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Preframe Techniques in Constructive Locale Theory

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Abstract

Our work is entirely constructive; none of the mathematics developed uses the excluded middle or any choice axioms. No use is made of a natural numbers object.

We get a glimpse of the parallel between the preframe approach and the SUP-lattice approach to locale theory by developing the preframe coverage theorem and the SUP-lattice coverage theorem side by side and as examples of a generalized coverage theorem.

Proper locale maps and open locale maps are defined and seen to be *parallel*. We argue that the compact regular locales are parallel to the discrete locales. It is an examination of this parallel that is the driving force behind the thesis.

We proceed with examples: relational composition in **Set** can be expressed as a statement about discrete locales; we then appeal to our parallel and examine relational composition of closed relations of compact regular locales. A technical achievement of the thesis is the discovery of a preframe formula for this relational composition.

We use this formula to investigate ordered compact regular locales (where the order is required to be closed). We find that Banaschewski and Brümmer's compact regular biframes (*Stably continuous frames* [Math. Proc. Camb. Phil. Soc. (1988) **104** 7-19]) are equivalent to the compact regular posets with closed partial order. We also find that the ordered Stone locales are equivalent to the coherent locales. This is a localic, and so constructive, version of Priestley's duality.

Further, using this relational composition, we can define the Hausdorff systems as the proper parallel to Vickers' continuous information systems (*Information systems for continuous posets* [Theoretical Computer Science **114** (1993) 201-229]) The category of continuous information systems is shown by Vickers to be equivalent to the (constructively) completely distributive lattices; we prove the proper parallel to this result which is that the Hausdorff systems are equivalent to the stably locally compact locales. This last result can be viewed as an extension of Priestley's duality.

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To Olly, Sarah, Richard, Tobias, Sean and Francine

Introduction

Say we are given a topological space X and are required to describe the set of opens of the product space $X \times X$. The obvious answer is to look at the following subsets of $X \times X$:

$$U \times V = \{(u, v)|u \in U, v \in V\}$$

where U, V are arbitary opens of X. We note that the collection of all such sets i.e.

$$\beta \equiv \{U \times V | U, V \text{ are open subsets of } X\}$$

is closed under finite intersections. (Since $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.) So β forms a basis for a topology. We form the whole topology by taking all unions of sets of the form $U \times V$, i.e. by taking the least sub(SUP-lattice) of $P(X \times X)$ generated by β. (Recall that a SUP-lattice is a poset with arbitary joins, and so the union operation tells as that $P(A)$ is a SUP-lattice for any set A.)

There is, however, a parallel solution to this problem. Look at the following subsets of $X \times X$:

$$U \invamp V \equiv \{(u, v)|u \in U \text{ or } v \in V\}$$

where again U, V are open subsets of X. It is easy to check that $(U_1 \invamp V_1) \cup (U_2 \invamp V_2) = (U_1 \cup U_2) \invamp (V_1 \cup V_2)$, and so we conclude that the collection

$$\gamma \equiv \{U \invamp V | U, V \text{ are open subsets of } X\}$$

is closed under finite unions. We want to generate a topology from γ, and so we need a collection of subsets (of $X \times X$) that is closed under arbitary unions and finite intersetions. It is a well known (lattice theoretic) fact that an arbitary union can always be expressed as a directed union of finite unions. For if $(B_i)_{i \in I}$ is a collection of subsets of some set A, then

$$\cup_{i \in I} B_i = \bigcup^{\uparrow}_{\{\bar{I} | \bar{I} \subseteq I, \bar{I} \text{ finite }\}} (\cup_{i \in \bar{I}} B_i)$$

The $\uparrow$ on $\cup$ indicates that the union is a union of a directed set. i.e. the set is non-empty and if a, b are in the set then there exists c in the set such that $a, b \subseteq c$.

Now γ is closed under finite unions, so all we need to do is close it with respect to directed unions and finite intersections in order to create a topology. Define τ to be the collection of all directed unions of finites intersections of elements of γ. It can be seen that τ is closed under directed unions and finite intersections. i.e. it is a sub*preframe* of $P(X \times X)$. Clearly it is the least subprefame of $P(X \times X)$, containing γ and finally (by distributivety of $P(X \times X)$) τ is closed under finite

unions. So τ forms a topology.
We have now defined two topologies for $X \times X$; one is the least sub(SUP-lattice) of $P(X \times X)$ containing all the sets $U \times V$ for U, V open in X, and the other is the least subpreframe of $P(X \times X)$ containing all the sets of the form $U \invamp V$ for U, V open in X.
But

$$\begin{gathered} U \invamp V = (U \times X) \cup (X \times U) \\ U \times V = (U \invamp \phi) \cap (\phi \invamp V) \end{gathered}$$

and so a short proof allows us to conlcude that these two toplogies are the same. We could hase used either approach in order to define the product topology.

The example just given is the most straightforward way of describing the parallel which forms the backbone to this thesis: there are two ways of looking at any topology, as a free SUP-lattice or as a free preframe.
However it must be emphasised that the work presented here is not about topological spaces. The example above is couched in topological language in order to make it more accessible: this is a thesis about locale theory.

Locale Theory

The first thing to say about locales is that they are like topological spaces. Locale theory is defined so that we can treat locales as if they are topological spaces: we talk of sublocales (cf subspaces), special cases being dense, closed and open sublocales (cf dense, closed and open subspaces). We talk of continuous maps between locales (cf continuous maps between topological spaces), special cases being proper maps and open maps (cf proper and open continuous functions between spaces). We talk of compact locales (cf compact topological spaces), and similarly most of the usual separation axioms on topological spaces have their localic translations: e.g. we talk of compact Hausdorff locales and discrete locales (cf compact Hausdorff spaces and discrete spaces).

This analogy between locale theory and topological space theory is not exact: if it were locale theory and topological space theory would be indistinguishable and so locale theory would be redundant.

What exists is a translating device between the two theories: whenever we are given a locale X there is a topological space ptX naturally associated with it. And whenever we are given a topological space Y there is a locale ΩY naturally associated with it. Categorically what this means is that there is a pair of functors going inbetween the category **Loc** of locales and the category **Sp** of topological spaces.

$$\begin{gathered} pt : \mathbf{Loc} \longrightarrow \mathbf{Sp} \\ \Omega : \mathbf{Sp} \longrightarrow \mathbf{Loc} \end{gathered}$$

Now say we are given a locale X and we translate it into a space (ptX) and then translate it back into a locale (ΩptX): do we come back to the same locale? Similarly, if we are given a space Y, is $pt\Omega Y$ the same thing (up to isomorphism) as Y? The answer is no, in general, since if we did get the same thing then the translation would be exact.
However the collection of all topological spaces Y such that $pt\Omega Y$ is the same thing as Y is important: we shall call these the *sober* spaces. Similarly the collection of

locales X such that ΩptX is X is important: these are the *spatial* locales. What is important about these collections is that if we restrict our attention to the sober spaces and to the spatial locales then the restricted translations are exact i.e. the theory of sober spaces and the theory of spatial locales are the same. Categorically this means that there is an equivalence

$$\mathbf{SLoc} \cong \mathbf{Sob}$$

where **Sloc** is the category of spatial locales and **Sob** is the category of sober spaces. So the next question is: how many spaces are sober? i.e. is the collection of sober spaces large enough to include most of the examples of topological spaces that are actually used in practice? The answer to this question, fortunately for locale theory, is yes.

> "... in effect, one sacrifices a small amount of pathology (non-sober spaces) in order to achieve a category that is more smoothly and purely 'topological' than the category of spaces itself. " [Joh85]

This is a good reason to take a serious look at locale theory: in practice when we study topological spaces we are almost always looking at sober spaces and so we might as well be working within the category of locales.
There are, however, much more compelling reasons why the category of locales should be considered the correct framework within which to do topology: the study of locales is, in a sense, logically purer than the study of topological spaces. Proving results in locale theory requires less axioms of our mathematics than the corresponding proofs in topological space theory.
A discussion of these axioms and how the need for them is removed by looking at locale theory will lead us to a point where the results of this thesis start.

Axioms

The law of excluded middle has a long history in mathematics. It is widely accepted as being true. Our intuitions about the real world indicate that statements are either true or false and so it understandable that the statment

$$(\forall p)(p \vee \neg p)$$

has been allowed as an axiom of our mathematics. In the work that follows we prove results and develop some theory that *does not* require this axiom to be true. Mathematics without this axiom (the intuitionistic approach) has a long history aswell. Earlier this century Brouwer and Heyting both tried to develop an intuitistionistic version of mathematics (for a good introduction look at [TD88]). It is the relatively new idea of a topos however that gives us some more impetus for taking the intuitionistic approach seriously.

Toposes are mathematical universes. Some toposes are Boolean (satisfy the law of excluded middle) but there are enough non-Boolean naturally occuring toposes to make it clear that there are important mathematical universes where the law of excluded middle fails. So if we want to be sure that our mathematics can be carried out in *any* topos (=mathematical universe) then we must make sure that it is not dependent on the law of excluded middle.

Very often the dependence of a topological proof on the law of excluded middle vanishes when we translate it into a proof about locales. This is one of the pay-offs of locale theory. We achieve a proof that is logically purer: it can be carried out in any topos. Interestingly enough the fact that dependence on excluded middle vanishes is really only the icing on the cake: historically the reason why mathematicians looked at locales was to avoid dependence on an axiom that has an even more tenuous connection with reality: the axiom of choice.

The axiom of choice states that if X_i is a collection of non-empty sets (where i ranges over some indexing set I) then the product $\prod_i X_i$ is non-empty. One may or may not find this axiom in agreement with ones intuitions of how infinite products of sets should behave. Certainly this axiom caused many more logical 'waves' when its importance to mathematics was discovered than did the law of excluded middle. But it was found that a lot of mathematical results used it: one of the most famous examples being the proof that the product of compact topological spaces is always compact (this is Tyhchonoff's theorem; recall that a topological space X is compact if for any directed collection of opens $(U_i)_{i\in I}$ we have that $X \subseteq \cup_i^{\uparrow} U_i$ implies that $X \subseteq U_i$ for some $i \in I$). Indeed it was shown that some of these results not only used the axiom of choice but they *needed* it, i.e. an assumption of the result leads to a proof of the axiom of choice. Given this fact and the general usefulness of the axiom it is understandable that certain pathologies that could be derived from it (e.g. the Tarski-Banach paradox, see pp. 3-6 of [Jec73]) were ignored . Indeed the task of developing a 'choice free' mathematics would seem impossible given the dependency results just referred to: if we want the Tychonoff theorem (and for any useful topology we most certainly do) then we need the axiom of choice.
Unless we change the definition of topology.
This is exactly what we do when we move to locales. By tampering slightly with the definition of a topological space we achieve a new category in which to carry out our topological results. Crucially we find that the Tychonoff theorem can now be proved *without* the axiom of choice. The mathematics of locale theory is 'choice free'.

Of course the question remains as to whether locale theory is really topology. One of the main problems of locale theory is to translate the ideas, concepts and finally results of topological space theory. The translating device referred to earlier does not completely solve this problem. An aim of locale theory and of this thesis is to carry out this translation further.

If we take another look at the Tychonoff theorem, and in particular the definition of compactness we see that it is a 'preframe' result; it is saying something about *directed* unions. Also, it is dependent on the definition of product spaces. As we have shown, (in the first part of this introduction) there are two equivalent ways of defining such products. This fact has a localic analogue: a product locale (indeed any locale) can be treated as a free SUP-lattice or as a free preframe. As with toplogical spaces it was the SUP-lattice definition that was originally accepted as *the* definition of a product locale and when Johnstone originally proved the Tychonoff theorem for locales (in [Joh81]) he used the SUP-lattice definition of the product. But the Tychonoff theorem is a 'preframe' result and so it is pleasing to note that once the equivalent preframe definition of a product locale had been worked out ([JV91]), the proof of the Tychonoff theorem was greatly simplified. This exemplifies a lot of the work that will take place in this thesis: if we are dealing with a result about compactness we need to look at the locales concerned as free preframes rather than as free SUP-lattices. Once the preframe definition is taken the algebraic manipulations become a lot easier.

The parallel between the SUP-lattice approach and the preframe approach leads naturally to the consideration of two classes of locales: the compact Hausdorff locales and the discrete locales. These turn out to be parallel to each other in much the same way that the SUP-lattice and the preframe definitions are parallel. The details of how these two approaches fit together, applications of them (such as a constructive proof that the category of compact Hausdorff locales is regular), and how knowledge about theorems on one side of the parallel can help us prove parallel results on the other side forms the core of this thesis.

Technical Introduction

Chapter 1 is devoted to the basics of locale theory. The first section is devoted to mathematical ground rules. All results are constructive: we are working in an arbitary topos. Or, more succintly, no use is made of any of the choice axioms or the excluded middle. It is sometimes not completely clear what the word 'finite' means in an arbitary topos and so some effort is taken to clarify that we mean Kuratowski finite.

We do not assume a natural number object in our topos. So care is needed to make that we can define the free Boolean algebra on a distributive lattice; we adapt Vickers' congruence preorders ([6.2.3] of [Vic89]) in order to prove that such a free Boolean algebra exists. Later on in the chapter care is also needed to make sure that the Prime Ideal Theorem can be disccused without assuming the excluded middle (since usual statments of the theorem contain a negation). We introduce the constructive prime ideal theorem.

In Chapter 2 there are two new offerings. Firstly there is the realization that Kříž's precongruences [Kříž86] can also be used on preframes. It is easy to see what a preocngruence on a preframe should be, and we have a preframe universal result which is just a restating of Kříž's frame universal result. This preframe universal result essentially tells us that preframe presentations present; and it is this fact that enables us to view frames as preframes. i.e. to construct frame coproduct from preframe tensor and to prove a preframe version of the coverage theorem.
The next offering is a generalized coverage theorem. This theorem is a statement about any symmetric monoidal closed category $\mathcal{C}$: it shows us how coequalizers can be constructed in the category of monoids over $\mathcal{C}$ from coequalizers in $\mathcal{C}$. Given further assumptions on $\mathcal{C}$ (for instance that a free commutative monoid can be constructed on any $\mathcal{C}$ object and $\mathcal{C}$ has image factorizations) we prove a result which can be viewed as a converse to the coverage theorem: coequalizers in $\mathcal{C}$ can be calculated as images of certain coequalizers in the category of commutative monoids over $\mathcal{C}$. Standing alone both these results are straightforward to prove. They are interesting in this context because from them we can discover a plethora of other results. The main results are the coverage theorems: not only do we get the SUP-lattice and the preframe versions of the coverage theorem we also get a coverage theorem for quantales and rings. Because of the converse of the coverage theorem we are able, from the construction of coequalizers in the category of SUP-lattices, to construct coequalizers in the category of directed complete partial orders (=dcpos). The coverage theorem applied to dcpos then implies that coequalizers exists in the category of preframes. i.e. we have with these results reproved that preframe presentations present.

What is being offered here doesn't add any new mathematical results. Once the 'Preframe Presentations Present' paper [JV91] is understood we know that the category of preframes has coequalizers, and this fact for dcpos is of course well known. What we now have is an ability to see that all these theorems stem from the same results that can be proved when you consider the category of commutative monoids over any symmetric monoidal closed category $\mathcal{C}$. i.e. they are all variations on the same theme, the theme being that there are ways of lifting and droping coequalizers between the category $\mathcal{C}$ and the category of commutative monoids over $\mathcal{C}$.

Chapter 3 introduces proper and open maps between locales. We prove some basic (well knwon) results about them. The investigation is much as in Joyal and Tierney's paper *An extension of the Galois theory of Grothendieck* [JT84] the only new aspect being that we develop the theory of open and proper maps side by side. So it is quite clear, for instance, that the proof that proper maps are pullback stable is really just a repetition of the proof that open maps are pullback stable but with 'has a left adjoint which is a SUP-lattice homomorphism' being replaced with 'has a right adjoint which is a preframe homomorphism'. The proper results are proved in [Ver92]; the novelty is with our program of 'parallel proofs for parallel results'. Towards the end of the Chapter we prove that the discrete locales are those whose finite diagonals are open and the compact regular locales are those whose finite diagonals are proper. The former result is in [JT84] and the latter result is in Vermeulen's paper 'Some Constructive Results Related to Compactness' [Ver91]. Our proof doesn't follow his: we use the preframe techniques that have been developed in Chapter 2. Given this last result it should be understandable why, for the rest of the text, we refer to the compact regular locales as the compact Hausdorff locales.

Another reason to state and prove these results side by side is to fix in the reader's mind the idea that the compact Hausdorff locales are parallel to the discrete locales in much the same way that the proper maps are parallel to the open maps. As an aside we present an argument which shows that the constructive prime ideal theorem is parallel to the excluded middle. We then check that the compact Hausdorff locales form a regular category. Classically this fact follows from the regularity of the category of compact Hausdorff spaces.

Once it is known that the compact Hausdorff locales form a regular category we can immediately deduce that there is an allegory whose objects are compact Hausdorff locales and whose morphisms are closed relations. Composition is given by relational composition. We are of course assuming familiarity with the work explained in Chapter 1.5 of Freyd and Scedrov's book 'Categories Allegories'; therein is an explanation of how to construct an allegory of objects and relations from any regular category. This leads us neatly to the main technical insight of the thesis which is that we can find a formula for relational compostion between closed sublocales of compact Hausdorff locales. Chapter 4 starts with a description of this formula.

Further there is the realization that just as spatially (when we are dealing with relations on sets) we can talk about 'lower closure of a subset with respect to a relation', 'a relation is transitive/symmetric/interpolative' etc we can state the same notions for our allegory of compact Hausdorff locales and relations. In this case lower closure (with respect to some closed relation) will correspond to an endomorphism on the set of closed sublocales (a closed sublocale is taken to its lower closure). The formula for relational compostion allows us to express this endomorphism as a particular preframe endomorphism on the frame of opens of the compact Hausdorff locale. In fact, just as in the spatial case where there is a well known correspondence between arbitary relations on a set and SUP-lattice endomorphisms on the power set we are able to find a bijection between preframe endomorphisms and closed relations. This fact, expressed in generality, can be viewed as a categorical equivalence: the category of compact Hausdorff locales and formally reversed preframe maps between them is equivalent to the allegory of compact Hausdorff relations. Stated as an equivalence this result is new, however it should be noted that the essence (i.e. the correpondence between preframe homomorphisms on the frame of opens of compact Hausdorff locales and closed relations) can be found in a result of Vickers' ([Vic94]) which states that if X is a compact Hausdorff locale then,

$$P_U(X) \cong \$^X$$

where P_U is the upper power locale construction and \$ is the Sierpinsksi locale (i.e. the locale whose frame of opens is the free frame on the terminal object of our background topos). This correspondence between preframe homomorphisms and closed relations is used again and again. Essentially it is used to turn spatial intuitions about what is going on into formulas about opens.

In Chapter 5 we look at ordered locales. Just as in ordered topological space theory we find that the locales of interest are the compact Hausdorff ones. The formulas that we have developed allow us to neatly reprove some basic results from ordered toplogical space theory in a localic context. In particular we show that there is a localic analogue to the result: *if X is a compact order-Hausdorff poset then the sets of the form $U \cap V$, where U is an open upper set and V is an open lower set, form a base for the topology on X*. This leads us to the new conclusion that Banaschewski and Brümmer's category of compact regular biframes is dual to the category of compact order-Hausdorff localic posets with order preserving locale maps. This fact will be reused in Chapter 8 when we are looking at stably locally compact locales.

Chapter 6 is called 'Localic Priestley Duality'. It contains a proof that the category of coherent locales is equivalent to the category ordered Stone locales. Classically the ordered Stone locales are just the ordered Stone spaces which are, by Priestley's original result, equivalent to the spectral spaces. This is one of the main results of the thesis: we have taken a well known classical topological result and proved it in a localic context. Some work has already been done in this direction: in Jean Pretorius' thesis [Pre93] there is a constructive proof that the coherent locales are equivalent to a particular category which is classically equivalent to the ordered Stone spaces. So what is new is the realization that this 'particular category' is equivalent to the ordered Stone locales i.e. it is the localic analog to the ordered Stone spaces. We prove localic Priestley duality directly rather than go via Pretorius' result.

Chapter 7 can roughly be understood as 'extending Priestley's duality'. Infact, the problem of extending from a categorical point of view can be solved with a few remarks: Banaschewksi and Brümmer [BB88] prove that the compact regular biframes are dual to the stably locally compact locales with semi-proper maps and we have seen (Chapter 5) that the compact regular biframes are dual to the compact order-Hausdorff posets; so the compact order-Hausdorff posets are equivalent to the stably locally compact locales with semi-proper maps. But ordered Stone locales form a full subcategory of compact order-Hausdorff posets, and coherent locales form a full subcategory of stably locally compact locales with semi-proper maps: we have extended Priestley's duality.
This extension relies on constructing a compact order-Hausdorff poset from a stably locally compact locale. Instead of going via Banaschewski and Brümmer's construction [BB88] (which relies on the excluded middle in Lemma 3), we give a new construction which reduces the amount of algebra required. However the main thrust of

the chapter is about a set of categorical equivalences which are between categories that have similar objects to compact order-Hausdorff posets and stably locally compact locales, but which have very different morphims. Here motivation is important: we are trying to discover the proper parallel to Vickers' results about continuous information systems [Vic93]. Given that these results can be viewed as statements about the allegory of sets and relations then it is clear what the proper parallels should be. We discover a new proof which is easily seen to be the proper parallel to the result that the category of continuous information systems and approximable mappings is dually equivalent to the category of completely distributive lattices and frame homomorphisms. It is also shown that variations of this equivalence (changing approximable maps to lower aprroximable semi-mappings and Lawson maps) have proper parallels. We derive the proper parallel to Hoffman-Lawson duality on continuous posets.

Chapter 1

Locale Theory

1.1 Introduction

In this chapter we give an introduction to locale theory. Our main purpose is to set notation and to reemphasise the constructivety of our results. The reader is assumed to know what meets and joins on posets are, and what a distributive lattice and a Boolean algebra is. We define the category of locales and remind the reader how the *pt* and Ω functors relate locales to the category of topological spaces. We discuss how the algebraic dcpos and the continuous posets can be viewed as locales that are constructively spatial. We develop the locale theory and introduce the constructive prime ideal theorem which is classically equivalent to the ordinary prime ideal theorem. We check that some well known classes of locales (e.g. the Stone locales) are spatial if and only if the constructive prime ideal theorem is true. Apart from the use of congruence preorders and the introduction of the constructive prime ideal theorem all the results of this chapter are well known.

1.2 Mathematical Ground Rules

Essentially we work in an arbitrary topos. Rather than go into the details of this we simply assume that we have sets, functions and subsets and manipulate them in the usual way that is taught to first year undergraduates *except* we do not allow use of the law of excluded middle or any of the choice axioms.

For motivation we will occasionally want to work *classically* i.e. we might want to assume that the excluded middle and/or some choice axiom is true. Whenever we are working classically a clear reference to this fact is made in the text.

The other piece of mathematical furniture that is to be removed is the natural numbers object. We remove it because we don't need it. All the proofs offered are free of any need to enumerate things or to rely on the naturals in some other way.

A consequence of working in an arbitrary topos is that we are forced to think more carefully about what it means for a set to be finite. We can no longer rely on just 'counting' the elements of it. In fact the definition of finite that we choose has the property that it is not the case that subsets of finite sets are necessarily finite. (For the details of this counter example see Exercise 9.2 of [Joh77].)

We use Kuratowski finite for our definition of finite. (As introduced by Kuratowski in [Kur20]; however see [KLM75] which examines the definition in the

context of an arbitrary topos.) We say that $\bar{A} \subseteq A$ is a *finite* subset of A if and only if it belongs to the free $\vee$-semilattice generated by A (viewed as a subset of PA). We can construct this free $\vee$-semilattice as the least subset X of PA such that (i) $\phi \in X$, (ii) if A_1,A_2 is in X then $A_1 \cup A_2$ is in X and (iii) the image of the singleton inclusion $\{\} : A \to PA$ is in X. We give this construction explicitly since the usual proof of a 'presentations of finite algebraic theories present' result requires the natural numbers.
It is not immediately apparent that the construction just given is the free join semilattice on X. To see that it is note that for any given function $f : X \to A$ where A is a join semilattice the set

$$\{\bar{I} \subseteq X | \vee \{f(i)|i \in \bar{I}\} \text{ exists}\}$$

contains all the singletons, the empty set and is closed under finite unions. So it contains FX and we can therefore define a function $\bar{f} : FX \to A$ such that $\bar{f} \circ \{\} = f$.
To check that $\bar{f}$ is the unique such join preserving map from FX to A, say $g : FX \to A$ is a join preserving map such that $g \circ \{\} = f$, then the set

$$\{I \subseteq X | I \in FX, \quad g(I) = \bar{f}(I)\}$$

contains singletons, the empty set and is closed under finite union. Hence it is the whole of FX. The proof that the the free semi-lattice on a set can be constructed in a topos without a natural numbers object is originally due to Mikkelsen.

Reassuringly we have now described all the machinery that is needed. i.e. sets, functions, subsets and the above definition of Kuratowski finite is enough of a mathematical foundation to prove the rest of the thesis.

We go through some basic consequences of these assumptions.

Lemma 1.2.1 1*, the terminal object in our background topos, is finite.*

Proof: 1 is the one element set, $1 = \{*\}$. We need to show that $1 \in F1$ where $F1$ is the free $\vee$-semilattice on 1. $F1$ is the intersection of all $X \subset P1$ which are closed under finite unions and which contain the image of $\{\} : 1 \to P1$. Any such X contains $\{*\} = 1$ and so $1 \in F1$ as required. $\square$

Lemma 1.2.2 *(Induction on finite sets) Say p is a proposition about finite subsets of some set X (i.e. $p \subseteq FX$) such that p is satisfied by the empty set and by all the singletons $\{x\}, x \in X$. If p also has the property that whenever p is satisfied by I and J then it is satisfied by $I \cup J$, then p is satisfied by all finite sets.*

Proof: The statement of the lemma tells us that $FX \subseteq p$ since FX is the least subset of PX satisfying conditions that are satisfied by p. $\square$

Lemma 1.2.3 *The product of two finite sets is finite. i.e. if $I \in FX$ and $J \in FY$ for two sets X, Y then $I \times J \in F(X \times Y)$*

Proof: Double induction. Consider the set:

$$\alpha \equiv \{I \times J | I \in FX, J \in FY\}$$

We need to show that if $\beta \subseteq P(X \times Y)$ is a set with the properties that

(i)$\{(x, y)\} \in \beta$ for every $x \in X$ and every $y \in Y$
(ii) $\phi \in \beta$
(iii) $A, B \in \beta$ then $A \cup B \in \beta$

then $\alpha \subseteq \beta$. First notice that certainly $\alpha_\phi, \alpha_{\{y\}} \subseteq \beta$ where

$$\alpha_\phi \equiv \{I \times \phi | I \in FX\} \ \alpha_{\{y\}} \equiv \{I \times \{y\} | I \in FX\}$$

The latter is by induction on FX. Finally for any $J \in FY$ define $\alpha_J = \{I \times J | I \in FX\}$. To prove that $\alpha \subseteq \beta$ clearly it is sufficient to verify that $\alpha_J \subseteq \beta$ for every finite J. But we can conclude 'for every finite J' by using using induction on FY. We have started this induction with the statement $\alpha_\phi, \alpha_{\{y\}} \subseteq \beta$ and shall now complete it by checking that $\alpha_{J_1}, \alpha_{J_2} \subseteq \beta$ implies $\alpha_{J_1 \cup J_2} \subseteq \beta$. This follows from the fact that β satisfies condition (iii) above. □

Lemma 1.2.4 *Say $f : A \to B$ is a function between sets A and B. Then the image of any finite subset of A is a finite subset of B.*

Proof: FA is the free join semilattice on the set A and so there exists a unique join preserving map Ff making the diagram

$$\begin{tikzcd} A \arrow[r, "\{\}"] \arrow[d, "f"] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\{\}"] & FB \end{tikzcd}$$

commute. But when we proved that FA is the free join semilattice on A we were able to give an explicit formula for Ff and from that formula it is clear that Ff is just the usual set theoretic image map. □

Lemma 1.2.5 *A join semilattice $(A, \vee, 0)$ has all finite joins.*

Proof: The set

$$\{I \in FA | \bigvee I \text{ exists }\}$$

contains the singletons and is closed under finite unions. Hence it is the whole of FA. □

It is an easy application of the induction lemma given above to prove for any distributive lattice A that

$$\forall I \subseteq A,\ I \text{ finite}, (\bigvee I) \wedge b = \bigvee \{a \wedge b | a \in I\}$$

(we know $\{a \wedge b | a \in I\}$ is finite since the image of any finite set is finite). Also note that $(FA)^{op}$ is the free meet semilattice on A, and so we see that meet semilattice $(A, \wedge, 1)$ has all finite meets in much the same way that we saw that any join semilattice has all finite joins. We now look at a slightly more complicated distributivity law:

Lemma 1.2.6 *If A is a distributive lattice and $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ are finite collections of elements of A. (I finite, or more precisely we assume $I \in FA$.) Then*

$$\wedge_{i \in I}(a_i \vee b_i) = \bigvee [(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i)]$$

where the join is taken over all pairs $J_1, J_2 \subseteq I$ such that $I = J_1 \cup J_2$, $\quad J_1, J_2$ finite.

Proof: We have assumed $I \in FA$ and so it is natural to go by the induction theorem already proven. The case when $I = \phi$ is trivial. Say $I = \{*\}$. We need to prove that

$$a_* \vee b_* = \bigvee[(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2})]$$

where the join is over pairs of subsets $J_1, J_2 \subseteq I$ such that $I \subseteq J_1 \cup J_2$. But

$$a_*, b_* \leq \bigvee[(\wedge_{i \in J_1} a_1) \wedge (\wedge_{i \in J_2} b_i)]$$

(take $J_1 = I \quad J_2 = \phi$ and then $J_1 = \phi \quad J_2 = I$). Say we are given $J_1, J_2 \subseteq I$, $I \subseteq J_1 \cup J_2$ then we will be done with the case $I = \{*\}$ if we can show

$$(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i) \leq a_* \vee b_*$$

Since $I \subseteq J_1 \cup J_2$ then either $* \in J_1$ or $* \in J_2$. In the former case we have

$$(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i) \leq a_*$$

and in the latter we have,

$$(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i) \leq b_*$$

And so

$$(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i) \leq a_* \vee b_*$$

as required.

Now say we are given two finite sets I_α, I_β (in FA) such that

$$\begin{array}{c} \wedge_{i \in I_\alpha}(a_i \vee b_i) = \bigvee[(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i)] \\ \wedge_{i \in I_\beta}(a_i \vee b_i) = \bigvee[(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i)] \end{array}$$

Then

$$\begin{aligned} \wedge_{i \in I_\alpha \cup I_\beta}(a_i \vee b_i) &= (\wedge_{i \in I_\alpha}(a_i \vee b_i)) \wedge (\wedge_{i \in I_\beta}(a_i \vee b_i)) \\ &= (\bigvee[(\wedge_{i \in J_1^\alpha} a_i) \wedge (\wedge_{i \in J_2^\alpha} b_i)] \wedge (\bigvee[(\wedge_{i \in J_1^\beta} a_i) \wedge (\wedge_{i \in J_2^\beta} b_i)]) \\ &= \bigvee[(\wedge_{i \in J_1^\alpha \cup J_1^\beta} a_i) \wedge (\wedge_{i \in J_2^\alpha \cup J_2^\beta} b_i)] \end{aligned}$$

where the last join is over quadruples $J_1^\alpha, J_2^\alpha (\subseteq I_\alpha), J_1^\beta, J_2^\beta (\subseteq I_\beta)$ such that $I_\alpha = J_1^\alpha \cup J_2^\alpha$ and $I_\beta = J_1^\beta \cup J_2^\beta$. We want this last line to be equal to

$$\bigvee_{I_\alpha \cup I_\beta = J_1 \cup J_2}[(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i)]$$

However for any J_1, J_2 in this last join set $J_i^\alpha = J_i \cap I_\alpha$ and set $J_i^\beta = J_i \cap I_\beta$ $(i = 1, 2)$. So J_i^α, J_i^β enjoy the property

$$\begin{array}{c} I_\alpha = J_1^\alpha \cup J_2^\alpha \\ I_\beta = J_1^\beta \cup J_2^\beta \end{array}$$

We see $J_i = J_i^\alpha \cup J_i^\beta$ for $i = 1, 2$ and so we see that

$$\bigvee[(\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i)] \quad \leq \quad \bigvee[(\wedge_{i \in J_1^\alpha \cup J_1^\beta} a_i) \wedge (\wedge_{i \in J_2^\alpha \cup J_2^\beta} b_i)]$$

The reverse inequality is easy. $\square$

1.3 The free Boolean algebra

We now address the question of constructing the free Boolean algebra on a distributive lattice. It is not possible in our context to use the usual finitary universal algebraic proof (e.g. Chapter 1 of [Joh87]) since this requires the natural numbers. We use a construction via congruence preorders which is equivalent to the more well known (e.g. [Pre93]) construction via congruences.

If D is a distributive lattice then $\precsim \subseteq D \times D$ is a congruence preorder if and only if it satisfies

$$\begin{array}{rcl} a \leq a_0 \precsim b_0 \leq b & \Rightarrow & a \precsim b \\ (\forall S \subseteq D \text{ finite}) \quad a \precsim b \quad \forall a \in S & \Rightarrow & \bigvee S \precsim b \\ (\forall S \subseteq D \text{ finite}) \quad a \precsim b \quad \forall b \in S & \Rightarrow & a \precsim \bigwedge S \\ a \precsim b, b \precsim c & \Rightarrow & a \precsim c \\ & a \precsim a & \end{array}$$

These were suggested to the author by Vickers and are an adaptation of his frame congruence preorders ([6.2.3] of [Vic89]).

Lemma 1.3.1 *There is an order preserving bijection between the poset of congruences on a distributive lattice and the poset of congruence preorders.*

Proof: Take a congruence $\equiv$ to the congruence preorder $\precsim$ where $a \precsim b \Leftrightarrow a \wedge b \equiv b$ and take a congruence preorder $\precsim$ to the congruence $\precsim \wedge \succsim$. $\Box$

Notice that the poset of congruence preorders on D (written $Con_P(D)$) has a least element $(\leq)$ and a greatest element $(D \times D)$.

Also notice that congruence preorders are closed under arbitrary intersection. It follows that the poset of congruence preorders has all joins. In particular it has finite joins. We prove that it is a distributive lattice:

Lemma 1.3.2 *$Con_P(D)$ is a distributive lattice.*

Proof: First note that it is sufficient to prove that for any $\precsim \in Con_P(D)$ the order preserving map

$$\precsim \cap (\_) : Con_P(D) \longrightarrow Con_P(D)$$

has a right adjoint. For then $\precsim \cap(\_)$ preserves arbitrary joins and so it certainly preserves finite joins. i.e. $Con_P(D)$ is distributive.

The right adjoint is given by

$$\precsim_0 \longmapsto \precsim / \precsim_0$$

where

$$\precsim / \precsim_0 \equiv \{(z, \bar{z}) | (z \wedge y) \precsim (\bar{z} \vee \bar{y}) \text{ whenever } y \precsim_0 \bar{y}\}. \ \Box$$

We construct the free Boolean algebra on a distributive lattice as a particular sublattice of $Con_P(D)$.

For all $a \in D$ define a pair of congruence preorders $\precsim_{[a,0]}, \precsim_{[1,b]}$ by

$$\begin{array}{rcl} x \precsim_{[a,0]} y & \Leftrightarrow & x \leq y \vee a \\ x \precsim_{[1,b]} y & \Leftrightarrow & x \wedge b \leq y \end{array}$$

Notice that

$$\begin{array}{c} \precsim_{[a,0]} \cap \precsim_{[1,a]} = \leq = 0_{Con_P(D)} \\ \text{and } \precsim_{[a,0]} \vee \precsim_{[1,a]} = D \times D = 1_{Con_P(D)} \end{array}$$

To see the latter note that

$$a \precsim_{[a,0]} 0 \text{ and } 1 \precsim_{[1,a]} a$$

and so $(a,0),(1,a) \in \precsim_{[a,0]} \vee \precsim_{[1,a]}$. But then $(1,0) \in \precsim_{[a,0]} \vee \precsim_{[1,a]}$ by transitivity of congruence preorders.

Thus $\precsim_{[a,0]}$ and $\precsim_{[1,b]}$ are complemented elements of $Con_P(D)$ for every a,b. It is easy to check, in any distributive lattice, that finite joins and finite meets of complemented elements are complemented. Define

$$\precsim_{[a,b]} \equiv \precsim_{[a,0]} \vee \precsim_{[1,b]}$$

So the set

$$B \equiv \{\wedge_{i \in I} \precsim_{[a_i,b_i]} | (a_i,b_i)_{i \in I} \text{ a finite collection of elements of } D\}$$

is a Boolean algebra. Any element of B can be expressed as

$$\wedge_{i \in I}(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]})$$

for some finite collection $(a_i,b_i)_{i \in I}$, where $\neg$ is the Boolean algebra negation.

There is a distributive lattice inclusion: $i : D \hookrightarrow B$ given by $i(a) = \precsim_{[a,0]}$.

Say $f : D \to \bar{B}$ is a distributive lattice homomorphism to some Boolean algebra $\bar{B}$. If we have found two finite sets of elements $\{a_i, b_i | i \in I\}, \{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ such that $\wedge_i(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]}) = \wedge_{\bar{i}}(\precsim_{[\bar{a}_{\bar{i}},0]} \vee \neg \precsim_{[\bar{b}_{\bar{i}},0]})$, we would like to check,

Lemma 1.3.3 $\wedge_i(fa_i \vee \neg fb_i) = \wedge_{\bar{i}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$

(For then it will be 'safe' to define $\phi : B \to \bar{B}$ by $\phi(\precsim) = \wedge_i(fa_i \vee \neg fb_i)$ for any collection $\{a_i, b_i | i \in I\}$ such that $\precsim = \wedge_i[\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]}]$.)

