N \otimes C, P] \otimes B \to [N \otimes C, P]$ with $\widehat{\beta} = \mathrm{eval} \circ (1_{[N,P]} \otimes n_B \otimes 1_C)$. Indeed, by our assumption, operation of tensoring preserves equalizers which implies that both rows of the diagram below are equalizer diagrams and hence there exists a unique morphism $\bar{\alpha} \colon \underline{Hom}_C(N, P) \otimes B \to \underline{Hom}_C(N, P)$ which makes the left square commute:

$$\begin{tikzcd}
\underline{Hom}_C(N, P) \otimes B \arrow[r, tail, "e \otimes 1_B"] \arrow[d, dashed, "\bar{\alpha}"'] & {[N, P] \otimes B} \arrow[r, shift left, "\partial_0 \otimes 1_B"] \arrow[r, shift right, "\partial_1 \otimes 1_B"'] \arrow[d, "\alpha"'] & {[N \otimes C, P] \otimes B} \arrow[d, "\beta"] \\
\underline{Hom}_C(N, P) \arrow[r, tail, "e"] & {[N, P]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & {[N \otimes C, P]}
\end{tikzcd}$$

When $\mathcal{V} = \mathcal{A}\mathrm{b}$, $\bar{\alpha}\ (f, b)\ n = f(b \centerdot n)$, and $\bar{\alpha}$ gives $\underline{Hom}_C(N, P)$ the structure of right B-module. Moreover, one can prove

$$\mathcal{M}\mathrm{od}_C(\mathcal{V})(M \otimes_B N, P) \cong \mathcal{M}\mathrm{od}_B(\mathcal{V})(M, \underline{Hom}_C(N, P))$$

natural in A, B, C which establishes internal *Hom-tensor adjunction*

$$- \otimes_B N \dashv \underline{Hom_C}(N, -) \tag{A.17}$$

A.9 Proofs from Chapter 2

For the sake of self-sufficiency, we present the proofs of some of the statements made in § 2.3. The statements are well-known and classical.

PROOF (Example 2.3.6: cod-cartesian morphisms). Consider diagram (2.12). We need to prove that the morphism $\langle g, f \rangle \colon \gamma' \to \gamma$ sitting over f is cartesian. Suppose $\langle g', f' \rangle \colon \gamma' \to u$ with $f \circ h = f'$ for some $h \colon B'' \to B$. These equations render the following diagram commutative:

$$\begin{tikzcd}
Z \arrow[r, "g'"] \arrow[d, "\gamma''"'] & X \arrow[d, "\gamma"] \\
B'' \arrow[r, "f'"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B'' \arrow[d, "h"'] \arrow[dr, "f'"] & \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

Using the universal property of pullback diagram (2.12), we find a unique morphism $k \colon Z \to Y$ which renders both the top triangle and the left square commuting:

$$\begin{tikzcd}
Z \arrow[d, "\gamma''"'] \arrow[dr, dotted, "k"'] \arrow[drr, bend left, "g'"] & & \\
B'' \arrow[dr, "h"'] & Y \arrow[r, "g"] \arrow[d, "\gamma'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\gamma"] \\
& B' \arrow[r, "f"'] & B
\end{tikzcd}$$

The morphism $\langle k, h \rangle \colon \gamma'' \to \gamma$ is the unique lift of $h \colon B'' \to B'$ we desired. The reverse direction is just the definition of being precartesian.

PROOF (Proposition 2.3.12). *Necessity:* Suppose $(P \colon \mathcal{E} \to \mathcal{B}, \mathfrak{c})$ is a cloven prefibration and a morphism $f \colon A \to PX$ is given in $\mathcal{B}$. Let $\widetilde{f}$ be a precartesian lift of f in the cleavage. Let $u \colon Z \to X$ be any morphism and let $h \colon PZ \to A$ with

$f \circ h = Pu$. Take $\tilde{h}$ to be a precartesian lift of h in the cleavage. Since, under the assumption of proposition, precartesian morphisms are closed under composition, we know that $\tilde{f} \circ \tilde{h}$ is again precartesian. Now, since $P(\tilde{f} \circ \tilde{h}) = f \circ h = Pu$, then u factors through $\tilde{f} \circ \tilde{h}$ via a unique vertical morphism w. Define $v := \tilde{h} \circ w$. Then $\tilde{f} \circ v = u$ and $Pv = h$. This proves existence of factorisation of u through $\tilde{f}$.

