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Spectral Bundles as Fibrations & Opfibrations

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Topos approaches

Given a quantum system — e.g. C^*-algebra A

- define a topos
- define — an object — discrete locale on object X: internal frame $= PX$
  — an internal locale

as spectrum

What is the topos?

Derives from idea of context = classical point of view

Restrict to commuting observables — commutative subalgebra C of A

Get a classical state space Σ_C — state determines values of all observables in C

Topos = category of sheaves over space of states

Bundles

= map thought of as space (fibre) indexed by base points

Joyal-Tierney: equivalence between

- internal locales in topos $\mathcal{E}$
- localic geometric morphisms $\mathcal{F} \to \mathcal{E}$
- maps $Y \to X$ (If $\mathcal{E} = Sh(X)$) — X, Y point-free spaces

$\therefore$ Known spectral gadgets expressible as

spectral bundles

$$\begin{array}{ll} \Sigma & \text{– fibre over } C = \text{spectrum } \Sigma_C \\ \downarrow & \\ B & \text{– space of contexts } C \end{array}$$

$\mathcal{C}(A)$ = poset of commutative subalgebras of A

	Isham, Butterfield, Döring — **Imperial**	**Nijmegen** — Heunen, Landsman, Spitters
Topos	$[\mathcal{C}(A)^{op}, \mathrm{Set}]$	$[\mathcal{C}(A), \mathrm{Set}]$
Context space	Filt $\mathcal{C}(A)$	Idl $\mathcal{C}(A)$
	Scott topology	Scott topology
Contains	$\mathcal{C}(A)$ coAlexandrov	$\mathcal{C}(A)$ Alexandrov
	topologies	topologies

Finite dimensional $A = M_n(\mathbb{C})$

Caspers, Heunen, Landsman, Spitters

$C \cong \mathbb{C}^m$, $m = \dim C = \Sigma_C$

C structured by m s.a. indecomposable idempotents, orthogonal & summing to 1

In A: m s.a. idempotents (projectors) P_i

$P_i P_j = 0 \quad (i \neq j), \quad \sum_i P_i = 1$

$\therefore \mathcal{C}(A) = \{$such projector sequences$\}$ modulo permutations

Projector sequence $\approx$ flag in $\mathbb{C}^n$ – subspaces

$0 = W_0 \leq W_1 \leq \ldots \leq W_m = \mathbb{C}^n$

$W_i = W_{i-1} \oplus \mathrm{Im}\, P_i$

Closer analysis

Consider trace sequence $t_i = \mathrm{tr}\, P_i = \mathrm{rank}\, P_i$

For given trace sequence:

corresponding flags form a compact manifold

$$\therefore \mathcal{C}(A) = \coprod_{\text{trace sequences } t} \text{Flag manifold} / \sim$$

for trace-preserving permutations of P_is

Point of bundle space

= pair (projector sequence, choice of projector)

e.g. $n = 2$ (a qbit)

Trace sequences

2 One projector sequence (Id)

Singleton fibre

$$1 \to 1$$

11 Projector sequences $(P, \mathrm{Id} - P)$

P a trace 1 projector - points of Bloch sphere S^2

$S^2/\sim = \mathbb{R}P^2$ real projective plane

$$S^2 \to \mathbb{R}P^2$$

Topology?

Imperial, Nijmegen:

each flag manifold gets discrete topology

(co) Alexandrov topology on $\mathcal{C}(A)$

- context refinement ~~decompose~~ decompose projector

Raynaud: manifold topology

- point-free, constructive
- first get regular spaces
- use extended Priestley duality to add context refinement as specialization order
- get stably locally compact space

writing up

Sheaf toposes

Imperial, Nijmegen ⇒ presheaf toposes

Raynaud ⇒ sheaf topos

Need calculational techniques for them

Avoid explicit calculations of sheaves.

Work with bundles in category of locales

Use geometric logic to restore points

Geometric mathematics

Space = geometric theory

point = model of theory

Map = geometric transformation of points to points

Bundle = geometric transformation of points to spaces

base points fibres

Realization of fibrewise topology

Coarse graining

For contexts $C \subseteq D$

coarse fine

have map $\Sigma_C \leftarrow \Sigma_D$

Gelfand-Naimark spectrum is contravariant

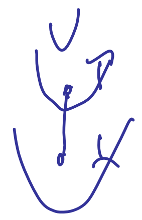
Express this more generally?