Proof: To conclude that $\wedge_I(fa_i \vee \neg fb_i) \leq \wedge_{\bar{I}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$ we need to prove that for every $\bar{i}$ and for every pair $J_1, J_2 \subseteq I$ with $I \subseteq J_1 \cup J_2$ we have

$$(\wedge_{i \in J_1} fa_i) \wedge (\wedge_{i \in J_2} \neg fb_i) \leq (f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$$

This relies on the finite distributivity law of Lemma [1.2.6] being applied to the meet $\wedge_i(fa_i \vee \neg fb_i)$. But the last inequality can be manipulated to

$$f((\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i) \leq f((\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i))$$

and so we want to check:

$$(\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i \leq (\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i) \qquad -(*)$$

But the assumption

$$\wedge_i(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]}) \leq \wedge_{\bar{i}}(\precsim_{[\bar{a}_{\bar{i}},0]} \vee \neg \precsim_{[\bar{b}_{\bar{i}},0]})$$

can via the same manipulations be shown to imply:

$$(\wedge_{i\in J_1} \precsim_{[a_i,0]} \wedge \precsim_{[\bar{b}_{\bar{i}},0]}) \vee \vee_{i\in J_2} \precsim_{[b_i,0]} \leq (\precsim_{[\bar{a}_{\bar{i}},0]} \wedge \precsim_{[\bar{b}_{\bar{i}},0]}) \vee (\vee_{i\in J_2} \precsim_{[b_i,0]}).$$

$(*)$ follows since i is a distributive lattice inclusion. $\square$

We check that ϕ, so defined, preserves finite meets. For if
$\precsim_1 = \wedge_{i\in I}(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]})$ and $\precsim_2 = \wedge_{i\in \bar{I}}(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]}) \quad \Rightarrow$
$\precsim_1 \wedge \precsim_2 = \wedge_{I\cup \bar{I}}(\precsim_{[a_i,0]} \vee \neg \precsim_{[b_i,0]})$. So

$$\begin{aligned}
\phi(\precsim_1 \wedge \precsim_2) &= \wedge_{I\cup\bar{I}}(fa_i \vee \neg fb_i) \\
&= [\wedge_{i\in I}(fa_i \vee \neg fb_i)] \wedge [\wedge_{i\in\bar{I}}(fa_i \vee \neg fb_i)] \\
&= \phi(\precsim_1) \wedge \phi(\precsim_2)
\end{aligned}$$

Similarly for $\vee$s.
Hence ϕ is the unique Boolean algebra homomorphism from B to $\bar{B}$ that satisfies the condition that $\phi \circ i = f$. i.e. B is the free Boolean algebra on the distributive lattice D.

We have one final use for our congruence preorders which is to show how they can be used to form the quotient of a distributive lattice by an ideal. An ideal I of a distributive lattice D is a subset of D which satisfies:

$$\begin{array}{ll}
(i) & I \text{ is lower closed. i.e. } \downarrow I = I, \\
(ii) & 0 \in I \\
(iii) & a, b \in I \text{ implies } a \vee b \in I
\end{array}$$

It follows immediately that for any ideal I the set

$$\precsim_I \equiv \{(x,y) | \exists i \in I \quad x \leq y \vee i\}$$

is a congruence preorder. We now quotient by the corresponding congruence, i.e. we define an equivalence relation $\equiv_I$ on D by $a \equiv_I b$ if and only if $a \precsim_I b$ and $b \precsim_I a$. Then the set of equivalence classes, $D/\equiv_I$, is a distributive lattice. The equivalence class of an element a in D is denoted by $[a]$. So there is a distributive lattice surjection $[\_] : D \to D/\equiv_I$. Given this construction we have

Lemma 1.3.4 *(i)* $[a] = [0]$ *if and only if* $a \in I$
(ii) For any second distributive lattice $\bar{D}$ *there is a bijection between the distributive lattice homomorphisms* $f : D/\equiv \to \bar{D}$ *and the distributive lattice homomorphisms* $\bar{f} : D \to \bar{D}$ *with the property that* $\bar{f}(a) = 0 \quad \forall a \in I$. *The bijection is given by*

$$f \longmapsto f \circ [\_]$$

Proof: (i) Say $a \in I$. Then $a \leq 0 \vee i$ for some $i \in I$ and $0 \leq a \vee i$ for some $i \in I$. i.e. $a \precsim_I 0$ and $0 \precsim_I a$ and so $a \equiv_I 0$. i.e. $[a] = [0]$.

Conversely if $[a] = [0]$ then $a \equiv_I 0$. So $a \precsim_I 0$. Hence $a \leq 0 \vee i$ for some $i \in I$. Therefore $a \in I$ as I is lower closed.

(ii) Say $f : D/\equiv_I \to \bar{D}$ is given. Then for all $i \in I$ $(f \circ [\_])(i) = f([i]) = f([0]) = 0$. Say $\bar{f} : D \to \bar{D}$ has property $\bar{f}(i) = 0$ for every $i \in I$. Define $f : D/\equiv_I \to \bar{D}$ by $f[a] = \bar{f}(a)$. This is well defined for if $[a_1] = [a_2]$ then $a_1 \precsim_I a_2$ and so there exists $i \in I$ such that $a_1 \leq a_2 \vee i$.

$$\bar{f}(a_1) \leq \bar{f}(a_2 \vee i) = \bar{f}(a_2) \vee \bar{f}(i) = \bar{f}(a_2) \vee 0 = \bar{f}(a_2)$$

Similarly $\bar{f}(a_2) \leq \bar{f}(a_1)$. It is also easy to see that f so defined is a distributive lattice homomorphism. Hence

$$f \longmapsto f \circ [\_]$$

is a surjection. Finally say

$$f_1 \circ [\_] = f_2 \circ [\_]$$

Then $f_1 = f_2$ since $[\_]$ is a surjection (surjections are epimorphisms). Hence $f \mapsto f \circ [\_]$ is a bijection. $\square$

1.4 Directed subsets

Alongside the finite subsets we have another important class of subsets, the *directed* subsets. These are particular subsets of posets.

Definition: A subset $\bar{A}$ of a poset A is said to be *directed* if and only if (i) $\exists a \in \bar{A}$ (ii) $\forall b, c \in \bar{A} \quad \exists d \in \bar{A}$ such that $b \leq d$ and $c \leq d$.

We use the up-arrow $\uparrow$ in $\bar{A} \subseteq^{\uparrow} A$ to denote the fact that $\bar{A}$ is a directed subset of A. Notice that a lower closed subset of a distributive lattice is an ideal if and only if it is a directed subset. We use the notation $\bigvee^{\uparrow}$ to denote the join of a set that is directed. A complete lattice is a poset with all joins.

Lemma 1.4.1 *Any join $\vee\bar{A}$ defined on a complete lattice A can be expressed as a directed join of finite joins of elements of $\bar{A}$.*

Proof: The set $\aleph \cong \{\vee B | B \subseteq \bar{A}, \quad B \in FA\}$ is a directed subset of A. Clearly $\bigvee^{\uparrow} \aleph = \vee\bar{A}$. $\square$

A poset is called a dcpo (directed complete partial order) if and only if all directed subsets have joins. A function between posets is a dcpo homomorphism iff it preserves directed joins. We have defined the category **dcpo**. If $x, y \in A$ for some dcpo A then we say that x is *way below* y and write $x \ll y$ iff for all directed $S \subseteq^{\uparrow} A$ if $y \leq \bigvee^{\uparrow} S$ then $x \leq s$ for some $s \in S$. An element $x \in A$ that is way below itself ($x \ll x$) is said to be *compact*. The set of directed lower subsets of a poset A is called the *ideal completion* of A and it is denoted $Idl(A)$. $Idl(A)$ is always a dcpo and there is a poset inclusion $\downarrow: A \to Idl(A)$ which takes an element of A to the set of elements lower than it in the order. $IdlA$ is the free dcpo on the poset A. The set of all dcpos of the form $IdlA$ for some poset A is important. They are called the *algebraic* dcpos. Given an algebraic dcpo an isomorphic copy of the poset of which it is an ideal completion can be found as the poset of compact elements. i.e. for every algebraic dcpo A if K_A is the poset of compact elements then $A \cong Idl(K_A)$ (where $\cong$ of course denotes the existence of an order preserving isomorphism between the two posets). Further if $IdlK_1 \cong IdlK_2$ then $K_1 \cong K_2$. We use **alg-dcpo** to denote the full subcategory of **dcpo** whose objects are the algebraic dcpos. Another characterization of the algebraic dcpos is the following: a dcpo A is algebraic iff $\forall a \in A$

(i) $\{b | b \ll b, b \leq a\}$ is directed
(ii) $\bigvee^{\uparrow}\{b | b \ll b, b \leq a\} = a$

A class larger than the class of algebraic dcpos is the class of *continuous posets*. A dcpo A is a continuous poset (or sometimes 'is continuous') if and only if

$$\text{(i) } \{b|b \ll a\} \text{ is directed for every } a \in A.$$
$$\text{(ii) } \bigvee^{\uparrow}\{b|b \ll a\} = a \quad \forall a \in A$$

Recall that if A, B are two objects of a category $\mathcal{C}$ then we say that A *is a retract of* B if and only if there are two maps $i : A \to B$, $p : B \to A$ in $\mathcal{C}$ such that $p \circ i = Id$. The following result is implicit in [Sco72]:

Lemma 1.4.2 (Scott) *A dcpo A is a continuous poset if and only if there exists an algebraic dcpo B such that A is a retract of B in* **dcpo**.

Proof: Say A is a continuous poset. Then $\twoheaddownarrow : A \to IdlA$ given by

$$\twoheaddownarrow(a) = \{b|b \ll a\}$$

is a dcpo map to an algebraic dcpo. But $\bigvee^{\uparrow} : IdlA \to A$ is also a dcpo map (it is left adjoint to $\downarrow$ and so preserves all joins) and $\bigvee^{\uparrow} \circ \twoheaddownarrow = Id$ by the definition of a continuous poset. Hence A is the retract of an algebraic dcpo.

Conversely say A is a retract of B, an algebraic dcpo. Certainly B is a continuous poset. So there exists dcpo maps $i : A \to B$ and $p : B \to A$ with the property $p \circ i = Id$. I claim that

$$a \ll_A \bar{a} \quad \Leftrightarrow \quad \exists \bar{b} \in B \quad a \leq p(\bar{b}) \quad \bar{b} \ll_B i(\bar{a})$$

Say $a \ll_A \bar{a}$ then since $i(\bar{a}) = \bigvee^{\uparrow}\{\bar{b}|\bar{b} \ll_B i(\bar{a})\}$, we can apply p to both sides and find that

$$\begin{aligned} \bar{a} = pi(\bar{a}) &= p(\bigvee^{\uparrow}\{\bar{b}|\bar{b} \ll_B i(\bar{a})\} \\ &= \bigvee^{\uparrow}\{p(\bar{b})|\bar{b} \ll_B i(\bar{a})\} \end{aligned}$$

and so $a \leq p(\bar{b})$ for some $\bar{b} \ll_B i(\bar{a})$.
Conversely say there exists $\bar{b} \in B$ such that $a \leq p(\bar{b})$ and $\bar{b} \ll_B i(\bar{a})$, and say $\bar{a} \leq \bigvee^{\uparrow} S$ for some $S \subseteq^{\uparrow} A$. Then

$$\begin{aligned} i(\bar{a}) &\leq i(\bigvee^{\uparrow} S) \\ &= \bigvee^{\uparrow}\{i(s)|s \in S\} \end{aligned}$$

Hence $\bar{b} \leq i(s)$ for some $s \in S$. We find that $a \leq s$ by applying p to both sides of this last conclusion. So I have verified my claim.
Notice that this claim in particular shows that if $\bar{a} \in A$ and $b \in B$ then $b \ll_B i(\bar{a})$ implies $p(b) \ll_A \bar{a}$. And so for any $\bar{a} \in A$

$$\begin{aligned} \bar{a} = pi(\bar{a}) &= p(\bigvee^{\uparrow}\{b|b \ll_B i(\bar{a})\} \\ &= \bigvee^{\uparrow}\{p(b)|b \ll_B i(\bar{a})\} \\ &= \bigvee\{a|a \ll_A \bar{a}\} \end{aligned}$$

Finally we need to check that the set $\{a|a \ll_A \bar{a}\}$ is directed for every $\bar{a} \in A$. This follows as an application of the claim from the fact that $\{b|b \ll_B \bar{b}\}$ is directed for every $\bar{b} \in B$. $\Box$

For technical use later we have

Lemma 1.4.3 *In a continuous lattice A the way below relation $\ll$ is interpolative. i.e. if $a \ll b$ then there exists c such that $a \ll c \ll b$.*

Proof: Define $S = \{d \in A|(\exists c \in A)(d \ll c \ll b)\}$. It follows that S is directed and

$$b \leq \bigvee^{\uparrow} S \ \Box$$

For more background on continuous posets consult 2.1 VII of [Joh82].

1.5 The Category Loc

A *frame* is a poset with all joins and finite meets such that the arbitrary joins distribute over finite meets. i.e. for any subset S of the frame and for any element a we have

$$\bigvee S \wedge a = \bigvee \{s \wedge a | s \in S\}$$

An example of a frame is the set of opens of a topological space. Frame homomorphisms are required to preserve finite meets and arbitrary joins. Given any continuous function $f : X \to Y$ for topological spaces X and Y it is clear that the inverse image of f is a frame homomorphism from the opens of Y to the opens of X. i.e.

$$f^{-1} : \Omega X \to \Omega Y$$

is a frame homomorphism where ΩX is the frames of opens of X and ΩY is the frame of opens of Y. We define **Loc**, the category of *locales*, to be the opposite of the category frames (=**Frm**). What has just been described is a functor from the category of topological spaces (**Sp**) to the category of locales:

$$\Omega{:}\mathbf{Sp}{\longrightarrow}\mathbf{Loc}$$

Having just given the impression that we shall talk about the locale ΩX we now confuse the reader by fixing a different notation for locales which will seem perverse to the newcomer: we shall talk about the locale X, but whenever we do any manipulations on it we shall talk about the corresponding frame of opens ΩX. The reason for doing this is to make sure that the discussions of locales and the discussions of frames are kept separate. Clearly the distinction is only mathematically important when we are dealing with the morphisms, but having a different notation for the objects will make it clearer which category we are working in. It will be tremendously helpful to talk about pullbacks and products of locales since these can be visualised as topological pullbacks and products and so having a distinct notation will help reinforce the spatial intuitions that are behind the localic results. Of course all this will seem like an irritating syntactic distraction for the newcomer.

If $f : X \to Y$ is a locale map between locales X and Y then we write Ωf for the corresponding frame homomorphism from ΩY to ΩX. Notice that since Ωf preserves arbitrary joins it has a right adjoint. This right adjoint is denoted $\forall_f$ and is given by the formula:

$$\begin{array}{rcl} \forall_f : \Omega X & \longrightarrow & \Omega Y \\ a & \longmapsto & \bigvee^{\uparrow} \{b | \Omega f(b) \leq a\} \end{array}$$

If Ωf has a left adjoint it is denoted by $\exists_f$.

The subobject classifier is a frame. If we assume the excluded middle it is the frame of two elements: true and false. In an arbitrary topos it is well known that the subobject classifier is the power set of the terminal object (i.e. $P1$ where $1 = \{*\}$) and clearly any power set is a frame with the order given by ordinary subset inclusion. In fact

Lemma 1.5.1 *Ω, the subobject classifier, is initial in the category of frames.*

Before proof let us make a seemingly innocuous observation: if $T \in P1$ then

$$T = \bigcup\{\{*\} | * \in T\}.$$

(Certainly $\cup\{\{*\}|* \in T\} \subseteq T$. Conversely for any $x \in T$ we have $x = *$. Hence $* \in T$ and so $x \in \cup\{\{*\}|* \in T\}$.) Expressed as a fact about the frame of opens of the locale it reads $\forall i \in \Omega$

$$i = \bigvee\{1 | 1 \leq i\}$$

This will be used a lot when reasoning about Ω. It corresponds to the idea of concluding that two propositions are equal whenever they logically imply each other.
Proof that Ω is initial: Say X is a locale. Define $! : X \to 1$ by

$$\begin{array}{rcl} \Omega! : \Omega & \longrightarrow & \Omega X \\ T & \longmapsto & \bigvee\{1_{\Omega X} | * \in T\} \end{array}$$

(Recall $\Omega = P\{*\}$.) Clearly $\Omega!$ preserves finite meets and arbitrary joins. Say $\Omega f : P\{*\} \longrightarrow \Omega X$ is some frame homomorphism. Then $\forall T \subseteq \{*\}$,

$$\begin{array}{rcl} \Omega f(T) & = & \Omega f \bigcup\{\{*\}|* \in T\} \\ & = & \bigvee\{\Omega f\{*\}|* \in T\} \\ & = & \bigvee\{\Omega f 1_\Omega | * \in T\} \\ & = & \bigvee\{1_{\Omega X}|* \in T\} = \Omega!(T) \quad \Box \end{array}$$

We use 1 to denote the locale corresponding to the frame Ω.

Given a locale X we can construct a topological space ptX ('point' X). The underlying set of ptX is given by

$$\{p | p : 1 \to X \quad p \text{ a locale map }\}$$

These ps are called the *points* of the locale X. (Not to be confused with the elements $a \in \Omega X$; they are the opens of the locale X.) The points of X correspond to frame homomorphisms from ΩX to Ω.

Notice that if $p_1 : 1 \to X$, $p_2 : 1 \to X$ are two points of some locale X then since $i = \bigvee\{1|1 \leq i\}$ for any $i \in \Omega$ we have that for any $a \in \Omega X$

$$\Omega p_1(a) = \bigvee\{1 | 1 \leq \Omega p_1(a)\}$$

It follows that if we know that for all $a \in \Omega X \quad \Omega p_1(a) = 1 \quad \Leftrightarrow \quad \Omega p_2(a) = 1$ then $p_1 = p_2$. It follows that a point is uniquely determined by the true kernel of its corresponding frame homomorphism.
The topology on this set of points is given by all sets of the form:

$$\{p | \Omega p(a) = 1\}$$

where a ranges over all elements of the frame ΩX and where 1 is the top element of the subobject classifier Ω. That this set forms a topology follows easily enough from the fact that Ωp is a frame homomorphism for any point p.

If $f : X \to Y$ is a locale map then composition of arrows in **Loc** clearly defines a function from the underlying set of ptX to the underlying set of ptY; it is easy to see that this function is continuous and so we can view pt as a functor:

$$pt{:}\mathbf{Loc}{\longrightarrow}\mathbf{Sp}$$

Theorem 1.5.1 *pt is right adjoint to Ω.*

Proof: Define a natural transformation $\eta : Id \stackrel{\cdot}{\to} pt\Omega$ by

$$\begin{array}{rcl} \eta_X : X & \longrightarrow & pt\Omega X \\ x & \longmapsto & f_x \end{array}$$

Where $f_x(U) = \bigcup\{\{*\}|x \in U\}$. So $f_x(U) = 1 \quad \Leftrightarrow \quad x \in U$, and from now on we will define points by simply giving the true kernel of the corresponding frame homomorphism. The reader can check that (i) f_x is a frame homomorphism for every x, (ii) η_X is continuous for every space X and (iii) η is a natural transformation.
To define a natural transformation $\epsilon : \Omega pt \stackrel{\cdot}{\to} Id$ we need to define a map

$$\epsilon_Y : \Omega ptY \longrightarrow Y$$

in **Loc** for every locale Y. We define a class of frame homomorphisms by

$$\begin{array}{rcl} \Omega\epsilon_Y : \Omega Y & \longrightarrow & \Omega ptY \\ a & \longmapsto & \{p|\Omega p(a) = 1\} \end{array}$$

Warning: notation does clash here. When the functor Ω is applied to the space X we get a locale ΩX. However the frame of opens of this locale is denoted by ΩX rather than $\Omega\Omega X$.
The reader can check that $\Omega\epsilon_Y$ is a frame homomorphism for every Y and that ϵ, so defined, is a natural transformation.
So to verify that $\Omega \dashv pt$ we just need to check the triangular equalities for η and ϵ. We first examine

$$\begin{tikzcd} \Omega X \arrow[r, "\Omega\eta_X"] \arrow[dr, "id"'] & \Omega pt\Omega X \arrow[d, "\epsilon_{\Omega X}"] \\ & \Omega X \end{tikzcd}$$

This amounts to checking that

$$\eta^{-1}\Omega\epsilon_{\Omega X}(U) = U \qquad \forall U \in \Omega X$$

i.e. that $\eta^{-1}\{p|\Omega p(U) = 1\} = U$. But

$$\begin{array}{rcl} x \in \eta^{-1}\{p|\Omega p(U) = 1\} & \Leftrightarrow & f_x \in \{p|\Omega p(U) = 1\} \\ & \Leftrightarrow & f_x(U) = 1 \quad \Leftrightarrow \quad x \in U \end{array}$$

The other triangular equality is

$$\begin{tikzcd} ptY \arrow[r, "\eta_{ptY}"] \arrow[dr, "id"'] & pt\Omega ptY \arrow[d, "pt\epsilon_Y"] \\ & ptY \end{tikzcd}$$

Say $\bar{p} \in ptY$. So $\bar{p} : 1 \to Y$ is a locale map. Then $\eta_{ptY}(\bar{p})$ is a locale map from 1 to ΩptY. It is given by the function $p_{\bar{p}} : \Omega ptY \to \Omega$ where

$$p_{\bar{p}}(U) = 1 \quad \Leftrightarrow \bar{p} \in U$$

$pt\epsilon_Y$ takes $p_{\bar{p}}$ to the composition

$$\Omega Y \xrightarrow{\Omega\epsilon_Y} \Omega ptY \xrightarrow{p_{\bar{p}}} \Omega$$

But $\forall a \in \Omega Y$

$$\begin{aligned} p_{\bar{p}}\Omega\epsilon_Y(a) = 1 &\Leftrightarrow p_{\bar{p}}\{p|\Omega p(a) = 1\} = 1 \\ &\Leftrightarrow \bar{p} \in \{p|\Omega p(a) = 1\} \\ &\Leftrightarrow \Omega\bar{p}(a) = 1 \end{aligned}$$

Thus $pt\epsilon_Y \circ \eta_{ptY}(\bar{p}) = \bar{p}$. $\Box$

A short note is appropriate at this point to the effect that 'category theory is constructive'; to conclude that the triangular equalities are enough to imply an adjunction we are of course assuming the well known categorical proof which verifies this fact. This categorical proof (see [Mac71] p81 theorem 2(v)) is easily seen to be constructive (it does not rely on the excluded middle) and so our overall proof that $\Omega \dashv pt$ is constructive. At a couple of other points in the thesis we will say 'by a well known categorical result...', and in all cases the proof being referred to is constructive.

We say that a locale X is *spatial* if and only if ΩptX is isomorphic (via the unit of the adjunction) to X and that a space Y is *sober* if and only if $pt\Omega Y$ is isomorphic to Y via the counit. Crucially: 'most' spaces are sober and so we can view the category of locales as a sensible (almost) generalisation of topological spaces. Further, in practice, most locales are spatial and so the category of locales is (in practice) not a massive generalization of the category of spaces.

Theorem 1.5.2 *The retracts of spatial locales are spatial.*

Proof: This is really just a piece of category theory. Say Y is spatial; i.e. ϵ_Y is an isomorphism in the category **Loc**. Let X be a retract of Y; say there exists $i : X \hookrightarrow Y$ and $p : Y \to X$ with the property that $p \circ i = 1$. I claim that

$$\epsilon_X^{-1} = \Omega pt(p) \circ \epsilon_Y^{-1} \circ i$$

For

$$\begin{aligned} \epsilon_X \circ \Omega pt(p) \circ \epsilon_Y^{-1} \circ i &= p \circ \epsilon_Y \circ \epsilon_Y^{-1} \circ i \\ &= p \circ i = 1, \\ \Omega pt(p) \circ \epsilon_Y^{-1} \circ i \circ \epsilon_X &= \Omega pt(p) \circ \epsilon_Y^{-1} \circ \epsilon_Y \circ \Omega pt(i) \\ &= \Omega pt(p) \circ \Omega pt(i) \\ &= \Omega pt(p \circ i) = \Omega pt(1) = 1 \qquad \Box \end{aligned}$$

1.6 Some Constructively Spatial Locales

We now look at an example of the $\Omega \dashv pt$ adjunction being applied to certain subclasses of locales and spaces. It will be useful to recall that for any topological space X we can define a specialization order between the points of the space:
$x_1 \sqsubseteq x_2$ if and only if

$$\forall U \in \Omega X \quad x_1 \in U \quad \Rightarrow \quad x_2 \in U$$

Notice that a simple argument proves that any continuous function between spaces preserves the specialization order.
Given an algebraic dcpo X we say that $U \subseteq X$ is *Scott open* iff $\uparrow U = U$ (i.e. $\forall x \in U$ if $y \geq x$ then $y \in U$; U is *upper closed*) and for every directed subset $S \subseteq^{\uparrow} X$ if $\bigvee^{\uparrow} S \in U$ then $\exists s \in S$ such that $s \in U$. The set of Scott open subsets of a dcpo X is denoted ΣX. It is a frame with the order given by subset inclusion.

Theorem 1.6.1 *If X is an algebraic dcpo then ΣX is isomorphic as a poset to $\mathcal{A}(K_X)$ where K_X is the poset of compact elements of X and $\mathcal{A}(K_X)$ is the set of all upper closed subsets of K_X.*

Proof: Clearly the maps

$$\begin{array}{rcl} \phi : \Sigma X & \longrightarrow & \mathcal{A}(K_X) \\ U & \longmapsto & \{k \in K_X | k \in U\} \\ \psi : \mathcal{A}(K_X) & \longrightarrow & \Sigma X \\ V & \longmapsto & \bigcup_{k \in V} \uparrow k \end{array}$$

preserve order. Trivially $\phi\psi(V) = V$ for all $V \subseteq K_X$ with $\uparrow V = V$.

We show $\psi\phi(U) = U$ for every Scott open U. Now

$$\psi\phi(U) \subseteq U$$

since U is upper. In the other direction recall that for every $x \in U$

$$x = \bigvee\nolimits^{\uparrow}\{k | k \in K_X \quad k \leq x\}$$

since X is algebraic. But U is Scott open and so there exists $k \leq x$ such that $k \in K_X \cap U$. i.e. $k \in \psi(U)$. Hence

$$x \in \bigcup\nolimits_{k \in \psi(U)} \uparrow k = \psi\phi(U) \ \Box$$

We call a topological space $(X, \Omega X)$ *Scott* if and only if X has a partial order on it which makes it into an algebraic dcpo and $\Omega X = \Sigma X$. Let **ScottSp** be the full subcategory of **Sp** whose objects are all the Scott spaces.

Lemma 1.6.1 *If X is a Scott space then the order of the dcpo is the specialization order.*

Proof: Say $x_1 \leq x_2$ in the dcpo order and $x_1 \in U$ for some Scott open U. Then $x_2 \in U$ since Scott opens are upper closed. Hence $x_1 \sqsubseteq x_2$ in the specialization order.
Conversely say $x_1 \sqsubseteq x_2$ in the specialization order. Then if $k \leq x_1$ for some compact k we see that $x_1 \in \uparrow k$. But $\uparrow k$ is a Scott open since k is compact, and we find that $x_2 \in \uparrow k$ by the definition of specialization order. i.e. $k \leq x_2$ for every compact k less than x_1. But x_1 is the join of all compact elements less than it, and so $x_1 \leq x_2$ in the dcpo order. $\Box$

Lemma 1.6.2 alg-dcpo≅ScottSp

Proof: Clearly, by definition, both these categories essentially share the same objects. All that remains is to check that directed join preserving functions between dcpos correspond to continuous function between Scott spaces.
Say $f : X \to Y$ is a directed join preserving function between dcpos X and Y. Say $U \subseteq Y$ is Scott open. Certainly $f^{-1}U$ is upper (N.B. it is easy to check that if f preserves directed joins then it preserves order, for if $x \leq y$ then $\{x, y\}$ is directed). Now say $S \subseteq^{\uparrow} X$ and $\bigvee^{\uparrow} S \in f^{-1}U$. Then $f(\bigvee^{\uparrow} S) \in U \quad \Rightarrow \quad \bigvee^{\uparrow}\{fs|s \in S\} \in U$ and so there exists an s in S such that $fs \in U$. Hence there exists an s in $f^{-1}U$ and we see that $f^{-1}U$ is Scott open. So $f : X \to Y$ is a continuous function.
Conversely say $f : X \longrightarrow Y$ is a continuous function between Scott spaces. So we know that it preserves the specialization order by an earlier remark, and since we have a lemma to the effect that the specialization order and the dcpo order coincide in this case we know that f preserves the dcpo order. Hence if $S \subseteq^{\uparrow} X$ is a directed subset of X we have that

(i) $\{fs|s \in S\}$ is a directed subset of Y
(ii) $\bigvee^{\uparrow}\{fs|s \in S\} \leq f(\bigvee^{\uparrow} S)$.

Say $k \leq f(\bigvee^{\uparrow} S)$ (k compact). Then $\uparrow k$ is open in Y as it is Scott open. Thus $f^{-1}(\uparrow k) \in \Omega X$. But $\bigvee^{\uparrow} S \in f^{-1}(\uparrow k)$ and so $\exists s \in S$ such that $s \in f^{-1}(\uparrow k) \quad \Rightarrow$ $k \leq fs \leq \bigvee^{\uparrow}\{fs|s \in S\}$. Hence $f(\bigvee^{\uparrow} S) \leq \bigvee^{\uparrow}\{fs|s \in S\}$ since every element of Y is the join of compact elements less than it. □

Thus **ScottSp** is just the full subcategory of dcpos given by the algebraic dcpos. But what are the locales that are going to correspond to the Scott spaces? They are the Alexandrov locales. A locale X is said to be *Alexandrov* if and only if $\Omega X = \mathcal{A}(K)$ for some poset K. Let **AlexLoc** be the full subcategory of **Loc** consisting of those locales which are Alexandrov.

Theorem 1.6.2 *pt, Ω define an equivalence* **ScottSp** $\cong$ **AlexLoc**.

Some work has been done already in the proof of Lemma [1.6.1]. This allowed us to conclude $\Sigma Idl(K) \cong \mathcal{A}(K)$ for any poset K. All we need to do is prove that Scott spaces are sober and Alexandrov locales are spatial.

Scott spaces are sober. We need to check that $\eta_X : X \to pt\Omega X$ $(x \mapsto p_x)$ is a homeomorphism between topological spaces for any Scott space X. Recall that $\Omega p_x(U) = 1 \quad \Leftrightarrow \quad x \in U$.
Say $\Omega p : \Omega X \to \Omega$ is the frame homomorphism corresponding to some point p of X. We know $\Omega X \cong \mathcal{A}(K_X)$ where K_X is the set of compact elements of X. Define $I_p \subseteq K_X$ by

$$I_p \equiv \{k|\Omega p(\uparrow k) = 1\}$$

Now certainly $\Omega p(K_X) = 1$. But $K_X = \bigcup\{\uparrow k|k \in K_X\}$. And so the following are equivalent,

$$\begin{array}{rcl} \{*\} = 1_{\Omega} & = & \bigcup\{\Omega p(\uparrow k)|k \in K_X\} \\ * & \in & \Omega p(\uparrow k) \text{ for some } k \in K_X \\ 1 & = & \Omega p(\uparrow k) \text{ for some } k \in K_X \\ k & \in & I_p \text{ for some } k \in K_X \\ & . & \end{array}$$

i.e. I_p is nonempty.
Say $k_1, k_2 \in I_p \quad \Rightarrow \quad \Omega p(\uparrow k_1) = 1, \Omega p(\uparrow k_2) = 1$. Then $1 \subseteq \Omega p(\uparrow k_1) \cap \Omega p(\uparrow k_2)$. i.e. $1 \subseteq \Omega p(\uparrow k_1 \cap \uparrow k_2)$. But

$$\uparrow k_1 \cap \uparrow k_2 = \bigcup \{\uparrow k | k_1, k_2 \leq k, \quad k \in K_X\}$$

and so by a similar argument (i.e. using the facts that $1_\Omega = \{*\}$ and join is given by union in Ω) we get that $\Omega p(\uparrow k) = 1$ for some $k \in K_X$ with $k_1, k_2 \leq k$. i.e. $k \in I_p$ and I_p is seen to be directed. i.e. $I_p \in IdlK_X \cong X$.
Thus $f : p \mapsto I_p$ is a function from the space $pt\Omega X$ to the space X. Is it continuous? Say $U \subseteq X$ is an open subset of X. Then for any $p \in f^{-1}U$ we have $I_p \in U$. But

$$I_p = \bigvee^{\uparrow} \{k | k \in I_p\}$$

and U is Scott open, so there exists k in I_p such that $k \in U$. Therefore
$1 = \Omega p(\uparrow k) \subseteq \Omega p(U)$. Hence $\Omega p(U) = 1$.
Conversely say $\Omega p(U) = 1$.

$$U = \bigcup \{\uparrow k | k \in U\}$$

Hence (again using the fact that $1_\Omega = \{*\}$) there exists $k \in U$ with $\Omega p(\uparrow k) = 1$. So $k \in I_p$ and hence $I_p \in U$ since U is upper closed. This last implies $p \in f^{-1}U$. It follows that

$$p \in f^{-1}U \quad \Leftrightarrow \quad \Omega p(U) = 1$$

i.e. $f^{-1}U = \{p | \Omega p(U) = 1\}$, and so f^{-1} is open implying that f is continuous. Notice we have also verified that $I_p \in U \quad \Leftrightarrow_* \Omega p(U) = 1$.
We check that $f \circ \eta_X(x) = x \quad \forall x \in X$ and $\eta_X \circ f(p) = p \quad \forall p \in pt\Omega X$ and so conclude that any Scott space is sober.

$$\begin{aligned} f \circ \eta_X(x) &= f(p_x) = I_{p_x} \\ &= \{k | \Omega p_x(\uparrow k) = 1\} \\ &= \{k | x \in \uparrow k\} \\ &= \{k | k \leq x\} \end{aligned}$$

But the ideal of the last line corresponds to x under the isomorphism $IdlK_X \cong X$.

$$\begin{aligned} (\eta_X \circ f(p))(U) = 1 &\Leftrightarrow p_{I_p}(U) = 1 \\ &\Leftrightarrow I_p \in U \\ &\Leftrightarrow \Omega p(U) = 1 \end{aligned}$$

The last equivalence is by the observation $(*)$ above. Hence $\eta_X \circ f = Id$ and $f \circ \eta_X = Id$.

Alexandrov locales are spatial: The frame homomorphism corresponding to $\epsilon_Y : \Omega pt Y \to Y$ is given by $\Omega \epsilon_Y(a) = \{p | \Omega p(a) = 1\}$. Clearly $\Omega \epsilon_Y$ is a surjective frame homomorphism. We would like to prove that it is injective whenever $\Omega Y = \mathcal{A}(K)$ for some poset K.
Say we have $a, b \in \Omega Y$ with the property that $\forall p : 1 \to Y$ (i.e. for all points p of Y) we have $\Omega p(a) = 1 \quad \Leftrightarrow \quad \Omega p(b) = 1$. Does this imply $a = b$?
Well $a = T$ for some $T \subseteq K \quad \uparrow T = T$ and $b = S$ for some $S \subseteq K \quad \uparrow S = S$ since $\Omega Y = \mathcal{A}(K)$ for some poset K.
Say $k \in K$. Define $\Omega p_k : \Omega Y \to \Omega$ by $\Omega p_k(\bar{T}) = 1 \quad \Leftrightarrow \quad k \in \bar{T}$ for all $\bar{T}$ in ΩY. Now say $k \in T$. Then $\Omega p_k(T) = 1$. Thus $\Omega p_k(S) = 1$. Thus $k \in S$. Hence $T \subseteq S$. Symmetrically we get $S \subseteq T$. So $S = T$ and $\Omega \epsilon_Y$ is injective. Alexandrov locales

are spatial. □

To a certain extent this example is forced. There is no real reason to investigate the Scott spaces, other than that by looking at them it is clear that we can use the pt, Ω adjunction in order to prove the result of interest, namely that the algebraic dcpos as a full subcategory of all dcpos is equivalent to the Alexandrov locales. (And even this is not the most straightforward way of looking at the result: we can't justify looking at locales unless we are trying to model a particular class of spaces and we have just said that we are not really looking spaces, we are looking at dcpos. The result, most simply stated, is a statement to the effect that the category whose objects are $\mathcal{A}(K)$ for posets K and whose morphisms are frame homomorphisms between them is dual to the full subcategory of dcpos consisting of the algebraic dcpos.) However there are reasons to examine this particular example of the pt, Ω adjunction in action over others: it is constructive. Thus, in our current constructive framework, we are permitted to make statements like '...if X is an Alexandrov locale and $x \in X$ then...' since we know that we constructively have points.

However most proofs that particular classes of locales are spatial (and hence can be thought of as spaces) are classical: they require some choice axioms. We will see these proofs in the final section of this chapter.

A special case of the Alexandrov locales is important: the discrete locales. These are defined as those locales whose frame of opens are the upper completions ($\mathcal{A}$) of discrete posets. A poset is discrete iff $x \leq y$ implies $x = y$. We use **DisLoc** to denote the full subcategory of **Loc** consisting of the discrete locales. All discrete locales are spatial since the Alexandrov locales are spatial.
Clearly the discrete locales are exactly those locales X such that $\Omega X = PA$ for some set A, and spatially we are thinking of the discrete spaces. A restriction of the equivalence **alg-dcpo**$\cong$**AlexLoc** to the discrete locales shows us that **Set** $\cong$ **DisLoc** where **Set** is the underlying topos. To see this last conclusion note that $K \cong Idl(K)$ if K is a discrete poset.