$$\begin{tikzcd}
Z \arrow[d, dotted, "w"'] \arrow[drr, "u"] & & & & PZ \arrow[dr, "h"'] \arrow[drr, "P(u)"] & & \\
\mathfrak{c}_h\mathfrak{c}_f X \arrow[r, "\tilde{h}"'] & \mathfrak{c}_f X \arrow[r, "\tilde{f}"'] & X & \mapsto & & A \arrow[r, "f"'] & PX
\end{tikzcd}$$

For the uniqueness, if v' is another such morphism then $\tilde{h} \circ w' = v'$ for a unique vertical w', because we have $Pv' = P\tilde{h} = h$ and $\tilde{h}$ is precartesian. Now, $\tilde{f} \circ \tilde{h} \circ w' = u$ which implies $w' = w$ and thence $v' = v$.

Sufficiency: Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a fibration and $u\colon Y \to X$ and $u'\colon Z \to Y$ are both precartesian morphisms in $\mathcal{E}$. We want to prove their composition is again precartesian. To this end, take a morphism $r\colon W \to X$ with $Pr = f \circ g$ where $f = Pu$ and $g = Pv$. We select $\tilde{f}$ and $\tilde{g}$ as the cartesian lifts of f and g in the cleavage, respectively. By precartesianness of u, f, u', f', there are unique vertical isomorphisms v and v' such that $\tilde{f} = u \circ v$ and $\tilde{f}' = u' \circ v'$. By Proposition 2.3.3, v is a cartesian morphism and and by Lemma 2.3.5, $\tilde{f} \circ v^{-1} \circ \tilde{f}'$, which lies over $r\colon W \to X$, is cartesian. Thus, there is a unique vertical morphisms w such that $\tilde{f} \circ v^{-1} \circ \tilde{f}' \circ w = r$. Let $w' := v' \circ w$. We have

$$u \circ u' \circ w' = u \circ u' \circ v' \circ w = u \circ \tilde{f}' \circ w = \tilde{f} \circ v^{-1} \circ \tilde{f}' \circ w = r,$$

and moreover, since v' is invertible, uniqueness of w guarantees uniqueness w' satisfying equation above. Therefore, $u \circ u'$ is indeed precartesian.

$$\begin{tikzcd}
W \arrow[dd, dotted, bend right, "w"'] \arrow[drr, "r"] & & \\
Z \arrow[r, "u'"] & Y \arrow[r, "u"] & X \\
\mathfrak{c}_{f'} Y \arrow[u, dotted, "v'"] \arrow[ur, "\tilde{f}'"'] & \mathfrak{c}_f X \arrow[u, dotted, "v"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}C \arrow[r, "f'"'] & B \arrow[r, "f"'] & A\end{tikzcd}$$

PROOF (Proposition 2.3.14). We define the right adjoint S_X of P_X on objects of $\mathcal{B}/PX$ by $S_X(A \xrightarrow{f} PX) := \mathfrak{c}_f X \xrightarrow{\widetilde{f}} X$. Thanks to the universal property of $\widetilde{f}$, this extends to a functor: for a morphism g between f_0 and f_1 in $\mathcal{B}/PX$, by cartesianness of $\widetilde{f_1}$, we define $S_X(g)$ as the unique morphism in $\mathcal{E}$ which makes the left triangle in below commute.

$$\begin{tikzcd}
\mathfrak{c}_{f_0}X \arrow[rr, dashed, "S_X(g)"] \arrow[dr, "\widetilde{f_0}"'] & & \mathfrak{c}_{f_1}X \arrow[dl, "\widetilde{f_1}"] & \mapsto & A \arrow[rr, "g"] \arrow[dr, "f_0"'] & & B \arrow[dl, "f_1"] \\
& X & & & & PX &
\end{tikzcd}$$

So, indeed $S_X(g)$ is a morphisms in $\mathcal{E}/X$. The unit of adjunction $P_X \dashv S_X$ is the natural transformation $\eta_X \colon 1_{\mathcal{E}/X} \Rightarrow S_X \circ P_X$ which is defined on component $u \colon Y \to X$ as the unique vertical morphism which makes the diagram below commute.

$$\begin{tikzcd}
Y \arrow[d, dashed, "\eta_X(u)"'] \arrow[dr, "u"] & \\
S_{Pu}(X) \arrow[r, "\widetilde{Pu}"'] & X
\end{tikzcd}$$

Also, it is readily observed that the counit is identity, and the adjunction triangle identities hold.