Relate context refinement to specialization order

x, y points of space X

$x \sqsubseteq y \iff$ for all opens U,
x in $U \to y$ in U

$$1 \underset{x}{\overset{y}{\rightrightarrows}} X \xrightarrow{U} \$ \qquad (x \sqsubseteq y)$$



$Ux = \top \to Uy = \top$

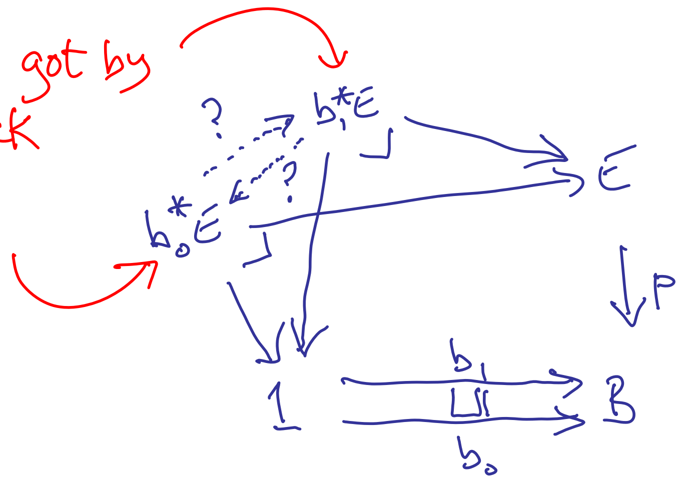
More generally: $Y \underset{f}{\overset{g}{\rightrightarrows}} X$

$f \sqsubseteq g \iff \forall U \; f^*U \leq g^*U$

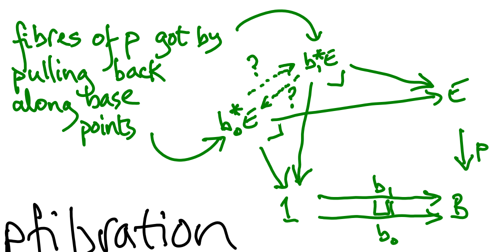
Loc is order enriched hence a 2-category

Specialization and bundles

fibres of p got by pulling back along base points



Intuition



p is a —

fibration   opfibration

if whenever $b_0 \sqsubseteq b_1$, get

restriction map

$$b_0^*E \xleftarrow{r_{b_0b_1}} b_1^*E$$

corestriction map

$$b_0^*E \xrightarrow{c_{b_0b_1}} b_1^*E$$

Problems

1. Suppose B lacks points?
2. Will any maps do here?

Use generalized points

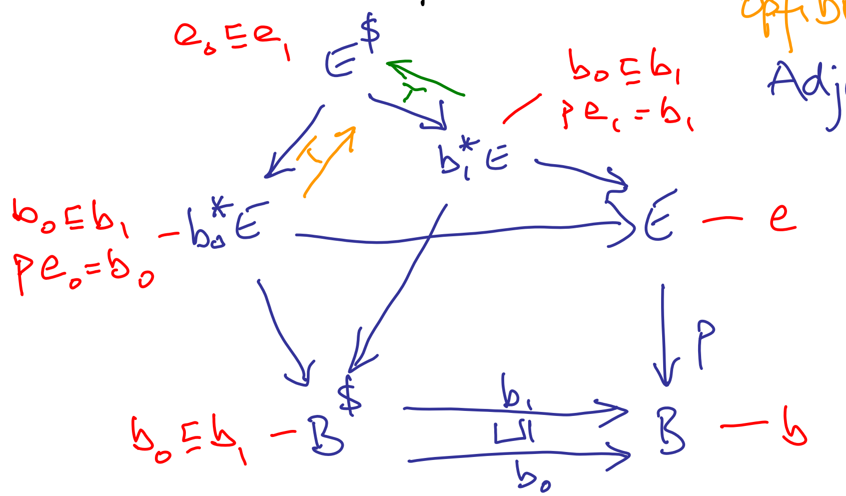
p fibration

opfibration

Adjunction is over

b_1^*E

b_0^*E



- Generic diagram covers all generalized points
- Maps between fibres are canonical