We now turn to the retracts of the Alexandrov locales. These are spatial by Theorem [1.5.2], and we might hope that they correspond to the continuous posets given that we know that the continuous posets are the retracts of the algebraic dcpos and the algebraic dcpos correspond to the Alexandrov locales. Indeed this fact can be verified (we point the reader to [Vic93] for a formal proof however). The rest of this section contains a discussion of another characterization of the class of localic retracts of the Alexandrov locales. They are the *completely distributive locales.* i.e. those locales whose frame of opens is a completely distributive lattice. The usual definition of a completely distributive lattice is roughly 'arbitrary joins distribute over arbitrary meets'. Technically this amounts to the statement: if $\{J_i | i \in I\}$ is an indexed family of sets then

$$\bigwedge\{\bigvee J_i | i \in I\} = \bigvee\{\wedge\{f(i) | i \in I\} | f \in F\}$$

where $F = \{f : I \to \coprod_{i \in I} J_i | f(i) \in J_i \quad \forall i\}$.

However showing results about completely distributive lattices with this definition can often require the axiom of choice: e.g. showing that the opposite of a completely distributive lattice is completely distributive requires the axiom of choice (e.g. lemma VII (1.10) of [Joh82]). Fawcett, Roseburgh and Wood address the problem of trying to find a constructive version of the complete distributivity axiom. They say that a complete lattice A is constructively completely distributive if and only if the join map $\bigvee : \mathcal{D}(A) \to A$ (where $\mathcal{D}$ denotes the action of taking all lower closed subsets) has a left adjoint. We see ([FW90],[RW91]) that the notions of constructive complete distributivity and ordinary complete distributivity coincide if and only if we assume the axiom of choice.

It might appear that a definition in terms of the existence of an adjoint is out of step with some of our other definitions; however note that a dcpo A is continuous if and only if $\bigvee^{\uparrow} : IdlA \to A$ has a left adjoint.

As an aside it is worth mentioning that the opposite of a constructively completely distributive lattice can be proven to be constructively completely distributive if *and only if* we assume the excluded middle. Thus we can translate the excluded middle into a statement about constructively completely distributive lattices. See [RW91].

We say that a locale X is CCD (constructively completely distributive) if and only if ΩX is a constructively completely distributive lattice. Let **CCDLoc** denote the full subcategory of **Loc** whose objects are CCD.

Theorem 1.6.3 *A locale X is CCD if and only if it is the retract of some Alexandrov locale.*

Proof: Consult [Vic93]. □

1.7 Locale Theory

The preceding discussion about the $\Omega \dashv pt$ adjunction is just a piece of history. It serves to convince the doubtful reader that the category of locales is a plausible environment in which to do topological space theory. From now on we shall take this motivation for granted, forget that spaces ever existed and develop locale theory as if it was topological space theory. Occasionally the topological intuitions behind what we do are explicitly referred to but mostly this is done implicitly through the choices we make of topological adjectives used to describe localic concepts. For more motivation consult [Joh82], [Isb72] and [Joh91].

1.7.1 Sublocales

If $X_0 \rightarrowtail X$ is a subspace inclusion, then its inverse image (going to the subspace topology) is a surjection. We take this as our definition of a sublocale: a locale map $X_0 \to X$ is a sublocale if and only if the corresponding frame homomorphism is a surjection. The sublocales form a poset which is denoted by $Sub(X)$.
There are two important classes of sublocales: the closed sublocales and the open sublocales. The spatial intuition behind these classes of sublocales is the idea of closed and open subspaces.
Given a locale X and an element a of ΩX we can define two surjections away from ΩX.
Open:

$$\begin{array}{rcl} \Omega X & \longrightarrow & \downarrow a \\ b & \longmapsto & a \wedge b \end{array}$$

and closed:

$$\begin{array}{rcl} \Omega X & \longrightarrow & \uparrow a \\ b & \longrightarrow & a \vee b \end{array}$$

Within the category of locales we use the expressions

$$\begin{array}{c} a \rightarrowtail X \\ \neg a \rightarrowtail X \end{array}$$

to refer to the locale maps corresponding to these two frame surjections. Spatially when we write $\neg a \hookrightarrow X$ we are thinking of the closed subspace corresponding to the set theoretic complement of the open a.

Notice that we can take the closure of any sublocale. The closure of $X_0 \hookrightarrow X$ is

$$\neg\forall_i(0) \hookrightarrow X$$

Lemma 1.7.1 *For any sublocale $i : X_0 \hookrightarrow X$ and closed sublocale $\neg a \hookrightarrow X$*

$$X_0 \leq_{Sub(X)} \neg a \quad \Leftrightarrow \quad \neg\forall_i(0) \leq_{Sub(X)} \neg a$$

Proof: First note that $X_0 \leq_{Sub(X)} \neg\forall_i(0)$, for we can define a frame homomorphism,

$$\begin{array}{rcl} \Omega n : \uparrow \forall_i(0) & \rightarrow & \Omega X_0 \\ \forall_i(0) \vee a & \mapsto & \Omega i(a). \end{array}$$

(This is well defined since $\Omega i \forall_i(0) = 0$.) Also note that the diagram

$$\begin{tikzcd} X_o \arrow[rr, "n"] \arrow[dr, "i"] & & \neg\forall_i(0) \arrow[dl] \\ & X & \end{tikzcd}$$

commutes in **Loc** proving $X_0 \leq_{Sub(X)} \neg\forall_i(0)$.

Further note $\neg\forall_i(0) \leq \neg a$ if and only if $a \leq \forall_i(0)$. (Essentially because

$$\begin{array}{rcl} \Omega n_a : \uparrow a & \longrightarrow & \forall_i(0) \\ a \vee b & \longmapsto & \forall_i(0) \vee b \end{array}$$

is a well defined frame homomorphism if and only if $a \leq \forall_i(0)$.)
But $a \leq \forall_i(0)$ if and only if

$$\begin{array}{rcl} \Omega p : \uparrow a & \longrightarrow & \Omega X_0 \\ a \vee b & \longmapsto & \Omega i(a) \end{array}$$

is a well defined frame homomorphism and so

$$X_0 \leq \neg a \quad \Leftrightarrow \quad \neg\forall_i(0) \leq \neg a$$

as required. $\square$

1.7.2 Denseness

A locale map $f : X \to Y$ is dense if and only if $\forall a \in \Omega Y(\Omega f(a) = 0 \Rightarrow a = 0)$. It is clear from the formula for the right adjoint to Ωf that density of f is just the assertion that $\forall_f(0) = 0$.
If $f : X_0 \rightarrowtail X$ is some sublocale of X then it is a dense sublocale of its closure.

If $a, b \in \Omega X$ for some locale X then $a \to b \in \Omega X$ is given by the formula

$$a \to b = \bigvee^{\uparrow}\{c | a \wedge c \leq b\}$$

$\to$ is the well known *Heyting arrow* (see I 1.10 of [Joh82]); it enjoys the property that for any $a, b, c \in \Omega X$

$$a \wedge b \leq c \quad \Leftrightarrow \quad a \leq b \to c$$

We introduce it here since it is needed in the following example of a dense sublocale: given any locale X define a new locale $X_{\neg\neg}$ by $\Omega(X_{\neg\neg}) = \{a \in \Omega X | \neg\neg a = a\}$ where $\neg$ is the Heyting negation, i.e. $\neg a = a \to 0$. Notice that the map

$$\begin{array}{ccc} \Omega X & \longrightarrow & \Omega X_{\neg\neg} \\ a & \longmapsto & \neg\neg a \end{array}$$

is a surjective frame homomorphism and so we have a sublocale $X_{\neg\neg} \hookrightarrow X$. The fact that $(\neg\neg a = 0 \Rightarrow a = 0)$ means that this inclusion is dense. Indeed it is the least dense sublocale of X. It is not the case that all topological spaces have least dense subspaces.

1.7.3 Separation axioms

A locale X is said to be *compact* if whenever we have a directed subset S of ΩX such that the join of S is the top element of X then the top element of X is in S. Clearly this is the localic analogy to the spatial idea of compactness.
Given two elements a, b of a frame ΩX we say $a \lhd b$ (*a well inside b*) if and only if $\exists\, c \in \Omega X$ such that

$$\begin{array}{rcl} a \wedge c & = & 0 \\ b \vee c & = & 1 \end{array}$$

Lemma 1.7.2 *$a \lhd b \quad \Leftrightarrow \quad \neg a \vee b = 1$ where $\neg a$ is the Heyting negation of a. i.e. $\neg a = \bigvee^{\uparrow}\{\bar{a} | \bar{a} \wedge a = 0\}$.*

Proof If $a \lhd b$ then there exists c with $a \wedge c = 0$ and $b \vee c =$. But $a \wedge c = 0$ implies that $c \leq \neg a$ since $\neg a = \bigvee^{\uparrow}\{\bar{c} | \bar{c} \wedge a = 0\}$. Hence $\neg a \vee b = 1$.
If $\neg a \vee b = 1$ then certainly $a \lhd b$ since $a \wedge \neg a$ is always equal to 0. $\Box$

We say that a locale X is *regular* if and only if $\forall a \in \Omega X$

$$a = \bigvee^{\uparrow}\{b | b \lhd a\}$$

Recall that a topological space X is regular if and only if for every closed F and every $x \notin F$ there are disjoint opens U, V such the $F \subseteq U$ and $x \in V$. This condition implies and is implied by the condition: for every open W

$$W = \bigcup^{\uparrow}\{V | V \lhd W\}$$

i.e. a topological space is regular if and only if the locale whose frame of opens are the opens of the space is regular.

Theorem 1.7.1 **(a)**: *A sublocale of a regular locale is regular.*
(b): *A closed sublocale of a compact locale is compact.*
(c): *A compact sublocale of a regular locale is closed.*

Proof: **(a)** Say $i : X_0 \hookrightarrow X$ is a sublocale such that X is regular. Clearly $a \lhd b \quad \Rightarrow \quad \Omega i(a) \lhd \Omega i(b)$. If $a \in \Omega X_0$ then $a = \Omega i(a_0)$ for some a_0 in ΩX. But

$$a_0 = \bigvee^{\uparrow}\{b|b \lhd a_0\}$$

Hence

$$\begin{aligned} a = \Omega i(a_0) &= \bigvee^{\uparrow}\{\Omega i(b)|b \lhd a_0\} \\ &\leq \bigvee^{\uparrow}\{c|c \lhd \Omega i(a_0)\} \\ &\leq \Omega(a_0) = a \end{aligned}$$

(b) Say $\neg a \hookrightarrow x$ is a closed sublocale of X and X compact. So $\Omega(\neg a) = \uparrow a$. Say $S \subseteq^{\uparrow} \uparrow a$ and $\bigvee^{\uparrow} S = 1_{\uparrow a} = 1_{\Omega X}$. Then $S \subseteq^{\uparrow} \Omega X$ and $\bigvee^{\uparrow} S = 1_{\Omega X}$. Hence $\exists s \in S$ such that $s = 1_{\Omega X} = 1_{\uparrow a}$ i.e. $\uparrow a$ is the frame of opens of a compact locale. i.e. $\neg a$ is compact.
(c) Say $i : X_0 \hookrightarrow X$ is a sublocale such that X_0 is compact and X is regular. We know that i can be factored as

$$X_0 \hookrightarrow \neg\forall_i(0) \hookrightarrow X$$

where the first part of the composition is dense. By **(a)** we know that $\neg\forall_i(0)$ is regular, and so we can conclude our result provided we show that if $i : X_0 \hookrightarrow X$ is also dense then it is an isomorphism.
First we check that $\forall a \in \Omega X$ if $\Omega i(a) = 1$ then $a = 1$. Certainly $a = \bigvee^{\uparrow}\{b|b \lhd a\}$ since X is regular. So

$$1 = \Omega i(a) = \bigvee^{\uparrow}\{\Omega i(b)|b \lhd a\}$$

Hence $\exists b \lhd a$ such that $\Omega i(b) = 1$ (as X_0 is compact). Thus $\exists c \quad b \wedge c = 0 \quad a \vee c = 1$. Thus $\Omega i(c) = \Omega i(b) \wedge \Omega i(c) = \Omega i(b \wedge c) = 0$. But this implies $\forall_i \Omega i(c) = 0$ as $\forall_i(0) = 0$ since i is assumed to be dense. And so $c = 0$ because $c \leq \forall_i \Omega i(c)$. We conclude $a = 1$ as $a = a \vee 0 = a \vee c = 1$.
We want to prove that Ωi is an injection for then we can conclude that i is a locale isomorphism. Say $\Omega i(b_1) = \Omega i(b_2)$. It is sufficient to prove for all $a \in \Omega X$ that

$$a \lhd b_1 \quad \Leftrightarrow \quad a \lhd b_2$$

in order to conclude $b_1 = b_2$ since X is regular.
But

$$\begin{aligned} a \lhd b_1 &\Leftrightarrow \neg a \vee b_1 = 1 \\ &\Leftrightarrow \Omega i(\neg a \vee b_1) = 1 \\ &\Leftrightarrow \Omega i(\neg a) \vee \Omega i(b_1) = 1 \\ &\Leftrightarrow \Omega i(\neg a) \vee \Omega i(b_2) = 1 \\ &\Leftrightarrow \Omega i(\neg a \vee b_2) = 1 \\ &\Leftrightarrow \neg a \vee b_2 = 1 \\ &\Leftrightarrow a \lhd b_2 \qquad \Box \end{aligned}$$

We say a locale X is *locally compact* if and only if for every $a \in \Omega X$ we have that

$$a = \bigvee \{b | b \ll a\}$$

So a locale X is locally compact if and only if ΩX is a continuous poset. Spatially we are thinking of the locally compact spaces.
X is said to be *stably locally compact* if and only if (it is locally compact and) the $\ll$ relation satisfies

$$\begin{array}{ll} (i) & 1 \ll 1 \text{ i.e. } X \text{ is compact} \\ (ii) & a \ll b_1, a \ll b_2 \;\Rightarrow\; a \ll b_1 \wedge b_2 \end{array}$$

where a, b_1, b_2 are arbitrary elements of ΩX.
Banaschewski and Brümmer ([BB88]) describe these locales as corresponding to the most reasonable not necessarily Hausdorff compact spaces.

Theorem 1.7.2 *Any compact regular locale is stably locally compact.*

Proof: It is sufficient to prove that for any compact regular X if $a, b \in X$ then

$$a \lhd b \quad \Leftrightarrow \quad a \ll b$$

(For from the definition of $\lhd$ it is easy to see that $1 \lhd 1$ and $a \lhd b_1, b_2 \quad \Rightarrow$ $a \lhd b_1 \wedge b_2$.) Say $a \lhd b$ and $b \leq \bigvee^{\uparrow} S$. Then $\neg a \vee b \geq 1$ and so

$$1 \leq \neg a \vee \bigvee^{\uparrow} S = \bigvee^{\uparrow} \{\neg a \vee s | s \in S\}$$

Thus $1 \leq \neg a \vee s$ for some s by compactness. Hence $a \leq s$ for some $s \in S$ and we conclude $a \ll b$.
Conversely say $a \ll b$. $b = \bigvee^{\uparrow} \{b_1 | b_1 \lhd b\}$ since X is regular. Therefore $a \leq b_1$ for some $b_1 \lhd b$. Hence $a \lhd b$. $\Box$
Another example of a stably locally compact locale is a coherent locale; spatially we are thinking of the coherent (or spectral) spaces. A locale X is said to be *coherent* iff

$$\begin{array}{lll} (i) & 1 \ll 1 & \\ (ii) & \forall k_1, k_2 \in \Omega X \text{ if } k_1 \ll k_1 \text{ and } k_2 \ll k_2 \text{ then } k_1 \wedge k_2 \ll k_1 \wedge k_2. & \\ (iii) & \forall a \in \Omega X \qquad a = \bigvee^{\uparrow} \{k | k \ll k, k \leq a\} & \end{array}$$

We use $K\Omega X$ to denote the subset of compact opens of a locale X. i.e. $K\Omega X \equiv \{k \in \Omega X | k \ll k\}$. So (i) and (ii) are saying that compact opens are closed under the formation of meets and (iii) is saying that every open is the join of compact opens less than it.
From the above definition of a coherent locale it is immediate that coherent locales are stably locally compact.
Just as algebraic dcpos can also be defined as those dcpos which are ideal completions of posets we find that

Theorem 1.7.3 *A locale X is coherent if and only if $\Omega X \cong Idl(D)$ for some distributive lattice D.*

Proof: What is needed is a repetition of the proof that a dcpo is algebraic if and only if it is the ideal completion of its compact elements. We only need to further check that the compact elements form a distributive lattice. It is trivial to check that the least element is compact and that if a_1, a_2 are compact then so is $a_1 \vee a_2$. Further, closure of compact opens under finite intersection is part of the definition of X being coherent; so the compact elements form a subdistributive lattice of ΩX. $\Box$

Just as the continuous posets are the retracts of the algebraic dcpos, we find a similar result applies to the stably locally compact locales:

Theorem 1.7.4 *A locale X is stably locally compact if and only if it is the retract in* **Loc** *of some coherent locale Y.*

Proof: Say X is stably locally compact. Then ΩX is a continuous poset. But the fact that any such poset is the retract of its ideal completion is seen in the proof [1.4.2] (which showed us that the continuous posets are exactly the retracts of the algebraic dcpos). The dcpo maps that prove that this retract exists are $\twoheaddownarrow : \Omega X \to Idl\Omega X$ and $\bigvee^{\uparrow} : Idl\Omega X \to \Omega X$.

However $\bigvee^{\uparrow}$ is left adjoint to $\downarrow : \Omega X \to Idl\Omega X$ and so preserves joins. $\twoheaddownarrow$ is left adjoint to $\bigvee^{\uparrow}$ and so $\bigvee^{\uparrow}$ preserves meets. Hence $\bigvee^{\uparrow}$ is a frame homomorphism. But $\twoheaddownarrow$, as a left adjoint, preserves all joins and the fact that it preserves finite meets follows from the conditions (i) and (ii) in the definition of stably locally compact above. Hence ΩX is the retract in **Frm** of the frame of opens of some coherent locale. Hence X is the retract in **Loc** of some coherent locale.

In the other direction say X is the retract of some coherent locale Y. Then there is a retract diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d, hook, "\Omega p"'] & \Omega X \\
\Omega Y \arrow[ur, "\Omega i"] &
\end{tikzcd}$$

in **Frm**. ΩY is an algebraic dcpo and so ΩX is a continuous poset by [1.4.2]. We only have to check the stability conditions (i),(ii) in order to verify that X is stably locally compact.

But recall the claim of the proof of [1.4.2] which showed us:

$$a \ll_{\Omega X} \bar{a} \text{ if and only if } \exists \bar{b} \in \Omega Y \quad a \leq \Omega i(\bar{b}) \quad \bar{b} \ll_{\Omega Y} \Omega p(\bar{a})$$

The stability conditions for X follow from the fact that they hold for Y. $\Box$

Finally, just as the ideal completion of a poset is the free dcpo over that poset we find that the ideal completion of a distributive lattice is the free frame over that distributive lattice. The proof follows the same route: if $f : D \to \Omega X$ is a distributive lattice homomorphism to some frame ΩX then the frame homomorphism corresponding to it is given by: $\Omega p : IdlD \to \Omega X$ where $\Omega p(I) = \bigvee^{\uparrow}\{f(k) | k \in I\}$. In the other direction a frame homomorphism from $IdlD$ to ΩX is taken to its restriction to compact opens.

A map $f : X \to Y$ between stably locally compact locales is said to be *semi-proper* if and only if Ωf preserves the way below relation $\ll$. Define **CohLoc**, the category of coherent locales, to have coherent locales as objects and semi-proper maps as morphisms. Clearly the maps between coherent locales that we are looking at here are those which preserve the compact opens; they are defined in [Joh82] as the coherent maps.

What is the class of locales which are both compact regular and coherent? These are called the *Stone* locales. Before we offer some alternative characterisations of them we need to define what it means for a locale to be zero-dimensional. A locale X is *zero-dimensional* if and only if for every a in ΩX we have that

$$a = \bigvee^{\uparrow}\{\bar{a} | \exists c \quad \bar{a} \wedge c = 0 \quad \bar{a} \vee c = 1 \quad \bar{a} \leq a\}$$

Of course we refer to elements $\bar{a} \in \Omega X$ as *complemented* if and only if there exists some $c \in \Omega X$ such that $\bar{a} \wedge c = 0$ and $\bar{a} \vee c = 1$. Notice that an open $\bar{a}$ is complemented iff $\bar{a} \lhd \bar{a}$. Further notice that the set of all complemented opens (denoted $(\Omega X)^c$) forms a Boolean algebra. So the zero-dimensionality condition could equally well have been written: every open is the join of complemented opens less than it.

Theorem 1.7.5 *The following are equivalent for any locale X.*

(i) *X is Stone.*

(ii) *X is compact and zero-dimensional.*

(iii) *ΩX is the ideal completion of some Boolean algebra.*

Proof:
(i)⇒(ii). $\forall a, b \in \Omega X$ we know $a \lhd b \Leftrightarrow a \ll b$ since ΩX is compact regular. But X is coherent so $\forall a \in \Omega X$

$$a = \bigvee^{\uparrow}\{\bar{a} | \bar{a} \ll \bar{a} \quad \bar{a} \leq a\}$$
$$\Rightarrow \quad a = \bigvee^{\uparrow}\{\bar{a} | \bar{a} \lhd \bar{a} \quad \bar{a} \leq a\}$$

However '$\bar{a} \lhd \bar{a}$' is just the same as saying 'a is complemented'.
(ii)⇒(iii). As X is compact we know that whenever $\bar{a}$ is complemented (i.e. whenever $\bar{a} \lhd \bar{a}$) we have that $\bar{a} \ll \bar{a}$. i.e. $\bar{a}$ is compact. So in the presence of compactness the zero-dimensionality condition implies that every open is the join of compact elements lower than it. But in the other direction if $\bar{a} \ll \bar{a}$ then because $\bar{a} = \bigvee^{\uparrow}\{a_0 | a_0 \lhd a_0 \quad a_0 \leq \bar{a}\}$ we have that $\bar{a} \leq a_0 \lhd a_0 \leq \bar{a}$ for some a_0. Hence $a_0 = \bar{a}$ and the complemented opens coincide with the compact opens. The complemented opens are certainly closed under meet and so we know that X is coherent: it is the ideal completion of its compact opens. i.e. it is the ideal completion of its complemented opens. But these form a Boolean algebra.
(iii)⇒(i). $\forall a \in \Omega X$ we know $a = \bigvee^{\uparrow}\{k | k \ll k \quad k \leq a\}$. We also know that the set $\{k | k \ll k\}$ is a Boolean algebra. So if $k \ll k$ then there exists c such that $k \wedge c = 0$ and $k \vee c = 1$. It follows that if k is less than a then $k \wedge c = 0$ and $a \vee c = 1$. i.e. $k \lhd a$. Hence

$$a = \bigvee^{\uparrow}\{b | b \lhd a\} \qquad \forall a \in \Omega X$$

i.e. ΩX is regular. Certainly X is (compact and) coherent since Boolean algebras are distributive lattices.

1.8 The Constructive Prime Ideal Theorem

The Prime Ideal Theorem (PIT) is the statement: for every distributive lattice D, provided D is not trivial (i.e. provided $D \neq \{*\}$) then there exists an ideal $I \subseteq D$

with the property that if $a \wedge b \in I$ then either $a \in I$ or $b \in I$ and $1 \notin I$. i.e. I is a prime ideal.
The prime ideal theorem is well known, classically, to be a weak form of the axiom of choice (see e.g. Chapter 7 of [Joh87]). Assuming the excluded middle (so the subobject classifier is $\{0,1\}$) if $f : D \to \Omega$ is a distributive lattice homomorphism then the set $\{a | f(a) = 0\}$ is a prime ideal. Certainly it is an ideal. If $f(a \wedge b) = 0$ and we find that both $f(a) \neq 0$ and $f(b) \neq 0$ then we can from these conclude that $f(a \wedge b) \neq 0$. But we are assuming the excluded middle so we can use this contradiction to conclude that either $f(a) = 0$ or $f(b) = 0$. Thus $\{a | f(a) = 0\}$ is a prime ideal for any distributive lattice homomorphism $f : D \to \Omega$. This argument works in the other direction: any prime ideal $I \subseteq D$ gives rise to a distributive lattice homomorphism $f : D \to \Omega$ with the property that $f(a) = 0$ if and only if $a \in I$.
Hence, if we are in a Boolean topos and so can use the excluded middle, we can find an equivalent form of the PIT: for every distributive lattice D provided $D \neq \{*\}$ then there exists a distributive lattice homomorphism $f : D \to \Omega$. However we are let down by the condition $D \neq \{*\}$ which (although possible to define in a general topos via Heyting negation) is clearly undesirable in our constructive context. However the above observations help us home in on the following statement which will make sense in any topos:
Constructive Prime Ideal Theorem (CPIT): For every distributive lattice D if $a \in D$ has the property that $f(a) = 0$ for every distributive lattice homomorphism $f : D \to \Omega$ then $a = 0$.
(I'd like to thank Till Plewe for helping me towards this definition.)

Theorem 1.8.1 *CPIT $\Leftrightarrow$ PIT in a Boolean topos. i.e. if we are allowed the excluded middle then the prime ideal theorem and the constructive prime ideal theorem are logically equivalent.*

Proof: Assume CPIT and say we are given some distributive lattice D which is not trivial. Then $1 \neq 0$ in D and so by CPIT there exists $f : D \to \Omega$. i.e. we have verified PIT.
Conversely say we are given a distributive lattice D and $a \in D$ has the property that $\forall f : D \to \Omega$, $f(a) = 0$. Say $a \neq 0$. Then the distributive lattice $\downarrow a$ is non-trivial and so there exists a distributive lattice homomorphism ($\bar{f}$ say) from it to Ω. Set $f = \bar{f} \circ c$ where c is the distributive lattice homomorphism from D to $\downarrow a$ given by $c(b) = a \wedge b$. Clearly $f(a) = \bar{f}(1_{\downarrow a}) = 1 \neq 0$ contradicting our assumption about a. Hence $a = 0$. $\Box$

We now note that just as the prime ideal theorem is well known to be equivalent to the statement 'every non-trivial Boolean algebra has a prime ideal' there is a similar constructively equivalent way of stating the constructive prime ideal theorem:

Lemma 1.8.1 *CPIT is equivalent to the statement: for every Boolean algebra B if $b \in B$ is an element that satisfies $f(b) = 0$ for every Boolean lattice homomorphism $f : B \to \Omega$ then $b = 0$.*

Proof: Clearly CPIT implies this statement. Conversely assume the statement holds for every Boolean algebra B. Say we are given a distributive lattice D and some $a \in D$ with the property that $f(a) = 0$ for every $f : D \to \Omega$. Then let $i : D \hookrightarrow B$ be the inclusion of D into the free Boolean algebra over it. It follows that $\bar{f}(ia) = 0$ for every Boolean homomorphism $\bar{f}$ from B to Ω. Hence $i(a) = 0$ by the assumption of the statement. Hence a is zero as i is an injection. $\Box$

We can now forget about the excluded middle and Boolean toposes. They were only introduced in order to verify that our choice for the constructive prime ideal theorem was reasonable.

Theorem 1.8.2 *In any topos if CPIT holds then all coherent locales are spatial.*

Proof: Say X is a coherent locale. Notice that the frame homomorphism corresponding to the counit of the adjunction is a surjection. It is given by

$$\begin{array}{rcl} \Omega\epsilon_X : \Omega X & \to & \Omega ptX \\ I & \mapsto & \{p|\Omega p(I) = 1\} \end{array}$$

We want to show that this surjection is an injection for every coherent X. Say

$$\{p|\Omega p(I) = 1\} = \{p|\Omega p(J) = 1\}$$

for some $I, J \in \Omega X \cong Idl(K\Omega X)$. This implies that for every point p, $\Omega p(I)$ and $\Omega p(J)$ are the same element of the subobject classifier Ω (recall that $i = \bigvee\{1|1 \leq i\}$ for every $i \in \Omega$). It follows that $\Omega p(I) \subseteq \Omega p(J)$ and in particular that if $\Omega p(J) = 0$ then $\Omega p(I) = 0$.
Recall that any distributive lattice can be quotiented by an ideal (Lemma [1.3.4]). We quotient $K\Omega X$ by J. So $[b] = 0 \Leftrightarrow b \in J \quad \forall b \in K\Omega X$ and there is a one to one correspondence between distributive lattice homomorphisms $f : K\Omega X \to \Omega$ which satisfy $f(b) = 0$ for all $b \in J$ and all distributive lattice homomorphisms $\bar{f} : K\Omega X/\equiv_J \to \Omega$. It follows, from the fact that $\Omega X \cong Idl(K\Omega X)$ is the free frame over the distributive lattice $K\Omega X$ that there is a one to one correspondence between points, p, of X satisfying $\Omega p(J) = 0$ and distributive lattice homomorphisms from $K\Omega X/\equiv_J$ to Ω.
Now to verify $I \subseteq J$ it is sufficient to check that $\forall a \in I$ and $\forall \bar{f} : K\Omega X/\equiv_J \to \Omega$

$$\bar{f}[a] = 0$$

for then by CPIT $[a] = 0$ i.e. $a \in J$.
However $\bar{f}[a] = 0 \Leftrightarrow \Omega p(\downarrow a) = 0$ where p is the point corresponding to $\bar{f}$ (which must satisfy $\Omega p(J) = 0$). But $\Omega p(\downarrow a) \subseteq \Omega p(I) \subseteq \Omega p(J) = 0$. □

Recall from Theorem [1.5.2] that the retracts of all spatial locales are spatial. It follows immediately that provided CPIT holds (a) all stably locally compact locales and (b) all compact regular locales are spatial. It is also worth saying that therefore the Stone locales are spatial (if we assume CPIT) for we have

Theorem 1.8.3 *In any topos if the Stone locales are spatial then the constructive prime ideal theorem is true.*

Proof: Say B is a Boolean algebra and $b \in B$ has the property that for every Boolean map $f : B \to \Omega$, $f(b) = 0$. It follows that for every such f, $f(\neg b) = 1$. There is a one to one correspondence between these functions f and points of the Stone locale whose frame of opens is given $IdlB$ since $IdlB$ is the free frame over the Boolean algebra B. It follows that for every point p of this locale $\Omega p(\downarrow \neg b) = 1$ Thus

$$\{p|\Omega p(\downarrow \neg b) = 1\} = \{p|\Omega p(\downarrow 1) = 1\}.$$

But we are assuming that the Stone locales are spatial and so this condition implies that $\downarrow \neg b = \downarrow 1$. Hence $\neg b = 1$, hence $b = 0$ and so by Lemma [1.8.1] the constructive prime ideal theorem is verified.□

Chapter 2

Preframes and the Generalized Coverage Theorem

2.1 Introduction

This chapter is more lattice theoretic than localic. We give a description of preframes (as introduced by Banaschewski [Ban88]), and show how they form a symmetric monoidal closed category. We prove this by adapting Křížʼs precongruences to the context of preframes. We recall [JT84] that the category of SUP-lattices is symmetric monoidal closed. Further analogies between SUP-lattices and preframes become clear: frames can be viewed both as special types of monoids in the symmetric monoidal category of preframes and as special types of monoids in the symmetric monoidal category of SUP-lattices. The latter fact is shown in Joyal and Tierney [JT84], the former in Johnstone and Vickers [JV91]. Moreover frame coproduct (=locale product) can be viewed as either tensor within the category of preframes or as tensor in the category of SUP-lattices. This is the localic version of the motivating example which is described in the introduction to the thesis. The usefulness of this result is seen immediately with a proof of the localic Tychonoff theorem.

Not only can we view locale products in these different ways, the same applies to all locale limits: in particular frame coequalizers (=locale equalizers) can be viewed as particular SUP-lattice coequalizers and as particular preframe coequalizers. Both these facts stem from a general categorical result about any symmetric monoidal closed category. We call this result the generalized coverage theorem and note that it has an ʻoppositeʼ. The end of the chapter is about applications of the generalized coverage theorem (and its opposite). In particular the name of the theorem is justified: it covers both the preframe version and Johnstoneʼs original (SUP-lattice) version of the coverage theorem. With the help of its ʻoppositeʼ we are able to deduce the fact that preframes have coequalizers from the fact that SUP-lattices have coequalizers.

2.2 Preframes

Johnstone's coverage theorem [Joh82] gives us a concrete description of the frame corresponding to a set of generators and frame relations. The fact that such a frame exists can be verified easily enough by constructing the free frame on the generators and then quotienting by the least congruence containing the relations. However the advantage of the coverage theorem is that it gives us a concrete description of the frame being presented. Hence we have a concrete description of arbitrary frame coproduct, and this can then be used to prove that the coproduct of compact frames is compact. In other words the product of compact locales is compact (i.e. localic Tychonoff theorem). It was observed in Abramsky and Vickers' work on quantales ([AV93]) that the real content of the coverage theorem is the fact that the frame being presented is isomorphic to the free SUP-lattice on another set of generators and relations. This ability to describe frames as particular quotients of free SUP-lattices is useful in the context of quantales since there one is often trying to find SUP-lattice homomorphisms away from a particular frame. In fact the coverage theorem extends very naturally to become a statement about how to present quantales as particular SUP-lattices.

The proof of the localic Tychonoff theorem using Johnstone's original description of the coproduct frame (see III 1.7 of [Joh82]) is far from straightforward. Many attempts were made to simplify e.g.[Ban88], [JV91]. In [JV91] the authors develop the theory of preframes, and check that given a set of generators and preframe relations then the preframe being presented is well defined. It is then possible to find a preframe version of the coverage theorem: it states that given a set of generators and frame relations then the frame being presented is isomorphic to the preframe being presented by some other set of generators and relations. Just as was done with the original coverage theorem this preframe version can be used to give an explicit description of the coproduct of frames. Only now the coproduct is presented as a preframe and since we know that a frame is compact if and only if a particular preframe homomorphism exists with the frame as its domain, the proof of the localic Tychonoff theorem becomes much simpler. This is what motivates us to look at preframes.

A *preframe* is a poset with directed joins and finite meets such that the directed joins distribute over the finite meets. A preframe homomorphism preserves directed joins and finite meets. The name 'preframe' was introduced by Banaschewski in his paper "Another look at the localic Tychonoff theorem" [Ban88], although these objects had already been looked at by Gierz et al as meet continuous semilattices [GHKLM80].
We aim to show that the category **PreFrm** of preframes is symmetric monoidal closed. Instead of just constructing a tensor product in **PreFrm** we address the more general question of whether preframe presentations present. i.e. if we are given a set G of generators and a set R of preframe equations of elements of G is the preframe

$$\mathrm{PreFrm}\!< G|R >$$

well defined?
It is true that such a general presentation presents [JV91] though for our purposes we only need to show that a smaller class of presentations present. We aim to check that for any meet semilattice S,

$$\mathrm{PreFrm} < S(\text{qua meet semilattice})| \vee^{\uparrow} X = \vee^{\uparrow} Y \quad (X,Y) \in R >$$

is well defined; where R is a set of pairs (X, Y) with X and Y directed subsets of S.

A note on notation is appropriate: the expression 'qua meet semilattice' is shorthand for saying that the equations

$$\begin{aligned} a \wedge b &= a \wedge_S b \quad \forall a, b \in S \\ 1 &= 1_S \end{aligned}$$

must be added to the presentation. This is saying that what is true in the semilattice must be inherited by the preframe being presented. The meaning of the expressions 'qua preframe', 'qua frame' etc should now be clear.