To prove that Grothendieck fibrations are stable under pullback, we are going to use the following result which is a direct application of example 2.3.6 combined with Proposition 2.3.5.

COROLLARY A.9.1. Suppose the following cubic diagram is commutative, and moreover, the side faces corresponding to $u_0 \to u_1$ and $u_2 \to u_3$, and the front face corresponding to $u_1 \to u_3$ in $(\mathcal{C} \downarrow \mathcal{C})$ are cartesian squares. By 2.3.5, the diagonal face

$u_0 \to u_3$ is cartesian square which in turns implies that the rear square $u_0 \to u_2$ is also cartesian.

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd, "u_0"] \arrow[dr, dotted] & & F \arrow[dl] \arrow[dd, "u_2"] \\
G \arrow[rr] \arrow[dd, "u_1"'] & & H \arrow[dd, "u_3"] & \\
& A \arrow[rr, "f_0"] \arrow[dl, "g_0"'] \arrow[dr, dotted] & & B \arrow[dl, "g_1"] \\
C \arrow[rr, "f_1"'] & & D &
\end{tikzcd}$$

From this we deduce that

COROLLARY A.9.2. For a fibration $P\colon \mathcal{E} \to \mathcal{B}$, and a morphism $u\colon J \to I$ of $\mathcal{B}$, reindexing along u preserves pullbacks: it takes pullbacks in $\mathcal{E}_I$ to pullbacks in $\mathcal{E}_J$.

PROOF (Proposition 2.3.16). We first prove that fibrations are closed under composition. Let $\langle Q, \mathfrak{c}' \rangle\colon \mathcal{F} \to \mathcal{E}$ and $\langle P, \mathfrak{c} \rangle\colon \mathcal{E} \to \mathcal{B}$ be cloven fibrations. Assume an object Y in $\mathcal{F}$ and a morphism $f\colon A \to PQ(Y)$ in $\mathcal{B}$.

$$\begin{tikzcd}
\mathfrak{c}'_{\tilde{f}}(Y) \arrow[r, dashed, "\tilde{\tilde{f}}"] & Y & \mathcal{F} \arrow[d, "Q"] \\
\mathfrak{c}_f(QY) \arrow[r, dashed, "\tilde{f}"] & QY & \mathcal{E} \arrow[d, "P"] \\
A \arrow[r, "f"] & PQY & \mathcal{B}
\end{tikzcd}$$

Cleavage $\mathfrak{c}$ In the diagram above, $\tilde{f}$ is a lift of f with codomain QY in $\mathfrak{c}$, and $\tilde{\tilde{f}}$ is a lift of $\tilde{f}$ with codomain Y in $\mathfrak{c}'$. We now show that the morphism $\tilde{\tilde{f}}$ is $P \circ Q$-cartesian. Because $\tilde{f}$ and $\tilde{\tilde{f}}$ are cartesian morphisms, Proposition 2.3.3 implies that for every

Z in $\mathcal{F}$, the left and right commutative squares in below are pullbacks. By pasting them, we have the outer commuting rectangle as a pullback, for each Z in $\mathcal{F}$.

$$\begin{tikzcd}
\mathcal{F}(Z, \mathfrak{c}'_{\tilde{f}}(Y)) \arrow[r, "Q"] \arrow[d, "\tilde{\tilde{f}}\circ-"'] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{E}(QZ, \mathfrak{c}_f(QY)) \arrow[r, "P"] \arrow[d, "\widetilde{f}\circ-"] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{B}(PQZ, A) \arrow[d, "f\circ-"] \\
\mathcal{F}(Z, Y) \arrow[r, "Q"'] & \mathcal{E}(QZ, QY) \arrow[r, "P"'] & \mathcal{B}(PQZ, PQY)
\end{tikzcd} \tag{A.18}$$

So, we can take the $\mathfrak{c}'$ of Q to be also the cleavage for $P \circ Q$ and with this choice of cleavge $P \circ Q$ becomes a cloven fibration.

