LOCALES ARE NOT POINTLESS

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ABSTRACT

The Kripke-Joyal semantics is used to interpret the fragment of intuitionistic logic containing $\wedge, \rightarrow$ and $\forall$ in the category of locales. An axiomatic theory is developed that can be interpreted soundly in two ways, using either lower or upper powerlocales, so that pairs of separate results can be proved as single formal theorems. Openness and properness of maps between locales are characterized by descriptions using the logic, and it is proved that a locale is open iff its lower powerlocale has a greatest point. The entire account is constructive and holds for locales over any topos.

1 Introduction

"Topology without points" is the clarion call of locale theory, yet it is usually hard to develop locale-theoretic results without some conception of what the points ought to be, if only we were allowed to see them. Indeed one often has the feeling that a frame-theoretic proof is no more than a mask for a natural argument using points.

To be sure, the *global* points of a locale D, *i.e.,* maps from 1 to D, are in general insufficient. However, if one considers generalized points in the form of maps to D from an arbitrary locale then one can properly use descriptions in terms of points. For instance, the universal characterization of a product $D \times E$ can be stated as "the points of $D \times E$ are the pairs (x, y) where x and y are points of D and E respectively".

The aim of this paper is to present some illustrative results that indicate a framework for using points to reason about locales. It brings together three main features.

First, we use the Kripke-Joyal semantics to support an interpretation of a logic with $\wedge, \rightarrow$ and $\forall$.

Second, we accept as fundamental the order enrichment in the category of **Loc** locales, so the logic includes the specialization order $\sqsubseteq$ as well as equality.

Third, we use the lower and upper powerlocales P_L and P_U to access within **Loc** functions between frames that are more general than the frame homomorphisms — namely, homomorphisms of suplattices and preframes respectively.

Our main new results here, such as they are, concern open and proper maps: we show, for example, that a locale map $f : D \to E$ is open iff $P_L f$ has a right adjoint that makes a certain diagram commute, and is proper iff $P_U f$ has a left adjoint that makes a corresponding diagram (replacing P_L by P_U and making other changes) commute. Our axiomatic approach puts these results into a common abstract context of order-enriched category with powerobject monad and there supports a reasoning style using points.

We shall take as a standing assumption for all our axiomatics that we are given a poset-enriched category $\mathcal{C}$, with order $\sqsubseteq$. The primitive paradigm for $\mathcal{C}$ is **Pos**, the category of posets, with monotone functions as morphisms, ordered pointwise; but later we shall also examine **Loc** and $\mathbf{Loc}^{co}$.

We shall also assume that $\mathcal{C}$ has finite limits in a 2-categorical sense: it has products and inserters (see Power and Robinson [15]).

Products must respect the order, which makes them slightly stronger than ordinary categorical products. For instance for a binary product $D_1 \times D_2$ with projections fst and snd, we have for any other object E that the canonical function from $\mathcal{C}(E, D_1 \times D_2)$ to $\mathcal{C}(E, D_1) \times \mathcal{C}(E, D_2)$ is an *order-isomorphism* (not just a bijection).

An inserter is a 2-categorical analogue of an equalizer: it's a universal solution to the problem of ordering two parallel morphisms. If $f, g : D \to E$, then the inserter $i : I \to D$ has $i; f \sqsubseteq i; g$ and is such that for any object F, the canonical function from $\mathcal{C}(F, I)$ to $\{i' \in \mathcal{C}(F, D) : i'; f \sqsubseteq i'; g\}$ is an order isomorphism.

In **Pos**, products are Cartesian products and the inserter of $f, g : P \to Q$ is $\{p \in P : f(p) \sqsubseteq g(p)\}$ with the order inherited from P.

1.1 Constructivity

Our standing references for locale theory are Johnstone [8] and Vickers [17] (we shall follow the notation of the latter). However, we take much more care to argue constructively so that our results hold for locales over any topos. Indeed, the results on open locales (see Theorem 4.9) are of little interest otherwise, for classically all locales are open. Such issues are dealt with in Joyal and Tierney [12].

2 Outline of Point Logic

We outline the Kripke-Joyal semantics, which is sound for intuitionistic logic. However, the interpretation of some of the connectives relies on fixing a notion of "cover" and it is not yet obvious to me what this should be in the category of locales. Hence, for the purpose of this paper, the important part is the fragment of intuitionistic logic whose connectives are $\wedge$, $\rightarrow$ and $\forall$ ($=$ and $\sqsubseteq$ are treated as extralogical symbols), since the semantics of these is independent of the cover. We continue to work in an unspecified category $\mathcal{C}$.

A *type* is to be interpreted as an object of $\mathcal{C}$. A *term* of type D is to be interpreted as a morphism targeted at D, which we shall call a *point* of D; its source — corresponding to the free variables in the term — is its *stage of definition*. A formula with free variables of type D (using a product to deal with more than one free variable) is to be interpreted as a *sieve* (or *crible*) at D, that is to say a family P of morphisms targeted at D and closed under precomposition — if $x \in P$ and $\alpha; x$ is defined, then $\alpha; x \in P$. If $x : \sigma \rightarrow D$ is a point of D, and P is a sieve at D, then $P(x)$ (or more formally, $\sigma \Vdash P(x)$) is to mean that $x \in P$. It remains to interpret the logical connectives as operations on sieves.

$\wedge$: Conjunction is intersection of sieves.

$\rightarrow$: If $x : \sigma \rightarrow D$ is a point of D, then let us write $\bullet x$ for the sieve at D generated by x. Then $x \in P \rightarrow Q$ iff $\bullet x \subseteq P \rightarrow Q$, which must be iff $\bullet x \cap P \subseteq Q$ (to ensure that intuitionistic logic is interpreted soundly). Hence

$$x \in P \rightarrow Q \text{ iff for all } \alpha : \tau \rightarrow \sigma, \text{ if } \alpha; x \in P \text{ then } \alpha; x \in Q.$$

This is often expressed as follows:

$$\sigma \Vdash (P \rightarrow Q)(x) \text{ iff for all } \alpha : \tau \rightarrow \sigma, \text{ if } \tau \Vdash P(\alpha; x) \text{ then } \tau \Vdash Q(\alpha; x).$$

$\forall$: Let $f : D \rightarrow E$ be a map. If Q is a sieve at E, then $f^*Q = \{x : x; f \in Q\}$ defines the sieve operation that corresponds to substitution — f^*Q interprets $Q(fx)$. We write $\forall_f$ for the right adjoint for f^*. Then $y \in \forall_f P \iff \bullet y \subseteq \forall_f P \iff f^*(\bullet y) \subseteq P$. If $y : \tau \rightarrow E$, then this condition says that whenever we have $\alpha : \sigma \rightarrow \tau$ and $x : \sigma \rightarrow D$ with $x; f = \alpha; y$, then $x \in P$.