It is an easy exercise in the definition of what it means for a presentation to present to check that we can further assume that the X and Ys in R are lower closed and that R satisfies the following meet stability condition:

$$(\forall a \in S)[(X, Y) \in R \quad \Rightarrow \quad (\{x \wedge a | x \in X\}, \{y \wedge a | y \in Y\}) \in R]$$

2.3 Precongruences

These were introduced by Kříž [Kříž86] in his study of the completion of a uniform locale. Given a frame ΩX a *precongruence*, R, on it is a subset

$$R \subseteq \Omega X \times \Omega X$$

such that whenever aRb we have that the set

$$\{u | (a \wedge u) R (b \wedge u)\}$$

is a join basis for ΩX. i.e. $\forall c \in \Omega X \quad c = \bigvee U$ where $U \subseteq \{u | (a \wedge u) R (b \wedge u)\}$. Of course this does not imply that a precongruence satisfies any of the axioms of being an equivalence relation.
We say that $u \in \Omega X$ is *R-coherent* if and only if whenever aRb then

$$(a \leq u) \quad \Leftrightarrow \quad (b \leq u)$$

The set of R-coherent elements is clearly closed under all meets. Further we have that if u is R-coherent and $c \in \Omega X$ then $c \to u$ is R-coherent. For if aRb then $\exists Q \subseteq \{v | (v \wedge a) R (v \wedge b)\}$ such that $\bigvee Q = c$. Then

$$\begin{aligned} a \leq c \to u &\Leftrightarrow a \wedge c \leq u \\ &\Leftrightarrow a \wedge q \leq u \quad \forall q \in Q \\ &\Leftrightarrow b \wedge q \leq u \quad \forall q \in Q \\ &\Leftrightarrow b \wedge c \leq u \\ &\Leftrightarrow b \leq c \to u \end{aligned}$$

It is a well known fact (see e.g. [6.2.8] of [Vic89]) that a subset A_0 of a frame (ΩX) is a surjective image (via the map $a \mapsto \wedge\{b \in A_0 | a \leq b\}$) of that frame if it is closed under all meets and is closed under the Heyting arrow in the manner described above. i.e. $(\forall u \in A_0)(\forall c \in \Omega X)(c \to u \in A_0)$. So if we define $\Omega X(R)$ to be the set of R-coherent elements of ΩX then we see that there is a frame surjection $\theta_R : \Omega X \to \Omega X(R)$. $\theta_R(a)$ is given by $\wedge\{u | a \leq u \quad u\ R\text{-coherent}\ \}$ and so $a \leq \theta_R(a)\ \forall a$. Also, joins on $\Omega X(R)$ are calculated as follows:

$$\bigvee_{\Omega X(R)} T = \theta_R(\bigvee T)$$

for all $T \subseteq \Omega X(R)$

The map θ_R is universal in the following sense:

Theorem 2.3.1 (Kříž) *Given a frame ΩX with a precongruence R on it any frame homomorphism $\Omega f : \Omega X \to \Omega Y$ satisfying $(aRb \Rightarrow \Omega f a = \Omega f b)$ factors (uniquely) through θ_R.*

Proof: Clearly it is enough to prove that

$$\forall a \in \Omega X \quad \Omega f \theta_R(a) = \Omega f(a) \qquad (*)$$

Set $s(a) = \bigvee\{b \in \Omega X | \Omega f(b) \leq \Omega f(a)\}$. Then if $\bar{a} R \bar{b}$ we see that

$$\begin{array}{rl} & \bar{a} \leq s(a) \\ \Leftrightarrow & \Omega f(\bar{a}) \leq \Omega f(a) \\ \Leftrightarrow & \Omega f(\bar{b}) \leq \Omega f(a) \\ \Leftrightarrow & \bar{b} \leq s(a) \end{array}$$

i.e. $s(a)$ is R-coherent, and so $\theta_R(s(a)) = s(a)$. Hence the fact that $a \leq s(a)$ implies $\theta_R(a) \leq s(a)$. Clearly, by the fact that Ωf preserves joins, we have

$$\Omega f s(a) \leq \Omega f(a).$$

And so $\Omega f(\theta_R(a)) \leq \Omega f(a)$ from which $(*)$ follows as θ_R is inflationary. □

The idea of prenuclei was introduced by Banaschewski ([Ban88]) to help with his proof of a localic version of Tychonoff's theorem. $\nu_0 : \Omega X \to \Omega X$ is a *prenucleus* if
(1) it is monotone
(2) $a \leq \nu_0(a) \quad \forall a \in \Omega X$
(3) $\nu_0(a) \wedge b \leq \nu_0(a \wedge b) \quad \forall a, b \in \Omega X$.
Condition (2) implies that the set of ν_0-fixed elements of ΩX is closed under arbitrary meets. Say $\nu_0(u) = u$ and $c \in \Omega X$, then $\nu_0(c \to u) \leq c \to u$ iff $c \wedge \nu_0(c \to u) \leq u$. But $c \wedge \nu_0(c \to u) \leq \nu_0(c \wedge (c \to u)) \leq \nu_0(u) \leq u$ and so the set of ν_0-fixed elements is the frame of opens of a sublocale by the same reasoning that allowed us to conclude that $\Omega X(R)$ is the frame of opens of a sublocale.
Given a prenucleus $\nu_0 : \Omega X \to \Omega X$ define $R_{\nu_0} \subseteq \Omega X \times \Omega X$ by

$$aR_{\nu_0}b \quad \Leftrightarrow \quad (\forall u \in \Omega X)[(\nu_0 u = u) \Rightarrow (a \leq u \Leftrightarrow b \leq u)]$$

Notice from this definition that $\nu_0(u) R_{\nu_0} u \quad \forall u$.

Lemma 2.3.1 *R_{ν_0} is a precongruence.*

Proof: Assume $aR_{\nu_0}b$.
I claim that $\{v | (a \wedge v) R_{\nu_0} (b \wedge v)\}$ is the whole of ΩX and so certainly is a join basis for ΩX.
So I need to prove, given an arbitrary $v \in \Omega X$, that if $u \in \Omega X$ satisfies $\nu_0(u) = u$ then

$$(a \wedge v) \leq u \quad \Leftrightarrow \quad (b \wedge v) \leq u$$

But $(a \wedge v \leq u \quad \Leftrightarrow \quad a \leq v \to u)$ and u ν_0-fixed $\Rightarrow \quad (v \to u)$ ν_0-fixed (see above).
So $(a \leq v \to u \quad \Leftrightarrow \quad b \leq v \to u \quad \Leftrightarrow b \wedge v \leq u)$ as required. □

Crucially we find that the set of R_{ν_0}-coherent elements is the same as the set of ν_0-fixed elements. One way round of this implication is obvious from the definition of R_{ν_0}: if u is ν_0-fixed then it is R_{ν_0}-coherent. Conversely say u is R_{ν_0}-coherent. We know that $\nu_0(u) R_{\nu_0} u$, and so $\nu_0(u) \leq u \Leftrightarrow u \leq u$. Hence $\nu_0(u) = u$.
I am not sure of the extent to which precongruences and prenuclei are the same thing. Certainly they are used in the same way: Kříž's universal theorem above having an identical form to Banaschewski's lemma 1 in [Ban88].
Given a precongruence R the mapping

$$u \longmapsto u \vee \bigvee\{a \wedge b | \exists c, cRa, c \wedge b \leq u\}$$

is a prenucleus, although (the trivial) proof of this fact doesn't require R to be a precongruence: it could be any subset of $\Omega X \times \Omega X$.
Also the precongruences R_{ν_0} that we get from prenuclei cannot cover all possible congruences. We saw that $\nu_0(u) R_{\nu_0} u$ for every $u \in \Omega X$, but the definition of precongruences allows for the empty precongruence. We leave these theoretical discussions aside and use precongruences only in what follows.

For any meet semilattice A let νA be the set of lower closed subsets of A. It is well known that νA is the free frame over the semilattice A.

Theorem 2.3.2 *Given a preframe A the set*

$$R_A \equiv \{(X, \downarrow \vee^{\uparrow} X) | X \text{ a directed lower subset of } A\}$$

is a precongruence on νA. Moreover $\nu A(R_A)$ is the free frame over the preframe A.

Remark: It is easy to see that the R_A-coherent elements of νA are exactly the Scott closed subsets of A. i.e. the classical complements of the Scott opens.
Proof: That R_A is a precongruence is quite straight forward: it is well known that the set of sets of the form $\downarrow a$ is a join basis for νA and since

$$\begin{aligned}
\downarrow a \cap X &= \{x \wedge a | x \in X\} \\
\downarrow a \cap \downarrow \bigvee^{\uparrow} X &= \downarrow \bigvee^{\uparrow}\{x \wedge a | x \in X\}
\end{aligned}$$

for any lower closed directed X we have that

$$(\downarrow a \cap X) R_A (\downarrow a \cap \downarrow \bigvee^{\uparrow} X)$$

for every a.
We now note that the composite $A \xrightarrow{\downarrow} \nu A \xrightarrow{\theta_{R_A}} \nu A(R_A)$ is a preframe homomorphism. To see this say we are given $X \subseteq^{\uparrow} A$ which is lower closed and directed. We need to prove that

$$\theta_{R_A} \downarrow \bigvee^{\uparrow} X = \bigvee^{\uparrow}_{\nu A(R_A)} \{\theta_{R_A} \downarrow x | x \in X\}$$

But θ_{R_A} is a frame homomorphism and so

$$\begin{aligned}
\bigvee^{\uparrow}_{\nu A(R_A)} \{\theta_{R_A} \downarrow x | x \in X\} &= \theta_{R_A} \bigcup^{\uparrow}\{\downarrow x | x \in X\} \\
&= \theta_{R_A} X
\end{aligned}$$

But we know that $\theta_{R_A} \downarrow \bigvee^{\uparrow} X = \theta_{R_A} X$ from Kříž's universal theorem. Hence $\theta_{R_A} \circ \downarrow$ is a preframe homomorphism.
Now say we are given some preframe homomorphism $f : A \to B$ where B is some frame. Since f is a meet semilattice homomorphism we know that it will factor (uniquely) through $\downarrow$. i.e. $\exists! \quad \bar{f} : \nu A \to B$ (a frame hom.) such that $\bar{f} \circ \downarrow = f$.
$\bar{f}$ is given by $\bar{f}(Y) = \bigvee_B \{f(y) | y \in Y\}$. All we need to do (to check that $\nu A(R_A)$ is the free frame on A) is verify that $\bar{f}$ satisfies the precondition of Kříž's universal theorem; for then $\bar{f}$ will factor through θ_{R_A}. i.e. we need that if URV then $\bar{f}U = \bar{f}V$. But this amounts to showing for any (lower) directed X that

$$\bar{f}X = \bar{f} \downarrow \bigvee^{\uparrow} X$$

i.e. that $\bigvee^{\uparrow}\{fx | x \in X\} = f \bigvee^{\uparrow} X$, which follows at once since f is a preframe homomorphism. $\Box$

We can also define precongruences on preframes; and this will give rise to a universal theorem identical to Kříž's except that the word 'frame' is replaced with the word 'preframe'. From this new universal theorem the fact that preframe presentations present will follow as an easy corollary. Proof of this new theorem relies on applying Kříž's universal theorem.
Given a preframe A a precongruence on A is a subset $R \subseteq A \times A$ such that if aRb then $\{u | (a \wedge u)R(b \wedge u)\}$ is a *directed* join basis for A. i.e. $\forall a \in A$ there exists $U \subseteq^{\uparrow} \{u | (a \wedge u)R(b \wedge u)\}$ such that $a = \bigvee^{\uparrow} U$.
Say we are given a preframe A with a precongruence R on it. Then this precongruence gives rise to a precongruence on the free frame on A in the following way: $\bar{R} \subseteq \nu A(R_A) \times \nu A(R_A)$ is defined to be $\{(\downarrow a, \downarrow b) | aRb\}$. We must check that $\bar{R}$ is a precongruence. Say $\downarrow a \bar{R} \downarrow b$. Now $\forall U \in \nu A(R_A)$ we have $U = \bigcup_{u \in U} \downarrow u$ and so by applying $\theta_{R_A} : \nu A \to \nu A(R_A)$ we see that $U = \bigvee_{\nu A(R_A)} \{\downarrow u | u \in U\}$. Hence to conclude that $\bar{R}$ is a precongruence we must but check that $\downarrow u$ is a $\nu A(R_A)$-join of elements $V \in \nu A(R_A)$ such that $(\downarrow a \cap V)\bar{R}(\downarrow b \cap V)$ for any $u \in A$.
Since $u \in A$ and aRb we know (by definition of precongruence on a preframe) that $u = \bigvee^{\uparrow} Q$ for some Q such that $(a \wedge q)R(b \wedge q) \quad \forall q \in Q$. We know that $\downarrow : A \longrightarrow \nu A(R_A)$ is a preframe homomorphism and so

$$\downarrow u = \bigvee^{\uparrow}_{\nu A(R_A)} \{\downarrow q | q \in Q\}$$

But $(a \wedge q)R(b \wedge q)$ implies $\downarrow (a \wedge q) \bar{R} \downarrow (b \wedge q)$ and so $(\downarrow a) \wedge (\downarrow q) \bar{R} (\downarrow b) \wedge (\downarrow q)$. Thus $\downarrow a$ is a join of elements $V \in \nu A(R_A)$ such that $(\downarrow a \cap V)\bar{R}(\downarrow b \cap V)$ as required. Hence $\bar{R}$ is a precongruence on $\nu A(R_A)$. This construction (of $\bar{R}$ from R) will be used in

Theorem 2.3.3 *If R is a precongruence on a preframe A then there exists an arrow $c : A \to C$ in the category of preframes which is universal amongst arrows with the property $aRb \Rightarrow c(a) = c(b)$.*

Proof: We know (see above) that $\bar{R} \equiv \{(\downarrow a, \downarrow b) | aRb\}$ is a precongruence on the free frame on A, $\nu A(R_A)$ and so there is a frame homomorphism

$$\theta_{\bar{R}} : \nu A(R_A) \longrightarrow \nu A(R_A)(\bar{R})$$

The map $\downarrow : A \longrightarrow \nu A(R_A)$ is a preframe injection. Define C to be the least subpreframe of $\nu A(R_A)(\bar{R})$ generated by the image of $\{\downarrow a | a \in A\}$ under $\theta_{\bar{R}}$. Clearly the map $c : A \to C$ defined by $a \mapsto \theta_{\bar{R}} \downarrow a$ is a preframe homomorphism. In fact it is easy to see that c is a preframe epimorphism. Also note that if aRb then $\downarrow a \bar{R} \downarrow b$ and so $\theta_{\bar{R}}(\downarrow a) = \theta_{\bar{R}}(\downarrow b)$ by Kříž's universal theorem, and so $c(a) = c(b)$.
Now say we are given $f : A \to B$, an arrow in **PreFrm** which satisfies $aRb \quad \Rightarrow$

$fa = fb$.
The inclusion $\downarrow: B \to \nu B(R_B)$ of B into its free frame is a preframe homomorphism and so the composite $\downarrow \circ f$ must factor through the inclusion of A into its free frame. i.e. there exists $\bar{f} : \nu A(R_A) \to \nu B(R_B)$ a frame homomorphism making

$$\begin{tikzcd} A \arrow[r, hook, "\downarrow"] \arrow[d, "f"] & \nu A(R_A) \arrow[d, "\bar{f}"] \\ B \arrow[r, hook, "\downarrow"] & \nu B(R_B) \end{tikzcd}$$

commute.
Say $\downarrow a \bar{R} \downarrow b$. Then aRb and so $fa = fb$. So certainly $\downarrow fa = \downarrow fb$ i.e. $\bar{f} \downarrow a = \bar{f} \downarrow b$. It follows from Kříž's universal theorem that there exists $\bar{g} : \nu A(R_A)(\bar{R}) \to \nu B(R_B)$ a frame homomorphism such that $\bar{g} \circ \theta_{\bar{R}} = \bar{f}$. It follows at once that

$$\bar{g} \circ \theta_{\bar{R}} \downarrow a = \bar{f} \downarrow a = \downarrow fa$$

and so the set $\bar{g}^{-1}\{\downarrow b | b \in B\}$ is a subpreframe of $\nu A(R_A)(\bar{R})$ which contains the set $\{\theta_{\bar{R}} \downarrow a | a \in A\}$. Hence it contains C. It follows that $\bar{g}$ restricts to a function from C to $\{\downarrow b | b \in B\} \cong B$. So there is a preframe $g : C \to B$ with $g \circ c = f$ as required. The uniqueness of such a g is immediate from our remark earlier that c is a preframe epimorphism. $\square$

Notation: By analogy to Kříž's result we call the C above $A(R)$ and we use θ_R to denote the preframe map $c : A \to C$.

2.4 Presentations

For a meet semilattice S recall that $IdlS$ is the set of lower directed subsets of S. It can be checked that $IdlS$ is the free preframe on the meet semilattice S. We are now in a position to prove:

Theorem 2.4.1 *If S is a meet semilattice and R is a set of pairs (X, Y) where X, Y are directed lower closed subsets of S and R satisfies the following meet stability condition:*

$$(\forall a \in S)[(X, Y) \in R \quad \Rightarrow \quad (\{x \wedge a | x \in X\}, \{y \wedge a | y \in Y\}) \in R]$$

then

$$\textit{PreFrm}\langle S \text{ (qua meet semilattice) } | \textstyle\bigvee^{\uparrow} X = \bigvee^{\uparrow} Y \quad (X, Y) \in R \rangle$$

is well defined.

Proof: The set $\{\downarrow s | s \in S\}$ is a directed join basis for $Idl(S)$ and so the conditions on R given in the statement of the theorem imply that R is a precongruence on the preframe $Idl(S)$. We check that

$$Idl(S)(R) \cong \text{PreFrm}\langle S \text{ (qua meet-semilattice) } | \textstyle\bigvee^{\uparrow} X = \bigvee^{\uparrow} Y \quad (X, Y) \in R \rangle$$

$IdlS$ is the free preframe on S and so given any meet semilattice homomorphism $s : S \to B$ to some preframe B which satisfies $\bigvee^{\uparrow}_B \{s(x) | x \in X\} = \bigvee^{\uparrow}_B \{s(y) | y \in Y\}$ for every $(X, Y) \in R$ we know that s factors uniquely through $\downarrow: S \to Idl(S)$. i.e. there exists $\bar{s} : Idl(S) \to B$ such that $\bar{s} \circ \downarrow = s$. But XRY implies $\bar{s}(X) = \bar{s}(Y)$ and so $\bar{s}$ factors through $\theta_R : Idl(S) \to Idl(S)(R)$. $\Box$

The rest of this section and Section 2.6 to follow spell out the consequences of the fact that preframe presentations present and as such are repetitions of the results of [JV91].

Now that Theorem [2.4.1] is proven we try out some examples. As with any presentable algebraic theory we have a tensor product. Given A and B there is a preframe $A \otimes B$ with a preframe bihomomorphism
$\otimes : A \times B \to A \otimes B$ which is universal amongst all such bihomomorphisms. So set

$$
\begin{aligned}
S \equiv \wedge - \mathbf{SLat} < a\otimes b, a \in A, b \in B | (a\otimes b_1) \wedge (a\otimes b_2) &= a\otimes(b_1 \wedge b_2) \quad a \in A, b_1, b_2 \in B \\
(a_1\otimes b) \wedge (a_2\otimes b) &= (a_1 \wedge a_2)\otimes b \quad a_1, a_2 \in A, b \in B \\
1 &= 1\otimes b \quad \forall b \in B \\
1 &= a\otimes 1 \quad \forall a \in A >
\end{aligned}
$$

and define the tensor by:

$$
\begin{aligned}
A \otimes B \equiv \mathbf{PreFrm} < S \text{ qua meet-semilattice} | \bigvee_i^{\uparrow}(a_i\otimes b) &= \bigvee_i^{\uparrow} a_i\otimes b \quad \forall (a_i) \subseteq^{\uparrow} A, \forall b \in B \\
\bigvee_i^{\uparrow}(a\otimes b_i) &= a\otimes \bigvee_i^{\uparrow} b_i \quad \forall a \in A, (b_i) \subseteq^{\uparrow} B >
\end{aligned}
$$

Clearly $A\otimes(\_)$ is left adjoint to the function space functor $[A \to \_]$:**PreFrm**$\to$**PreFrm**. In fact

Theorem 2.4.2 **PreFrm** *is a symmetric monoidal closed category.*

Proof: The fact that presentations are well defined is the real 'work' of this theorem. We use this proof to check that the subobject classifier (i.e. the power set of 1) is the unit of the tensor. We define two functions

$$
\begin{aligned}
p : A &\to A \otimes \Omega \\
a &\mapsto a\otimes 0 \\
q : A \otimes \Omega &\to A \\
\text{by } (a\otimes i) &\mapsto \bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\})
\end{aligned}
$$

Clearly p is a preframe homomorphism. Assume for the moment that $(a, i) \mapsto \bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\})$ is a preframe bihomomorphism.

$$
\begin{aligned}
qp(a) &= q(a\otimes 0) \\
&= \bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq 0\}) \\
&= a
\end{aligned}
$$

We also want that $pq(a \otimes i) = a \otimes i$.

$$\begin{aligned} pq(a \otimes i) &= p \bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\}) \\ &= (\bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\})) \otimes 0 \\ &= \bigvee^{\uparrow}(\{a \otimes 0\} \cup \{1 | 1 \leq i\}) \\ &= \bigvee^{\uparrow}(\{a \otimes 0\} \cup \{a \otimes 1 | 1 \leq i\}) \\ &= a \otimes \bigvee^{\uparrow}(\{0\} \cup \{1 | 1 \leq i\}) \\ &= a \otimes i \end{aligned}$$

To prove $i \leq \bigvee^{\uparrow}(\{0\} \cup \{1 | 1 \leq i\})$ recall from Chapter 1 that it is sufficient to check that $i = 1$ implies $1 = \bigvee^{\uparrow}(\{0\} \cup \{1 | 1 \leq i\})$. We now check that $(a, i) \mapsto \bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\})$ is a preframe bihomomorphism in order to be sure that q is well defined. Fix $i \in \Omega$. Clearly $\bigvee^{\uparrow}\{1\} \cup \{1_A | 1 \leq i\} = 1$. Say $a, b \in A$.

$$\begin{aligned} &\bigvee^{\uparrow}(\{a\} \cup \{1_A | 1 \leq i\}) \wedge \bigvee^{\uparrow}(\{b\} \cup \{1_A | 1 \leq i\}) \\ = &\bigvee^{\uparrow}(\{a \wedge b\} \cup \{b | 1 \leq i\} \cup \{a | 1 \leq i\} \cup \{1 | 1 \leq i\}) \\ = &\bigvee^{\uparrow}(\{a \wedge b\} \cup \{1 | 1 \leq i\}) \end{aligned}$$

So $((\_), i) \mapsto \bigvee^{\uparrow}(\{\_\} \cup \{1 | 1 \leq i\})$ preserves finite meets.

Say $T \subseteq^{\uparrow} A$ then $\forall t \in T$ certainly

$$t \leq \bigvee^{\uparrow}(\{t\} \cup \{1 | 1 \leq i\})$$

hence $\bigvee^{\uparrow} T \leq \bigvee^{\uparrow}_t(\bigvee^{\uparrow}(\{t\} \cup \{1 | 1 \leq i\}))$ and so an examination of cases tells us

$$\bigvee^{\uparrow}(\{\bigvee^{\uparrow} T\} \cup \{1 | 1 \leq i\}) \leq \bigvee^{\uparrow}_t(\bigvee^{\uparrow}\{t\} \cup \{1 | 1 \leq i\}).$$

N.B. non-emptiness of T is needed. Hence $((\_), i) \mapsto \bigvee^{\uparrow}(\{\_\} \cup \{1 | 1 \leq i\})$ preserves directed joins.
The fact that for any $i, j \in \Omega, a \in A$ we have

$$\begin{aligned} &\bigvee^{\uparrow}(\{a\} \cup \{1 | 1 \leq i \wedge j\}) \\ = &\bigvee^{\uparrow}(\{a\} \cup \{1 | 1 \leq i\}) \wedge \bigvee^{\uparrow}(\{a\} \cup \{1 | 1 \leq j\}) \end{aligned}$$

is easy enough to see: use distributivity of directed joins over finite meets and note that the sets $\{a\}$ and $\{a\} \cup \{a | 1 \leq i\} \cup \{a | 1 \leq j\}$ are the same. Finally for any a the function $i \mapsto \bigvee^{\uparrow}(\{a\} \cup \{1 | 1 \leq i\}$ preserves directed joins. This follows from compactness of Ω. $\Box$

We will need to construct some infinite coproducts of preframes when we prove the localic Tychonoff theorem in Section 2.8. We have

Theorem 2.4.3 PreFrm *is cocomplete.*

Proof: Again the 'work' has been done with the presentation result. Say $D : J \rightarrow$**PreFrm** is a diagram of preframes. Define

$$S \equiv \wedge - \mathbf{SLat} < \coprod_{i \in ObJ} D(i) | \quad 1 = 1_{D(i)} \quad \forall i$$
$$a \wedge b = a \wedge_{D(i)} b \quad \forall a, b \in D(i) \quad \forall i$$
$$a = D(f)(a) \quad \forall a \in D(i) \quad \forall f : i \rightarrow j \in \mathcal{M}(J) >$$

Then the preframe colimit is given by:

$$A \equiv \mathbf{PreFrm} < S \text{ qua meet semilattice } | \bigvee^{\uparrow} T = \bigvee^{\uparrow}_{D(i)} T \quad \forall T \subseteq^{\uparrow} D(i) \quad \forall i > \Box$$

2.5 The Generalized Coverage Theorem

We have a symmetric monoidal category **PreFrm**. Over any symmetric monoidal category $\mathcal{C}$ we can construct **CMon**($\mathcal{C}$), the category of commutative monoids on the tensor of $\mathcal{C}$. We will find that frames can be characterised as special types of objects in **CMon(PreFrm)**. In the next section we will then be able to use the following results to give us facts about frames. We need the following well known (see e.g. lemma 4.1 of [JV91]) general result about symmetric monoidal categories,

Theorem 2.5.1 **CMon**($\mathcal{C}$) *has finite coproducts. They are given by tensor (and unit).*

Proof: Say $(A, *_A, e_A), (B, *_B, e_B)$ are two objects of **CMon**($\mathcal{C}$), define $* : (A \otimes B) \otimes (A \otimes B) \rightarrow (A \otimes B)$ to be the composite

$$(A \otimes B) \otimes (A \otimes B) \stackrel{\cong}{\rightarrow} (A \otimes A) \otimes (B \otimes B) \stackrel{*_A \otimes *_B}{\longrightarrow} A \otimes B$$

and $e : \Omega \rightarrow A \otimes B$ to be

$$\Omega \stackrel{\cong}{\rightarrow} \Omega \otimes \Omega \stackrel{e_A \otimes e_B}{\longrightarrow} A \otimes B.$$

From these definitions it is easily established that $\otimes$ can be viewed as a functor **CMon**($\mathcal{C}$)$\times$**CMon**($\mathcal{C}$)$\longrightarrow$**CMon**($\mathcal{C}$). If $\otimes$ is left adjoint to the diagonal functor

$$\Delta : \mathbf{CMon}(\mathcal{C}) \longrightarrow \mathbf{CMon}(\mathcal{C}) \times \mathbf{CMon}(\mathcal{C})$$

then $\otimes$ is a coproduct operation.
Given a commutative monoid $(A, *_A, e_A)$ the map $*_A : A \otimes A \rightarrow A$ can be viewed as a natural transformation from $\otimes\Delta$ to Id and given a pair of commutative monoids $(A, *_A, e_A)$ and $(B, *_B, e_B)$ the maps

$$A \stackrel{\cong}{\rightarrow} A \otimes \Omega \stackrel{1 \otimes e_B}{\longrightarrow} A \otimes B$$
$$B \stackrel{\cong}{\rightarrow} \Omega \otimes B \stackrel{e_A \otimes 1}{\longrightarrow} A \otimes B$$

define a natural transformation from Id to $\Delta\otimes$.
That these natural transformations satisfy the triangle equalities for $\otimes$ being left adjoint to Δ follows from the fact that e is a unit. So $\otimes \dashv \Delta$ as required.
That $(\Omega, \cong, Id)$ is initial in **CMon**($\mathcal{C}$) requires a similar manipulation. $\Box$

It is *not* the case that we can extend the above theorem to non-commutative monoids. i.e. coproduct in **Mon** ($\mathcal{C}$), the category of monoids over $\mathcal{C}$, is not given by tensor. The above proof breaks down since $*_A : A \otimes A \rightarrow A$ is not a monoid homomorphism from $A \otimes A$ to A unless A is a commutative monoid.
As for a concrete counter example we look at the case where $\mathcal{C}$=**Ab**, Abelian groups. Then **CMon(Ab)** is the category **CRng** of commutative rings and **Mon(Ab)** is

the category **Rng** of rings. Say R is a ring and $x, y \in R$ have the property that $xy \neq yx$. There is a unique ring homomorphism (f) from the commutative ring $Z[x]$ of polynomials over x to R that maps the polynomial x to x, and similarly there is a ring homomorphism (g) from $Z[y]$ to R that maps y to y. Now

$$Z[x] \otimes Z[y] = Z[x, y]$$

where $Z[x, y]$ is the commutative ring of polynomials over the set $\{x, y\}$. So *if* this tensor gave coproduct in the category of rings we would find that there is a ring homomorphism from $Z[x, y]$ to R corresponding to f, g. The image of this ring homomorphism would be a commutative subring of R. This contradicts the fact that $xy \neq yx$. In the context of a counter example it is appropriate to use the excluded middle: if a theorem is not true classically it certainly won't be true constructively. However, more subtly, the reader should be aware that whenever we make the assertion '**Ab** is monoidal closed', we are assuming a natural numbers object. This is because we need a natural numbers object in order to prove that Abelian group presentations present.

If we may assume further that $\mathcal{C}$ is symmetric monoidal *closed* (i.e. that $\forall A \in Ob(\mathcal{C})$ $A \otimes (\_) \dashv [A \to \_]$) then we have another result about the creation of colimits:

Theorem 2.5.2 *The forgetful functor F :$\mathbf{CMon}(\mathcal{C}) \to \mathcal{C}$ creates all filtered colimits.*

Proof: Say $D : J \longrightarrow \mathbf{CMon}(\mathcal{C})$ is a filtered diagram in $\mathbf{CMon}(\mathcal{C})$. Since $\otimes$ preserves colimits in each of its coordinates we can do the following manipulations:

$$\begin{aligned} colim_i FD(i) \otimes colim_j FD(j) &\cong colim_i(FD(i) \otimes colim_j FD(j)) \\ &\cong colim_i(colim_j(FD(i) \otimes FD(j))) \\ &\cong colim_{(i,j)} FD(i) \otimes FD(j) \end{aligned}$$

But from a piece of well known 'abstract nonsense' we know that

$$colim_{(i,j)}(FD(i) \otimes FD(j)) \cong colim_i(FD(i) \otimes FD(i))$$

since J is a filtered category and so the monoid operation $*_{D(i)}$ on the $D(i)$s induce a function

$$*_D : colim_i FD(i) \otimes colim_i FD(i) \to colim_i FD(i)$$

As for a unit on $colim_i FD(i)$ note that the composite

$$\Omega \xrightarrow{e_{D(i)}} FD(i) \xrightarrow{\coprod_{FD(i)}} colim_i FD(i)$$

(where the $\coprod_{FD(i)}$ is an edge of the colimit cocone on FD) is the same for every i (use filteredness of J) and so define a unit (e_D) for $colim_i FD(i)$. It is then easy to check that $(colim_i FD(i), *_D, e_D)$ is the colimit of D in $\mathbf{CMon}(\mathcal{C})$. $\square$

So to complete our discussion about the existence of colimits in the category $\mathbf{CMon}(\mathcal{C})$ all we need to do is find out whether coequalizers exists or not. It turns out that the we have a more general theorem relating the existence of coequalizers in $\mathcal{C}$ to the existence of coequalizers in $\mathbf{Mon}(\mathcal{C})$, the category of monoids over $\mathcal{C}$. Compare this to our examination of finite coproducts above; there we saw that the description of coproducts in terms of tensor did not extend to the non-commutative case.

Theorem 2.5.3 (The generalized coverage theorem) *If $\mathcal{C}$ is a symmetric monoidal closed category and*

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B)$$

is a diagram in Mon($\mathcal{C}$) then if $c : B \to C$ is the coequalizer of

$$B \otimes A \otimes B \underset{*(1 \otimes g \otimes 1)}{\overset{*(1 \otimes f \otimes 1)}{\rightrightarrows}} B$$

(where $*$ is ternary multiplication induced by $*_B$) then C can be given a monoid structure $(C, *_C, e_C)$ such that

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B) \xrightarrow{c} (C, *_C, e_C)$$

is a coequalizer diagram in Mon($\mathcal{C}$).

Proof: The definition of e_C is just the composite $c \circ e_B$. Defining $*_C$ is a little more involved. Since $\mathcal{C}$ is closed we know that the endofunctor $(\_) \otimes B$ preserves coequalizers, hence the diagram

$$B \otimes A \otimes B \otimes B \underset{*(1 \otimes g \otimes 1) \otimes 1}{\overset{*(1 \otimes f \otimes 1) \otimes 1}{\rightrightarrows}} B \otimes B \xrightarrow{c \otimes 1} C \otimes B$$

is a coequalizer diagram in $\mathcal{C}$. But by associativity of the commutative monoid B the morphisms $*(1 \otimes f \otimes 1) \otimes 1$ and $*(1 \otimes g \otimes 1) \otimes 1$ are equalized by the morphism

$$B \otimes B \xrightarrow{*_B} B \xrightarrow{c} C$$

and so there exists a (unique) map $R : C \otimes B \to C$ such that $R(c \otimes 1) = c *_B$.
But we have two commutative squares:

$$\begin{tikzcd}
B \otimes B \otimes A \otimes B \arrow[r, shift left, "1 \otimes *(1 \otimes f \otimes 1)"] \arrow[r, shift right, "1 \otimes *(1 \otimes g \otimes 1)"'] \arrow[d, "c \otimes 1 \otimes 1 \otimes 1"'] & B \otimes B \arrow[d, "c \otimes 1"] \\
C \otimes B \otimes A \otimes B \arrow[r, shift left, "1 \otimes *(1 \otimes f \otimes 1)"] \arrow[r, shift right, "1 \otimes *(1 \otimes g \otimes 1)"'] & C \otimes B
\end{tikzcd}$$

$R(c \otimes 1)$ equalizes the top row and so since $c \otimes 1 \otimes 1 \otimes 1$ is an epimorphism (as c is) we know that R will equalize the bottom row. Hence it will factor through the coequalizer of the bottom row. But the coequalizer of the bottom row is $1 \otimes c : C \otimes B \to C \otimes C$ since $C \otimes (\_)$ preserves coequalizers. Hence $\exists *_C : C \otimes C \to C$ such that $R = *_C \circ (1 \otimes c)$. It is now a routine exercise to check that $(C, *_C, e_C)$ is a monoid, that c is a commutative monoid homomorphism and that

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B) \xrightarrow{c} (C, *_C, e_C)$$

is a coequalizer diagram in $\mathbf{Mon}(\mathcal{C})$ as required. For instance since $R = *_C(1 \otimes c)$ we have that $*_C(c \otimes c) = *_C(1 \otimes c)(c \otimes 1) = R(c \otimes 1) = c *_B$. i.e. c is a monoid homomorphism. Also $(c \otimes c \otimes c)$ is epic and so associativity for $*_C$ follows from associativity of $*_B$. $\Box$

As an immediate example we can use the above to construct coequalizers in the category **Rng** of rings. If

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is a digram in **Rng**, then it is well known that its coequalizer is given by taking the quotient of B by the two sided ideal generated by $\{f(a) - g(a) | a \in A\}$. However this two sided ideal is given by

$$I = \{\Sigma b_i(f-g)(a_i)c_i | a_i \in A, \quad b_i, c_i \in B\}$$

But the ring B/I is found by taking the quotient in **Ab**, and it is clear from the above expression for I that the Abelian group B/I is the coequalizer in **Ab** of

$$B \otimes A \otimes B \underset{*(1 \otimes g \otimes 1)}{\overset{*(1 \otimes f \otimes 1)}{\rightrightarrows}} B$$

As another application we have restriction to the commutative case. In the proof of the theorem it is a triviality to check that if B is a commutative monoid then so is the monoid structure constructed on C. Hence we are able to lift coequalizers from $\mathcal{C}$ to $\mathbf{CMon}(\mathcal{C})$. In fact most of our examples will be commutative, and in these cases the following simplification of the generalized coverage theorem is appropriate:

Theorem 2.5.4 *If $\mathcal{C}$ is a symmetric monoidal closed category and*

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B)$$

is a diagram in CMon($\mathcal{C}$) then if $c : B \to C$ is the coequalizer of

$$A \otimes B \underset{*_B(g \otimes 1)}{\overset{*_B(f \otimes 1)}{\rightrightarrows}} B$$

then C can be given a commutative monoid structure $(C, *_C, e_C)$ such that

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B) \xrightarrow{c} (C, *_C, e_C)$$

is a coequalizer diagram in CMon($\mathcal{C}$). $\Box$

A detailed discussion of why [2.5.3] is called the generalized coverage theorem is omitted until Section 2.9. There we will need a theorem that goes in the opposite direction; a theorem which shows how to find coequalizers in $\mathcal{C}$ given coequalizers in some category that behaves like $\mathbf{CMon}(\mathcal{C})$. The forgetful functor going from $\mathbf{CMon}(\mathcal{C})$ to $\mathcal{C}$ has a left adjoint if and only if free commutative monoids can be found on $\mathcal{C}$ objects. We find, opposite to the coverage theorem, that if there is some category $\mathcal{D}$ and a faithful functor U from $\mathcal{D}$ to $\mathcal{C}$ which has a left adjoint then coequalizers in $\mathcal{C}$ can be constructed from particular coequalizers in $\mathcal{D}$ provided we also know that $\mathcal{C}$ has finite limits and image factorisations (see e.g. 1.51 of [FS90] for a definition of image factorization). We know from Theorem [2.3.2] how to construct the free frame on a preframe and so we know that the forgetful functor from **Frm** to **PreFrm** has a left adjoint. It is easy to construct finite limits and image factorisations in the category **PreFrm** of preframes (for the latter just take the subpreframe generated by the set theoretic image of the function to be factorized) so the next theorem will prove that **PreFrm** has coequalizers from an assumption that **Frm** has coequalizers. Indeed the proof to follow is really just a repetition of the preframe version of Kříž's universal Theorem [2.3.3] (which itself is just a manipulation of the proof in [JV91] that preframe presentations present).

Theorem 2.5.5 *If $\mathcal{C}$ has finite limits and image factorisations, and there is some category $\mathcal{D}$ with a faithful functor $U : \mathcal{D} \to \mathcal{C}$ which has a left adjoint F then for any diagram*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in $\mathcal{C}$ its coequalizer is given by the image factorization of $B \overset{\eta_B}{\rightarrowtail} UFB \overset{Ue}{\twoheadrightarrow} UE$ where $FB \overset{e}{\twoheadrightarrow} E$ is the coequalizer in $\mathcal{D}$ of

$$FA \underset{Fg}{\overset{Ff}{\rightrightarrows}} FB$$

Proof: Let the image factorization described in the statement be written $q : B \to e[B]$. Say there is a morphism $B \overset{\bar{e}}{\to} \bar{E}$ in $\mathcal{C}$ such that $\bar{e}f = \bar{e}g$. So certainly $F\bar{e}Ff = F\bar{e}Fg$ and so there is a morphism d of $\mathcal{D}$

$$d : E \longrightarrow F\bar{E}$$

such that $de = F\bar{e}$. Pull the monomorphism $\eta_{\bar{E}}$ back along Ud to find a monomorphism $i : J \rightarrowtail UE$. But from the pullback diagram we see that the map $B \overset{\eta_B}{\rightarrowtail} UFB \overset{Ue}{\twoheadrightarrow} UE$ factors through i since:

$$\begin{aligned} Ud \circ Ue \circ \eta_{\bar{B}} &= U(d \circ e) \circ \eta_{\bar{B}} \\ &= UF\bar{e} \circ \eta_{\bar{B}} \\ &= \eta_{\bar{E}} \circ \bar{e} \end{aligned}$$

and hence the subobject J contains the subobject $e[B]$. So there is a map $\bar{d}$ from $e[B]$ to $\bar{E}$ such that $\bar{d}q = \bar{e}$. Uniqueness of $\bar{d}$ follows if q is an epimorphism; but we have equalizers in $\mathcal{C}$ and so the cover q is an epimorphism. □

2.6 Frames as commutative monoids

We first introduce the more well known way of looking at frames as commutative monoids i.e. as SUP-lattices with a monoid structure given by meet. Of course a SUP-lattice is a complete poset, i.e. a poset with all joins. SUP-lattice homomorphisms preserve all joins. We have defined the category **SUP**.