Fundamental results

See Johnstone "Fibrations & partial products in a 2-category"

If a bundle $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ corresponds to —

- a discrete internal locale then p is an opfibration

  p is a local homeomorphism

- a compact regular internal locale then p is a fibration

Consequence : Imperial

Seek spectral object in topos

$\Rightarrow$ discrete locale

$\Rightarrow$ bundle an opfibration

$C \sqsubseteq D \Rightarrow$ get $\Sigma_C \to \Sigma_D$

$\therefore \quad \sqsubseteq \quad = \quad \supseteq$

- use filter completion

  topos is $[\mathcal{G}(A)^{op}, \mathrm{Set}]$

Consequence : Nijmegen

Seek compact regular locale in topos

- Gelfand-Naimark spectrum

  Banaschewski Mulvey

  $\Rightarrow$ bundle a fibration

$C \sqsubseteq D \Rightarrow$ get $\Sigma_C \leftarrow \Sigma_D$

$\therefore \quad \sqsubseteq \quad = \quad \subseteq$

- use ideal completion

  topos is $[\mathcal{G}(A), \mathrm{Set}]$

Variance is consequence of the kind of spectrum being sought

Specialization on E

$\begin{array}{c} E \\ p \downarrow \\ B \end{array}$

Typical situation:

- know B
- know fibres of p — fibrewise topology

Write e in E as (b, x) $\quad b$ in B $\quad x$ in E_b

What is specialization order on E?

If p is a fibration,

$(b_0, x_0) \sqsubseteq (b_1, x_1) \iff$

$b_0 \sqsubseteq b_1$ and $x_0 \sqsubseteq r_{b_0 b_1}(x_1)$

Similarly for opfibrations

e.g. Valuation locale $\mathfrak{V}$

$\mathfrak{V}X$ = space of valuations on X
regular measures

Monad on Loc cf. Giry monad

Topos valid $\Rightarrow$ monad $\mathfrak{V}_B$ on Loc/B apply in sh(B)

Geometric – preserved under pb of bundles
$\Rightarrow$ works fibrewise

$\mathfrak{V}_B\begin{pmatrix} \Sigma \\ \downarrow \\ B \end{pmatrix}$ important in quantum work.
What is specialization order on bundle space?

Theorem (Fauser, Vickers)

Any geometric space transformer preserves fibrations & opfibrations e.g. $\mathfrak{V}$, powerlocales
(in general 2-category $\mathcal{C}$ with suitable limits)

- "Geometric" in terms of arrow 2-category $\mathcal{C}^\downarrow$
- Extend Street's $\mathcal{L}_B$ to $\mathcal{L}_\bullet$ on $\mathcal{C}^\downarrow$
- Opfibration = pseudoalgebra for $\mathcal{L}_\bullet$ for which structure is identity on base
- Transformer lifts to pseudoalgebras

(use existence of a certain 2-natural transformation)

"Geometric" in terms of arrow category

T: Loc $\to$ Loc
- topos-valid, hence T_B: Loc/B $\to$ Loc/B

Geometricity

$$\begin{tikzcd}
 & T_{B'}f^*E \cong f^*T_B E \arrow[rr] \arrow[dd] & & T_B E \arrow[dd] \\
f^*E \arrow[rr] \arrow[dr] & & E \arrow[dr, "p"] & \\
 & B' \arrow[rr, "f"] & & B
\end{tikzcd}$$

Coherence properties for $\cong$?

Extend to Loc$^\downarrow$

Townsend, Vickers

$T_\bullet$: Loc$^\downarrow$ $\to$ Loc$^\downarrow$

sufficient conditions on T to make this work

$$T_\bullet \left(\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ p_1\downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array} \right)$$

Change of base is incorporated into structure

$$= \begin{tikzcd}
T_{B_1}E_1 \arrow[r, "T_{B_1}\langle p_1, g\rangle"] \arrow[rr, bend right, "T_{fg}"'] \arrow[d] & T_{B_1}f^*E_2 \cong f^*T_{B_2}E_2 \arrow[r] & T_{B_2}E_2 \arrow[d] \\
B_1 \arrow[rr, "f"] & & B_2
\end{tikzcd}$$