The familiar application to logic is obtained when f is a product projection, say $f : D \times E \rightarrow E$. $(f^*Q)(x, y)$ then represents $Q(y)$ with an unused free variable x, and $(\forall_f P)(y)$ is $\forall x.P(x, y)$. We have $\tau \Vdash \forall x.P(x, y)$ iff for every $\alpha : \sigma \to \tau$ and for every $x : \sigma \to D$ we have $\sigma \Vdash P(x, \alpha; y)$.

If $P(x)$ and $Q(x)$ are formulae in x of type D (denoting sieves at D), let us write $P(x) \models_{x:D} Q(x)$ iff the sieve for $P(x)$ is included in that for $Q(x)$; we shall also write $\models_{x:D} Q(x)$ iff the sieve for $Q(x)$ comprises all morphisms targeted at D. It is immediate from the adjunctions that —

- $P(x) \models_{x:D} Q(x) \wedge Q'(x)$ iff $P(x) \models_{x:D} Q(x)$ and $P(x) \models_{x:D} Q'(x)$
- $P(x) \models_{x:D} Q(x) \to R(x)$ iff $P(x) \wedge Q(x) \models_{x:D} R(x)$
- $P(y) \models_{y:E} \forall x : D.Q(x, y)$ iff $P(y) \models_{x:D,y:E} Q(x, y)$

2.1 Representable Sieves

A sieve is *representable* iff it is of the form $\bullet x$ for some morphism x — x can be thought of as a generic point for the property corresponding to the sieve. The semantic concept of representability blurs the syntactic distinction between terms and predicates: every point represents a sieve, and some sieves are represented by points. The use of representatives (generic points) simplifies the reasoning, since $\bullet x \subseteq P$ iff $x \in P$. It also takes the sting out of the fact that since **Loc** is large, the sieves are proper classes.

We note that —

- If x and y are points of D, then $\bullet x \wedge \bullet y$ is represented by the pullback of x and y.
- If x is a point of D, and $f : E \to D$, then $f^*(\bullet x)$ is represented by the pullback $f^* x$.
- The equality predicate on D, a sieve at D^2, is represented by the diagonal $\Delta : D \to D^2$.
- The inequality predicate $\sqsubseteq$ on D, a sieve at D^2, is represented by the inserter for the two projections from D^2 to D.

3 The Abstract 2-Categorical Axioms

3.1 The Powerobjects

Axiom 3.1 *$\mathcal{C}$ is equipped with a KZ monad $\mathcal{L}$ with a unit $\downarrow$ and multiplication $\bigsqcup$.*

When $\mathcal{C}$ is **Pos**, $\mathcal{L}P$ is the set of lower closed subsets of P, ordered by $\subseteq$. $\downarrow$: $P \to \mathcal{L}P$ maps x to the principal ideal $\downarrow x$, and $\bigsqcup : \mathcal{L}\mathcal{L}P \to \mathcal{L}P$ is union. (Strictly speaking one should write "$\downarrow_P$" and "$\bigsqcup_P$" for "$\downarrow$" and "$\bigsqcup$" here, but I shall frequently omit such subscripts when there is no risk of confusion.) "KZ" means that $\bigsqcup$ is left adjoint to $\downarrow_{\mathcal{L}P}$, and in fact it follows that for any $\mathcal{L}$-algebra P, the structure map from $\mathcal{L}$ to P is left adjoint to $\downarrow_P$ and hence is uniquely determined by P. It also follows that $\bigsqcup$ is *right* adjoint to $\mathcal{L}\downarrow_P$. (See Kock [13].)

Definition 3.2 *If $x : D$ and $Y : \mathcal{L}D$ (*i.e.*, for some stage σ we have morphisms $x : \sigma \to D$ and $Y : \sigma \to \mathcal{L}D$ in $\mathcal{C}$) then we write $x \in Y$ iff $\downarrow x \sqsubseteq X$.*

In **Pos**, $\in$ is just elementwise elementhood.

Axiom 3.3 *For every D,*

$\models \forall x, y : D.(x \in\downarrow y \leftrightarrow x \sqsubseteq y)$

It is not hard to see that this means the components $\downarrow$ of the unit of the monad are monic in a 2-categorical sense: that $x;\downarrow\sqsubseteq y;\downarrow$ implies $x \sqsubseteq y$.

Axiom 3.4 *If $f : D \to E$ is a morphism, then*

$\models \forall X : \mathcal{L}D.\forall Y : \mathcal{L}E.(\mathcal{L}f(X) \sqsubseteq Y \leftrightarrow \forall x : D.(x \in X \to f(x) \in Y))$

Let us make the meaning of this more explicit. By the remarks of Section 2, we find that this axiom is equivalent to the conjunction of two others:

$$\mathcal{L}f(X) \sqsubseteq Y \wedge x \in X \models_{X:\mathcal{L}D, Y:\mathcal{L}E, x:X} f(x) \in Y$$
$$\forall x : D.(x \in X \to f(x) \in Y) \models_{X:\mathcal{L}D, Y:\mathcal{L}E} \mathcal{L}f(X) \sqsubseteq Y$$

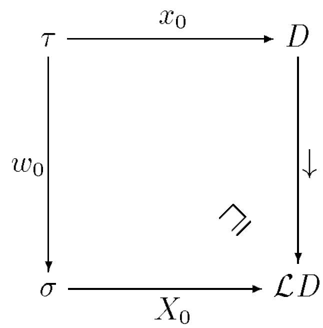
The first of these holds automatically: for, given the premises, we have $\downarrow f(x) = \mathcal{L}f(\downarrow x) \sqsubseteq \mathcal{L}f(X) \sqsubseteq Y$. However, the second takes a little unraveling. Let $\langle X_0, Y_0\rangle : \sigma \to \mathcal{L}D \times \mathcal{L}E$. Then $\langle X_0, Y_0\rangle$ is in the sieve for $\forall x : D.(x \in X \to f(x) \in Y)$ iff id_σ is in $\langle X_0, Y_0\rangle^*(\forall x : D.(x \in X \to f(x) \in Y))$, *i.e.*,

$$\models_{w:\sigma} \forall x : D.(x \in X_0(w) \to f(x) \in Y_0(w))$$

i.e.,

$$x \in X_0(w) \models_{w:\sigma, x:D} f(x) \in Y_0(w)$$

Now the sieve for $x \in X_0(w)$ (at $D \times \sigma$) is represented by $\langle x_0, w_0\rangle$ in the comma square



Hence our condition $x \in X_0(w) \models_{w:\sigma,x:D} f(x) \in Y_0(w)$ holds iff $x_0; f; \downarrow \sqsubseteq w_0; Y_0$, and our axiom requires that this holds exactly when $X_0; \mathcal{L}f \sqsubseteq Y_0$.