The fact that **SUP** has coequalizers is shown in [JT84]. In Proposition 4.3 of Chapter 1 they show that if R is any subset of $M \times M$ where M is a SUP-lattice then the quotient of M by the congruence generated by R is given by the set

$$Q = \{x \in M | \forall (z_1, z_2) \in R, \quad z_1 \leq x \quad \Leftrightarrow \quad z_2 \leq x\}$$

(cf R-coherent elements). So if

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a pair of arrows in **SUP** then use the relation $\{(fb, gb)|b \in B\}$ to define the coequalizer of f and g. Clearly we can also use this general construct to describe tensor product of SUP-lattices and so we see that **SUP** is a symmetric monoidal closed category with coequalizers.

Now say we are given a commutative monoid $(A, *, e_A)$ over a SUP-lattice A which is also a semilattice. i.e. $*$ is idempotent. We can then give A a second order with which the $*$ operation becomes meet. This second order will not necessarily coincide with $\leq_A$. However the two orders will coincide if (and only if) $a \leq_A e_A$ for every $a \in A$. For if we assume $a \leq_A e_A$ for every $a \in A$ then since $*$ is monotone in both its coordinates we know

$$\begin{array}{rcl} *_A(a \otimes b) & \leq_A & *_A(a \otimes e_A) \\ & = & a \\ *_A(a \otimes b) & \leq_A & *_A(e_A \otimes b) \\ & = & b \end{array}$$

for every a, b. Further if $c \leq_A a, b$ then $c = *(c \otimes c) \leq_A *(a \otimes b)$ and so $*$ is meet with respect to the order $\leq_A$. Clearly such a commutative monoid will be a frame.

So frames are particular types of commutative monoids over **SUP**. A (commutative) monoid $(A, *_A, e_A)$ is a frame if and only if (1) $a \leq e_A \quad \forall a \in A$ and (2) $*_A(a \otimes a) = a \quad \forall a \in A$. The first equation tells us that e_A is the top element of A. We find ([JV91]) that this result has a 'preframe parallel':

Theorem 2.6.1 *The category of frames is isomorphic to the full subcategory of CMon(***PreFrm***) consisting of all objects $(A, *, e)$ satisfying*

$$\begin{array}{ll} (1) & e(0) \leq a \quad \forall A \\ (2) & *(a \otimes a) = a \end{array}$$

Proof: Say A is a frame. Then

$$\vee : A \times A \to A$$

is clearly a preframe bihomomorphism. It is easy to check that

$$\begin{array}{rcl} e : \Omega & \longrightarrow & A \\ i & \longmapsto & \bigvee^{\uparrow}(\{0\} \cup \{1|1 \leq i\}) \end{array}$$

is a preframe homomorphism (Ω is compact) and that this makes $(A, \vee, e)$ into a commutative monoid which satisfies (1) and (2).
Conversely say $(A, *, e)$ is a commutative monoid which satisfies (1) and (2). Certainly $e(0)$ is 0_A and so A has a least element. We check that $*(a \otimes b)$ is the least upper bound of a and b in A. The fact that e is a unit tells us that $a = *(a \otimes e(0)) \quad (\forall a)$. But $a \otimes e(0) \leq a \otimes b \quad \forall b$ and so $a, b \leq *(a \otimes b)$.

Now say $a, b \leq c$ then $a \otimes b \leq c \otimes c$ and so $*(a \otimes b) \leq *(c \otimes c) = c$. $\square$

Frames can thus be viewed as SUP-lattices with a particular monoid structure (corresponding to meet) or they can be viewed as preframes with a monoid structure giving a finitary join operation.
Say $(A, *_A, e_A), (B, *_B, e_B)$ are two commutative monoids in **PreFrm**. We know that their coproduct in **CMon(PreFrm)** is given by

$$(A \otimes B, *, e)$$

where $* : (A \otimes B) \otimes (A \otimes B) \stackrel{\cong}{\rightarrow} (A \otimes A) \otimes (B \otimes B) \stackrel{*_A \otimes *_B}{\longrightarrow} A \otimes B$ and

$$e : \Omega \stackrel{\cong}{\rightarrow} \Omega \otimes \Omega \stackrel{e_A \otimes e_B}{\longrightarrow} A \otimes B$$

Now $\forall a \in A, b \in B$ we have

$$\begin{aligned} e(0) &= (e_A \otimes e_B)(0 \otimes 0) \\ &= e_A(0) \otimes e_B(0) \leq a \otimes b \end{aligned}$$

if $e_A(0) \leq a \quad \forall a$ and $e_B(0) \leq b \quad \forall b$ So if A, B are frames then the set

$$\{u \in A \otimes B | e_A(0) \otimes e_B(0) \leq u\}$$

is a subpreframe of $A \otimes B$ that contains all the generators of $A \otimes B$ and so is the whole of $A \otimes B$. Hence, if A, B are frames then $A \otimes B$ has a least element: $0_A \otimes 0_B$.

$$\begin{aligned} *((a \otimes b) \otimes (a \otimes b)) &= (*_A \otimes *_B)((a \otimes a) \otimes (b \otimes b)) \\ &= (*_A(a \otimes a)) \otimes (*_B(b \otimes b)) \\ &= a \otimes b \end{aligned}$$

if $*_A(a \otimes a) = a \quad \forall a$ and $*_B(b \otimes b) = b \quad \forall b$.
Notice that the equation $*((a \otimes b) \otimes (a \otimes b)) = a \otimes b$ is enough to tell us that $*(u \otimes u) = u$ for any $u \in A \otimes B$. This is because the set

$$\{u \in A \otimes B | *(u \otimes u) = u\}$$

is a subpreframe of $A \otimes B$ and contains all the generators of $A \otimes B$.
Proof that it is a subpreframe: Certainly $*(1 \otimes 1) = 1$. Say u, v satisfy $*(u \otimes u) = u$ and $*(v \otimes v) = v$. Then

$$\begin{aligned} *((u \wedge v) \otimes (u \wedge v)) &= *((u \otimes u) \wedge (v \otimes v) \wedge (u \otimes v) \wedge (v \otimes u)) \\ &\leq *((u \otimes u) \wedge (v \otimes v)) \\ &= *(u \otimes u) \wedge *(v \otimes v) = u \wedge v \end{aligned}$$

In the other direction

$$u \wedge v = *((u \wedge v) \otimes 0) \leq *((u \wedge v) \otimes (u \wedge v))$$

Say $T \subseteq^{\uparrow} A \otimes B$ is such that $*(t \otimes t) = t$ for all $t \in T$. Then for all $t \in T$:

$$\begin{aligned} t = *(t \otimes t) &\leq *(\bigvee^{\uparrow} T \otimes t) \\ &\leq *(\bigvee^{\uparrow} T \otimes \bigvee^{\uparrow} T) \end{aligned}$$

Hence $\bigvee^{\uparrow} T \leq *(\bigvee^{\uparrow} T \otimes \bigvee^{\uparrow} T)$. Conversely

$$\begin{aligned} *(\bigvee^{\uparrow} T \otimes \bigvee^{\uparrow} T) &= \bigvee^{\uparrow}_{t} *(t \otimes \bigvee^{\uparrow} T) \\ &= \bigvee^{\uparrow}_{(t,\bar{t}) \in T \times T} *(t \otimes \bar{t}) \\ &\leq \bigvee^{\uparrow}_{t \in T} *(t \otimes t) = \bigvee^{\uparrow} T \end{aligned}$$

where the penultimate implication is by directedness of T. □

So the above shows us that if $(A, *_A, e_A), (B, *_B, e_B)$ are both frames then their coproduct in **CMon**(**PreFrm**) is also a frame. i.e. frame coproduct is given by preframe tensor.

Theorem 2.6.2 **Loc** *has finite products. If X, Y are two locales then the frame of opens of their product is given by:*

$$\Omega(X \times Y) \cong \Omega X \otimes \Omega Y$$

where the tensor $\otimes$ is either preframe tensor or SUP-lattice tensor.

Proof: We have shown the result for the preframe tensor. The result for the SUP-lattice tensor (is well known and) follows exactly the same path. It relies on the characterization of frames as those members A of **CMon**(**SUP**) which satisfy $a \leq e_A(1)\ \forall a \in A$ and $*_A(a \otimes a) = a$ for all $a \in A$. Note that the proof that the set $\{u | *(u \otimes u) = u\}$ is a subSUP-lattice is less intricate. □

The 'creation of colimits' results of the previous section also preserves the frame structure:

Theorem 2.6.3 F :**Frm**$\to$**PreFrm** *creates filtered colimits*

Proof: Say $D : J \to$**CMon**(**PreFrm**) is such that its image is contained within **Frm** and J is filtered. So $D(i) = (FD(i), *_i, e_i)$ is a frame for every object i of J. We saw in the last section that $colim D = (colim FD, *, e)$ where
$* : colim FD \otimes colim FD \to colim FD$ is such that

$$\begin{tikzcd} FD(i) \otimes FD(i) \arrow[r, "*_i"] \arrow[d, "\lambda_i \otimes \lambda_i"'] & FD(i) \arrow[d, "\lambda_i"] \\ colim FD \otimes colim FD \arrow[r, "*"] & colim FD \end{tikzcd}$$

commutes for every i,
and $e : \Omega \to colim FD$ is such that

$$\begin{tikzcd} \Omega \arrow[r, "e_i"] \arrow[dr, "e"'] & FD(i) \arrow[d, "\lambda_i"] \\ & colim FD \end{tikzcd}$$

commutes for every i.
Now recall that

$$colimFD = \mathbf{PreFrm}< \coprod_i FD(i)|R >$$

for suitable R (see Theorem [2.4.3]) and $\lambda_i : FD(i) \to colimFD$ is given by $a \mapsto a$. So to prove $e(0) \leq u \quad \forall u \in colimFD$ all we need to do is check that

$$e(0) \leq a \quad \forall a \in \coprod_i FD(i)$$

Say $a \in FD(i)$ then

$$e(0) = \lambda_i e_i(0) = e_i(0) \leq a$$

and so $e(0) \leq u \quad \forall u \in colimFD$.
Similarly to see that $*(u \curlyvee u) = u \quad \forall u \in colimFD$ simply note that $*_i(a \curlyvee a) = a$ whenever $a \in FD(i)$. □

Again the SUP-lattice parallel can be checked by an identical method and we can write up both results as facts about locales:

Theorem 2.6.4 **Loc** *has cofiltered limits. If* $D : J \longrightarrow$ **Loc** *is a cofiltered diagram of locales then*

$$\begin{aligned} \Omega lim_J D &\cong \mathbf{PreFrm} < \coprod_i FD(i)|R_{PreFrm} > \\ &\cong \mathbf{SUP} < \coprod_i FD(i)|R_{SUP} > \end{aligned}$$

for suitable Rs. □

Theorem 2.6.5 *If*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is a diagram in **Frm** *then the preframe coequalizer of*

$$A \otimes B \underset{*_B(g \otimes 1)}{\overset{*_B(f \otimes 1)}{\rightrightarrows}} B$$

is a frame, and is the coequalizer of f and g in **Frm**.

Proof: As in the last proof the concrete construction of the coequalizer enables us to check the commutative monoid structure defined on it via Theorem [2.5.4] satisfies the conditions (1) and (2).
Say $c : B \to C$ is the preframe coequalizer of $*_B(f \otimes 1), *_B(g \otimes 1)$. Then

$$\{a \in C | e_C(0) \leq a\}$$
$$\{a \in C | *_C (a \curlyvee a) = a\}$$

are both subpreframes of C and c factors through both of them since B is a frame. Hence they are both the whole of C. □

It should be apparent that this last result could also have been written with SUP-lattices in place of preframes. The localic conclusion is:

Theorem 2.6.6 **Loc** *has equalizers. If*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is a diagram in **Loc** *then the equalizer, E, is given by*

$$\begin{aligned} \Omega E &\cong \textbf{PreFrm} < \Omega X \text{ (qua preframe)} | \Omega f(b) \vee a = \Omega g(b) \vee a \quad \forall a \in \Omega X, b \in \Omega Y > \\ &\cong \textbf{SUP} < \Omega X \text{ (qua SUP-lattice)} | \Omega f(b) \wedge a = \Omega g(b) \wedge a \quad \forall a \in \Omega X, b \in \Omega Y > \end{aligned}$$ □

We will discuss how this last theorem is just the preframe version and the SUP-lattice version of the coverage theorem in Section 2.9.

When it comes to discuss the pullback stability of proper and open locale maps in the next chapter it will be useful to have the corollary:

Corollary 2.6.1 **Loc** *has pullbacks. If*

$$\begin{array}{ccc} W & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a pullback diagram in **Loc** *then*

$$\Omega W \cong PreFrm< \Omega X \otimes \Omega Y \text{ (qua preframe) } |(\Omega f(c) \vee a) \otimes b = a \otimes (\Omega g(c) \vee b) \\ \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z >$$

and

$$\Omega W \cong SUP< \Omega X \otimes \Omega Y \text{ (qua SUP-lattice) } |(\Omega f(c) \wedge a) \otimes b = a \otimes (\Omega g(c) \wedge b) \\ \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z >$$

(where the tensor is SUP-lattice tensor in the second equation and prefame tensor in the first).

Proof: A pushout is just a particular kind of coequalizer. The corollary is an application of the last result. □

2.7 Applications in Loc

The following lemma shows us how the two descriptions of locale product given in the last section lead to two very different formulas for the closure of the diagonal of a locale. The new preframe version of this formula will be used extensively later on.

Lemma 2.7.1 *If X is any locale then the closure of the diagonal $\Delta : X \hookrightarrow X \times X$ is given by the closed sublocale*

$$\neg\# \hookrightarrow X \times X$$

where $\# \in \Omega(X \times X)$ is given by

$$\# = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes b_i) | \wedge_{i\in I} (a_i \vee b_i) = 0 \ I \textit{ finite } \}$$

and equivalently by

$$\# = \bigvee\{a \otimes b | a \wedge b = 0\}.$$

This preframe formula for # can be found in [Vic94].
Proof: From Section 1.7.1 we know that if $i : Y \hookrightarrow X$ is a sublocale then its closure is given by

$$\neg\forall_i(0) \hookrightarrow X$$

and so all that we are doing is checking that $\forall_\Delta(0) = \#$
We prove the first claim of the theorem by looking at the case where $\Omega\Delta : \Omega X \otimes \Omega X \to \Omega X$ is given by the unique *preframe* homomorphism which sends $a \otimes b$ to $a \vee b$. It follows that

$$\forall_\Delta(0) = \bigvee^{\uparrow}\{J | \Omega\Delta(J) = 0\}$$

The result then follows quite clearly from the fact that for every J in $\Omega X \otimes \Omega X$

$$J = \bigvee_j^{\uparrow} \wedge_{i \in I_j} (a_i^j \otimes b_i^j)$$

for some suitable collection of a_i^j, b_i^js (where all the I_js are finite). This is because the set of all elements of this form forms a subpreframe of $\Omega X \otimes \Omega X$ which contains all the generators of the tensor.

As for the SUP-lattice presentation of the closure of the diagonal we use the same argument. Success of this argument hinges on the fact that the set of all elements of $\Omega X \otimes \Omega X$ (=SUP-lattice tensor) of the form

$$\bigvee_{i\in I} a_i \otimes b_i$$

for some set I forms a subSUP-lattice of $\Omega X \otimes \Omega X$ which contains all the generators of the tensor and so is the whole of $\Omega X \otimes \Omega X$. $\Omega\Delta$ sends $a \otimes b$ to $a \wedge b$.

Notice also that these two parallel results are inter-provable; use the fact that $a \otimes b = (a \otimes 0) \wedge (0 \otimes b)$. For then $(a \vee 0) \wedge (0 \vee b) = 0$ if $a \wedge b = 0$ and so certainly

$$\bigvee\{a \otimes b | a \wedge b = 0\} \leq \bigvee^{\uparrow}\{\wedge_i(a_i \otimes b_i) | \wedge_{i\in I} (a_i \vee b_i) = 0 \ I \text{ finite } \}$$

In the other direction say $\wedge_{i\in I}(a_i \vee b_i) = 0$. Then $(\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2} b_i) = 0$ for every J_1, J_2 finite with $J_1, J_2 \subseteq I, \quad I \subseteq J_1 \cup J_2$ by the finite distributivity law of [1.2.6]. But by the same finite distributivety law (and the equation $a \otimes b = (a \otimes 1) \vee (1 \otimes b)$) we have

$$\begin{aligned}
\wedge_i(a_i \otimes b_i) &= \wedge_i((a_i \otimes 1) \vee (1 \otimes b_i)) \\
&= \bigvee[\wedge_{i\in J_1}(a_i \otimes 1) \wedge \wedge_{i\in J_2}(1 \otimes b_i)] \\
&= \bigvee[((\wedge_{i\in J_1} a_i) \otimes 1) \wedge (1 \otimes (\wedge_{i\in J_2} b_i))] \\
&= \bigvee(\wedge_{i\in J_1} a_i) \otimes (\wedge_{i\in J_2} b_i) \\
&\leq \bigvee\{a \otimes b | a \wedge b = 0\} \qquad \Box
\end{aligned}$$

Recall in Chapter 1 that we defined the specialization order on a space. The localic analogue is the specialization sublocale. It is clear that if, for any locale X, we define $\sqsubseteq \hookrightarrow X \times X$ by

$$\Omega(\sqsubseteq) \equiv \mathbf{Fr} < \Omega X \otimes \Omega X \text{ qua frame} | a \otimes 1 \leq 1 \otimes a \quad \forall a \in \Omega X >$$

then we will have captured the defining spatial characteristic of the specialization order (namely that $x \sqsubseteq y$ if and only if for every open a if $x \in a$ then $y \in a$). The tensor in the above is the SUP-lattice tensor. On the preframe side we have:

Lemma 2.7.2 *$\Omega(\sqsubseteq) \cong \mathbf{Fr} < \Omega X \otimes \Omega X$ qua frame$|a \parr 0 \leq 0 \parr a \quad \forall a \in \Omega X >$, where $\otimes$ is preframe tensor.*

Proof: Take $a \parr b$ to $(a \otimes 1) \vee (1 \otimes b)$ and $a \otimes b$ to $(a \parr 0) \wedge (0 \parr b)$. The relations are preserved and so these assignments define frame homomorphisms between the two presentations of $\Omega(\sqsubseteq)$. □

Lemma 2.7.3 *$\sqsubseteq \wedge \sqsupseteq = \Delta$, where $\wedge$ is meet in the poset $Sub(X \times X)$, and $\sqsupseteq \equiv \tau \circ \sqsubseteq$ (τ is the twist isomorphism $X \times X \to X \times X$).*

Proof: (We prove this fact using preframe techniques though SUP-lattice techniques could equally well have been used.)
Certainly $\Delta \leq_{Sub(X \times X)} \sqsubseteq$, since

$$\begin{aligned} \Omega l : \Omega(\sqsubseteq) &\longrightarrow \Omega X \\ a \parr b &\longmapsto a \vee b \end{aligned}$$

is clearly well defined and so

$$\begin{tikzcd} X \arrow[rr, "l"] \arrow[dr, "\Delta"'] & & \sqsubseteq \arrow[dl] \\ & X \times X & \end{tikzcd}$$

commutes.
Symmetrically $\Delta \leq (\sqsupseteq)$.
Say $z : Z \hookrightarrow X \times X$ is some sublocale with the property that

$$Z \leq_{Sub(X \times X)} (\sqsubseteq), \qquad Z \leq_{Sub(X \times X)} (\sqsupseteq)$$

So there exists $\Omega m : \Omega(\sqsubseteq) \to \Omega Z$ and $\Omega m_\tau : \Omega(\sqsupseteq) \to \Omega Z$ with

$$\Omega m(a \parr b) = \Omega z(a \parr b), \quad \Omega m_\tau(a \parr b) = \Omega z(a \parr b)$$

It follows that for all $b \in \Omega X$

$$\begin{aligned} \Omega z(b \parr 0) &= \Omega m(b \parr 0) \\ &\leq \Omega m(0 \parr b) = \Omega z(0 \parr b) \end{aligned}$$

and by the existence of m_τ we find

$$\begin{aligned} \Omega z(0 \parr b) &= \Omega m_\tau(0 \parr b) \\ &\leq \Omega m_\tau(b \parr 0) = \Omega z(b \parr 0) \end{aligned}$$

i.e. $\Omega z(b \parr 0) = \Omega z(0 \parr b)$ and so

$$\pi_1 \circ z = \pi_2 \circ z$$

Hence $Z \leq_{Sub(X \times X)} \Delta$. $\Box$
Of course this result is true spatially if (and only if) the topological space is T_0.

Our next comment is that we can now show that a locale map $f : X \to Y$ is a sublocale if and only if it is a regular monomorphism. This is a well known basic fact about locales and is equivalent to the statement that a frame homomorphism is a regular epimorphism if and only if it is a surjection. But since we have shown that **Frm** is suitably algebraic this follows at once. [For a proof notice that if $q : A \to C$ is a frame surjection then it is the coequalizer of

$$B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$$

where B is the congruence on A given by $\{(a_1, a_2) | q(a_1) = q(a_2)\}$. In the other direction we can use the coverage theorem with C=**SUP** to show that coequalizers in **Frm** are surjections since coequalizers in **SUP** are surjections.]

Inside **Frm** we then find that a homomorphism $h : A \to B$ can be factored as

$$A \xrightarrow{[\_]} (A/\equiv_h) \overset{i}{\hookrightarrow} B$$

where $[\_]$ is a surjection and $\equiv_h$ is the frame congruence $a_1 \equiv_h a_2$ if and only if $h(a_1) = h(a_2)$. This factorization enjoys the property that if h can also be factored as

$$A \xrightarrow{q} C \xrightarrow{l} B$$

for some surjection q then there is a frame homomorphism $k : C \to A/\equiv_h$ such that

$$k \circ q = [\_] \qquad i \circ k = l$$

Translated to a fact about locales this means that if $f : X \to Y$ is a locale map then it can be factored as

$$X \xrightarrow{q} f[X] \overset{i}{\hookrightarrow} Y$$

where q is an epimorphism and i is a regular monomorphism, and if f can also be factored as

$$X \xrightarrow{\bar{q}} Z \overset{\bar{i}}{\hookrightarrow} Y$$

where $\bar{i}$ is a regular monomorphism then there is a locale map $p : f[X] \to Z$ such that

$$p \circ q = \bar{q} \qquad \bar{i} \circ p = i$$

This result implies that any locale map factors uniquely (up to isomorphism) as an epimorphism followed by a regular monomorphism. This is a well known result of locale theory.

2.8 Tychonoff's theorem

The following proof is what appears in Johnstone and Vickers' paper [JV91].

Theorem 2.8.1 *The product of compact locales is compact*

Proof: We need to show, given a set $(A_i)_{i\in I}$ of compact frames, that their coproduct $\coprod_i A_i$ is compact.
It is well known that just as arbitrary joins can be written as directed joins of finite joins, arbitrary coproducts can be written as filtered colimits of finite coproducts.
We first check that finite coproducts of compact frames are compact. Since Ω is compact we know that nullary frame coproducts are compact. Say A, B are two compact frames. Then the functions

$$\begin{array}{rcl} A & \to & \Omega \\ a & \mapsto & (1 \le a) \\ B & \to & \Omega \\ b & \mapsto & (1 \le b) \end{array}$$

are both preframe homomorphisms and so

$$(a, b) \longmapsto (1 \le b) \vee (1 \le a)$$

is a preframe bihomomorphism from $A \times B$ to Ω and hence induces a preframe homomorphism $h : A \otimes B \to \Omega$. I claim that

$$\{u \in A \otimes B | h(u) = 1 \quad \Rightarrow \quad u = 1\}$$

is a subpreframe of $A \otimes B$ and contains all the generators $a \otimes b$ of $A \otimes B$. That it is a subpreframe is easy enough (Ω is compact!), and so we check that
$h(a \otimes b) = 1 \quad \Rightarrow \quad a \otimes b = 1$.
But $h(a \otimes b) = 1 \quad \Rightarrow \quad (1 \le a) \vee (1 \le b)$ and so $1 \le a \otimes b$ follows.
Hence $\forall u \in A \otimes B \quad h(u) = 1 \quad \Rightarrow u = 1$. Now say $S \subseteq^{\uparrow} A \otimes B$ has $\bigvee^{\uparrow} S = 1$. Then $h(\bigvee^{\uparrow} S) = 1 \quad \Rightarrow \quad \bigvee^{\uparrow}_{s\in S} h(s) = 1 \quad \Rightarrow \quad \exists s \in S \quad h(s) \quad \Rightarrow \quad s = 1$, and so $A \otimes B$ is compact.
Now, as we said above,

$$(\textstyle\coprod_i A_i) = colim_{\bar{I}}(\coprod_{i\in\bar{I}} A_i)$$

where $\bar{I}$ ranges over the finite subsets of I, and we've just checked that $\coprod_{i\in\bar{I}} A_i$ is compact for every such $\bar{I}$.
Since all such $\coprod_{i\in\bar{I}} A_i$ are compact we know that there are preframe homomorphisms

$$\begin{array}{rcl} h_{\bar{I}} : \coprod_{i\in\bar{I}} A_i & \longrightarrow & \Omega \\ u & \longmapsto & (1 \le u) \end{array}$$

and so (since as we saw above $colim_{\bar{I}}(\coprod_{i\in\bar{I}} A_i)$ is created from the preframe colimit) there exists

$$h : \textstyle\coprod_i A_i \to \Omega$$

a preframe homomorphism such that

$$\begin{tikzcd} \coprod_{i\in\bar{I}} A_i \arrow[d, "\lambda_{\bar{I}}"'] \arrow[dr, "h_{\bar{I}}"] & \\ \coprod_i A_i \arrow[r, "h"] & \Omega \end{tikzcd}$$

commutes for every $\bar{I}$.
As before all we need to do (to conclude that $\coprod_i A_i$ is compact) is check that the set

$$\{u \in \coprod_i A_i | h(u) = 1 \quad \Rightarrow \quad u = 1\}$$

is a subpreframe of $\coprod_i A_i$ which contains all the generators. It is certainly a subpreframe.
That it contains all the generators is easy enough since the set of generators is just the disjoint union of the $\coprod_{i \in \bar{I}} A_i$. $\square$

2.9 The Coverage Theorems

2.9.1 SUP-lattice version

We describe Johnstone's coverage theorem as stated in II 2.11 of [Joh82]. Given a meet semilattice A a function $C : A \to PPA$ is called a *coverage* if

$$\begin{gathered}\text{(i) } T \subseteq \downarrow a \quad \forall a \in A \quad \forall T \in C(a) \text{ and} \\ \text{(ii) } C \text{ is } \textit{meet stable}, \text{ i.e. } \forall a \in A, \forall T \in C(a), \forall b \in A \\ \{t \wedge b | t \in T\} \in C(a \wedge b)\end{gathered}$$

Define $C - Idl(A)$ to be the set of *C-ideals* of A: they are the lower closed subsets I of A such that $\forall a \in A, \forall T \in C(a)$ if $T \subseteq I$ then $a \in I$. If B is some frame then a function $f : A \to B$ is said to *take covers to joins* if $\forall a \in A, \forall T \in C(a)$,

$$\bigvee_B \{f\bar{a} | \bar{a} \in T\} = fa$$

Johnstone's coverage result is: the set of C-ideals on a coverage forms a frame and the map

$$A \xrightarrow{<\text{-}>} C - Idl(A)$$

which is defined to take $a \in A$ to the ideal generated by $\{a\}$, is the free semilattice homomorphism from A to a frame which takes covers to joins.

When Abramsky and Vickers were investigating quantales in [AV93] they found it useful to view the coverage theorem as the statement that certain frame presentation could equally be viewed as SUP-lattice presentations. Indeed in the 'Preframe Presentation Presents' paper [JV91] the *content* of the coverage result is stated as follows: given any meet semilattice A with a coverage on it then

$$\begin{gathered}\mathbf{Frm} < A \text{ (qua meet semilattice) } | a = \vee T \quad T \in C(a) > \\ \cong \mathbf{SUP} < A \text{ (qua poset) } | a = \vee T \quad T \in C(a) >\end{gathered}$$

We take Johnstone's coverage theorem to be this last result and prove that it implies and is implied by the SUP-lattice version of the generalized coverage theorem. This theorem then reads as the following coequalizer result: if

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \end{tikzcd}$$

is a diagram in **Frm** and if

$$\begin{tikzcd} B \otimes A \arrow[r, shift left, "\wedge(f \otimes 1)"] \arrow[r, shift right, "\wedge(g \otimes 1)"'] & A \arrow[r, "e"] & E \end{tikzcd} \qquad (*)$$

is a coequalizer diagram in **SUP** then

$$B \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{e} E$$

is a coequalizer diagram in **Frm**.
Intuitively the presence of $\wedge$ in $(*)$ corresponds to the meet stability condition that we have on coverages.
We now assume Johnstone's coverage theorem and try to prove this coequalizer result. Say we are given

$$B \overset{f}{\underset{g}{\rightrightarrows}} A$$

in **Frm**. Define a coverage on A as follows:

$$\begin{array}{cc} \{gb \wedge a \wedge fb\} \in C(fb \wedge a) & \forall b \in B, \forall a \in A \\ \{fb \wedge a \wedge gb\} \in C(gb \wedge a) & \forall b \in B, \forall a \in A \\ T \in C(\bigvee_A T) & \forall T \subseteq A \end{array}$$

(It is easy to check that this defines a coverage.)

But it is clear that with this coverage the coequalizer of

$$B \overset{f}{\underset{g}{\rightrightarrows}} A$$

(in **Frm**) must be the frame presented by

$$\mathbf{Frm}< A \text{ (qua meet semilattice) } | a = \vee T \quad T \in C(a) >$$

and also that the coequalizer of

$$B \otimes A \overset{\wedge(f \otimes 1)}{\underset{\wedge(g \otimes 1)}{\rightrightarrows}} A$$

(in **SUP**) must be the SUP-lattice presented by

$$\mathbf{SUP}< A \text{ (qua poset) } | a = \vee T \quad T \in C(a) >$$

so an assumption of the Johnstone's coverage theorem allows us to conclude the SUP-lattice version of the generalized coverage theorem.

Conversely let us assume the SUP-lattice version of the generalized coverage theorem i.e. the coequalizer result of the previous page. Say we are given a coverage $C : A \to PPA$ on some meetsemilattice A. Let DA be the set of lower closed subsets of A. It is clearly a frame where join is given by union and meet is given by intersection. It is also the free frame on the meet semilattice A, this has been remarked upon already just before Theorem [2.3.2]. Let B be the least frame congruence on $DA \times DA$ which contains $(\downarrow T, \downarrow a)$ for all pairs (T, a) such that $T \in C(a)$. So there are frame homomorphisms

$$B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} DA.$$

It is easy to see that if their coequalizer exists then it is

$$\mathbf{Frm}< A \text{ (qua meet semilattice) } | a = \vee T \quad T \in C(a) >.$$

But

Lemma 2.9.1 *The free SUP-lattice on A qua poset and the free frame on A qua meet semilattice are isomorphic.*

Proof: They are both given by DA. □
Because of this fact we know that there is a SUP-lattice morphism e from DA to the SUP-lattice E defined to be

$$\mathbf{SUP}< A \text{ (qua poset) } | a = \vee T \quad T \in C(a) >.$$

It is easy, using the meet stability property of coverages, to verify that

$$B \otimes DA \underset{\wedge(\pi_2 \otimes 1)}{\overset{\wedge(\pi_1 \otimes 1)}{\rightrightarrows}} DA \xrightarrow{\quad e \quad} E$$

is a coequalizer diagram in **SUP** and so Johnstone's coverage theorem will follow from the generalized coverage theorem.

2.9.2 Preframe version

Before we tackle the preframe version of the coverage theorem we need to make an observation about the free $\wedge$-semilattice on a poset.

Lemma 2.9.2 *Let A be a join semilattice. Then the free meet semilattice on A qua poset (i.e.* **SLat** $< A | a_1 \wedge a_2 = a_1$ *if* $a_1 \leq_A a_2 >$*) is a distributive lattice and is the free distributive lattice on A qua $\vee$-semilattice*
(i.e. **DLat** $< A | a_1 \vee a_2 = a_1 \vee_A a_2 \quad \forall a_1, a_2 \in A, \quad 0 = 0_A >$*).*

Proof: (This proof also gives a concrete description of $\wedge$-**Slat**$< A$ qua poset$>$.) If $T, S \in FA$ (i.e. if T, S are finite subsets of A) then we write

$$S \precsim_U T$$

if and only if $\forall t \in T$ there exists $s \in S$ such that $s \leq_A t$. ($\precsim_U$ is the upper or Smyth preorder.) $FA / \precsim_U$ (i.e. FA quotiented by this preorder) is the free $\wedge$-semilattice on A qua poset. A is injected into $FA/ \precsim_U$ by $a \mapsto [\{a\}]$. If $[S], [T]$ are two elements of $FA/ \precsim_U$ then

$$[S] \wedge [T] = [S \cup T].$$

This is easily verified using the fact that $[S] \leq [T]$ in $FA/ \precsim_U$ if and only if $S \precsim_U T$.

If A is a join semilattice then

$$[S] \vee [T] = [\{s \vee t | (s,t) \in S \times T\}]$$

and so $FA/ \precsim_U$ is a join semilattice. As for distributivity notice that

$$([S] \vee [T]) \wedge [V] = [\{s \vee t | s \in S, t \in T\} \cup V]$$

and

$$([S] \wedge [V]) \vee ([T] \wedge [V]) = [\{\bar{s} \vee \bar{t} | \bar{s} \in S \cup V, \bar{t} \in T \cup V\}]$$

It is easy to see,

$$\begin{array}{rcl} \{\bar{s} \vee \bar{t} | \bar{s} \in S \cup V, \bar{t} \in T \cup V\} & \precsim_U & \{s \vee t | s \in S, t \in T\} \cup V \\ \{s \vee t | s \in S, t \in T\} \cup V & \precsim_U & \{\bar{s} \vee \bar{t} | \bar{s} \in S \cup V, \bar{t} \in T \cup V\}, \end{array}$$

the latter by observing that $(S \cup V) \times (T \cup V) \subseteq (S \times T) \cup (A \times V) \cup (V \times A)$.

That $FA/\precsim_U$ is the free distributive lattice on A qua $\vee$-semilattice follows a simple manipulation: say $f : A \to B$ is a $\vee$-preserving function to a distributive lattice B. Then there exists a unique meet preserving $\bar{f} : FA/\precsim_U \to B$ such that $\bar{f} \circ [\{\_\}] = f$. Clearly for any $a, b \in A$

$$\begin{array}{rcl} \bar{f}([\{a\}] \vee [\{b\}]) & = & \bar{f}[\{a \vee b\}] \\ & = & f(a \vee b) \\ & = & f(a) \vee f(b) \\ & = & \bar{f}[\{a\}] \vee \bar{f}[\{b\}] \end{array}$$

and so $\bar{f}([S] \vee [T]) = \bar{f}([S]) \vee \bar{f}([T])$ follows since for any $V \in FA$ we have

$$[V] = \wedge_{v \in V}[\{v\}]. \qquad \square$$

The preframe coverage theorem (5.1 of [JV91]) is as follows: let A be a join semilattice and let C be a set of preframe relations of the form

$$\wedge S \leq \bigvee\nolimits^{\uparrow}_{i \in I} \wedge S_i$$

(where S, S_i are finite subsets of A and $\{\wedge S_i | i \in I\} \subseteq^{\uparrow} A$) which are *join stable.* This means that if $x \in A$ and $\wedge S \leq \bigvee^{\uparrow}_i \wedge S_i$ is in C than

$$\wedge\{x \vee y : y \in S\} \leq \bigvee\nolimits^{\uparrow}_i \wedge\{x \vee y : y \in S_i\}$$

is also in C. Then

$$\mathbf{PreFrm} < A \text{ (qua poset) } |C> \cong \mathbf{Frm} < A \text{ (qua } \vee\text{-semilattice)} |C>$$

the generators corresponding under the isomorphism in the obvious way.

The preframe version of the generalized coverage theorem is the following coequalizer result: if

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a diagram in **Frm** and if

$$B \otimes A \underset{\vee(g \otimes 1)}{\overset{\vee(f \otimes 1)}{\rightrightarrows}} A \xrightarrow{e} E$$

is a coequalizer diagram in **PreFrm** then

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{e} E$$

is a coequalizer diagram in **Frm**.