Next, we prove that fibrations are closed under pullback. Consider a (strict) pullback diagram in $\mathfrak{Cat}$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "L"] \arrow[d, "Q"'] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd} \tag{A.19}$$

where P is a Grothendieck fibration. we want to show that Q is a Grothendieck fibration as well. Let $g\colon C \to QY$ be a morphism in $\mathcal{C}$. So, $F(g)\colon F(C) \to PL(Y)$, and it has a cartesian lift $\widetilde{F(g)}\colon X \to L(Y)$ in $\mathcal{E}$. Now, since $P(\widetilde{F(g)}) = F(g)$, we obtain a unique morphism $\tilde{g}\colon W \to Y$ in $\mathcal{F}$ with $Q(\tilde{g}) = g$ and $L(\tilde{g}) = \widetilde{F(g)}$. In particular, $L(W) = X$ and $Q(W) = C$. It remains to show that $\tilde{g}$ is cartesian. For every Z in $\mathcal{F}$, we form the commutative cube below.

$$\begin{tikzcd}
& \mathcal{F}(Z, W) \arrow[rr, "\tilde{g}\circ-"] \arrow[dl, "L_{Z,W}"'] \arrow[dd] & & \mathcal{F}(Z, Y) \arrow[dl, "Q_{Z,Y}"] \arrow[dd, "Q_{Z,Y}"] \\
\mathcal{E}(LZ, LW) \arrow[rr] \arrow[dd, "P_{LZ,LW}"'] & & \mathcal{E}(LZ, LY) \arrow[dd] & \\
& \mathcal{C}(QZ, QW) \arrow[rr, "g\circ-"] \arrow[dl] & & \mathcal{C}(QZ, QY) \arrow[dl, "F_{QZ,QY}"] \\
\mathcal{B}(FQZ, FQW) \arrow[rr, "F(g)\circ-"'] & & \mathcal{B}(PLZ, PLY) &
\end{tikzcd}$$

The left and right faces are cartesian squares of sets since the diagram A.19 is a pullback square. The front face is also a cartesian square since P is a fibration. Hence, the back face is also cartesian by A.9.1 and this implies that Q is a Grothendieck fibration.

PROOF (Proposition 2.3.32). To prove this, take any morphism $(i,f):(V,B)\to(U,A)$ in $\mathbb{P}\rtimes\mathcal{B}$. We show that is is cartesian. Take any morphism $(k,g):(W,C)\to(U,A)$ in $\mathbb{P}\rtimes\mathcal{B}$ with $i\circ j=k$ in $\mathcal{B}$. Now since $\mathbb{P}$ takes values in $\mathfrak{Grpd}$, f and g are isomorphisms and we define $h\colon C\to j^*B$ as $h:=j^*(f)^{-1}\circ\phi_{i,j}(A)^{-1}\circ g$. Obviously h is an isomorphism and (j,h) is the unique morphism in $\mathbb{P}\rtimes\mathcal{B}$ which lies over j and makes the upper triangle (in diagram below) commute .

$$\begin{tikzcd}
(W,C) \arrow[d, mapsto] \arrow[dr, dotted, "{(j,h)}"'] \arrow[drr, "{(k,g)}"] & & \\
W \arrow[dr, "j"'] \arrow[drr, "k"] & (V,B) \arrow[r, "{(i,f)}"'] \arrow[d, mapsto] & (U,A) \arrow[d, mapsto] \\
& V \arrow[r, "i"'] & U
\end{tikzcd}\;.$$

A.10 Pseudo Algebras and KZ-monads

DEFINITION A.10.1. Let $\mathfrak{K}$ be a 2-category and $(T\colon\mathfrak{K}\to\mathfrak{K}, i\colon 1\Rightarrow T, m\colon T^2\Rightarrow T)$ a strict 2-monad on $\mathfrak{K}$. A **pseudo algebra** of T consists of

(PSDALG1) an object A in $\mathfrak{K}$,

(PSDALG2) a morphism $\mathfrak{a}\colon TA\to A$ which we call the *structure map*, and

(PSDALG3) and invertible 2-morphisms $\zeta\colon 1_A\cong\mathfrak{a}\circ i_A$ and $\theta\colon\mathfrak{a}\circ T\mathfrak{a}\cong\mathfrak{a}\circ m_A$

$$\begin{tikzcd}
A \arrow[d, "i_A"'] \arrow[dr, "1"] \arrow[dr, phantom, "\Swarrow\zeta"', near end] & \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{A.20}$$

subject to the following compatibility axiom:

- *Weak associativity* expressed by the equality of pasting diagrams[7]

$$\begin{tikzcd}
T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "m_{TA}"'] & T^2A \arrow[d, "m_A"] \arrow[dr, bend left, "T\mathfrak{a}"] & \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] & TA \arrow[l, phantom, "\overset{\theta}{\Leftarrow}"] \arrow[dl, bend left, "a"] \\
TA \arrow[r, "\mathfrak{a}"'] & A &
\end{tikzcd}
=
\begin{tikzcd}
 & T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "Tm_A"'] \arrow[dl, bend right, "m_{TA}"'] \arrow[dr, phantom, "T\theta\Downarrow"] & T^2A \arrow[d, "T\mathfrak{a}"] \\
T^2A \arrow[dr, bend right, "m_A"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
 & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{A.21}$$

In equations, that is

$$(\theta \centerdot m_{TA}) \circ (\theta \centerdot T^2\mathfrak{a}) = (\theta \centerdot Tm_A) \circ (\mathfrak{a} \centerdot T\theta)$$

- *Weak unicity* expressed by the equality of pasting diagrams:

$$\begin{tikzcd}
TA \arrow[r, "1_{TA}"] \arrow[d, "Ti_A"'] \arrow[dd, bend right=60, "1_{TA}"'] \arrow[dr, phantom, "T.\zeta\Downarrow"] & TA \arrow[d, "1_{TA}"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"'] \arrow[dd, bend right=60, "1_{TA}"'] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\underset{\zeta}{\Leftarrow}"] & {} \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{A.22}$$

In equation, that is

$$(\theta \centerdot Ti_A) \circ (\mathfrak{a} \centerdot T\zeta) = \mathrm{id}_{\mathfrak{a}} = (\theta \centerdot i_{TA}) \circ (\zeta \centerdot \mathfrak{a})$$

DEFINITION A.10.2. We call pseudo algebra $\mathfrak{a}$ **splitting** whenever θ is identity and **normal** when ζ is identity.

Trivially, $(m_A, \mathrm{id}, \mathrm{id})$ is a splitting normal (aka strict) algebra.

---

[7] Note that the top left square on the LHS commutes due to the naturality condition of m, and the most left semi-circle on the RHS commutes due to the monad law.

Definition A.10.3. Suppose $(\mathfrak{a}, \zeta_A, \theta_A)\colon TA \to A$ and $(\mathfrak{b}, \zeta_B, \theta_B)\colon TB \to B$ are pseudo algebras of a 2-monad $T\colon \mathfrak{K} \to \mathfrak{K}$. A **lax morphism** from $\mathfrak{a}$ to $\mathfrak{b}$ consists of a morphism $f\colon A \to B$ and a 2-morphism

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

subject to two compatibility conditions:

(L1) First, that ζ commutes with $\check{f}$, expressed by

$$\begin{tikzcd}
 & A \arrow[dl, "i_A"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[d, "\mathfrak{a}"'] \arrow[r, phantom, "\zeta\Uparrow"] & {} \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[rr, "f"] \arrow[dl, "i_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[rr, "Tf"'] \arrow[d, "\mathfrak{a}"'] \arrow[drr, phantom, "\check{f}\Downarrow"] & & TB \arrow[d, "\mathfrak{b}"'] \arrow[r, phantom, "\zeta\Uparrow"] & {} \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

that is

$$f \centerdot \zeta_A = (\check{f} \centerdot i_A) \circ (\zeta_B \centerdot f)$$

(L2) And, second, θ commutes with $\check{f}$ expressed by

$$\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dl, "T\mathfrak{a}"'] \arrow[dd, "Tf"] \arrow[drr, phantom, "\Swarrow T\check{f}"] & & T^2B \arrow[dl, "T\mathfrak{b}"] \\
TA \arrow[rr, "Tf"] \arrow[dd, "\mathfrak{a}"'] \arrow[dr, phantom, "\theta\Downarrow"] & & TB \arrow[dd, "\mathfrak{b}"] & \\
 & TA \arrow[dl, "\mathfrak{a}"'] \arrow[r, phantom, "\check{f}\Downarrow"] & {} & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dd, "m_A"'] & & T^2B \arrow[dl, "T\mathfrak{b}"'] \arrow[dd, "m_B"] \\
 & & TB \arrow[dd] \arrow[r, phantom, "\theta\Downarrow"] & {} \\
 & TA \arrow[rr, "Tf"] \arrow[dl, "\mathfrak{a}"'] \arrow[d, phantom, "\Swarrow \check{f}"] & & TB \arrow[dl, "\mathfrak{b}"] \\
A \arrow[rr, "f"'] & {} & B &
\end{tikzcd}$$

that is

$$(f \centerdot \theta_A) \circ (\check{f} \centerdot T\mathfrak{a}) \circ (\mathfrak{b} \centerdot T\check{f}) = (\check{f} \centerdot m_A) \circ (\theta_B \centerdot T^2 f)$$

REMARK A.10.4. A **colax T-morphism** is a lax T^{co}-morphism where $T^{\mathrm{co}} \colon \mathfrak{K}^{\mathrm{co}} \to \mathfrak{K}^{\mathrm{co}}$.