This is a non-trivial property of $\mathcal{C}$, though it is easy to prove for **Pos**. In the context of locales, it is harder and requires sharp application of Johnstone's coverage theorem and its preframe analogue. Note also that if points are interpreted as *global* points, *i.e.,* maps from 1, then it is patently false: take $D = E$ (with f the identity) to be a non-trivial locale with no global points at all. Classically the global points of $P_L D$ are in order-reversing bijection with the opens of D, so there are distinct points X and Y of $P_L D$ that cannot be distinguished by global points of D.

Axiom 3.5 *Let D and E be objects of $\mathcal{C}$. Then there is a morphism $\times : \mathcal{L}D \times \mathcal{L}E \to \mathcal{L}(D \times E)$ such that*

$$\models \forall x : D. \forall y : E. \forall X : \mathcal{L}D. \forall Y : \mathcal{L}E. ((x, y) \in X \times Y \leftrightarrow x \in X \wedge y \in Y)$$

(We write $X \times Y$ for $\times(X, Y)$.)
For **Pos**, $\times$ is just Cartesian product.

The following result, in addition to putting the condition of Axiom 3.5 in categorical form, also shows that it characterizes $\times$ uniquely.

Proposition 3.6 *Axiom 3.5 is equivalent to $\mathcal{L}$ preserving finite products "up to right adjoint", in other words that $\langle \mathcal{L}\mathrm{fst}, \mathcal{L}\mathrm{snd} \rangle : \mathcal{L}(D \times E) \to \mathcal{L}D \times \mathcal{L}E$ always has a right adjoint, which is $\times$.*

Proof There are two inequalities for the adjunction: $\times; \langle \mathcal{L}\mathrm{fst}, \mathcal{L}\mathrm{snd} \rangle \sqsubseteq \mathrm{id}$ and $\langle \mathcal{L}\mathrm{fst}, \mathcal{L}\mathrm{snd} \rangle; \times \sqsupseteq \mathrm{id}$. Consider the first. It is equivalent to $\times; \mathcal{L}\mathrm{fst} \sqsubseteq \mathrm{fst}$ with a similar inequality for snd, in other words for all tuples of points $(X : \mathcal{L}D, Y : \mathcal{L}E)$ we want $\mathcal{L}\mathrm{fst}(X \times Y) \sqsubseteq X$ and $\mathcal{L}\mathrm{snd}(X \times Y) \sqsubseteq Y$. By Axiom 3.4 (with fst or snd for f), this is equivalent to the $\to$ half of Axiom 3.5.

Now consider the second inequality. Again by Axiom 3.4, but this time with id_D for f, we find the inequation is equivalent to

$$(x, y) \in Z \models_{x:D, y:E, Z:\mathcal{L}(D \times E)} (x, y) \in \mathcal{L}\mathrm{fst}(Z) \times \mathcal{L}\mathrm{snd}(Z) \qquad (*)$$

Suppose we do have Axiom 3.5. Using Axiom 3.4 we have that

$$(x, y) \in Z \models_{x:D,y:E,Z:\mathcal{L}(D\times E)} x \in \mathcal{L}\mathrm{fst}(Z) \wedge y \in \mathcal{L}\mathrm{snd}(Z)$$

and combining this with Axiom 3.5 gives us $(*)$. Conversely, if we have $(*)$ and $x \in X \wedge y \in Y$, then putting $Z = \downarrow (x, y)$ in $(*)$ tells us that

$$\begin{aligned}(x, y) \in \mathcal{L}\mathrm{fst}(\downarrow (x, y)) \times \mathcal{L}\mathrm{snd}(\downarrow (x, y)) &= \downarrow x \times \downarrow y \text{ (by naturality of } \downarrow) \\ &\sqsubseteq X \times Y\end{aligned}$$

□

This binary case clearly extends to finite products of more than two objects, but let us note also the nullary case, which can be proved from the KZ structure.

Proposition 3.7 *The right adjoint of* $! : \mathcal{L}1 \to 1$ *is* $\downarrow_1$.

Proof 1 is an $\mathcal{L}$-algebra, and by the KZ property its structure map (necessarily !) is left adjoint to $\downarrow_1$. □

Definition 3.8 *Let* $f : D \to E$ *be a morphism in* $\mathcal{C}$. *Then* f *is* semi-upper *iff there is a morphism* $f^{-1} : \mathcal{L}E \to \mathcal{L}D$ *satisfying*

$$\models \forall x : D. \forall Y : \mathcal{L}E.(x \in f^{-1}(Y) \leftrightarrow f(x) \in Y)$$

Again, the following result shows that the condition of Definition 3.8 characterizes f^{-1} uniquely.

Proposition 3.9 $f^{-1} : \mathcal{L}E \to \mathcal{L}D$ *satisfies the given condition in Definition 3.8 iff it is right adjoint to* $\mathcal{L}f : \mathcal{L}D \to \mathcal{L}E$.

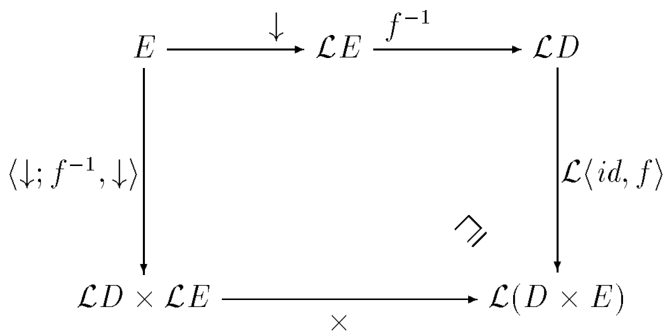
Proof As in 3.6, consider the inequalities for the adjunction: $f^{-1}; \mathcal{L}f \sqsubseteq \mathrm{id}$ and $\mathrm{id} \sqsubseteq \mathcal{L}f; f^{-1}$. By 3.4, the first is equivalent to the $\to$ direction in 3.8 and the second is equivalent to

$$x \in X \models_{x:D,X:\mathcal{L}D} x \in f^{-1}(\mathcal{L}f(X)) \qquad (*)$$

We get from 3.8 to $(*)$ by putting $Y = \mathcal{L}f(X)$ in 3.8, and the converse by putting $X = \downarrow x$ in $(*)$. □

In **Pos**, as it happens, *every* morphism is a semi-upper. In **Loc**, the semi-upper maps are those for which Ωf has a left adjoint $\exists_f$, while in $\mathbf{Loc}^{co}$ they are the perfect maps (those for which the right adjoint of Ωf preserves directed joins).