Let us assume the preframe coverage theorem. Say we are given

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in **Frm**. Define C, a set of preframe relations on A, as follows:

$$\bigvee_A^{\uparrow} J \leq \bigvee^{\uparrow}\{j|j \in J\}$$

for every directed $J \subseteq^{\uparrow} A$ and

$$a_1 \wedge a_2 \leq a_1 \wedge_A a_2 \quad \forall a_1, a_2 \in A$$

and $\forall b \in B, \forall a \in A$

$$\begin{array}{c} fb \vee a \leq gb \vee a \\ gb \vee a \leq fb \vee a \end{array}$$

It is easy to see that C is join stable. It is also easy to see that

$$\mathbf{PreFrm} < A \text{ (qua poset) } |C>$$

is the coequalizer of

$$B \otimes A \underset{\vee(g \otimes 1)}{\overset{\vee(f \otimes 1)}{\rightrightarrows}} A$$

in **PreFrm** and that

$$\mathbf{Frm}< A \text{ (qua } \vee\text{-semilattice) } |C>$$

is the coequalizer of

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in **Frm**. Hence the preframe version of the generalized coverage theorem follows from the preframe coverage theorem.

If we look at the case of the preframe coverage theorem when C is the empty set, it is then the statement that the free preframe on a poset A is equal to the free frame on the join semilattice A if A is indeed a join semilattice. But such a free preframe can be seen to be the ideal completion of the free semilattice on the poset A, and such a free frame can be seen to be the ideal completion of the free distributive lattice on the join semilattice A. But since Lemma [2.9.2] showed us that the free semilattice and the free distributive lattice just described are the same we know that their ideal completions are isomorphic. Hence we have proven the preframe coverage theorem in the case when C is empty. i.e.

Lemma 2.9.3 *Let A be a join semilattice. Then the free preframe on a A qua poset is isomorphic to the free frame on A qua join semilattice.* $\square$

Given a join semilattice A we will call the free frame on it K_A. The fact that it is also a free preframe will help us prove that the preframe version of the generalized coverage theorem implies the preframe coverage theorem.

Say we are given a join semilattice A and a join stable collection of preframe relations C. Let $j : A \rightarrowtail K_A$ denote the inclusion of generators. Let B be the least frame congruence on K_A which contains all the pairs

$$(\wedge_{K_A}\{js : s \in S\}, (\wedge_{K_A}\{js : s \in S\}) \wedge_{K_A} (\bigvee_i^{\uparrow} \wedge_{K_A}\{js|s \in S_i\}))$$

So there are two frame inclusions

$$B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} K_A$$

and it is easy to see that their coequalizer in **Frm** is **Frm**$< A$ (qua $\vee$-semilattice)$|C >$. Further more once we view K_A as the free preframe on A (qua poset) then it can be seen that the coequalizer of

$$B \otimes K_A \underset{\vee(\pi_2 \otimes 1)}{\overset{\vee(\pi_1 \otimes 1)}{\rightrightarrows}} K_A$$

is equal to **PreFrm**$< A$ (qua poset) $|C >$. Hence the preframe coverage theorem follows from the generalized coverage theorem.

Of course it is a matter of opinion as to whether the coequalizer results really capture the coverage theorems, particularly in view of the need for lemmas [2.9.1] and [2.9.2]. However both these lemmas seem to follow a general form; see the concluding remarks to this chapter.

2.9.3 Quantale version and general remarks

A quantale is a SUP-lattice A together with a monoidal structure

$$e \in A$$
$$* : A \times A \longrightarrow A$$

with the property that $*$ preserves arbitrary joins in both of its coordinates. In other words a quantale is an object of $\mathbf{Mon}(\mathcal{C})$ where $\mathcal{C}$ is the symmetric monoidal closed category of SUP-lattices. A good reference for quantales is [Ros90]. They are investigated in [AV93] as models for various process calculi. In that investigation a coverage theorem for quantales is developed. For simplicity we examine the commutative case although, with the obvious modifications, this analysis works for general quantales. Given a commutative monoid A we say that $C : A \to PPA$ is a coverage if and only if $\forall T \in C(a), \forall b \in A$

$$\{t *_A b | t \in T\} \in C(a *_A b).$$

The coverage theorem for quantales is then the statement that the presentation

$$\mathbf{Qu}< S \text{ (qua monoid) } | \vee T \geq a \quad \forall T \in C(a) >$$

is well defined and is isomorphic as a poset to

$$\mathbf{SUP}< S | \vee T \geq a \quad \forall T \in C(a) >.$$

The free SUP-lattice on a set S is the power set of S. But:

Lemma 2.9.4 *The free quantale on a monoid S (i.e.* **Qu**$<$ *S (qua monoid)* $>$*) is isomorphic as a poset to the free SUP-lattice on the set S.*

Proof: Both are given by PS where the monoid operation on PS is given by, (for $T_1, T_2 \subseteq S$)

$$T_1 * T_2 = \{t_1 * t_2 | t_1 \in T_1 \quad t_2 \in T_2\} \qquad \square$$

We now prove that the quantale coverage result is implied by the generalized coverage theorem applied to the category $\mathcal{C}$=**SUP**.

Given a coverage C on some commutative monoid S let B be the least quantale congruence on PS which contains the pair

$$(T, T \cup \{a\})$$

for every $T \in C(a)$.
We then have a pair of quantale maps

$$B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} PS$$

and it is clear that their coequalizer in **Qu** will be

$$\mathbf{Qu}< S \text{ (qua monoid) } | \vee T \geq a \quad T \in C(a) >$$

It is also clear that

$$\mathbf{SUP}< S | \vee T \geq a \quad T \in C(a) >$$

is the coequalizer of

$$B \otimes PS \overset{*(\pi_1 \otimes 1)}{\underset{*(\pi_2 \otimes 1)}{\rightrightarrows}} PS$$

in **SUP** and so the generalized coverage theorem implies the quantale coverage result.

It might be interesting, for further research, to look at **CMon**(**PreFrm**). We know that this category will have coequalizers, and indeed one can write a coverage theorem for it. Aside from these facts not much is known about this category as far as the author is aware. It might be possible to use it in much the same way that quantales were used as models for various process calculi in [AV93]. Restricting to the category of idempotent commutative preframe monoids recaptures the analysis of Section 2.6.

We now turn our attention to an application of the converse of the coverage theorem (Theorem [2.5.4]). We take $\mathcal{C}$=**dcpo**, the category of directed complete partial orders. It clearly has finite limits and image factorisations. The category $\mathcal{D}$ is taken to be SUP-lattices, which we know has coequalizers. Also it is easy to see that the forgetful functor from **SUP** to **dcpo** has a left adjoint F. Simply take

$$FA = \mathbf{SUP} < A \text{ (qua dcpo) } >$$

It follows at once that **dcpo** has coequalizers. From this we recover another well known fact:

Theorem 2.9.1 dcpo *is symmetric monoidal closed*

Proof: Say A, B are two dcpos. Then define C to be the least congruence on $Idl(A \times B)$ which contains the pairs:

$$\begin{array}{l} \bigvee^{\uparrow}_{t \in T} \downarrow (t, b) = \downarrow (\bigvee^{\uparrow} T, b), \forall T \subseteq^{\uparrow} A \quad \forall b \in B \\ \bigvee^{\uparrow}_{t \in T} \downarrow (a, t) = \downarrow (a, \bigvee^{\uparrow} T) \quad \forall a \in A, \forall T \subseteq^{\uparrow} B \end{array}$$

Then there are two dcpo homomorphisms:

$$C \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Idl(A \times B)$$

It is easy to see that $A \otimes B$ is the coequalizer of these two maps. $\Box$

The next step is to investigate **CMon**(**dcpo**). We know that this category has coequalizers, although it is when we restrict our attention to the idempotent commutative monoids that we get more interesting results. Provided we insist that the unit of the idempotent commutative monoid is the greatest element with respect to the original order on our dcpo A then, just as in the discussion preceding Theorem [2.6.1], we can see that the monoidal operation will be meet. Furthermore it is a meet which commutes with directed joins in both coordinates. i.e. A has finite meets and these meets distribute over directed joins: we have a preframe.

Further, just as in the discussion of Section 2.6, we can check that the colimits of these preframes are found by suitable **dcpo** constructions. In short preframes have coequalizers and a preframe tensor can be defined. i.e. by an application of the opposite of the generalized coverage theorem we find that **dcpo** is symmetric monoidal closed and if we follow this by an application of the generalized coverage theorem to **dcpo** we recover Theorem [2.4.1]: **PreFrm** has a coequalizers.

This analysis works another way as well: if **SUP** has coequalizers then the coverage theorem tells us **Frm** has coequalizers. An application of the opposite of the coverage theorem implies that **PreFrm** has coequalizers. Hence the existence of coequalizers can be chased throughout the square:

$$\begin{tikzcd} & \mathbf{Frm} & \\ \mathbf{SUP} \arrow[ur] & & \mathbf{PreFrm} \arrow[ul] \\ & \mathbf{dcpo} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Similarly (at a 'lower' level) existence of coequalizers can be chased around:

$$\begin{tikzcd} & \mathbf{DLat} & \\ \vee-\mathbf{SLat} \arrow[ur] & & \wedge-\mathbf{SLat} \arrow[ul] \\ & \mathbf{POS} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Using the converse of the coverage theorem we know that coequalizers can be dropped along each of the following:

$$\begin{tikzcd} \mathbf{Frm} & \mathbf{PreFrm} & \mathbf{SUP} & \mathbf{dcpo} \\ \mathbf{DLat} \arrow[u, "Idl"] & \wedge-\mathbf{SLat} \arrow[u, "Idl"] & \vee-\mathbf{SLat} \arrow[u, "Idl"] & \mathbf{POS} \arrow[u, "Idl"] \end{tikzcd}$$

We can also look at Lemma [2.9.2] in another way; it is just the statement that

$$\begin{tikzcd}
\mathbf{DLat} \arrow[r, "U"] & \wedge - \mathbf{SLat} \\
\vee - \mathbf{SLat} \arrow[u, "F"] \arrow[r, "U_\vee"] & \mathbf{POS} \arrow[u, "F_\wedge"']
\end{tikzcd}$$

commutes where the Fs are free functors and the Us are forgetful functors. Notice also that Lemma [2.9.1] follows from the same lemma but with $\wedge$ and $\vee$ interchanged. To see this last observation note that the free SUP lattice on A qua poset is given by $Idl(F_\vee A)$ where $F_\vee$ is the free $\vee$-semilattice on A qua poset. Also, if A is a meet semilattice then the free frame on A qua $\wedge$-semilattice is $Idl(D)$ where D is the free distributive lattice on A qua meet semilattice. So, these lemmas seem to follow from a sort of Beck-Chevalley condition.

The cube

$$\begin{tikzcd}
\mathbf{Frm} & & \mathbf{PreFrm} \arrow[ll] & \\
& \mathbf{SUP} \arrow[ul] & & \mathbf{dcpo} \arrow[ll] \arrow[ul] \\
\mathbf{DLat} \arrow[uu] & & \wedge - \mathbf{SLat} \arrow[ll] \arrow[uu] & \\
& \vee - \mathbf{SLat} \arrow[ul] \arrow[uu] & & \mathbf{POS} \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}$$

is a useful visualisation of the algebra underlying locale theory.

Finally, by Linton's theorem [Lin69], it is interesting to note that 'coequalizers are enough'. Once (reflexive) coequalizers can be found in a node $\mathcal{C}$ of the above cube then all colimits in $\mathcal{C}$ can be constructed by 'lifting' them from any node below $\mathcal{C}$. Also, the existence of reflexive coequalizers in $\mathbf{CMon}(\mathcal{C})$ can be found by the existence of reflexive coequalizers in $\mathcal{C}$ (see Exercise 0.1 of [Joh77]): the generalized coverage theorem, as a statement about the existence of coequalizers, can be recovered through this result.

Chapter 3

Open and Proper Maps

3.1 Introduction

We now return to our locale theory. Definitions of proper and open maps are given, and we see that these are just generalisations of closed and open sublocales. Basic results about these maps are proved side by side so that the similarities between the theories of the two classes should be apparent without too much comment. Importantly these classes of maps are closed under pullback. This fact had been observed by Joyal and Tierney in [JT84] for the class of open maps, and was used in their description of the discrete locales as those locales whose finite diagonals are open. We look at the equivalent result for proper maps and find a description for the compact regular locales (Vermeulen, [Ver91], noticed this description): they are those locales whose finite diagonals are proper. We can now justify the assertion made in the abstract that the category of discrete locales and the category of compact regular locales are parallel to each other. It is a trivial fact that the discrete locales form a regular category since they are equivalent to **Set**. We prove the parallel result: the compact regular locales form a regular category. Of course classically this is a well known consequence of Manes' theorem which states that the category of compact Hausdorff spaces is monadic over **Set** (see 2.4 III of [Joh82]). Apart from this last theorem the results of the chapter are in general known ([JT84] or [Ver92]), the novelty is in the presentation: parallel results are presented with parallel proofs based on the preframe techniques developed in the previous chapter.

3.2 Basic definitions and results

The importance of the next two definitions cannot be over emphasised:
Definition: $f : X \to Y$ is a map between locales. Then
f is open iff
(1) Ωf has a left adjoint $\exists_f$,
(2) $\exists_f$ is a SUP-lattice homomorphism,
(3) $\exists_f(a \wedge \Omega f b) = b \wedge \exists_f a \quad \forall a \in \Omega X, b \in \Omega Y$. (Frobenius condition.)
f is proper iff
(1) Ωf has a right adjoint $\forall_f$,
(2) $\forall_f$ is a preframe homomorphism,
(3) $\forall_f(a \vee \Omega f b) = b \vee \forall_f a \quad \forall a \in \Omega X, b \in \Omega Y$. (coFrobenius condition.)

Clearly condition (2) of the open definition and condition (1) of the proper definition are redundant. See [JT84] and [Ver92] for some alternative descriptions of the open and proper maps respectively. The classical intuition to apply is the idea of open and proper continuous maps between topological spaces. It is immediate that these two classes of maps are closed under composition. We develop the theories of open and proper locale maps side by side noting their similarities. We argue (by example) that the two theories are *parallel* to each other.

Lemma 3.2.1 *If X, Y are stably locally compact locales then $f : X \to Y$ is semi-proper if and only if it satisfies (2) in the definition of proper.*

Proof: Recall from the definition of **CohLoc** in Section 1.7.3 that f is semi-proper if and only if Ωf preserves $\ll$. If Ωf preserves $\ll$ then to prove that $\forall_f$ preserves directed joins it is sufficient to show that for every $b \in \Omega X$,

$$\forall_f(b) = \bigvee^{\uparrow}\{c|\Omega f(c) \ll b\}$$

However $\forall_f(b) = \bigvee^{\uparrow}\{c|c \ll \forall_f(b)\}$ since Y is stably locally compact, and $c \ll \forall_f(b)$ implies $\Omega f(c) \ll b$ since Ωf preserves $\ll$ and $\Omega f \forall_f(b) \leq b$. Trivially $\forall_f$ preserves finite joins since it has a left adjoint.

In the other direction say $\forall_f$ preserves directed joins. Then if $a \ll b$, $(a, b \in \Omega X)$ and $\Omega f(b) \leq \bigvee^{\uparrow} S$ for some $S \subseteq^{\uparrow} \Omega Y$ then we have the following implications:

$$\begin{array}{rcl} b & \leq & \forall_f(\overset{\uparrow}{\bigvee} S) \\ b & \leq & \overset{\uparrow}{\bigvee}\{\forall_f(s)|s \in S\} \\ a & \leq & \forall_f(s) \text{ some } s \in S \\ \Omega f(a) & \leq & s \text{ some } s \in S \end{array}$$

Hence $\Omega f(a) \ll \Omega f(b)$. $\Box$

Theorem 3.2.1 *A sublocale $i : X_0 \hookrightarrow X$ is closed if and only if it is proper as a locale map.*

Proof: Say $i : X_0 \hookrightarrow X$ is a closed sublocale. Then

$$\begin{array}{rcl} \Omega X & \longrightarrow & \uparrow \forall_i(0) \\ a & \longmapsto & \forall_i(0) \vee a \end{array}$$

corresponds to a sublocale of X isomorphic (in $Sub(X)$) to $i : X_0 \hookrightarrow X$. But $\forall a \in \Omega X$ and $\forall b \geq \forall_i(0)$ we have

$$\forall_i(0) \vee a \leq b \quad \Leftrightarrow \quad a \leq b$$

and so the inclusion of $\uparrow \forall_i(0)$ into ΩX is a (preframe homomorphism) right adjoint to

$$a \longmapsto \forall_i(0) \vee a$$

As for the coFrobenius condition it amounts to: $\forall a \in \Omega X \quad \forall b \geq \forall_i(0)$

$$(b \vee (\forall_i(0) \vee a) = a \vee b$$

in this case.

Conversely say $i : X_0 \hookrightarrow X$ is proper. We know i factors as

$$X_0 \hookrightarrow \neg\forall_i(0) \hookrightarrow X$$

(i.e. $X_0 \in Sub(X)$ is contained in its closure.) To check that X_0 is a closed sublocale it is sufficient to check that $\neg\forall_i(0) \leq_{Sub(X)} X_0$ and to see this it is sufficient to prove that

$$\begin{array}{rcl} \Omega X_0 & \longrightarrow & \uparrow \forall_i(0) \\ \Omega i(a) & \longmapsto & \forall_i(0) \vee a \end{array}$$

is a well defined frame homomorphism. It is well defined since

$$\forall_i(0 \vee \Omega i(a)) = a \vee \forall_i(0)$$

by the coFrobenius condition and is easily seen to be a frame homomorphism. □

Theorem 3.2.2 *A sublocale $i : X_0 \hookrightarrow X$ is open if and only if it is open as a map.*

Proof: Say $i : X_0 \hookrightarrow X$ is open. $(X_0 \hookrightarrow X) \cong (a \hookrightarrow X)$ in $Sub(X)$ for some $a \in \Omega X$. But

$$\begin{array}{rcl} \Omega X & \longrightarrow & \downarrow a \\ \bar{a} & \longmapsto & a \wedge \bar{a} \end{array}$$

has a left adjoint: the inclusion of $\downarrow a$ into ΩX. The Frobenius condition then reads: $\forall \bar{a} \in \Omega X, \quad \forall b \leq a$

$$b \wedge (\bar{a} \wedge a) = \bar{a} \wedge b$$

which is clearly true.
Conversely, say we have some open map $i : X_0 \hookrightarrow X$ which is also a sublocale. I claim it is equal (in $Sub(X)$) to the open sublocale:

$$\exists_i(1) \hookrightarrow X$$

To check $\exists_i(1) \leq_{Sub(X)} X_0$ we need to verify

$$\begin{array}{rcl} \Omega X_0 & \longrightarrow & \downarrow \exists_i(1) \\ \Omega i(a) & \longmapsto & \exists_i(1) \wedge a \end{array}$$

is well defined. But the Frobenius condition on i implies:

$$\exists_i(1 \wedge \Omega i(a)) = a \wedge \exists_i(1)$$

To check $X_0 \leq_{Sub(X)} \exists_i(1)$ we need to know that

$$\begin{array}{rcl} \downarrow \exists_i(1) & \longrightarrow & \Omega X_0 \\ \exists_i(1) \wedge a & \longmapsto & \Omega i(a) \end{array}$$

is well defined. It clearly is since $\Omega i \exists_i(1) = 1$. □.

We examine the case of locale maps to the terminal locale 1, i.e. we look at the maps $! : X \to 1$. In the case when codomain of our map is the terminal object 1 the Frobenius condition is automatic once the left adjoint to $\Omega!$ is found. We check

$$\exists_!(a \wedge \Omega!(i)) = i \wedge \exists_!(a)$$

(N.B. it is always the case that $\exists_f \Omega f a \leq a$. Hence all we ever need to check is $a \wedge \exists_f(b) \leq \exists_f(b \wedge \Omega f a)$.)
So we'd like to verify $i \wedge \exists_!(a) \leq \exists_!(a \wedge \Omega!(i))$. As usual when reasoning in Ω we have only to check that

$$i \wedge \exists_!(a) = 1 \quad \Rightarrow \quad \exists_!(a \wedge \Omega!(i)) = 1$$

But if $i \wedge \exists_!(a) = 1$ then $i = 1$ and $\exists_! a = 1$. Since $i = 1$ implies $\Omega! i = 1$ the result is seen to be trivial. What we have shown here is that for any locale X the unique map $! : X \to 1$ is open if and only if $\Omega!$ has a left adjoint.

A locale is said to be *open* if and only if $! : X \to 1$ is an open map. Notice that if we assume the excluded middle then $\exists_! : \Omega X \to \Omega$, a left adjoint to $\Omega!$, can always be defined:

$$\exists_!(a) = 0 \text{ if and only if } a = 0$$

and so (assuming the excluded middle) all locales are open.

We can apply a similar analysis to the proper maps whose codomain is the terminal locale and get a similar result: $! : X \to 1$ is proper if and only if $\forall_!$ is a preframe homomorphism (if and only if X is compact). To check this fact we only need to prove the coFrobenius condition from the assumption that $\forall_!$ is a preframe homomorphism. But $i \leq \forall_! \Omega!(i)$ for any i and so

$$i \vee \forall_!(a) \leq \forall_!(a \vee \Omega!(i))$$

For the opposite direction note that

$$\Omega!(i) = \bigvee\nolimits^{\uparrow}(\{0\} \cup \{1 | 1 \leq i\})$$

and so if $\forall_!(a \vee \Omega!(i)) = 1$ then $a \vee \Omega!(i) = 1$ i.e.

$$\begin{aligned} 1 &= a \vee \bigvee^{\uparrow}(\{0\} \cup \{1 | 1 \leq i\}) \\ &= \bigvee^{\uparrow}(\{a\} \cup \{1 | 1 \leq i\}) \end{aligned}$$

By applying $\forall_!$ to both sides we see

$$1 = \bigvee\nolimits^{\uparrow}(\{\forall_!(a)\} \cup \{1 | 1 \leq i\})$$

and so $1 \leq \forall_!(a)$ or $1 \leq i$, i.e. $1 \leq \forall_!(a) \vee i$.

3.3 Pullback stability

We have the definition: $f : X \to Y$ is a *surjection* if and only if Ωf is an injection (if and only if f is an epimorphism). A straightforward application of the Frobenius condition shows that any open $f : X \to Y$ is a surjection if and only if $\exists_f(1) = 1$, and similarly an application of the coFrobenius condition shows that any proper $f : X \to Y$ is a surjection if and only if $\forall_f(0) = 0$.
We will find that the theorems:

Theorem 3.3.1 *For any locale X, $X \cong 1 \quad \Leftrightarrow \quad ! : X \to 1$ and $\Delta : X \to X \times X$ are open surjections*

Theorem 3.3.2 *For any locale X, $X \cong 1 \quad \Leftrightarrow \quad ! : X \to 1$ and $\Delta : X \to X \times X$ are proper surjections*

share the same proof. In order to find this proof we need to check pullback stability for open and proper maps. We find that to prove these facts the SUP-lattice presentation of the pushout in frame corresponding to the pullback is used for the open result and the preframe presentation of the pushout in frame corresponding to the pullback is used for the proper result. We have:

Theorem 3.3.3 *If*

$$\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a pullback diagram in **Loc** *and g is proper then*

$$\text{(i) } p_1 \text{ is proper}$$
$$\text{(ii) } \forall_{p_1}\Omega p_2(b) = \Omega f\forall_g(b) \quad \forall b \in \Omega Y$$

From (ii) we see that $\forall_g(0) = 0$ implies $\forall_{p_1}(0) = 0$ and so the class of proper surjections is pullback stable.

Proof: We saw in the last chapter (Corollary [2.6.1]) that ΩW is isomorphic to

$$\text{PreFrm}< a\parr b \in A \otimes B \text{ (qua preframe) } |(\Omega f(c) \vee a)\parr b = a\parr(\Omega g(c) \vee b)$$
$$\forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z >$$

We define

$$\begin{array}{rcl} \forall_{p_1} : \Omega W & \longrightarrow & \Omega X \\ a\parr b & \longmapsto & a \vee \Omega f\forall_g(b) \end{array}$$

This clearly satisfies the 'qua preframe' conditions in the presentation of ΩW since $\forall_g$ is a preframe homomorphism. Given any $a \in \Omega X, b \in \Omega Y, c \in \Omega Z$ we need to check

$$(\Omega f(c) \vee a) \vee \Omega f\forall_g(b) = a \vee \Omega f\forall_g(\Omega g(c) \vee b)$$

But this follows from the coFrobenius condition which is satisfied by $\Omega g \dashv \forall_g$.
So $\forall_{p_1}$ is well defined. Is it right adjoint to Ωp_1?
Now $\forall a \in \Omega X, b \in \Omega Y$

$$\begin{array}{rcl} \forall_{p_1}\Omega p_1(a) & = & \forall_{p_1}(a\parr 0) \\ & = & a \vee \Omega f\forall_g(0) \\ & \geq & a \end{array}$$

and

$$\begin{array}{rcl} \Omega p_1\forall_{p_1}(a\parr b) & = & (a \vee \Omega f\forall_g(b))\parr 0 \\ & = & (a\parr 0) \vee (\Omega f\forall_g(b)\parr 0) \\ & = & (a\parr 0) \vee (0\parr\Omega g\forall_g b) \\ & \leq & (a\parr 0) \vee (0\parr b) = a\parr b \end{array}$$

Hence $\Omega p_1 \dashv \forall_{p_1}$.
We check the coFrobenius condition for this adjunction. i.e. for every $a, \bar{a} \in \Omega X$ and every $b \in \Omega Y$ we want

$$\forall_{p_1}((a\parr b) \vee \Omega p_1(\bar{a})) = \bar{a} \vee \forall_{p_1}(a\parr b)$$

Well,

$$\begin{aligned} LHS &= \forall_{p_1}((a \vee \bar{a}) \parr b) \\ &= (a \vee \bar{a}) \vee \Omega f \forall_g(b) \\ &= \bar{a} \vee (a \vee \Omega f \forall_g(b)) \\ &= \bar{a} \vee \forall_{p_1}(a \parr b). \end{aligned}$$

Finally given $b \in \Omega Y$

$$\begin{aligned} \forall_{p_1} \Omega p_2(b) &= \forall_{p_1}(0 \parr b) \\ &= \Omega f \forall_g(b) \end{aligned}$$

and so condition (ii) in the statement of the theorem is satisfied. □
This proof, via preframe techniques, is new. The SUP-lattice parallel to the last theorem is true and follows a similar proof. It is proved in [JT84].

Theorem 3.3.4 *If*

$$\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a pullback diagram in **Loc** *and g is open then*

(i) p_1 is open
(ii) $\exists_{p_1} \Omega p_2(b) = \Omega f \exists_g(b) \quad \forall b \in \Omega Y$

From (ii) we see that $\exists_g(1) = 1$ implies $\exists_{p_1}(1) = 1$ and so the class of open surjections is pullback stable.

Proof: We saw in the last chapter (Corollary [2.6.1]) that ΩW is isomorphic to

$$\text{SUP}< a \otimes b \in A \otimes B \text{ (qua SUP-lattice) } |(\Omega f(c) \wedge a) \otimes b = a \otimes (\Omega g(c) \wedge b) \\ \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z >$$

We define

$$\begin{aligned} \exists_{p_1} : \Omega W &\longrightarrow \Omega X \\ a \otimes b &\longmapsto a \wedge \Omega f \exists_g(b) \end{aligned}$$

This clearly satisfies the 'qua SUP-lattice' conditions in the presentation of ΩW since $\exists_g$ is a SUP-lattice homomorphism. Given any $a \in \Omega X, b \in \Omega Y, c \in \Omega Z$ we need to check

$$(\Omega f(c) \wedge a) \wedge \Omega f \exists_g(b) = a \wedge \Omega f \exists_g(\Omega g(c) \wedge b)$$

But this follows from the Frobenius condition which is satisfied by $\exists_g \dashv \Omega g$.
So $\exists_{p_1}$ is well defined. Is it left adjoint to Ωp_1?
Now $\forall a \in \Omega X, b \in \Omega Y$

$$\begin{aligned} \exists_{p_1} \Omega p_1(a) &= \exists_{p_1}(a \otimes 1) \\ &= a \wedge \Omega f \exists_g(1) \\ &\leq a \end{aligned}$$

and

$$\begin{aligned}
\Omega p_1 \exists_{p_1}(a \otimes b) &= (a \wedge \Omega f \exists_g(b)) \otimes 1 \\
&= (a \otimes 1) \wedge (\Omega f \exists_g(b) \otimes 1) \\
&= (a \otimes 1) \wedge (1 \otimes \Omega g \exists_g b) \\
&\geq (a \otimes 1) \wedge (1 \otimes b) = a \otimes b
\end{aligned}$$

Hence $\exists_{p_1} \dashv \Omega p_1$.
We check the Frobenius condition for this adjunction. i.e. for every $a, \bar{a} \in \Omega X$ and every $b \in \Omega Y$ we need

$$\exists_{p_1}((a \otimes b) \wedge \Omega p_1(\bar{a})) = \bar{a} \wedge \exists_{p_1}(a \otimes b)$$

Well

$$\begin{aligned}
LHS &= \exists_{p_1}((a \wedge \bar{a}) \otimes b) \\
&= (a \wedge \bar{a}) \wedge \Omega f \exists_g(b) \\
&= \bar{a} \wedge (a \wedge \Omega f \exists_g(b)) \\
&= \bar{a} \wedge \exists_{p_1}(a \otimes b)
\end{aligned}$$

Finally given $b \in \Omega Y$

$$\begin{aligned}
\exists_{p_1} \Omega p_2(b) &= \exists_{p_1}(1 \otimes b) \\
&= \Omega f \exists_g(b)
\end{aligned}$$

and so condition (ii) in the statement of the theorem is satisfied. □

We can now exploit the pullback stability of open surjections and the statement (ii) of the last theorem in order to show that open surjections are actually always coequalizers. Again the proper parallel follows an identical proof. The open result is in [JT84]. The proper parallel is proved in [Ver92]: his approach, however, follows a different line.

Lemma 3.3.1 *If $p : X \to Z$ is an open surjection then*

$$X \times_Z X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Z$$

is a coequalizer diagram in **Loc***.*

Proof: $pp_1 = pp_2$ by definition of pullback, hence all we need to do is show that any $f : X \to W$ with $fp_1 = fp_2$ factors through $p : X \to Z$.
So $\Omega p_1 \Omega f = \Omega p_2 \Omega f$ and it is sufficient to prove $\exists_p : \Omega X \to \Omega Z$ satisfies $\Omega p \exists_p \Omega fc = \Omega fc$ for every c, for then $\exists_p : Im(\Omega f) \longrightarrow \Omega Z$ has an inverse, Ωp, which is a frame homomorphism. And then $c \mapsto \exists_p \Omega fc$ will be a frame homomorphism from ΩW to ΩZ.
Hence it is sufficient to show $\Omega p \exists_p u = u$ for any u with $\Omega p_1 u = \Omega p_2 u$.

$$\begin{aligned}
\Omega p \exists_p u &= \exists_{p_1} \Omega p_2 u \text{ pullback result [3.3.4]} \\
&= \exists_{p_1} \Omega p_1 u \\
&= u
\end{aligned}$$

The last line is because Ωp_1 is a surjective open as it is the pullback of a surjective open. □

Lemma 3.3.2 *If $p : X \to Z$ is a proper surjection then*

$$X \times_Z X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Z$$

is a coequalizer digram in **Loc**.

Proof: $pp_1 = pp_2$ by the definition of pullback. Thus all we need to do is show that any $f : X \to W$ with $fp_1 = fp_2$ factors through $p : X \to Z$.
Say $\Omega p_1 \Omega f = \Omega p_2 \Omega f$. It is sufficient to prove $\forall_p : \Omega X \to \Omega Z$ has $\Omega p \forall_p \Omega f c = \Omega f c$ for every $c \in \Omega W$. For then $\forall_p : Im(\Omega f) \to \Omega Z$ has an inverse Ωp and so is a frame homomorphism. (Recall that $\forall_p \Omega p(a) = a \quad \forall a$ since p is a proper surjection). Hence it is sufficient to check that $\Omega p \forall_p u = u$ for any u with $\Omega p_1 u = \Omega p_2 u$. For any such u we have

$$\begin{aligned} \Omega p \forall_p u &= \forall_{p_1} \Omega p_2 u \text{ (pullback result [3.3.3])} \\ &= \forall_{p_1} \Omega p_1 u = u \end{aligned}$$

The last line is because Ωp_1 is a proper surjection since it is the pullback of a proper surjection. □
We can now prove Theorems [3.3.1] and [3.3.2] which gave two characterisations of the terminal locale. The proofs are so similar that we give but one,
Proof: Say $! : X \to 1$ and $\Delta : X \to X \times X$ are open surjections.

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

is a pullback. Hence

$$X \underset{1}{\overset{1}{\rightrightarrows}} X \xrightarrow{\Delta} X \times X$$

is a coequalizer and so Δ^{-1} exists.
But

$$\begin{tikzcd} X \times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "!"] \\ X \arrow[r, "!"] & 1 \end{tikzcd}$$

is a pullback. Hence

$$X \times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{!} 1$$

is a coequalizer. $\pi_1 = \pi_2$ since Δ^{-1} exists. Therefore $!^{-1}$ exists and so $X \cong 1$. □

The pullbacks of proper/open maps are proper/open; the pullback of a regular monomorphism is well known to be a regular monomorphism. Hence:

Lemma 3.3.3 *(i) The pullback of a closed sublocale is closed. Further, the pullback of $\neg a \hookrightarrow Y$ along $f : X \to Y$ is the closed sublocale $\neg \Omega f(a) \hookrightarrow X$.*

(ii) The pullback of an open sublocale is open. Further, the pullback of $a \hookrightarrow Y$ along $f : X \to Y$ is the open sublocale $\Omega f(a) \hookrightarrow X$. □

3.4 Discrete and compact regular locales

We will consider two full subcategories of locales: those whose finite diagonals (it suffices to consider $! : X \to 1$ and $\Delta : X \to X \times X$) are open, and those whose finite diagonals are proper. We prove that these two subcategories are in fact well known: the first is the category of discrete locales and the second is the category of compact regular locales. (So classically the second is the category of compact Hausdorff spaces.) A proof of these two facts will clearly need to follow different paths since the definitions of discrete and compact regular are not parallel to each other in any obvious way. We first tackle the proof of

Theorem 3.4.1 (Joyal and Tierney) *X is discrete $\Leftrightarrow$ $X \overset{\Delta}{\to} X \times X$ and $X \overset{!}{\to} 1$ are open.*

An 'open' lemma is needed first:

Lemma 3.4.1 *If $! : X \to 1$ is open then for any $S \subseteq \Omega X$*

$$\bigvee S = \bigvee \{s \in S | \exists_! s = 1\}$$

("you only have to worry about the elements that exist.")

Proof: Say $s \in S$ we need $s \leq \bigvee \{\bar{s} | \bar{s} \in S \quad \exists_! \bar{s} = 1\}$
We know $s \leq \Omega ! \exists_! s$ i.e. $s \wedge \Omega ! \exists_! s = s$ Hence

$$\begin{array}{rrcl} & s & \leq & \bigvee \{\bar{s} | \exists \bar{s} = 1\} \\ \Leftrightarrow & s \wedge \Omega ! \exists_! s & \leq & \bigvee \{\bar{s} | \exists \bar{s} = 1\} \\ \Leftrightarrow & \Omega ! \exists_! s & \leq & s \to \bigvee \{\bar{s} | \exists \bar{s} = 1\} \\ \Leftrightarrow & \exists_! s & \leq & \forall_! (s \to \bigvee \{\bar{s} | \exists \bar{s} = 1\}) \end{array}$$

To prove the last line we are reasoning in Ω and so must but prove $\exists_! s = 1 \Rightarrow \forall_! (s \to \bigvee \{\bar{s} | \exists \bar{s} = 1\}) = 1$. But this is trivial. □

There is an alternative description of the statement $\exists_!(s) = 1$. Following Johnstone we say $s \in \Omega X$ (for any locale X) is *positive* if and only if $\forall T \subseteq \Omega X$ if $s \leq \bigvee T$ then $\exists t \in T$. Clearly (for open X) if $\exists_!(s) = 1$ then s is positive. (For if $s \leq \bigvee T$ then $1 = \exists_!(s) \leq \exists_!(\bigvee T) = \bigvee_{t \in T} \exists_!(t)$ and so $\exists t \in T$ since $1 = \{*\}$ and so $* \in \exists_!(t)$ for some $t \in T$.)
Conversely if s is positive ($s \in \Omega X, \quad X$ open) then

$$s = \bigvee \{\bar{s} | \exists_!(\bar{s}) = 1, \quad \bar{s} \leq s\}$$

by the last lemma and so there exists $\bar{s} \leq s$ such that $\exists_!(\bar{s}) = 1$, hence $\exists_!(s) = 1$.

So the last lemma implies that if X is open then any $s \in \Omega X$ is the join of positive opens less than it. This result has a converse:

Lemma 3.4.2 *For any locale X if every $s \in \Omega X$ is the join of positive opens less than it then X is open.*

This lemma is in Johnstone's paper 'Open Locales and Exponentiation' ([Joh84]).
Proof: $\forall s \in \Omega X$ the statement

$$(\forall T)[(s \leq \bigvee T) \Rightarrow (\exists t \in T)]$$

can be viewed as an element of the subobject classifier (i.e. as a truth value). So we have a map

$$\begin{array}{rcl} \exists_! : \Omega X & \longrightarrow & \Omega \\ s & \longmapsto & (\forall T)[(s \leq \bigvee T) \Rightarrow (\exists t \in T)] \end{array}$$

Clearly $\exists_!$ preserves order.
We need to check $\exists_! \dashv \Omega !$. To check $\exists_! \Omega!(i) \leq i$ we must verify

$$\exists_! \Omega!(i) = 1 \quad \Rightarrow \quad i = 1$$

But $\exists_! \Omega!(i) = 1$ means $\Omega!(i)$ is positive. But $\Omega!(i) = \bigvee\{1 | 1 \leq i\}$ and so $1 \leq i$ as $\Omega!(i)$ is positive.