REMARK A.10.5. Lax morphisms of algebras (resp. pseudo algebras) of a 2-monad T can themselves be realized as algebras (resp. pseudo algebras) of 2-monad $\hat{T} := [\mathbf{2}, T]_l$ on the 2-category $[\mathbf{2}, \mathfrak{K}]_l$, the latter constructed as the cylinder 2-category $\mathbf{cyl}(\mathfrak{K})$ of Construction 1.4.12 (where 1-morphisms are given by lax squares).

$$\begin{tikzcd}
p \arrow[r, "l(\alpha\bullet k)"] \arrow[d, "\cong"'] \arrow[dd, bend right=70, "gk"'] \arrow[dd, phantom, shift right=6ex, "\cong"] \arrow[dr, phantom, "\cong"] & p \arrow[d, "\cong"] \arrow[dd, bend left=70, "fk"] \arrow[dd, phantom, shift left=6ex, "\cong"] \\
p \arrow[r, "l(\alpha)"'] \arrow[d, "gk_g"'] \arrow[dr, phantom, "\Downarrow \alpha\bullet\kappa"] & p \arrow[d, "fk_f"] \\
p \arrow[r, equal] & p
\end{tikzcd}
=
\begin{tikzcd}
p \arrow[r, "l(\alpha\bullet k)"] \arrow[d, "gk"'] \arrow[dr, phantom, "\Downarrow \alpha\bullet k"] & p \arrow[d, "fk"] \\
p \arrow[r, equal] & p
\end{tikzcd}$$

Similarly we obtain colax morphisms of algebras as algebras of a similar monad $\check{T}$ on the 2-category ${[\mathbf{2}, \mathfrak{K}]_l}^{\mathrm{op}}$.

PROOF (Lemma 2.4.9). We calculate the composite 2-cell

$$\begin{tikzcd}
TA \arrow[r, bend left=40, "i_{TA}"] \arrow[r, bend right=40, "Ti_A"'] \arrow[r, phantom, "\Downarrow \lambda_A"] & T^2A \arrow[r, bend left=40, "\mathfrak{a} \circ T\mathfrak{a}"] \arrow[r, bend right=40, "\mathfrak{a} \circ m_A"'] \arrow[r, phantom, "\Downarrow \theta"] & TA
\end{tikzcd}$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$\begin{tikzcd}
TA \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"] & A \arrow[d, "i_A"] \arrow[dd, bend left=70, "1_A"] \\
TA \arrow[r, "Ti_A"] \arrow[d, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\overset{\Leftarrow}{\zeta}"] & {} \\
TA \arrow[r, "1"'] & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}$$

$$\theta \mathbin{\scriptscriptstyle\blacksquare} \lambda_A = \zeta^{-1} \mathbin{\scriptscriptstyle\blacksquare} \mathfrak{a}$$

On the other hand, we can compose row-wise instead, and we get

$$\theta \mathbin{\scriptscriptstyle\blacksquare} \lambda_A = (\theta \mathbin{\scriptscriptstyle\blacksquare} Ti_A) \circ (\mathfrak{a} \circ T\mathfrak{a} \mathbin{\scriptscriptstyle\blacksquare} \lambda_A) = (\mathfrak{a} \mathbin{\scriptscriptstyle\blacksquare} T\zeta^{-1}) \circ (\mathfrak{a} \circ T\mathfrak{a} \mathbin{\scriptscriptstyle\blacksquare} \lambda_A)$$

Thus, in the end, we have

$$\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}", ""{name=U, below}] \arrow[r, bend right, "Ti_A"', ""{name=D, above}] \arrow[rr, bend right=60, "1"'] \arrow[from=U, to=D, Rightarrow, "\lambda_A"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] & A \\
 & {} & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & TA \arrow[dr, "\mathfrak{a}"] \arrow[d, phantom, "\zeta^{-1}\Downarrow"] & \\
TA \arrow[r, "\mathfrak{a}"] & A \arrow[ur, "i_A"] \arrow[rr, bend right, "1"'] & {} & A
\end{tikzcd}
\tag{A.23}$$

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