Definition 3.10 *Let* $f : D \to E$ *in* $\mathcal{C}$ *be a semi-upper. Then* f *is* upper *iff the following diagram commutes:*



The inequality shown exists in any case: for it is equivalent to

$$\mathcal{L}\langle \mathrm{id}, f\rangle(f^{-1}(\downarrow y)) \sqsubseteq f^{-1}(\downarrow y)\times \downarrow y$$

and by 3.4 this is equivalent to

$$x \in f^{-1}(\downarrow y) \models_{x:D,y:E} (x, f(x)) \in f^{-1}(\downarrow y)\times \downarrow y$$

which, in the light of 3.3, 3.5 and 3.8, is obvious.

Hence, the upperness amounts to the following condition, which expresses the opposite inequality:

$$f(x) \sqsubseteq y \wedge y' \sqsubseteq y \models_{x:D,y,y':E} (x, y') \in \mathcal{L}\langle \mathrm{id}, f\rangle(f^{-1}(\downarrow y))$$

The critical case is when $y' = y$, so the condition is equivalent to

$$f(x) \sqsubseteq y \models_{x:D,y:E} (x, y) \in \mathcal{L}\langle \mathrm{id}, f\rangle(f^{-1}(\downarrow y)) \qquad (*)$$

Let us examine this condition in **Pos**. It says that if $f(x) \sqsubseteq y$, then there is some $x' : D$ with $f(x') \sqsubseteq y$ and $(x, y) \sqsubseteq (x', f(x'))$. Hence $y = f(x')$ for some $x' \sqsupseteq x$. In other words for all x, f maps $\uparrow x$ onto $\uparrow f(x)$, so the direct image of any upper closed set of points is upper closed. This explains the terminology.

Definition 3.11 *An object D is* upper *iff $! : D \to 1$ is upper.*

Proposition 3.12 *An object D is upper iff $! : D \to 1$ is semi-upper.*

Proof We have our inverse image morphism $!^{-1} : \mathcal{L}1 \to \mathcal{L}D$ with $x \in !^{-1}(Y) \iff ! \in Y$, and we must show that the equation $(*)$ under 3.10 holds. y and $f(x)$ must be the unique point $!$ of 1, so we must show

$$\models_{x:D} (x, !) \in \mathcal{L}\langle \mathrm{id}, !\rangle(!^{-1}(\downarrow !))$$

Since $! \in \downarrow !$ we know that $x \in !^{-1}(\downarrow !)$, and it follows that

$$(x, !) = \langle \mathrm{id}, !\rangle(x) \in \mathcal{L}\langle \mathrm{id}, !\rangle(!^{-1}(\downarrow !))$$

□

Theorem 3.13 *An object D is upper iff $\mathcal{L}D$ has a greatest point,* i.e., $T : 1 \to \mathcal{L}D$ *satisfying*

$$\models_{x:D} x \in T$$

(It's not hard to see that this condition on T is equivalent to its being right adjoint to $! : \mathcal{L}D \to 1$.)

Proof

$\Rightarrow$: $T = \downarrow; !^{-1} : 1 \to \mathcal{L}D$. For any $x : D$ we have $x \in !^{-1}(\downarrow !)$ iff $! \in \downarrow !$, which is true.

$\Leftarrow$: Suppose $! : \mathcal{L}D \to 1$ as a right adjoint T. Define

$$!^{-1} = (\mathcal{L}T; \bigsqcup) : \mathcal{L}1 \to \mathcal{L}^2 D \to \mathcal{L}D$$

We show that $x \in \bigsqcup(\mathcal{L}T(Y)) \Leftrightarrow_{x:D,Y:\mathcal{L}1} ! \in Y$. One way, if $! \in Y$ then we have $x \in T = \bigsqcup \downarrow T = \bigsqcup \mathcal{L}T(\downarrow !) \sqsubseteq \bigsqcup(\mathcal{L}T(Y))$. The other, if $x \in \bigsqcup(\mathcal{L}T(Y))$ then

$$! \in \mathcal{L}!(\bigsqcup(\mathcal{L}T(Y))) = \bigsqcup \mathcal{L}^2!(\mathcal{L}T(Y)) = \bigsqcup \mathcal{L}(\mathcal{L}! \circ T)(Y) \sqsubseteq \bigsqcup \mathcal{L} \downarrow (Y) = Y$$

(We have $\mathcal{L}! \circ T \sqsubseteq \downarrow$ by 3.5 because if $x \in T$ then $! = !(x) \in \downarrow !$.)

$\square$

4 Loc and the Lower Powerlocale

We now turn to the case of **Loc**. It is easy to construct finite products and inserters in **Loc**; products are well-known, and an inserter I for $f, g : D \to E$ is defined by

$$\Omega I = \mathrm{Fr}\langle \Omega D (\text{qua Fr}) \mid \Omega f(b) \leq \Omega g(b) (b \in \Omega E)\rangle$$

Our $\mathcal{L}$ will be P_L.

Definition 4.1 *If D is a locale then $P_L D$, the* lower powerlocale *over D, is defined by*

$$\Omega P_L D = Fr\langle \diamond a (a \in \Omega D) \mid \diamond \bigvee_i a_i = \bigvee_i \diamond a_i \rangle$$

In other words, $\Omega P_L D$ is the free frame generated by ΩD qua suplattice.

A global point of $P_L D$ is a suplattice homomorphism from ΩD to Ω. Classically, these are in order-reversing bijection with the opens of D — such a homomorphism corresponds to the join of all the opens that it maps to **false** — and hence, classically, they can be identified with closed sublocales of D. This argument does not hold constructively, but nonetheless it is known (Bunge and Funk [4]) that there is an order-isomorphism between the global points of $P_L D$ and certain sublocales of D (technically, the weakly closed sublocales with open domain). This holds out the hope that set-theoretic intuitions might reasonably be applied to general points of $P_L D$, that in some sense they are determined by the points of D that they "contain". We shall justify this by proving (Proposition 4.4) that Axiom 3.4 holds.

P_L is a KZ-monad. P_L as a functor is defined by (for $f : D \to E$) $\Omega P_L f(\diamond b) = \diamond\Omega f(b)$, the unit $\downarrow$ by $\Omega\downarrow(\diamond a) = a$ and the multiplication by $\Omega\bigsqcup(\diamond a) = \diamond\diamond a$. It is easy to check that $\bigsqcup_D$ is a left adjoint to $\downarrow_{P_L D}$. Since $\diamond;\Omega\downarrow = \mathrm{id}_{\Omega D}$, it follows that $\Omega\downarrow$ is onto and Axiom 3.3 holds.