To see $a \leq \Omega!(\exists_!(a))$, i.e. that

$$a \leq \bigvee\{1 | 1 \leq \exists_!(a)\},$$

we use our assumption that a is the join of positive element less than it, i.e.

$$a = \bigvee\{\bar{a} | \exists_!(\bar{a}) = 1, \quad \bar{a} \leq a\}$$

Clearly $\exists_!(\bar{a}) = 1$ and $\bar{a} \leq a$ together imply $\exists_!(a) = 1$. $\Box$

Proof of Theorem [3.4.1]: Say $X \xrightarrow{\Delta} X \times X$ and $X \xrightarrow{!} 1$ are open.
We say for any $a \in \Omega X$ that a is an *atom* iff $a \times a \leq_{Sub(X \times X)} \Delta$ (iff $a \otimes a \leq \exists_\Delta(1)$) and $\exists_! a = 1$. (NB $a \times a$ is a sublocale of $X \times X$; it is easy to check that it is open and that the element of $\Omega(X \times X)$ that corresponds to it is $a \otimes a$.)
The composition of two open maps is open. Hence $\Omega \xrightarrow{\Omega !} \Omega X \xrightarrow{(\_) \wedge a} \downarrow a$ i.e. $!^a : a \to 1$ is open. The condition $\exists_!(a) = 1$ implies $\exists_{!^a}(1) = 1$. Hence $!^a$ is an open surjection for any atom a.
Further

$$\begin{tikzcd} a \arrow[r, "m"] \arrow[d, "\Delta_a"'] & X \arrow[d, "\Delta"] \\ a \times a \arrow[r, "m \times m"] & X \times X \end{tikzcd}$$

is a pullback since $m \times m$ is a monomorphism in **Loc**. Thus Δ_a is an open map.

$$\begin{array}{rcl} \exists_{\Delta_a}(1) & = & \exists_{\Delta_a} \Omega m(1) \\ & = & \Omega(m \times m)(\exists_\Delta(1)) \text{ pullback result [3.3.4]} \\ & \geq & \Omega(m \times m)(a \otimes a) = 1 \otimes 1 = 1 \end{array}$$

Hence Δ_a is an open surjection, and so by Theorem [4.3.1] $a \cong 1$. Also atoms behave as atoms should in the following way: if a_1, a_2 are two atoms with $a_1 \leq a_2$

then $a_1 = a_2$. [Prooflet: if $a_1 \leq a_2$ then there is a continuous map $a_1 \stackrel{q}{\to} a_2$ in $Sub(X)$. But $1 \cong a_1$ and $1 \cong a_2$ hence Ωq is easily checked to be a bijection as we must have $\Omega(!^{a_1}) = \Omega q \circ \Omega(!^{a_2})$ and $!^{a_1}, !^{a_2}$ are isomorphisms.]
Let A denote the set of atoms.
Define:

$$\begin{array}{rcl} \phi : \Omega X & \longrightarrow & PA \\ u & \longmapsto & \{a \in A | a \leq u\} \end{array}$$

ϕ clearly preserves finite meets. As for joins it is sufficient to check $a \leq \bigvee_{i \in I} u_i$ implies $\exists i \in I \quad a \leq u_i$ for any atom a.
Say $a \leq \bigvee_{i \in I} u_i$

$$\begin{array}{rcl} a \wedge \bigvee u_i = a & \Rightarrow & \exists_{!^a}(a \wedge \bigvee u_i) = \exists_{!^a}(a) \\ & \Rightarrow & \bigvee \exists_{!^a}(a \wedge u_i) = 1 \\ & \Rightarrow & \exists i \quad \exists_{!^a}(a \wedge u_i) = 1 = \exists_{!^a}(a) \text{ (reasoning in } \Omega) \\ & \Rightarrow & a \wedge u_i = a \text{ since } \exists_{!^a} = (\Omega(!^a))^{-1} \\ & \Rightarrow & a \leq u_i \end{array}$$

In fact ϕ has a left adjoint:

$$\begin{array}{rcl} \sigma : PA & \longrightarrow & \Omega X \\ I & \longmapsto & \bigvee \{a | a \in I\} \end{array}$$

We check $\phi\sigma(I) \subseteq I$.
Say $\bar{a} \in \phi\sigma(I)$ then $\bar{a} \leq \bigvee\{a | a \in I\}$ and so as above $\bar{a} \leq a$ for some $a \in I$. But then $\bar{a} = a$ by a property of atoms that we have just demonstrated.
Finally we must check that $u = \sigma\phi(u)$. i.e. $u = \bigvee\{a | a \leq u\}$.
First I claim that

$$\exists_\Delta(u) = \bigvee\{v \otimes v | \quad v \otimes v \leq \exists_\Delta(u)\}$$

Certainly:

$$\exists_\Delta(u) = \bigvee\{v \otimes w | \quad v \otimes w \leq \exists_\Delta(u)\}$$

But $v \otimes w \leq \exists_\Delta(u) \quad \Rightarrow v \otimes w \leq \exists_\Delta(1)$
i.e. $v \times w \leq \Delta$ in $Sub(X \times X)$.
$\Rightarrow \qquad v \times w = w \times v$
$\Rightarrow \qquad v \otimes w = w \otimes v$
Thus

$$\exists_\Delta(u) = \bigvee\{v \otimes v | \quad v \otimes v \leq \exists_\Delta(u)\}$$

Apply $\Omega\Delta$ to both sides and recall $u \leq \Omega\Delta\exists_\Delta(u)$ and that if $v \otimes v \leq \exists_\Delta(u)$ then $v \leq u$. [This is because $\exists_\Delta(u) \leq u \otimes u \quad \Leftrightarrow \quad u \leq \Omega\Delta(u \otimes u) = u$.]
We obtain

$$\begin{array}{rl} u = & \bigvee\{v | v \otimes v \leq \exists_\Delta(u)\} \\ = & \bigvee\{v | v \otimes v \leq \exists_\Delta(1) \quad v \leq u\} \end{array}$$

Which is seen by the 'open' Lemma [3.4.1] to imply

$$u = \bigvee\{v| \quad \exists_! v = 1 \quad v \otimes v \leq \exists_\Delta(1) \quad v \leq u\}$$
$$\text{i.e. } u = \bigvee\{a|a \text{ is an atom, } a \leq u\} \ \square$$

What follows now is a very different type of proof. It shows that just as the class of locales whose finite diagonals are open turns out to be well known (i.e. the discrete locales) so does the class of locales whose finite diagonals are proper: they are the compact regular locales. The proof to follow, via preframe techniques, is new.

Theorem 3.4.2 *For any locale X, X is compact regular if and only if $! : X \to 1$ and $\Delta : X \to X \times X$ are both proper.*

Proof: It is well known (see Johnstone [Joh82] III 1.3) that any regular locale is strongly Hausdorff i.e. has a closed diagonal. So we know that any regular locale X has $\Delta : X \to X \times X$ proper.
We have established already that $! : X \to 1$ is proper if and only if X is compact.
What needs to be proven is that if $\Delta : X \to X \times X$ and $! : X \to 1$ are proper then $\forall a \in \Omega X$

$$a \leq \bigvee^{\uparrow}\{c|c \lhd a\}$$

Since $\Delta : X \to X \times X$ is proper we know that for every $a, b \in \Omega X$

$$\forall_\Delta \Omega\Delta(a \otimes b) = \# \vee (a \otimes b)$$

where $\#$ is given by

$$\# = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes b_i)| \wedge_{i\in I} (a_i \vee b_i) = 0 \ I \text{ finite }\}$$

(Since $\Delta : X \to X \times X$ is just the closed sublocale $\neg\# \rightarrowtail X \times X$, see Lemma [2.7.1].) Now

$$\forall_\Delta \Omega\Delta(a \otimes b) = \bigvee^{\uparrow}\{I|\Omega\Delta(I) \leq a \vee b\} = \forall_\Delta \Omega\Delta(b \otimes a)$$

and so we see that for any a in ΩX

$$0 \otimes a \leq \# \vee a \otimes 0, \text{ i.e. } 0 \otimes a \leq \bigvee^{\uparrow}\{\wedge_{i\in I}((a_i \vee a) \otimes b_i)| \wedge_i (a_i \vee b_i) = 0\} \text{ - } (*)$$

We will use the fact that (for finite I),

$$\wedge_i(a_i \vee b_i) = \bigvee_{I \subseteq J_1 \cup J_2}((\wedge_{i\in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i))$$

where the J_1, J_2 are subsets of I. This finite distributivity rule shows us that if $\wedge_i(a_i \vee b_i) = 0$ then for all finite subsets $J_1, J_2 \subseteq I$ with $I \subseteq J_1 \cup J_2$ we have $(\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2} b_i) = 0$. We can also use the above distributivity and the rules relating $\otimes$ to $\otimes$, e.g. $a \otimes b = a \otimes 1 \vee 1 \otimes b$, to prove that

$$\wedge_i(a_i \otimes b_i) = \bigvee_{I \subseteq J_1 \cup J_2}[(\wedge_{i\in J_1} a_i) \otimes (\wedge_{i\in J_2} b_i)]$$

(see Lemma [2.7.1]). Now $\forall_!$ is a preframe homomorphism and so we can apply the composite

$$\Omega X \otimes \Omega X \xrightarrow{\forall_! \otimes 1} \Omega \otimes \Omega X \xrightarrow{\Omega! \otimes 1} \Omega X \otimes \Omega X \xrightarrow{\Omega\Delta} \Omega X$$

to both sides of $(*)$ to obtain

$$\begin{aligned} a &\leq \bigvee^{\uparrow}\{\Omega\Delta(\wedge_i(\Omega!\forall_!(a_i \vee a) \otimes b_i))| \wedge_i (a_i \vee b_i) = 0\} \\ &= \bigvee^{\uparrow}\{\Omega\Delta[\bigvee_{I\subseteq J_1 \cup J_2} [\wedge_{i\in J_1}(\Omega!\forall_!(a_i \vee a)) \otimes (\wedge_{i\in J_2}) b_i]]| \wedge_{i\in I} (a_i \vee b_i) = 0\} \\ &= \bigvee^{\uparrow}\{\bigvee_{I\subseteq J_1\cup J_2} [(\wedge_{i\in J_1}(\Omega!\forall_!(a_i \vee a))) \wedge (\wedge_{i\in J_2} b_i)]| \wedge_{i\in I} (a_i \vee b_i) = 0\} \end{aligned}$$

and so to prove that $a \leq \bigvee^{\uparrow}\{c|c \lhd a\}$ all we need do is check that

$$(\wedge_{i\in J_1}(\Omega!\forall_!(a_i \vee a))) \wedge (\wedge_{i\in J_2} b_i) \leq \bigvee\{c | c \lhd a\}$$

given any (finite) collection of a_is and b_is with $(\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2} b_i) = 0$. Now

$$\begin{aligned} \wedge_{i\in J_1}\Omega!\forall_!(a_i \vee a) &= \Omega!\forall_!((\wedge_{i\in J_1} a_i) \vee a) \\ \text{and} \quad \Omega!\forall_!(\alpha) &= \textstyle\bigvee_{\Omega X}\{1 | 1 \leq \alpha\} \text{ for any } \alpha \in \Omega X \end{aligned}$$

and so

$$\begin{aligned} &\wedge_{i\in J_1}(\Omega!\forall_!(a_i \vee a)) \wedge \wedge_{i\in J_2} b_i \\ &= \textstyle\bigvee_{\Omega X}\{\wedge_{i\in J_2} b_i | 1 \leq (\wedge_{i\in J_1} a_i) \vee a\} \end{aligned}$$

But for any $c \in \{\wedge_{i\in J_2} b_i | 1 \leq (\wedge_{i\in J_1} a_i) \vee a\}$ we have $c \lhd a$ and so

$$(\wedge_{i\in J_1}(\Omega!\forall_!(a_i \vee a)) \wedge (\wedge_{i\in J_2} b_i) \leq \bigvee\{c | c \lhd a\}$$

as required. □

Given this last result we now change our notation slightly and shall refer to the compact regular locales as the compact Hausdorff locales. The category of compact Hausdorff locales will be written **KHausLoc**. We have just shown that the compact Hausdorff locales are parallel to the discrete locales. Notice that if we were not working in a constructive context and were assuming the excluded middle then, since all locales would be open, such a parallel becomes invisible. It is only by working constructively that we can appreciate the full force of the parallel.

3.5 Historically Important Axioms

This section consists of an argument which shows that the constructive prime ideal theorem is parallel to the excluded middle. The section is separate from the rest of the work and is the only time that we use the points of a locale in a context that is not motivational. This result is new.

For any locale X consider the map

$$\begin{aligned} \phi_X : \Omega X &\longrightarrow PptX \\ a &\longmapsto \{p \in ptX | \Omega p(a) = 0\} \end{aligned}$$

It is order reversing. Consider the results:

(i) $\forall X$ compact Hausdorff, ϕ_X is an injection.

(ii) $\forall X$ discrete, ϕ_X is an injection.

We show that (i) is true if and only if the constructive prime ideal theorem (CPIT) is true and that (ii) is true if and only if the excluded middle holds. So we have found a result which is true if and only the excluded middle holds and whose proper parallel is true if and only if CPIT. The grander conclusion is that CPIT is 'parallel' to the excluded middle; though the reader is asked to bear in mind the fact that, so far, no formal definition has been given for our parallel.

Before proof we note that if ϕ_Y is an injection then so is ϕ_X for any retract X of Y. To see this say ϕ_Y is an injection and there exists $q : Y \to X$, $i : X \to Y$ such that $q \circ i = 1$. If $a, \bar{a} \in \Omega X$ satisfy

$$\{p \in ptX | \Omega p(a) = 0\} = \{p \in ptX | \Omega p(\bar{a}) = 0\}$$

then

$$\{\bar{p} \in ptY | \Omega\bar{p}(\Omega q(a)) = 0\} = \{\bar{p} \in ptY | \Omega\bar{p}(\Omega q(\bar{a})) = 0\}$$

and so as ϕ_Y is an injection we get $\Omega q(a) = \Omega q(\bar{a})$ allowing us $a = \bar{a}$ since $q \circ i = 1$. Hence ϕ_X is injective.

Proof that (i) $\Leftrightarrow$ CPIT: Assume CPIT. By the preceding remarks and the fact that all compact Hausdorff locales are stably locally compact (and the fact that the stably locally compact locales are the retracts of the coherent locales) it is clearly sufficient to prove ϕ_Y is an injection for every coherent Y in order to conclude that ϕ_X is an injection for all compact Hausdorff X.

Say Y is coherent and $I, J \in Idl(K\Omega Y)$ are such that

$$\{p \in ptY | \Omega p(I) = 0\} = \{p \in ptY | \Omega p(J) = 0\} \quad (*)$$

We prove $J \subseteq I$. Say $j \in J$. Clearly, by the assumption of CPIT and by Lemma [1.3.4] it is sufficient to prove $f[j] = 0$ for every distributive lattice homomorphism

$$f : K\Omega Y / \equiv_I \longrightarrow \Omega$$

in order to conclude $j \in I$. But every such f corresponds to a point, p, of Y with the property $\Omega p(I) = 0$. Hence $\Omega p(\downarrow j) = 0$ by $(*)$ and so $f[j] = 0$ as required. Thus $J \subseteq I$. $I \subseteq J$ follows symmetrically and so ϕ_Y is an injection for every coherent Y assuming CPIT.

Conversely assume ϕ_X is an injection for every compact Hausdorff X. To conclude CPIT it is sufficient (by Lemma [1.8.1]) to show that for every Boolean algebra B if $b \in B$ has the property that $f(b) = 0$ for every distributive lattice homomorphism $f : B \to \Omega$ then $b = 0$. Say $b \in B$ enjoys such a property. Set X to be the locale whose frame of opens is $Idl(B)$. So X is Stone and so is compact Hausdorff. Clearly

$$\{p \in ptX | \Omega p(\downarrow b) = 0\} = \{p \in ptX | \Omega p(0) = 0\}$$

by assumption about $b \in B$. Hence, since ϕ_X is an injection, we get $b = 0$. $\Box$

Proof that (ii) $\Leftrightarrow$ excluded middle holds: Recall that all discrete locales are constructively spatial (Section 1.6) and further that the frame homomorphism corresponding to the counit:

$$\begin{array}{rcl} \Omega\epsilon_X : \Omega X & \longrightarrow & PptX \\ a & \longmapsto & \{p | \Omega p(a) = 1\} \end{array}$$

is a surjection.

Assume the excluded middle. Say X is a discrete locale. Then $\Omega X = PA$ for some set A. It follows that for every $T \in \Omega X$

$$\{p \in ptX | \Omega p(T) = 0\} = \{p \in ptX | \Omega p(T^c) = 1\}$$

by the excluded middle (where T^c is the complement of T). If $\phi_X(T_1) = \phi_X(T_2)$ for some opens T_1, T_2. Then

$$\{p | \Omega p(T_1^c) = 1\} = \{p | \Omega p(T_2^c) = 1\}$$

and so by spatiality of X we have that $T_1^c = T_2^c$. Leading us to $T_1 = T_2$. Hence ϕ_X is injective. We conclude (using the excluded middle) that (ii) is true.

Conversely say ϕ_X is an injection. We know $PptX \cong \Omega X$. I claim that

$$\{p|\Omega p(a)=0\} = \neg\{p|\Omega p(a)=1\}$$

where $\neg$ is Heyting negation in $Ppt(X)$. It will then follow that $(\Omega\epsilon_X)^{-1}\circ\phi_X$ is Heyting negation on ΩX. Injectivety of ϕ_X will then imply injectivety of $\neg:\Omega X\to\Omega X$. But $\neg\neg\neg a=\neg a$ for any open of any frame and so $\neg\neg a=a$ for all $a\in\Omega X$ if $\neg$ is injective. So ΩX would then be Boolean for every discrete locale X, i.e. PA is Boolean for any set A. This implies the excluded middle is true in our topos.

Verifying the claim is straightforward. We need

$$\{p|\Omega p(a)=0\} = \bigcup\{T\in PptX|T\cap\{p|\Omega p(a)=1\}=\phi\}$$

The inclusion of the left hand side in the right hand side is trivial. Say $T\in PptX$ is such that

$$T\cap\{p|\Omega p(a)=1\}=\phi$$

Then $T=\{p|\Omega p(\bar a)=1\}$ for some $\bar a\in\Omega X$ since $\Omega\epsilon_X$ is a surjection to $PptX$. Thus $a\wedge\bar a=0$ by spatiality of ΩX (use $\{p|\Omega p(0)=1\}=\phi$). Thus for all $p\in T$

$$\begin{aligned}\Omega p(a)=\Omega p(a)\wedge 1 &= \Omega p(a)\wedge\Omega p(\bar a)\\ &= \Omega p(a\wedge\bar a)=\Omega p(0)=0\end{aligned}$$

Hence $T\subseteq\{p|\Omega p(a)=0\}$. □

3.6 Further results about proper and open maps

We now turn to the question of regularity of our two parallel categories (the discrete locales and the compact Hausdorff locales). We find that a proof that they are regular follows the same route. The fact that the category **DisLoc** of discrete locales is regular is of course known already since we know that it is equivalent to **Set** (where **Set** is our background topos). However the observation that the category **KHausLoc** of compact Hausdorff locales is regular will bear much fruit: we know from Freyd and Ščedrov ([FS90]) that any regular category gives rise to an allegory in the vein of 'sets and relations'. Along the way some more technical results about proper and open maps are shown.

Theorem 3.6.1 (Vermeulen) *If $Y\xrightarrow{f}X$ is a map between compact Hausdorff locales then f is proper.*

Proof:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "{(1,f)}"'] & X \arrow[d, "\Delta"] \\ Y\times X \arrow[r, "f\times 1"] & X\times X \end{tikzcd}$$

is a pullback square so $(1,f)$ is proper. But $Y\times X\xrightarrow{\pi_2}X$ is proper as it is the pullback of the proper map $Y\xrightarrow{!}1$. Properness is easily seen to be stable under composition. Hence $\pi_2\circ(1,f)$ is proper. i.e. f is proper. □

Notice that exactly the same proof proves that if $Y \xrightarrow{f} X$ is a map between discrete locales then f is open.

To check that **KHausLoc** is regular we need to check that any $f : X \to Y$ with X, Y compact Hausdorff has a factorization as a cover followed by monomorphism. Certainly it has a factorization in **Loc** as an epimorphism followed by a regular monomorphism: $X \xrightarrow{q} f[X] \xrightarrow{i} Y$ (see Section 2.7) We offer a
Proof that $f[X]$ is compact Hausdorff: [N.B. this result can be generalized in the obvious way i.e. we only really need X compact and Y Hausdorff.]

$$\begin{array}{ccc} f[X] & \xrightarrow{i} & Y \\ \Big\downarrow{\scriptstyle \Delta_{f[X]}} & & \Big\downarrow{\scriptstyle \Delta_Y} \\ f[X] \times f[X] & \xrightarrow{i \times i} & Y \times Y \end{array}$$

is a pullback square and so $\Delta_{f[X]}$ is proper.
To prove that $! : f[X] \to 1$ is proper we appeal to the following general result: if $X \xrightarrow{q} Y \xrightarrow{f} Z$ in **Loc** are such that $f'(= f \circ q)$ is proper and q is a surjection then f is proper. Take the case $f = !^{f[X]}$ and $f' = !^X$ to prove that $f[X]$ is compact. The proof of this general result is straightforward, can be found in [Ver92] and requires the following manipulations: (note that since q is surjective $\forall_q \Omega q(d) = d \quad \forall d$)
Say $S \subseteq^{\uparrow} \Omega Y$,

$$\begin{aligned} \forall_f \bigvee^{\uparrow} S &= \forall_f \forall_q \Omega q(\bigvee^{\uparrow} S) \\ &= \forall_{f'} \Omega q(\bigvee^{\uparrow} S) \\ &= \forall_{f'} \bigvee^{\uparrow} \{\Omega q d | d \in S\} \\ &= \bigvee^{\uparrow} \{\forall_{f'} \Omega q d | d \in S\} \\ &= \bigvee^{\uparrow} \{\forall_f \forall_q \Omega q d | d \in S\} \\ &= \bigvee^{\uparrow} \{\forall_f d | d \in S\} \end{aligned}$$

and

$$\begin{aligned} \forall_f (a \vee \Omega f b) &= \forall_f \forall_q \Omega q (a \vee \Omega f b) \\ &= \forall_{f'} (\Omega q a \vee \Omega f' b) \\ &= \forall_{f'} \Omega q a \vee b = \forall_f a \vee b. \qquad \square \end{aligned}$$

Similarly if $X \xrightarrow{q} f[X] \xrightarrow{i} Y$ is the epi/regular mono decomposition of $X \xrightarrow{f} Y$, and X, Y are discrete, then so is $f[X]$. As before we see straight away that $\Delta_{f[X]}$ is open since it is a pullback of the open $Y \xrightarrow{\Delta} Y \times Y$. That $! : f[X] \to 1$ is open then follows exactly as before from:

Lemma 3.6.1 *If X, Y, Z are locales and $X \xrightarrow{q} Y \xrightarrow{f} Z$ is such that $f'(= f \circ q)$ is open and q is surjective (i.e. epi in* **Loc**, *i.e. Ωq injective) then f is open.*

This result can be found as Proposition 1.2 VII of [JT84].
Proof: Define

$$\begin{array}{rcl} \exists_f : \Omega Y & \to & \Omega Z \\ y & \mapsto & \exists_{f'} \Omega q y \end{array}$$

Hence

$$\begin{array}{rcl} \exists_f y \leq z & \Leftrightarrow & \exists_{f'} \Omega q y \leq z \\ & \Leftrightarrow & \Omega q y \leq \Omega f' z \\ & \Leftrightarrow & \Omega q y \leq \Omega q \Omega f z \\ & \Leftrightarrow & y \leq \Omega f z \qquad (\Omega q \text{ inj.}) \end{array}$$

and so $\exists_f \dashv \Omega f$.
Also

$$\begin{array}{rcl} \exists_f (y \wedge \Omega f z) & = & \exists_{f'} (\Omega q y \wedge \Omega f' z) \\ & = & \exists_{f'} \Omega q y \wedge z = \exists_f y \wedge z \end{array}$$

and so f is open. $\Box$
Heading towards a proof of regularity of **KHausLoc** (and **DisLoc**) we need some technical lemmas:

Lemma 3.6.2 *If $X \xrightarrow{f} Y$ and $\bar{X} \xrightarrow{\bar{f}} \bar{Y}$ are two open(proper) maps then*

$$X \times \bar{X} \xrightarrow{f \times \bar{f}} Y \times \bar{Y}$$

is open(proper).
Proof: Take $\exists_{f \times \bar{f}} (a \otimes \bar{a}) = \exists_f a \otimes \exists_{\bar{f}} \bar{a}$. (Use SUP-lattice definition of tensor product.) Take $\forall_{f \times \bar{f}} (a \otimes \bar{a}) = \forall_f a \otimes \forall_{\bar{f}} \bar{a}$. (Use preframe definition of tensor product.) $\Box$

Lemma 3.6.3 KHausLoc$\subseteq$Loc *is closed under the formation of finite limits in* **Loc**. *(i.e. the inclusion functor creates finites limits.)*

Notice that exactly the same proof (to follow) demonstrates that **DisLoc$\subseteq$Loc** is closed under finite limits.
Proof: The terminal locale 1 is compact Hausdorff. We first check that if X, Y are compact Hausdorff then so is $X \times Y$. $X \times Y \xrightarrow{\pi_1} Y$ is proper since it is the pullback of the proper map $X \xrightarrow{!} 1$. Hence composition with the proper $Y \xrightarrow{!} 1$ proves that $! : X \times Y \to 1$ is proper.
It is straightforward to check that

$$\begin{tikzcd} X \times Y \arrow[r, "Id"] \arrow[d, "\Delta"'] & X \times Y \arrow[d, "\Delta^X \times \Delta^Y"] \\ (X \times Y) \times (X \times Y) \arrow[r, "i"] & (X \times X) \times (Y \times Y) \end{tikzcd}$$

is a pullback, where i is the obvious twist isomorphism. It follows that Δ is proper, and so $X \times Y$ is compact Hausdorff.
Say now that we are given an equalizer diagram

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in **Loc**, where X and Y are compact Hausdorff. First note that e is proper since it is the pullback of the proper map Δ^Y along (f, g). Thus since $E \overset{!}{\to} 1$ can be factored as $E \overset{e}{\to} X \overset{!}{\to} 1$ we know that $!^E$ is proper. Further

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "\Delta"'] & X \arrow[d, "\Delta"] \\ E \times E \arrow[r, "e \times e"] & X \times X \end{tikzcd}$$

is a pullback since e is mono. Hence Δ^E is proper and so E is a compact Hausdorff locale. □

Theorem 3.6.2 *If $X \overset{m}{\to} Y$ is a monomorphism in* **KHausLoc** *then m is a regular monomorphism in* **Loc**.

Proof: $X \overset{m}{\to} Y$ can be factored as $X \overset{q}{\to} m[X] \overset{i}{\to} Y$ where q is a proper surjection. But by a corollary to the pullback result (Lemma [3.3.2]) we know that for any proper surjection q

$$X \times_{m[X]} X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{q} m[X]$$

is a coequalizer diagram in **Loc**. By the results that we've just proven we know that this diagram is in fact inside **KHausLoc**. Hence $mp_1 = mp_2 \quad \Rightarrow$
$p_1 = p_2 \quad \Rightarrow \quad q$ is an isomorphism. Thus m is regular since i is. □
This last result is really all we need to check that **KHausLoc** is regular. To prove that a category is regular one needs to check that (it has finite limits and) for any $f : X \to Y$ there is an image factorization

$$X \overset{q}{\to} f[X] \overset{i}{\hookrightarrow} Y$$

and such a factorization is pullback stable (see [FS90] or [BGO71]). But what we have shown above is that the usual epi/regular mono decomposition in **Loc** gives rise to such an image factorization. It is then easy to see that the covers are the proper surjections and we know that these are pullback stable. We have proven:

Theorem 3.6.3 **KHausLoc** *is regular.*□

Also, as another corollary to [3.6.2], notice that subobjects in **KHausLoc** (i.e. monomorphisms in **KHausLoc**) are exactly the closed sublocales of compact Hausdorff locales. Certainly they are proper; but we need [3.6.2] in order to conclude that these subobjects are actually sublocales. Hence they are proper maps and are sublocale maps. i.e. they are closed (use Theorem [3.2.1]).

Chapter 4

Compact Hausdorff Relations

4.1 Introduction

We establish the existence, via Freyd and Ščedrov's definitions ([FS90]), of a category of compact Hausdorff relations (parallel to the category of sets and relations; composition is given by relational composition). We then give a much more concrete description of what this category is like i.e. we give an explicit definition of a function that defines relational composition of closed sublocales.

We find that there is a bijection between the closed sublocales of a locale product $X \times Y$ (where X and Y are compact Hausdorff) and preframe homomorphisms from ΩY to ΩX. This result is used to establish an equivalence between the category of compact Hausdorff locales with closed relations and another category whose morphisms are much more concrete. The connection between preframe homomorphisms and closed sublocales will be exploited considerable in the rest of this work, in particular we are able to use the function that defines relational composition of closed sublocales to turn our spatial intuitions (about relational composition of closed subspaces) into suitable preframe formulas.

Although the results presented here are new we do find some of the corollaries to them in Vickers' paper [Vic94]. The thesis is, form now on, entirely concerned with the proper side of our parallel i.e. preframe techniques. However we will not prove results in isolation, the open parallels of our results (which are all known) are stated for completeness.

4.2 Relational composition

If $\mathcal{C}$ is a regular category and

$$P \overset{(p_1,p_2)}{\hookrightarrow} X \times Y$$

$$Q \overset{(q_1,q_2)}{\hookrightarrow} Y \times Z$$

are monics in $\mathcal{C}$, then the relational composition of P and Q ($Q \circ P$) is given as follows: form the pullback

$$\begin{tikzcd} P \times_Y Q \arrow[r, "a_2"] \arrow[d, "a_1"'] & Q \arrow[d, "q_1"] \\ P \arrow[r, "p_2"] & Y \end{tikzcd}$$

then $Q \circ P$ is defined to be the image of

$$P \times_Y Q \xrightarrow{(p_1 a_1) \times (q_2 a_2)} X \times Y$$

If $\mathcal{C}$ is just **Set** then the pullback $P \times_Y Q$ would be the set

$$\{(x, y, \bar{y}, \bar{z}) | (x, y) \in P, (\bar{y}, \bar{z}) \in Q, y = \bar{y}\}.$$

The function $(p_1 a_1) \times (q_2 a_2)$ is given by

$$(x, y, \bar{y}, \bar{z}) \longmapsto (x, \bar{z})$$

and so its image is

$$\{(x, \bar{z}) | \exists y \ (x, y) \in P, (y, \bar{z}) \in Q\}$$

which is the usual definition of relational composition of subsets.
Given a general (regular) $\mathcal{C}$ we can now form the category $\mathcal{REL}(\mathcal{C})$ with $\mathcal{C}$-objects as objects and relations as morphisms. Composition is given by relational composition and the identity on an object is the diagonal. In fact $\mathcal{REL}(\mathcal{C})$ is an allegory in the sense of Freyd and Ščedrov [FS90] (although see [BGO71] for an earlier description of $\mathcal{REL}$).

We will use the category $\mathcal{REL}(\mathbf{KHausLoc})$ a lot in what follows and shall call it **KHausRel**.

The definition of relational composition as given above doesn't give us much of an algebraic handle. In order to find such an algebraic handle we continue with our spatial intuition. Say X, Y, Z are spaces and$R_1 \subseteq X \times Y, R_2 \subseteq Y \times Z$ are both closed. So $R_i = \neg I_i$ where $\neg$ is set theoretic complement and the I_is are open. (We are only looking at the spatial case in order to justify the choice of formula to follow and so are at liberty to use the excluded middle.)
We want $R_2 \circ R_1$ to be closed and so to define $\circ$ all we need define is some function

$$* : \Omega(X \times Y) \times \Omega(Y \times Z) \to \Omega(X \times Z)$$

such that $R_2 \circ R_1 = \neg * (I_1, I_2)$. Given the facts about preframe tensors discussed in Chapter 2 it should be clear that we only need be concerned with the cases

$$I_1 = U_1 \parr V_1 \qquad I_2 = V_2 \parr W_2$$

for some opens U_1, V_1, V_2, W_2. We know $(x, z) \in R_2 \circ R_1$ iff $\exists y \quad xR_1y \quad yR_2z$. Hence $(x, z) \in *(I_1, I_2)$ iff $\forall y \quad (x \neg R_1 y) \vee (y \neg R_2 z)$. Hence

$$\begin{aligned} (x, z) \in *(I_1, I_2) &\Leftrightarrow \forall y((x, y) \in I_1) \vee ((y, z) \in I_2) \\ &\Leftrightarrow \forall y(x \in U_1 \vee y \in V_1 \vee y \in V_2 \vee z \in W_2) \\ &\Leftrightarrow (x \in U_1 \vee z \in W_2) \vee Y \subseteq V_1 \cup V_2 \\ &\Leftrightarrow (x, z) \in U_1 \parr W_2 \vee Y \subseteq V_1 \cup V_2 \end{aligned}$$

Now say $R_1 \rightarrowtail X \times Y, R_2 \rightarrowtail Y \times Z$ are closed sublocales. Define

$$R_2 \circ R_1 = \neg * (a_{R_1}, a_{R_2})$$

where a_{R_i} is the open corresponding to the closed sublocale R_i and
$* : \Omega(X \times Y) \times \Omega(Y \times Z) \to \Omega(X \times Z)$ is defined on generators as

$$*(a_1 \otimes b_1, b_2 \otimes c_2) = a_1 \otimes c_2 \vee \Omega!(1 \leq b_1 \vee b_2)$$

In fact we have to factor $*$ through $\bar{*}$:

$$\begin{array}{rcl} \bar{*} : \Omega X \otimes \Omega Y \otimes \Omega Z & \to & \Omega X \otimes \Omega Z \\ a \otimes b \otimes c & \mapsto & a \otimes c \vee \Omega!(1 \leq b) \end{array}$$

since to make sure that we are defining a function we need to define it on all generators of some tensor. We need to check that $\bar{*}$ is well defined. i.e. that

$$(a, b, c) \mapsto a \otimes c \vee \Omega!(1 \leq b)$$

is a preframe trihomomorphism. This follows from the compactness of ΩY. Then take $*(I_1, I_2) = \bar{*}(\coprod_{12} I_1 \vee \coprod_{23} I_2)$ where the $\coprod$s are frame coprojections.

Theorem 4.2.1 *If X, Y, Z are compact Hausdorff locales then the function*

$$\begin{array}{rcl} \Omega X \times \Omega Y \times \Omega Z & \longrightarrow & \Omega X \otimes \Omega Z \\ (a, b, c) & \longmapsto & (a \otimes c) \vee \Omega!(1 \leq b) \end{array}$$

is a preframe trihomomorphism and so induces a preframe homomorphism

$$\bar{*} : \Omega X \otimes \Omega Y \otimes \Omega Z \longrightarrow \Omega X \otimes \Omega Z$$

There are preframe homomorphisms

$$\begin{array}{rcl} \Omega(\pi_{12}) : \Omega X \otimes \Omega Y & \longrightarrow & \Omega X \otimes \Omega Y \otimes \Omega Z \\ a \otimes b & \longmapsto & a \otimes b \otimes 0 \\ \Omega(\pi_{23}) : \Omega Y \otimes \Omega Z & \longrightarrow & \Omega X \otimes \Omega Y \otimes Z \\ b \otimes c & \longmapsto & 0 \otimes b \otimes c \end{array}$$

Define $* : (\Omega X \otimes \Omega Y) \times (\Omega Y \otimes \Omega Z) \longrightarrow \Omega X \otimes \Omega Z$ by $I * J = \bar{*}(\Omega\pi_{12} I \vee \Omega\pi_{23} J)$, then if $\neg I \rightarrowtail X \times Y, \neg J \rightarrowtail Y \times Z$ are two monics in **KHausLoc** *then their relational composition is given by*

$$\neg(I * J).$$

Before proof we find an alternative formula for $\bar{*}$. Note that for $a \in \Omega X, b \in \Omega Y$, $c \in \Omega Z$

$$\begin{array}{rcl} \Omega\pi_{13}(\bar{*}(a \otimes b \otimes c)) & = & \Omega\pi_{13}(a \otimes c \vee \Omega!(1 \leq b)) \\ & = & a \otimes 0 \otimes c \vee \bigvee\{1 | 1 \leq b\} \\ & \leq & a \otimes b \otimes c \end{array}$$

Thus $\Omega\pi_{13}(\bar{*}(I)) \leq I$ for all $I \in \Omega X \otimes \Omega Y \otimes \Omega Z$. And

$$\begin{array}{rcl} \bar{*}\Omega\pi_{13}(a \otimes c) & = & \bar{*}(a \otimes 0 \otimes c) \\ & = & (a \otimes c) \vee \Omega!(1 \leq 0) \\ & \geq & a \otimes c \end{array}$$

and so $J \leq \bar{*}\Omega\pi_{13}(J)$ for all $J \in \Omega X \otimes \Omega Z$. Hence $\bar{*}$ is right adjoint to $\Omega\pi_{13}$ i.e. $\bar{*} = \forall_{\pi_{13}}$.