Geometric transformer on $\mathcal{C}$

Define as

- 2-endofunctor $T_\bullet$ on $\mathcal{C}^\downarrow$
- Acts as identity on base
- "Preserves horizontality" –

if
$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$
is pb, so is result of applying $T_\bullet$

Street

- Gave analogous definition for arbitrary 2-category $\mathcal{C}$
- Showed p an opfibration if
  - as object of $\mathcal{C}/B$
  - carries pseudo algebra structure for a certain 2-monad $\mathcal{L}_B$
- Similar result by duality for fibrations

$\mathcal{L}_B : \mathcal{C}/B \to \mathcal{C}/B$ – idea

$$\mathcal{L}_B\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \left\{ \begin{tikzcd} b_0^* E \arrow[r, "p^* b_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B^\$ \arrow[r, "b_0"'] \arrow[d, "b_1"] & B \\ B & \end{tikzcd} \right.$$

Pseudoalgebra structure gives

$$\begin{tikzcd} p_0^* E \arrow[r, "\alpha"] \arrow[d] & E \arrow[dd, "p"] \\ B^\$ \arrow[dr, "b_1"'] & \\ & B \end{tikzcd}$$

Combine α with $p^* b_0$ –

get $p_0^* E \longrightarrow E^\$$

Extend Street's $\mathcal{L}_B$ to $\mathcal{L}_\bullet$ on $\mathcal{C}^\downarrow$

- same on objects
- on morphisms must deal with change of base $B_1 \to B_2$
- monad unit, multiplication
  - same definition
  - show 2-naturality w.r.t. new morphisms
- monad equations – same proof
- Opfibration = pseudoalgebra for $\mathcal{L}_\bullet$ for which structure is identity on base – clear

$T_\bullet$ lifts to pseudoalgebras for $\mathcal{L}_\bullet$

Define 2-natural transformation

$$\psi : \mathcal{L}_\bullet T_\bullet \to T_\bullet \mathcal{L}_\bullet$$

satisfying certain properties

1-cat case (Applegate): $T_\bullet$ lifts to algebras

$$\begin{tikzcd} \mathcal{L}_\bullet P \arrow[d, "\alpha"'] \\ P \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}_\bullet T_\bullet P \arrow[r, "\psi"] & T_\bullet \mathcal{L}_\bullet P \arrow[d, "T_\bullet \alpha"] \\ & T_\bullet P \end{tikzcd}$$

Marmolejo – generalizes to 2-categories & pseudoalgebras

other generalizations also known

Defining $\psi : \mathcal{L}_\bullet T_\bullet \begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \to T_\bullet \mathcal{L}_\bullet \begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix}$

$$\mathcal{L}_\bullet \begin{pmatrix} T_B E \\ \downarrow \\ B \end{pmatrix} = \begin{tikzcd} b_0^* T_B E \arrow[d] \\ B^\$ \arrow[d] \\ B \end{tikzcd}$$

$$\mathcal{L}_B \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \left\{ \begin{tikzcd} b_0^* E \arrow[r, "p^* b_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B^\$ \arrow[r, "b_0"] \arrow[d, "b_1"] & B \\ B & \end{tikzcd} \right.$$

For $T_\bullet \mathcal{L}_\bullet$: apply $T_\bullet$ to

$$\mathcal{L}_B E = \begin{tikzcd} & b_0^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ B & B^\$ \arrow[l, "b_1"'] \arrow[r, "b_0"] & B \end{tikzcd}$$

$$\begin{tikzcd} b_0^* T_B E \arrow[rr, bend left, "\cong"] & T_B \mathcal{L}_B E \arrow[d] & T_{B^\$} b_0^* E \arrow[l, "T_{b_1} ="'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T_B E \arrow[d] \\ & B & B^\$ \arrow[l, "b_1"'] \arrow[r, "b_0"] & B \end{tikzcd}$$

Summary

- (Op)fibrational describes how specialization order between base points gives maps between fibres
- Different quantum topos approaches have implicitly chosen to be either opfibrational (Imperial) or fibrational (Nijmegen, Birmingham)
- You can stick to your choice if you use geometric constructions