Lemma 4.2 *$\diamond : \Omega D \to \Omega P_L D$ is left adjoint to $\Omega\downarrow$.*

Proof $\Omega\downarrow(\diamond a) = a$. For the other composition, because $\diamond$ and $\Omega\downarrow$ are both suplattice homomorphisms it suffices to check the inequality on basic opens $\bigwedge_i \diamond a_i$ of $P_L D$. $\diamond(\Omega\downarrow(\bigwedge_i \diamond a_i)) = \diamond(\bigwedge_i \Omega\downarrow(\diamond a_i)) = \diamond(\bigwedge_i a_i) \le \bigwedge_i \diamond a_i$. $\square$

A map from D to $P_L E$ is just a suplattice homomorphism from ΩE to ΩD, and a crucial tool is a sharpening of Johnstone's [8] coverage theorem for frames that allows us to describe suplattice homomorphisms between frames. The sharpening is discussed in detail in Abramsky and Vickers [1] and we shall merely summarize it here.

Theorem 4.3 *(Johnstone's Coverage Theorem) Let S be a meet semilattice, and let C — "covers" — be a relation between $\wp S$ and S such that —*

- *if $x \in X\ C\ u$ then $x \le u$* (i.e., *if $X\ C\ u$ then $X \subseteq\downarrow u$)*
- *if $X\ C\ u$ and $s \in S$ then $\{x \wedge s : x \in X\}\ C\ u \wedge s$*

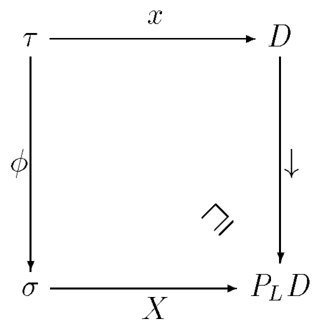
(Any presentation of a frame by generators and relations can be manipulated into this form.)

Then $Fr\langle S(qua\ \wedge\text{-semilattice}) \mid u \le \bigvee X (X\ C\ u)\rangle$
$\cong SupLat\langle S(qua\ poset) \mid u \le \bigvee X(X\ C\ u)\rangle$

Proposition 4.4 *(Axiom 3.4) If $f : D \to E$ is a map of locales, then*

$$\models \forall X : P_L D. \forall Y : P_L E. (P_L f(X) \sqsubseteq Y \leftrightarrow \forall x : D. (x \in X \to f(x) \in Y))$$

Proof As discussed in Section 3, we consider points X and Y of $P_L D$ and $P_L E$ (at stage σ), and consider the comma square



We first find — and this does not depend on Y — a left adjoint $\exists_\phi$ for $\Omega\phi$. (In fact, though we shan't need this, ϕ is open.)

$$\Omega\tau \;\cong\; \mathrm{Fr}\langle \Omega D, \Omega\sigma(\text{qua Fr}) \mid a \otimes \mathbf{true} \leq \mathbf{true} \otimes \Omega X(a)\rangle$$

(Here we use the usual notation for opens of $D \times \sigma$, and also abuse notation by treating ΩX as a suplattice homomorphism from ΩD to $\Omega\sigma$.)

$$\begin{aligned} &\cong \mathrm{Fr}\langle \Omega D \times \Omega\sigma(\text{qua } \wedge\text{-semilattice}) \mid && \otimes \text{ bilinear w.r.t } \textstyle\bigvee \\ & && a \wedge u \otimes v \leq a \wedge u \otimes \Omega X(a) \wedge v\rangle \\ &\cong \mathrm{SupLat}\langle \Omega D \times \Omega\sigma \text{ (qua poset)} \mid && \text{same relations}\rangle \end{aligned}$$

It follows that we can define a suplattice homomorphism $\exists_\phi : \Omega\tau \to \Omega\sigma$ by $\exists_\phi(u \otimes v) = \Omega X(u) \wedge v$, and it is easy to show the inequations to make it left adjoint to $\Omega\phi$. (To show further that ϕ is open, one checks the Frobenius identity: $\exists_\phi(u \otimes v \wedge \mathbf{true} \otimes b) = \Omega X(u) \wedge v \wedge b = \exists_\phi(u \otimes v) \wedge b$.)

Now we need to show that if $x; f; \downarrow \sqsubseteq \phi; Y$ then $X; P_L f \sqsubseteq Y$. In $\Omega\tau$, $\Omega f(a) \otimes \mathbf{true} \leq \mathbf{true} \otimes \Omega Y(a)$ for all $a \in \Omega D$. Applying $\exists_\phi$, we get $\Omega X(\Omega f(a)) \leq \Omega X(\mathbf{true}) \wedge \Omega Y(a) \leq \Omega Y(a)$, and hence $X; P_L f \sqsubseteq Y$. □

Proposition 4.5 *(Axiom 3.5) P_L preserves finite products "up to right adjoint".*

Proof Let D and E be locales. We want a suplattice homomorphism $\phi = \diamond; \Omega(\times)$ from $\Omega(D \times E)$ to $\Omega(P_L D \times P_L E)$. Because $\Omega(D \times E)$ is the suplattice tensor product of ΩD and ΩE, this amounts to a suplattice-bilinear function from $\Omega D \times \Omega E$ to $\Omega(P_L D \times P_L E)$, so it suffices to define $\phi(a \otimes b) = \diamond a \otimes \diamond b$ and check bilinearity (which is obvious).

To show that $\times$ is right adjoint to $\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle$, we must show that $\Omega(\times)$ is *left* adjoint to $\Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle$, in other words that $\Omega(\times); \Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle \geq \mathrm{id}$ and $\Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle; \Omega(\times) \leq \mathrm{id}$. For the former, it suffices to show that $\phi; \langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle \geq \diamond$:

$$\begin{aligned}
\Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle(\phi(a\otimes b)) &= \Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle(\diamond a\otimes\diamond b)\\
&= \diamond(a\otimes \mathbf{true})\wedge\diamond(\mathbf{true}\otimes b)\\
&\geq \diamond(a\otimes b)
\end{aligned}$$

For the latter, it suffices to check on the generators $\diamond a\otimes\mathbf{true}(a\in\Omega D)$ and $\mathbf{true}\otimes\diamond b(b\in\Omega E)$: *e.g.*,

$$\begin{aligned}
\Omega(\times)\circ\Omega\langle P_L\mathrm{fst}, P_L\mathrm{snd}\rangle(\diamond a\otimes\mathbf{true}) &= \Omega(\times)(\diamond(a\otimes\mathbf{true}))\\
&= \diamond a\otimes\diamond\mathbf{true}\leq\diamond a\otimes\mathbf{true}
\end{aligned}$$

□

4.1 Results on Open Maps and Locales

(See Joyal and Tierney [12] for the basic properties of open maps.)

Proposition 4.6 *Let $f : D\to E$ be a map of locales. Then f is semi-upper (in the sense of 3.8) iff Ωf has a left adjoint $\exists_f : \Omega D\to\Omega E$.*

Proof $\Leftarrow$: Given $\exists_f$, define $\Omega(f^{-1}) : \Omega P_L D\to\Omega P_L E$ a frame homomorphism such that $\diamond;\Omega(f^{-1}) = \exists_f;\diamond$.

$$\begin{aligned}
&\diamond;\Omega(f^{-1});\Omega P_L f = \exists_f;\diamond;\Omega P_L f = \exists_f;\Omega f;\diamond\geq\diamond && \text{hence } \Omega(f^{-1});\Omega P_L f\geq\mathrm{id}\\
&\diamond;\Omega P_L f;\Omega(f^{-1}) = \Omega f;\diamond;\Omega(f^{-1}) = \Omega f;\exists_f;\diamond\leq\diamond && \text{hence } \Omega P_L f;\Omega(f^{-1})\leq\mathrm{id}
\end{aligned}$$

Therefore $\Omega(f^{-1})$ is the left adjoint to $\Omega P_L f$, f^{-1} is right adjoint to $P_L f$.

$\Rightarrow$: Given f^{-1}, define $\exists_f = \diamond;\Omega(f^{-1});\Omega(\downarrow)$.

$$\begin{aligned}
\exists_f;\Omega f &= \diamond;\Omega(f^{-1});\Omega(\downarrow);\Omega f = \diamond;\Omega(f^{-1});\Omega P_L f;\Omega(\downarrow)\geq\diamond;\Omega(\downarrow) = \mathrm{id}\\
\Omega f;\exists_f &= \Omega f;\diamond;\Omega(f^{-1});\Omega(\downarrow) = \diamond;\Omega P_L f;\Omega(f^{-1});\Omega(\downarrow)\leq\diamond;\Omega(\downarrow) = \mathrm{id}
\end{aligned}$$

Therefore $\exists_f$ is the left adjoint to Ωf. □

Theorem 4.7 *$f : D\to E$ is upper (in the sense of the 3.10) iff it is open.*

Proof Consider $\diamond(a\otimes b)\in\Omega P_L(D\times E)$. Round the upper right path of the diagram from 3.10, we have

$$\diamond(a\otimes b)\mapsto\diamond(a\wedge\Omega f(b))\mapsto\diamond\exists_f(a\wedge\Omega f(b))\mapsto\exists_f(a\wedge\Omega f(b))$$

Round the lower left path,

$$\diamond(a\otimes b)\mapsto\diamond a\otimes\diamond b\mapsto\exists_f a\wedge b$$

Hence the diagram commutes iff for all $a\in\Omega D$ and $b\in\Omega E$, $\exists_f(a\wedge\Omega f(b)) = \exists_f a\wedge b$, in other words iff the Frobenius condition holds making f open. □

The general results of the previous section now give us the following:

Proposition 4.8 *A locale D is open iff $P_L! : P_L D \to P_L 1$ has a right adjoint (*i.e., *iff — as Joyal and Tierney [12] have already proved — $\Omega!$ has a left adjoint. The Frobenius condition follows automatically in this case.)*

Theorem 4.9 *A locale D is open iff $! : P_L D \to 1$ has a right adjoint.*

Classically, this result is trivial: all locales are open, and $P_L D$ always has a greatest point given by the suplattice homomorphisms $\Omega D \to \Omega$ under which all opens except **false** map to **true** (corresponding to the whole of D as a closed sublocale). But our argument — and in particular the sharpened coverage theorem — also holds constructively, when openness is a non-trivial property of locales.

5 $\mathbf{Loc}^{co}$ and the Upper Powerlocale

The theory with P_L replaced by the *upper* powerlocale, P_U, is very similar except that all the adjunctions work the opposite way round. We can bring this into the general theory by treating P_U as a monad on $\mathbf{Loc}^{co}$, *i.e.,* **Loc** with the order enrichment reversed. However, to avoid the confusion of having the two opposite orders, we prefer to dualize the axioms and results of Section 3.

Definition 5.1 *If D is a locale, then $P_U D$, the* upper powerlocale *over D, is defined by*

$$\Omega P_U D = Fr\langle \Box a (a \in \Omega D) \mid \Box \bigvee^{\uparrow}_i a_i = \bigvee^{\uparrow}_i \Box a_i, \Box \wedge_i a_i = \wedge_i \Box a_i \text{ (finite meet)}\rangle$$

In other words, $\Omega P_U D$ is the free frame generated by ΩD qua preframe. (A preframe is a poset with directed joins and finite meets, with binary meet distributing over the directed joins. A homomorphism of preframes preserves all directed joins and finite meets.)

A point of $P_U D$ is a preframe homomorphism from ΩD to Ω and these are equivalent to Scott open filters in ΩD. Classically, the Hofmann-Mislove [6] theorem (see also Vickers [17] for the remark that it doesn't depend on spatiality) tells us that these are in bijection with the compact saturated sets of global points of D, such a homomorphism corresponding to the intersection of all the extents of opens that it maps to **true**. This bijection is order reversing: the specialization order on $P_U D$ is the *superset* order on compact saturated subsets. Constructively, one has to replace the Hofmann-Mislove theorem by a result of Johnstone [9]: Scott open filters of ΩD are equivalent to compact fitted sublocales of D.

P_U is a co-KZ-monad. As a functor it is defined by (for $f : D \to E$) $\Omega P_U f(\Box b) = \Box \Omega f(b)$, the unit $\uparrow$ is $\Omega \uparrow (\Box a) = a$ and the multiplication $\bigsqcap$ is $(\Omega \bigsqcap)(\Box a) = \Box\Box a$. It is easy to check that $\bigsqcap_D$ is right adjoint to $\uparrow_{P_U D}$. Because $\Box; \Omega \uparrow = \mathrm{id}$, it follows that $\uparrow$ is a 2-categorical monic (Axiom 3.3).

Lemma 5.2 $\Box : \Omega D \to \Omega P_U D$ *is right adjoint to* $\Omega \uparrow$.

Proof $\Omega \uparrow (\Box a) = a$. For the other composition,

$\Box(\Omega\uparrow(\bigvee_i \Box a_i)) = \Box(\bigvee_i \Omega\uparrow(\Box a_i)) = \Box(\bigvee_i a_i) \geq \bigvee_i \Box a_i$ $\Box$

(Interestingly, this shows that $\Box$ preserves *all* meets, not just finite ones.)

A map from D to $P_U E$ is just a preframe homomorphism from ΩE to ΩD, and again we need techniques that allow us to describe preframe homomorphisms between frames. These are provided by Johnstone and Vickers [11], and it is worth pointing out that the arguments presented there (and those of Banaschewski [2] on which they rely) are constructively valid — they hold in any elementary topos. Let us briefly recall here a preframe version of the coverage theorem, and the preframe account of product locales.

Theorem 5.3 *(The Preframe Coverage Theorem) Let P be a join semilattice, and let C — "covers" — be a relation between $\wp\mathcal{F}P$ and $\mathcal{F}P$ ($\mathcal{F}$ for the finite power set) such that —*

- *if $T \in X\ C\ S$ then $T \leq_U S$* (i.e., $\forall s \in S.\exists t \in T.t \leq s$).
- *if $X\ C\ S$ then X is directed with respect to $\leq_U$.*
- *if $X\ C\ S$ and $u \in P$ then $\{\{x \vee u : x \in T\} : T \in X\} C \{x \vee u : x \in S\}$*

(Any presentation of a frame by generators and relations can be manipulated into this form.)

Then $Fr\langle P \text{(qua } \vee\text{-semilattice)} \mid \bigwedge S \leq \bigvee^{\uparrow}\{\bigwedge T : T \in X\}\ (X\ C\ S)\rangle$

$\cong PreFr\langle P \text{(qua poset)} \mid \bigwedge S \leq \bigvee^{\uparrow}\{\bigwedge T : T \in X\}\ (X\ C\ S)\rangle$

From this it can be proved that is that if D and E are locales, then $\Omega(D \times E)$ is a tensor product in a natural sense of ΩD and ΩE qua preframes. To define a preframe homomorphism out of $\Omega(D \times E)$, it suffices to define its values on the elements $a \otimes b = a \otimes \mathbf{true} \vee \mathbf{true} \otimes b$ and show that the resulting function from $\Omega D \times \Omega E$ is "preframe bilinear" — it preserves directed joins and finite meets in each of the arguments (when the other argument is held fixed).

Proposition 5.4 *(Axiom 3.4) If $f : D \to E$ is a map of locales, then*

$$\models \forall X : P_U D.\forall Y : P_U E.(P_U f(X) \sqsupseteq Y \leftrightarrow \forall x : D.(x \in X \to f(x) \in Y))$$

(Note: "$x \in X$" now means $X \sqsubseteq\uparrow x$.)

Proof The proof is not very different from the P_L case, but let us sketch it to illustrate the preframe techniques. Let X and Y be points at stage σ and let τ be the comma object that interprets $x \in X$ over σ.

$$\begin{aligned} \Omega\tau &\cong \mathrm{Fr}\langle \Omega D, \Omega\sigma \text{ (qua Fr)} \mid && \mathbf{false} \otimes \Omega X(a) \leq a \otimes \mathbf{false}\rangle \\ &\cong \mathrm{PreFr}\langle \Omega D \times \Omega\sigma \text{ (qua poset)} \mid && \otimes \text{ is bilinear w.r.t. } \textstyle\bigvee^{\uparrow} \text{ and } \wedge \\ & && a \vee u \otimes \Omega X(a) \vee v \leq a \vee u \otimes v\rangle \end{aligned}$$

It follows that we can define a preframe homomorphism $\alpha : \Omega\tau \to \Omega\sigma$ by $\alpha(u \otimes v) = \Omega X(u) \vee v$. (In fact α is the right adjoint of $\Omega\phi$, and indeed ϕ

is proper). Suppose also we have the commutative diagram corresponding to $x \in X \models_{x:D} f(x) \in Y$, which amounts to saying that in $\Omega\tau$, $\Omega f(a) \otimes \mathbf{false} \geq \mathbf{false} \otimes \Omega Y(a)$ for all $a \in \Omega D$. By applying α, we get $\Omega X(\Omega f(a)) \geq \Omega X(\mathbf{false}) \vee \Omega Y(a) \geq \Omega Y(a)$, and hence $X; P_U f \sqsupseteq Y$. $\square$

Proposition 5.5 *(Axiom 3.5) P_U preserves finite products "up to left adjoint".*

Proof Let D and E be locales. We want a preframe homomorphism $\phi = \square; \Omega(\times)$ from $\Omega(D \times E)$ to $\Omega(P_U D \times P_U E)$. Because $\Omega(D \times E)$ is a preframe tensor product of ΩD and ΩE, this amounts to a preframe-bilinear function from $\Omega D \times \Omega E$ to $\Omega(P_U D \times P_U E)$, so it suffices to define $\phi(a \otimes b) = \square a \otimes \square b$ and check bilinearity (which is obvious). The rest of the proof is just like that of Proposition 4.5, though of course the inequalities are reversed. $\square$

5.1 Results on Proper Maps

Note: The word "proper" has been applied to locale maps f in more than one sense. Hofmann and Lawson [5] use it to mean simply that the right adjoint $\forall_f$ of Ωf preserves directed joins (see Proposition 5.6 below), but we shall follow Vermeulen [16] in requiring in addition that a Frobenius condition be satisfied, $\forall_f(\Omega f(b) \vee a) = b \vee \forall_f(a)$. He shows that this definition is equivalent to D being compact over E when considered as a frame object in the category of sheaves over E. Such maps were called *perfect* in Johnstone [7], where *proper* was used in a third sense, namely that D is compact regular over E.

Proposition 5.6 *Let $f : D \to E$ be a map of locales. Then f is semi-upper in $\mathbf{Loc}^{co}$ iff the right adjoint of Ωf preserves directed joins.*

Proof We write $\forall_f$ for the right adjoint of Ωf.

$\Rightarrow$: Given f^{-1}, define $G = \square; \Omega(f^{-1}); \Omega(\uparrow)$.

$$G; \Omega f = \square; \Omega(f^{-1}); \Omega(\uparrow); \Omega f = \square; \Omega(f^{-1}); \Omega P_U f; \Omega(\uparrow) \leq \square; \Omega(\uparrow) = \mathrm{id}$$
$$\Omega f; G = \Omega f; \square; \Omega(f^{-1}); \Omega(\uparrow) = \square; \Omega P_U f; \Omega(f^{-1}); \Omega(\uparrow) \geq \square; \Omega(\uparrow) = \mathrm{id}$$

Therefore G is right adjoint to Ωf and so $G = \forall_f$. But G is a preframe homomorphism.

$\Leftarrow$: If $\forall_f$ preserves directed joins, then we can define $\Omega(f^{-1}) : \Omega P_U D \to \Omega P_U E$ a frame homomorphism such that $\square; \Omega(f^{-1}) = \forall_f; \square$.

$$\square; \Omega(f^{-1}); \Omega P_U f = \forall_f; \square; \Omega P_U f = \forall_f; \Omega f; \square \leq \square \text{ hence } \Omega(f^{-1}); \Omega P_U f \leq \mathrm{id}$$
$$\square; \Omega P_U f; \Omega(f^{-1}) = \Omega f; \square; \Omega(f^{-1}) = \Omega f; \forall_f; \square \geq \square \text{ hence } \Omega P_U f; \Omega(f^{-1}) \geq \mathrm{id}$$

Therefore $\Omega(f^{-1})$ is right adjoint to $\Omega P_U f$, f^{-1} is the left adjoint to $P_U f$. $\square$

Theorem 5.7 *$f : D \to E$ is upper in $\mathbf{Loc}^{co}$ iff it is proper.*

Proof The proof is just like that of Theorem 4.7: straightforward diagram chasing shows that it commutes iff the Frobenius condition holds. □

Theorem 5.8 *A locale D is compact iff $! : D \to 1$ is proper.*

Proof See Vermeulen [16]. □

As before, we can now apply the general results.

Proposition 5.9 *A locale D is compact iff $P_U! : P_U D \to P_U 1$ has a left adjoint.*

Theorem 5.10 *A locale D is compact iff $P_U D$ has a least point (*i.e., *$! : P_U D \to 1$ has a left adjoint).*

Even constructively, this result is not difficult. The frame homomorphism $\Omega ! : \Omega \to \Omega D$ always has a right adjoint, $\forall_!$, say, and we find $\mathbf{true} \le \forall_!(a)$ iff $\mathbf{true} \le a$, so the predicate $\forall_!$ corresponds to the subset $\{\mathbf{true}\}$ of ΩD. $\forall_!$ is a preframe homomorphism iff D is compact, and this is then the least point of $P_U D$. Classically, we have the Hofmann-Mislove theorem under which this least point of $P_U D$ is identified with the whole of D as a compact saturated set.

6 Conclusions

What we have presented here only touches the surface, and so obvious further work is to test the approach, to find a tidy axiomatization and to develop the logic so that one can reason very generally about locales in a point-based fashion.

That in itself is no light task, but I believe it is only preliminary to a much harder question of applying the ideas to *toposes*, thus bringing the idea of topos as generalized space much closer to the mathematical surface. Though I believe that for locales the interaction of P_L with P_U is potentially fruitful, I do not know what the topos-theoretic analogues of these powerlocales are. Plausible candidates are bagtoposes (Vickers [18], Johnstone [10]) and symmetric toposes (Bunge and Carboni [3]).

There are various other properties that it would seem desirable to axiomatize, though how they could be captured as a tidy system I don't know. For instance, the system should include stability under lax pullback or pullback of various classes of maps and some important properties of open maps axiomatized by Moerdijk [14]. Also, in an axiomatization that has two interacting monads corresponding to P_L and P_U, they should commute (Johnstone and Vickers [11] — the maps from D to $P_L P_U E$ correspond to functions from ΩE to ΩD that preserve directed joins).

We have not mentioned disjunction or existential quantification in our logic. To bring this into the Kripke-Joyal semantics requires a notion of covering, and the work of Till Plewe on "triquotient" maps of locales promises some relevance here.

Finally, let us mention a result that holds in **Loc** and $\mathbf{Loc}^{co}$, but *fails* in **Pos**. The map $\downarrow : D \to \mathcal{L}D$ ($\mathcal{L}$ being P_L or P_U) is a pullback of two equalizers, for the following parallel pairs sourced from $\mathcal{L}D$:

$$\begin{array}{ccc} & 1 & \\ {}^{!}\nearrow & & \searrow{}^{\downarrow} \\ \mathcal{L}D & \xrightarrow[\mathcal{L}!]{} & \mathcal{L}1 \end{array} \qquad \begin{array}{ccc} & \mathcal{L}D \times \mathcal{L}D & \\ {}^{\Delta}\nearrow & & \searrow{}^{\times} \\ \mathcal{L}D & \xrightarrow[\mathcal{L}\Delta]{} & \mathcal{L}(D \times D) \end{array}$$

The reason is that these diagrams require $\diamond$ to preserve nullary and binary meets, or $\Box$ to preserve nullary and binary joins. The result fails in **Pos** because it would say that a lower set is principal iff it is directed.

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References

[1] Samson Abramsky and Steven J. Vickers. Quantales, observational logic and process semantics. *Mathematical Structures in Computer Science*, 3:161–227, 1993.

[2] B. Banaschewski. Another look at the localic tychonoff theorem. *Commentationes Mathematicae Universitatis Carolinae*, 29:647–656, 1988.

[3] M. Bunge and A. Carboni. The symmetric topos. Technical Report 260, Dipartmento di Mathematica, Università di Genova, 1994.

[4] M. Bunge and J. Funk. Constructive theory of the lower power locale, 1994. Unpublished.

[5] K. H. Hofmann and J. D. Lawson. On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom. volume 27 of *Mathematik-Arbeitspapiere*, pages 143–160. 1982.

[6] K. H. Hofmann and M. W. Mislove. Local compactness and continuous lattices. volume 871 of *Springer Lecture Notes in Mathematics*, pages 209–248, 1981.

[7] Peter T. Johnstone. Factorization and pullback theorems for localic geometric morphisms. 1979.

[8] Peter T. Johnstone. *Stone Spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.

[9] Peter T. Johnstone. Vietoris locales and localic semi-lattices. In R.-E. Hoffmann and K. H. Hoffmann, editors, *Continuous lattices and their Applications*, pages 155–180. Marcel Dekker, 1985. Pure and Aplied Mathematics Vol. 101.

[10] Peter T. Johnstone. Partial products, bagdomains and hyperlocal toposes. In Michael Fourman, Peter T. Johnstone, and Andrew Pitts, editors, *Applications of Categories in Computer Science*, volume 177 of *LMS Lecture Note Series*, pages 315–339. Cambridge University Press, 1991.

[11] Peter T. Johnstone and Steven J. Vickers. Preframe presentations present. In Carboni, Pedicchio, and Rosolini, editors, *Category Theory - Proceedings, Como, 1990*, number 1488 in LNMS, pages 193–212. Springer Verlag, 1991.

[12] André Joyal and Miles Tierney. *An Extension of the Galois Theory of Grothendieck*, volume 309. 1984.

[13] A. Kock. Monads for which structures are adjoint to units, 1993. To appear.

[14] I. Moerdijk. Descent theory for toposes. *Bull. Soc. Math. Belgique*, 41, 1989.

[15] John Power and Edmund Robinson. A characterization of pie limits. *Math. Proc. Cam. Phil. Soc.*, 110:33–47, 1991.

[16] J. J. C. Vermeulen. Proper maps of locales. *Journal of Pure and Applied Algebra*, 92:79–107, 1994.

[17] Steven J. Vickers. *Topology Via Logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1988.

[18] Steven J. Vickers. Geometric theories and databases. In Fourman, Johnstone, and Pitts, editors, *Applications of Categories in Computer Science*, volume 177 of *LMS Lecture Note Series*, pages 288–314. Cambridge University Press, 1992.