Proof: For $I \in \Omega X \otimes \Omega Y$, $J \in \Omega Y \otimes \Omega Z$ we are trying to prove that
$\neg J \circ \neg I = \neg\forall_{\pi_{13}}(I \boxtimes 0 \vee 0 \boxtimes J)$.
It is easy to see that $(1 \otimes \Omega\Delta)(I \boxtimes 0) = I \quad (\Delta : Y \rightarrowtail Y \times Y)$ and so since
$I \boxtimes J = (I \boxtimes (0 \boxtimes 0) \vee (0 \boxtimes 0) \boxtimes J)$ we have to prove

$$\neg J \circ \neg I = \neg\forall_{\pi_{13}}(1 \otimes \Omega\Delta \otimes 1)(I \boxtimes J)$$

Now set $P \overset{(p_1,p_2)}{\rightarrowtail} X \times Y \equiv \neg I \rightarrowtail X \times Y, \quad Q \overset{(q_1,q_2)}{\rightarrowtail} Y \times Z \equiv \neg J \rightarrowtail Y \times Z$, and to define $Q \circ P$ we form the pullback:

$$\begin{tikzcd}
P \times_Y Q \arrow[r, "a_2"] \arrow[d, "a_1"'] & Q \arrow[d, "q_1"] \\
P \arrow[r, "p_2"] & Y
\end{tikzcd}$$

which is well know to be defined equivalently by the pullback

$$\begin{tikzcd}
P \times_Y Q \arrow[r, "p_2 a_1 = q_1 a_2"] \arrow[d, "(a_1, a_2)"'] & Y \arrow[d, hook, "\Delta"'] \\
P \times Q \arrow[r, "p_2 \times q_1"] & Y \times Y
\end{tikzcd}$$

$P \times_Y Q$ is a closed sublocale of $P \times Q$ (we are working in **KHausLoc**). The open corresponding to this closed sublocale is given by

$$\begin{array}{rl}
& \Omega(p_2 \times q_1)(\#) \\
= & (\Omega p_2) \otimes (\Omega q_1)(\#)
\end{array}$$

(see Lemma [3.3.3]). Now

$$\begin{array}{rcl}
\Omega p_2 : \Omega Y & \to & \Omega X \otimes \Omega Y \to\uparrow I \\
b & \mapsto & 0 \boxtimes b \mapsto I \vee 0 \boxtimes b \\
\Omega q_1 : \Omega Y & \to & \Omega Y \otimes \Omega Z \to\uparrow J \\
\bar{b} & \mapsto & \bar{b} \boxtimes 0 \mapsto J \vee \bar{b} \boxtimes 0
\end{array}$$

Recalling that

$$\# = \bigvee{}^{\uparrow}\{\wedge_i(b_i \boxtimes \bar{b}_i) | \wedge_{i \in I}(b_i \vee \bar{b}_i) = 0 \ I \text{ finite }\}$$

we can see that the open corresponding to the closed sublocale $P \times_Y Q$ is

$$(I \boxtimes J) \vee (0 \boxtimes \# \boxtimes 0)$$

The definition of $Q \circ P$ is that it is the image of the composition

$$P \times_Y Q \xhookrightarrow{(a_1 \times a_2)} P \times Q \xhookrightarrow{(p_1,p_2) \times (q_1,q_2)} X \times Y \times Y \times Z \xrightarrow{\pi_{14}} X \times Z$$

However $P \times_Y Q \rightarrowtail X \times Y \times Y \times Z$ is less than

$$X\times Y\times Z \overset{1\times\Delta\times 1}{\rightarrowtail} X\times Y\times Y\times Z$$

in the poset $Sub(X\times Y\times Y\times Z)$. (This is just the statement that $0\otimes\#\otimes 0\leq (I\otimes J)\vee(0\otimes\#\otimes 0)$.) And so $P\times_Y Q$ is a closed sublocale of $X\times Y\times Z$. The open corresponding to it is given by $(1\otimes\Omega\Delta\otimes 1)((I\otimes J)\vee(0\otimes\#\otimes 0))$ $=(1\otimes\Omega\Delta\otimes 1)(I\otimes J)$. So the image of the composition

$$P\times_Y Q \xhookrightarrow{(a_1\times a_2)} P\times Q \xhookrightarrow{(p_1,p_2)\times(q_1,q_2)} X\times Y\times Y\times Z \xrightarrow{\pi_{14}} X\times Z$$

is given by the image of

$$\neg(1\otimes\Omega\Delta\otimes 1)(I\otimes J)\rightarrowtail X\times Y\times Z\overset{\pi_{13}}{\longrightarrow} X\times Z$$

(since $\pi_{14}\circ(1\times\Delta\times 1)=\pi_{13}$) and the open corresponding to this image is

$$\forall_{\pi_{13}}(1\otimes\Omega\Delta\otimes 1)(I\otimes J).$$

To see this last line recall that the image of $f:X\to Y$ in **KHausLoc** is given by $\neg\forall_f(0)\hookrightarrow Y$. $\square$

Yet another formula for $*$ can be found:

$$\begin{aligned} *(a_1\otimes b_1,b_2\otimes c_2) &= (a_1\otimes c_2)\vee\Omega!(1\leq b_1\vee b_2)\\ &= a_1\otimes c_2\vee\bigvee^{\uparrow}(\{0\}\cup\{1|1\leq b_1\vee b_2\}\\ &= \bigvee^{\uparrow}(\{a_1\otimes c_2\}\cup\{1|1\leq b_1\vee b_2\}) \end{aligned}$$

Theorem 4.2.2 **KHausRel** *is a category.*

Proof: The reader may consult the proof that $\mathcal{REL}(\mathcal{C})$ is a category for any regular $\mathcal{C}$ (in [FS90] for example) in order to deduce that **KHausRel** is a category. We include the following direct proof for completeness.

The problem is to show associativity of $*$ and that $\#$ corresponds to the identity. For suitable $a_1\otimes b_1, b_2\otimes c_2, c_3\otimes d_3$ we find

$$\begin{aligned} *(a_1\otimes b_1,*(b_2\otimes c_2,c_3\otimes d_3)) &= *(a_1\otimes b_1,\bigvee^{\uparrow}(\{b_2\otimes d_3\}\cup\{1|1\leq c_2\vee c_2\})\\ &= \bigvee^{\uparrow}(\{*(a_1\otimes b_1,b_2\otimes d_3)\}\cup\{1|1\leq c_2\vee c_3\})\\ &= \bigvee^{\uparrow}(\{\bigvee^{\uparrow}(\{a_1\otimes d_3\}\cup\{1|1\leq b_1\vee b_2\})\}\cup\{1|1\leq c_2\vee c_3\})\\ &= \bigvee^{\uparrow}(\{a_1\otimes d_3\}\cup\{1|1\leq b_1\vee b_2\}\cup\{1|1\leq c_2\vee c_3\}) \end{aligned}$$

A similar manipulation on $*(*(a_1\otimes b_1,b_2\otimes c_2),c_3\otimes d_3)$ reduces it to the same term. # is given by the formula:

$$\#=\bigvee^{\uparrow}\{\wedge_i(b_i\otimes\bar{b}_i)|\wedge_i(b_i\vee\bar{b}_i)=0\}$$

We want $*(\#,b\otimes a)=b\otimes a$ for appropriate a,b.

$$\begin{aligned} *(\#, b \parr a) &= \bigvee^{\uparrow} \{ *(\wedge_i (b_i \parr \bar{b}_i), b \parr a) | \wedge_i (b_i \vee \bar{b}_i) = 0 \} \\ &= \bigvee^{\uparrow} \{ \wedge_i [(b_i \parr a) \vee \Omega!(1 \leq \bar{b}_i \vee b)] | \wedge_i (b_i \vee \bar{b}_i) = 0 \} \end{aligned}$$

Say $(b_i, \bar{b}_i)_{i \in I}$ is a finite collection of opens such that $\wedge_i (b_i \vee \bar{b}_i) = 0$. Using the finite distributivity law:

$$\wedge_i (b_i \vee \bar{b}_i) = \bigvee (\wedge_{i \in J_1} b_i) \wedge (\wedge_{i \in J_2} \bar{b}_i)$$

(where the join is over all pairs $J_1, J_2 \subseteq I$ such that J_1, J_2 are finite and $I \subseteq J_1 \cup J_2$) we see that $(\wedge_{i \in J_1} b_i) \wedge (\wedge_{i \in J_2} \bar{b}_i) = 0$ for every such pair. By applying the same finite distributivity law to the meet

$$\wedge_i [(b_i \parr a) \vee \Omega!(1 \leq \bar{b}_i \vee b)]$$

we find that to conclude $*(\#, b \parr a) \leq b \parr a$ it is sufficient to prove:

$$(\wedge_{i \in J_1} (b_i \parr a)) \wedge (\wedge_{i \in J_2} \Omega!(1 \leq \bar{b}_i \vee b)) \leq b \parr a$$

But

$$\begin{aligned} \wedge_{i \in J_2} \Omega!(1 \leq \bar{b}_i \vee b) &= \Omega!(1 \leq \wedge_{i \in J_2} \bar{b}_i \vee b) \\ &\leq \Omega!(\wedge_{i \in J_1} b_i \leq b) \end{aligned}$$

by the fact that $(\wedge_{i \in J_1} b_i) \wedge (\wedge_{i \in J_2} \bar{b}_i) = 0$. However for any opens c, d

$$c \wedge \Omega!(c \leq d) \leq d$$

(to see this formally note $\Omega!(c \leq d) = \bigvee \{1 | c \leq d\}$ and joins distribute over finite meets). Thus $*(\#, b \parr a) \leq b \parr a$.

Proving the opposite inequality requires an application of Theorem [3.4.2]: we need to know that compact Hausdorff locales are regular (as a separation axiom of course, rather than as a whole category!). i.e. we exploit the fact that for any open b,

$$b = \bigvee^{\uparrow} \{ b_0 | b_0 \lhd b \}$$

and so

$$b \parr a = \bigvee^{\uparrow} \{ b_0 \parr a | b_0 \lhd b \}$$

Say $b_0 \lhd b$. Then there exists c such that $b_0 \wedge c = 0$ and $1 \leq b \vee c$. So

$$\begin{aligned} b_0 \parr a &\leq (0 \parr a) \vee \Omega!(1 \leq c \vee b) \\ b_0 \parr a &\leq (b_0 \parr a) \vee \Omega!(1 \leq 0 \vee b) \end{aligned}$$

i.e.

$$b_0 \parr a \leq \wedge_{i \in \{1,2\}} [(b_i \parr a) \vee \Omega!(1 \leq \bar{b}_i \vee b)]$$

where $b_1 = 0, \bar{b}_1 = c, b_2 = b_0$ and $\bar{b}_2 = 0$. But

$$\begin{aligned} \wedge_{i \in \{1,2\}} (b_i \vee \bar{b}_i) &= (0 \vee c) \wedge (b_0 \vee 0) \\ &= c \wedge b_0 = 0 \end{aligned}$$

and so $b_0 \parr a \leq *(\#, b \parr a)$. Hence $b \parr a \leq *(\#, b \parr a)$. $\square$

We have an important technical lemma which will help us relate closed sublocales of $X \times Y$ to preframe homomorphisms $\Omega Y \to \Omega X$. Indeed will see that closed sublocales of $X \times Y$ and preframe homomorphisms $\Omega Y \to \Omega X$ are the same thing provided X, Y are compact Hausdorff.

Lemma 4.2.1 *If $f_1 : \Omega X \to \Omega \bar{X}, f_2 : \Omega Z \to \Omega \bar{Z}$ are preframe homomorphisms and $X, \bar{X}, Y, Z, \bar{Z}$ are compact Hausdorff locales and $I \in \Omega X \otimes \Omega Y, J \in \Omega Y \otimes \Omega Z$ then*

$$(f_1 \otimes f_2)(I * J) = (f_1 \otimes 1)(I) * (1 \otimes f_2)(J)$$

Proof: We first check the cases $I = a \parr b, J = \bar{b} \parr \bar{c}$.

$$\begin{aligned}
&(f_1 \otimes 1)(I) * (1 \otimes f_2)(J) \\
&= \bar{*}((f_1 a \parr b \parr 0) \vee (0 \parr \bar{b} \parr f_2 \bar{c})) \\
&= \bar{*}(f_1 a \parr (b \vee \bar{b}) \parr f_2 \bar{c}) \\
&= \bigvee^{\uparrow} (\{f_1 a \parr f_2 \bar{c}\} \cup \{1 | 1 \leq b \vee \bar{b}\}) \\
&= (f_1 \otimes f_2) \bigvee^{\uparrow} (\{a \parr \bar{c}\} \cup \{1 | 1 \leq b \vee \bar{b}\}) \\
&= (f_1 \otimes f_2)(I * J).
\end{aligned}$$

The result then follows for general I, J since $*$ is a preframe bihomomorphism. $\square$

We can interpret this lemma spatially. Recall that if $g : X \to Y$ is a locale map between compact Hausdorff locales then for any closed sublocale $\neg I \rightarrowtail X$ of X its image under g (written $g(\neg I)$) is given by $\neg \forall_g (I)$. So the lemma could have been stated: given $g_1 : X \to \bar{X}, g_2 : Z \to \bar{Z}$ with $X, \bar{X}, Y, Z, \bar{Z}$ compact Hausdorff then for any closed relations $\neg I \rightarrowtail X \times Y, \neg J \rightarrowtail Y \times Z$

$$(g_1 \times g_2)(\neg J \circ \neg I) = ((1 \times g_2)(\neg J)) \circ ((g_1 \times 1)(\neg I))$$

(Take $f_1 = \forall_{g_1}$ and $f_2 = \forall_{g_2}$ in the lemma.)

4.3 Axioms on relations

We would like to use our relational composition on compact Hausdorff locales in order to capture well known spatial ideas about sets and relations. Often when looking at the upper closure of a subset with respect to some relation R we are interested in the cases when R is a preorder, or a partial order, or transitive, or interpolative etc. These axioms can be expressed using relational composition:

$$\begin{aligned}
R \text{ reflexive} &\quad \Leftrightarrow \quad \Delta \subseteq R \\
R \text{ transitive} &\quad \Leftrightarrow \quad R \circ R \subseteq R \\
R \text{ interpolative} &\quad \Leftrightarrow \quad R \subseteq R \circ R \\
R \text{ antisymmetric} &\quad \Leftrightarrow \quad R \cap \tau R \subseteq \Delta
\end{aligned}$$

where Δ is the diagonal on X and τ is the twist isomorphism $X \times X \to X \times X$. The localic version of the above is clear: if X is a compact Hausdorff locale and R is a closed sublocale of $X \times X$ then we say

$$\begin{array}{rcl} R \text{ reflexive} & \Leftrightarrow & \Delta \leq R \\ R \text{ transitive} & \Leftrightarrow & R \circ R \leq R \\ R \text{ interpolative} & \Leftrightarrow & R \leq R \circ R \\ R \text{ antisymmetric} & \Leftrightarrow & R \wedge \tau R \leq \Delta. \end{array}$$

Where $\leq$ is the inclusion of closed sublocales and $\Delta : X \rightarrowtail X \times X$ is the (closed) diagonal. It is important to realize how these axioms are going to be used in practice. The diagonal is closed so,

$$\Delta = \neg\# \rightarrowtail X \times X$$

where $\# = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes b_i)| \wedge_{i \in I} (a_i \vee b_i) = 0, I \text{ finite }\}$. So if $R = a \otimes b$ then the antisymmetric axiom is the statement that for every collection $(a_i, b_i)_{i \in I}$ (I finite) with $\wedge_{i \in I}(a_i \vee b_i) = 0$ we have

$$\wedge_i(a_i \otimes b_i) \leq (a \otimes b) \vee (b \otimes a)$$

The order reverses since $\neg a \leq_{Sub(X)} \neg b$ if and only if $b \leq a$ for any $a, b \in \Omega X$.

Say R is some relation on a set X (so R is a subset of $X \times X$), then given any subset $\bar{X}$ of X we often want to look at the 'upper closure' of $\bar{X}$ with respect to R. i.e. the set

$$\{x \in X | \exists y \in \bar{X} \quad yRx\} \ (*)$$

Now $X \cong 1 \times X$ and so $PX \cong P(1 \times X)$. It is easy to see that the set (*) is the image under this last correspondence of the relational composition of $R \subseteq X \times X$ and $\{(*, x)| x \in \bar{X}\} \subseteq 1 \times X$ $(1 = \{*\})$. i.e. upper closure can be expressed via relational composition.
Say R is some closed relation on a compact Hausdorff locale X, and $\bar{X}$ is some closed sublocale of X (so $\bar{X} \rightarrowtail X = \neg a \rightarrowtail X$ for some $a \in \Omega X$) then we can define an R-upper closure of $\bar{X}$. Similarly to the discrete case just described closed sublocales of $1 \times X$ are in bijective correspondence with closed sublocales of X. But $1 \times \bar{X}$ is a closed sublocale of $1 \times X$, and so we take its relational composition with $R \rightarrowtail X \times X$ and then transform the sublocale of $1 \times X$ that we get to a sublocale of X. This defines the R-upper closure of $\bar{X}$. Symbolically the R-upper closure of $\bar{X}$ is

$$\pi_2(R \circ (1 \times \bar{X}))$$

(Recall $\pi_2 : 1 \times X \to X$ is an isomorphism.)

Symmetrically we can define the lower closure of a closed sublocale with respect to a closed relation.

We can also define the R-lower closure of a subset $\bar{Y}$ of some set Y if R is a relation on $X \times Y$ where Y is some other set. We are referring to the set

$$\{x \in X | \exists y \in \bar{Y} \quad xRy\}$$

Given a closed relation $R \rightarrowtail X \times Y$ where X, Y are compact Hausdorff locales and given $\bar{Y}$ a closed sublocale of Y we define the R-lower closure of $\bar{Y}$ to be the closed sublocale given by

$$\bar{Y} \circ R$$

This is, of course, an abuse of notation. $\bar{Y}$ is not a relation and the result of $\bar{Y} \circ R$ is not a closed sublocale, it is a closed relation. We are assuming that the relational composition $\circ$ is performed on $\bar{Y} \times 1 \rightarrowtail Y \times 1$, and that the result is composed with the isomorphism π_1 in order to obtain a sublocale of X.
This notion of R-lower closure with respect to some closed relation R on compact Hausdorff locales X, Y gives rise to a preframe morphism $\psi_R : \Omega Y \to \Omega X$. The procedure for defining ψ_R is: take $b \in \Omega Y$ then define ψ_R by $\neg\psi_R b =$ the lower closure of $\neg b$. We use the notation $R = \neg a_R \rightarrowtail X \times X$ in order to talk about the element of $\Omega(X) \otimes \Omega(X)$ corresponding to R. We can use the $*$ function to define ψ_R:

$$\psi_R : b \longmapsto a_R * b$$

N.B. this is an abuse of notation. $*$ cannot take b as one of its arguments, so really we are looking at the function

$$b \longmapsto (\Omega\pi_1)^{-1}(a_R * (b \otimes 0))$$

Where

$$\Omega\pi_1 : \Omega X \longrightarrow \Omega X \otimes \Omega$$

is the isomorphism $a \mapsto a \otimes 0$. It is clear from the definition of $*$ that ψ_R is a preframe homomorphism.

Moreover the assignment $a_R \mapsto \psi_R$ from $\Omega X \otimes \Omega Y$ to $\mathbf{PreFrm}(\Omega Y, \Omega X)$ is a preframe homomorphism. We aim to show that it is an isomorphism. Say we are given a preframe homomorphism $\psi : \Omega Y \to \Omega X$ we can define a closed sublocale $R_\psi = \neg a_\psi \rightarrowtail X \times Y$ by

$$a_\psi = (\psi \otimes 1)(\#)$$

Theorem 4.3.1 *If X, Y are compact Hausdorff locales then*

$$\mathbf{PreFrm}(\Omega Y, \Omega X) \cong \Omega X \otimes \Omega Y$$

as preframes.

Before the proof we need a technical lemma.

Lemma 4.3.1 *For any $I \in \Omega X \otimes \Omega Y$ (X, Y compact Hausdorff) the preframe homomorphism*

$$\begin{array}{rcl} \Omega Y \otimes \Omega Y & \longrightarrow & \Omega X \otimes \Omega Y \\ J & \longmapsto & I * J \end{array}$$

can be factored as

$$\Omega Y \otimes \Omega Y \xrightarrow{\Omega\pi_1 \otimes 1} \Omega Y \otimes \Omega \otimes \Omega Y \xrightarrow{(I * (\_)) \otimes 1} \Omega X \otimes \Omega \otimes \Omega Y \xrightarrow{(\Omega\pi_1)^{-1} \otimes 1} \Omega X \otimes \Omega Y$$

Proof: We need to check for any $J \in \Omega Y \otimes \Omega Y$ that

$$I * J = ((\Omega\pi_1)^{-1} \otimes 1)((I * (\_)) \otimes 1)(\Omega\pi_1 \otimes 1)(J)$$

As in technical Lemma [4.2.1] it is clearly sufficient to check the cases $J = b_1 \parr b_2$
$I = a \parr b$.
But then

$$\begin{array}{rcl}
LHS & = & (a \parr b) * (b_1 \parr b_2) \\
& = & \bigvee^{\uparrow}(\{a \parr b_2\} \cup \{1 | 1 \leq b_1 \vee b\}) \\
RHS & = & ((\Omega\pi_1)^{-1} \otimes 1)((a \parr b * (_)) \otimes 1)(b_1 \parr 0 \parr b_2) \\
& = & ((\Omega\pi_1)^{-1} \otimes 1)([(a \parr b) * (b_1 \parr 0)] \parr b_2) \\
& = & ((\Omega\pi_1)^{-1} \otimes 1)(\bigvee^{\uparrow}(\{a \parr 0\} \cup \{1 | 1 \leq b \vee b_1\}) \parr b_2) \\
& = & ((\Omega\pi_1)^{-1} \otimes 1) \bigvee^{\uparrow}(\{a \parr 0 \parr b_2\} \cup \{1 | 1 \leq b_1 \vee b_2\}) \\
& = & \bigvee^{\uparrow}(\{((\Omega\pi_1)^{-1} \otimes 1)(a \parr 0 \parr b_2)\} \cup \{1 | 1 \leq b_1 \vee b\}) \\
& = & \bigvee^{\uparrow}(\{a \parr b_2\} \cup \{1 | 1 \leq b_1 \vee b\}) \qquad \Box
\end{array}$$

Proof of Theorem [4.3.1] Define

$$\begin{array}{rcl}
\beta : \mathbf{PreFrm}(\Omega Y, \Omega X) & \longrightarrow & \Omega X \otimes \Omega Y \\
\psi & \longmapsto & (\psi \otimes 1)(\#) \\
\alpha : \Omega X \otimes \Omega Y & \longrightarrow & \mathbf{PreFrm}(\Omega Y, \Omega X) \\
I & \longmapsto & (b \mapsto (\Omega\pi_1)^{-1}(I * (b \parr 0)))
\end{array}$$

We need to check $\alpha \circ \beta = id$ and $\beta \circ \alpha = id$.

But $(\alpha(I) \otimes 1) = ((\Omega\pi_1)^{-1} \otimes 1)((I * (\_)) \otimes 1)(\Omega\pi_1 \otimes 1)$ by the definition of α. Hence $(\alpha(I) \otimes 1)(J) = I * J$ for every $J \in \Omega Y \otimes \Omega Y$ by the last lemma. It follows that $(\alpha(I) \otimes 1)(\#) = I * \#$. But $I * \# = I$ since the diagonal is the identity for relational composition. Hence $\beta \circ \alpha = id$.

On the other hand for any $a \in \Omega Y$ (and any $\psi \in \mathbf{PreFrm}(\Omega Y, \Omega X)$)

$$\begin{array}{rcl}
[(\alpha \circ \beta)(\psi)](a) & = & [\alpha((\psi \otimes 1)(\#))](a) \\
& = & (\Omega\pi_1)^{-1}((\psi \otimes 1)(\#) * (a \parr 0)) \\
& = & (\Omega\pi_1)^{-1}((\psi \otimes 1)(\#) * (1 \otimes 1)(a \parr 0)) \\
& = & (\Omega\pi_1)^{-1}(\psi \otimes 1)(\# * (a \parr 0)) \text{ by Lemma [4.2.1] with } f_1 = \psi, f_2 = 1 \\
& = & (\Omega\pi_1)^{-1}((\psi \otimes 1)(a \parr 0)) \\
& = & (\Omega\pi_1)^{-1}(\psi(a) \parr 0) \\
& = & \psi(a) \quad \Box
\end{array}$$

As an immediate corollary notice that a relation $R \hookrightarrow X \times X$ is reflexive if and only if

$$\psi_R(a) \leq a \quad \forall a \in \Omega X$$

The proof of Theorem [4.3.1] shows that there is an order reversing bijective correspondence between the closed relations on two compact Hausdorff locales X, Y and preframe homomorphisms from ΩY to ΩX. By looking at the SUP-lattice description of locales the above can be translated into a proof of

Theorem 4.3.2 *If X, Y are discrete locales then*

$$\mathbf{SUP}(\Omega Y, \Omega X) \cong \Omega X \otimes \Omega Y$$

as SUP-lattices.

Proof: As stated in the preamble we can repeat the above proof (of Theorem [4.3.1]) with SUP-lattice tensor in place of preframe tensor. However we know that the category of discrete locales is equivalent to the category of sets (=the background topos) and so we can offer a much more straightforward proof of this result. All we need to do is check that there is a one to one correspondence between the relations on two sets X, Y and SUP-lattice homomorphisms going from PY to PX. This is an elementary exercise. $\Box$

The last theorem and its proper analogue (Theorem [4.3.1]) can both be written as categorical equivalences. **KHausLoc** is the category of compact Hausdorff locales. We use

$$\mathbf{KHausLoc}_U$$

to denote the opposite of the category whose objects are the frames of opens of compact Hausdorff locales and whose maps are preframe homomorphisms. The open parallel is the category

$$\mathbf{DisLoc}_L$$

which is the opposite of the category whose objects are powers sets of sets (i.e. the frames of opens of discrete locales) and whose morphisms are SUP-lattice homomorphisms.

Theorem 4.3.3

$$\begin{aligned} \mathbf{KHausLoc}_U &\cong \mathbf{KHausRel} \\ \mathbf{DisLoc}_L &\cong \mathbf{Rel} \end{aligned}$$

Proof: We prove the proper parallel only. The problem is to check that relational composition is taken to function composition of the corresponding preframe maps. (For then since α and β are inverse to each other it will follow that β takes function composition to relational composition in an appropriate way.) Clearly it is sufficient to prove that

$$\alpha(I * J) = \alpha(I) \circ \alpha(J)$$

for all $I \in \Omega(X \times Y), J \in \Omega(Y \times Z)$. But if $c \in \Omega Z$ then

$$\alpha(I * J)(c) = (\Omega\pi_1)^{-1}(I * J * (c \otimes 0))$$

(recall that $*$ is associative). But

$$\begin{aligned} [\alpha(I) \circ \alpha(J)](c) &= \alpha(I)[(\Omega\pi_1)^{-1}(J * (c \otimes 0))] \\ &= (\Omega\pi_1)^{-1}(I * [(\Omega\pi_1)^{-1}(J * (c \otimes 0)) \otimes 0]) \end{aligned}$$

But $b \mapsto b \otimes 0$ is $\Omega\pi_1 : \Omega Y \to \Omega Y \otimes \Omega$ and so $[((\Omega\pi_1)^{-1}K) \otimes 0] = K$ for every $K \in \Omega Y \otimes \Omega$.
Hence

$$[\alpha(I) \circ \alpha(J)](c) = (\Omega\pi_1)^{-1}(I * J * (c \otimes 0)) \quad \Box$$

Corollary 4.3.1 *$(\mathbf{KHausLoc})_U$ and $(\mathbf{DisLoc})_L$ are both self dual.*

Proof: This result follows from the fact that **KHausRel** and **Rel** are both self dual. Their dualizing functor is effectively given by the twist isomorphism on the product of locales: $\tau_{X,Y} : X \times Y \longrightarrow Y \times X$. So a morphism $(\neg I \hookrightarrow X \times Y)$ of **KHausRel** is mapped to the morphism

$$\neg I \hookrightarrow X \times Y \xrightarrow{\tau} Y \times X$$

of $\mathbf{KHausRel}^{op}$. $\square$

We now fix some notation that will be used in the final three chapters. Say $R \hookrightarrow X \times X$ is a closed relation on a compact Hausdorff locale X.
Then $R = \neg a_R, \quad a_R \in \Omega X \otimes \Omega X$. The lower closure of closed sublocales is the function:

$$\begin{aligned} \Downarrow: CSub(X) &\longrightarrow CSub(X) \\ \neg a &\longmapsto \neg a \circ R \end{aligned}$$

(where $CSub(X)$=the closed sublocales of X). The upper closure is the function:

$$\begin{aligned} \Uparrow: CSub(X) &\longrightarrow CSub(X) \\ \neg a &\longmapsto R \circ \neg a \end{aligned}$$

But in practice (i.e. when it comes to algebraic manipulations) we are interested in the corresponding preframe homomorphisms.

$$\Downarrow^{op}: \Omega X \to \Omega X$$

is the unique preframe homomorphism such that

$$\Downarrow \neg a = \neg \Downarrow^{op} a \quad \forall a \in \Omega X$$

and

$$\Uparrow^{op}: \Omega X \to \Omega X$$

is the unique preframe homomorphism such that

$$\Uparrow \neg a = \neg \Uparrow^{op} a \quad \forall a \in \Omega X.$$

We choose the 'op' since $CSub(X) \cong \Omega X^{op}$ and so $\Downarrow$ is effectively a function from ΩX^{op} to ΩX^{op}. $\Downarrow^{op}$ is the same function but acting on (and going to) the opposite poset. So the analogy is with categorical notation: if $F : \mathcal{C} \to \mathcal{D}$ is a functor between categories then $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is the same functor but with the arrows of the domain and codomain formally reversed.

We can now write out some implications of Theorem [4.3.1] applied to the case $X = Y$: if R is a relation on X then we know

$$a_R = (\Downarrow^{op} \otimes 1)(\#).$$

But because of the duality referred to in the last corollary we see that

$$a_R = (1 \otimes \Uparrow^{op})(\#)$$

as well. Of course the general conclusion is that for any relation $R \hookrightarrow X \times Y$ not only $a_R = (\psi_R \otimes 1)(\#)$ but also

$$a_R = (1 \otimes \phi_R)(\#)$$

where $\phi_R : \Omega X \to \Omega Y$ is given by $\phi_R(a) = (\Omega\pi_2)^{-1}((0 \otimes a) * a_R)$.

We can also use the fact that relational composition corresponds to function composition to make the following conclusions: a relation $R \hookrightarrow X \times X$ is

$$\begin{array}{rcl} \text{transitive} & \Leftrightarrow & \Downarrow^{op}(a) \leq \Downarrow^{op} \circ \Downarrow^{op}(a) \quad \forall a \in \Omega X \\ \text{interpolative} & \Leftrightarrow & \Downarrow^{op} \circ \Downarrow^{op}(a) \leq \Downarrow^{op}(a) \quad \forall a \in \Omega X \\ \text{reflexive} & \Leftrightarrow & \Downarrow^{op}(a) \leq a \quad \forall a \in \Omega X. \end{array}$$

4.4 Notes

For the reader who knows what the upper/lower power locale monad is, note that the equivalences of Theorem [4.3.3] are saying that the allegory is equal to the full subcategory of the Kleisli category of the monad, consisting of all compact Hausdorff/discrete locales. Also, notice that the corollaries

Corollary 4.4.1 $P_U(X) \cong \$^X$ *for all compact Hausdorff* X

Corollary 4.4.2 $P_L(X) \cong \$^X$ *for all discrete* X

(which appear in [Vic94]) can easily be derived from Theorems [4.3.1] and [4.3.2] respectively.

Much more can be said about these monads (e.g. a discussion of the constructive points of the power locales). Most interestingly we see in [Vic95] that it might be possible to use them to formalize what is meant by our expression ‘parallel’.

Chapter 5

Ordered Locales

5.1 Spatial Intuitions

We begin the chapter by repeating some well known facts about ordered topological spaces, noting that the results we examine do not require the antisymmetry axiom for the order $\leq$. We then prove some new theorems which show that these results become more straightforward localically.
The topological exposition is based on the beginning of Chapter VII in [Joh82].
We are looking at classical topological space theory in order to inspire a constructive localic treatment to follow and so are free to use the excluded middle at this point.

Lemma 5.1.1 *Assume the excluded middle. Given a topological space X with a preorder $\leq$ on it, then $\leq$ is closed iff $\forall x, y \in X \quad x \not\leq y$ implies*

$$\exists U, V \subseteq X \text{ s.t. } x \in intU, y \in intV, U \cap V = \phi, \uparrow U = U, \downarrow V = V$$

Proof: ($\Rightarrow$) If $\leq$ is closed and $x \not\leq y$ then $\exists U_1, V_1$ open such that $U_1 \times V_1 \cap (\leq) = \phi$. Take $U = \uparrow U_1, V = \downarrow V_1$. The reverse implication is equally straight forward. □

Lemma 5.1.2 *Assume the excluded middle. If $(X, \leq)$ is a preordered topological space with $\leq$ closed, and if $K \subseteq X$ is compact then $\downarrow K, \uparrow K$ are closed.*

Proof: Say $x \in X - \downarrow K$ then for every $k \in K$ $x \not\leq k$ and so by the lemma above $\exists U_k$ upper closed and V_k lower closed s.t. $(x, k) \in intU_k \times intV_k$ and $U_k \cap V_k = \phi$. Clearly then $K \subseteq \cup_{i=1}^{n} V_{k_i}$ for some n and since $\cup V_{k_i}$ is lower closed $\downarrow K \subseteq \cup V_{k_i}$. Also since $U_{k_i} \cap V_{k_i} = \phi \quad \forall i$ we see that $\cap_i U_{k_i}$ is a neighbourhood of x disjoint from $\downarrow K$ hence $\downarrow K$ is closed. $\uparrow K$ is shown to be closed by a similar argument. □
Notice that the above shows us that if the preordered topological space is compact Hausdorff then the upper(lower) closure of closed subspaces is closed (provided the preorder is closed). The localic analogy here is clear: if we are assuming X is a compact Hausdorff locale it is a matter of definition that relational composition takes closeds to closeds (provided the relation is closed).

Corollary 5.1.1 *Assume the excluded middle. If $(X, \leq)$ is a compact Hausdorff topological space with a closed preorder $\leq$ then whenever $x \not\leq y$ we can find disjoint opens U and V such that U is upper closed and V is lower closed and $(x, y) \in U \times V$.*

Proof:$\uparrow x$ and $\downarrow y$ are closed (by the lemma since $\{x\}$ and $\{y\}$ are compact) and $\uparrow x \cap \downarrow y = \phi$. Hence since compact Hausdorff spaces are normal we know that $\exists$ disjoint opens U_1, V_1 such that $\uparrow x \subseteq U_1, \downarrow y \subseteq V_1$. Take

$$U = X - \downarrow (X - U_1) \quad (\subseteq U_1)$$
$$V = X - \uparrow (X - V_1) \quad (\subseteq V_1) \quad \Box$$

This last corollary may be written

$$\not\leq \subseteq \bigcup \{U \times V | \quad U \cap V = \phi \quad \uparrow U = U \quad \downarrow V = V \quad U, V \text{ open}\}$$

The opposite inclusion is trivial so we have the equation

$$\not\leq = \bigcup \{U \times V | \quad U \cap V = \phi \quad \uparrow U = U \quad \downarrow V = V \quad U, V \text{ open}\}$$

for any compact Hausdorff topological space X. Recall that classically a set is upper closed iff its complement is lower closed. So we guess that the condition $\uparrow U = U$ can be safely translated to the localic condition

$$\Downarrow \neg U =_{Sub(X)} \neg U$$

where $\Downarrow$ is the lower closure operation corresponding to the relation $\leq$.
The reasoning behind the localic form of the above corollary should now be clear:

Theorem 5.1.1 *If X is a compact Hausdorff locale and $\leq$ is a closed preorder on it (i.e. $(\leq) \circ (\leq) \leq (\leq)$ and $\Delta \leq (\leq)$) then*

$$a_{\leq} = \bigvee \{a \otimes b | \quad a \wedge b = 0 \quad \Downarrow^{op} a = a \quad \Uparrow^{op} b = b\}$$

where $\leq = \neg(a_{\leq})$.

Recall from the end of the last chapter that $\Downarrow^{op}$ is the preframe homomorphism from ΩX to ΩX which corresponds to the closed relation $\leq$, and $\Uparrow^{op}$ is the preframe homomorphism from ΩX to ΩX corresponding to the closed relation $\geq$. We saw that

$$a_{\leq} = (\Downarrow^{op} \otimes 1)(\#)$$

and noticed that the symmetrical result is true:

$$a_{\leq} = (1 \otimes \Uparrow^{op})(\#).$$

Proof of Theorem:
First note that for any open a of our compact Hausdorff locale X we have that

$$\Uparrow^{op} a \leq a \text{ and } \Downarrow^{op} a \leq a$$

This is simply a reflection of the fact that $\leq$ is postulated to be reflexive.
Now $(\leq) \circ (\leq) \leq (\leq)$ means

$$\begin{aligned}
a_{\leq} &\leq a_{\leq} * a_{\leq} \\
&= (\Downarrow^{op} \otimes 1)(\#) * (1 \otimes \Uparrow^{op})(\#) \\
&= (\Downarrow^{op} \otimes \Uparrow^{op})(\# * \#) \quad \text{Lemma [4.2.1]} \\
&= (\Downarrow^{op} \otimes \Uparrow^{op})(\#) \\
&= (\Downarrow^{op} \otimes \Uparrow^{op})(\vee^{\uparrow}\{\wedge_i a_i \otimes b_i | \wedge_i (a_i \vee b_i) = 0\}) \\
&= (\vee^{\uparrow}\{\wedge_i (\Downarrow^{op} a_i \otimes \Uparrow^{op} b_i) | \wedge_i (a_i \vee b_i) = 0\} \\
&= \vee\{\Downarrow^{op} a \otimes \Uparrow^{op} b | a \wedge b = 0\} \\
&\leq \vee\{\bar{a} \otimes \bar{b} | \bar{a} \wedge \bar{b} = 0 \quad \Downarrow^{op} \bar{a} = \bar{a} \quad \Uparrow^{op} \bar{b} = \bar{b}\}
\end{aligned}$$

The proof behind the penultimate line is a simple reworking of the proof that $\vee^{\uparrow}\{\wedge_i(a_i \otimes b_i)| \wedge_i (a_i \vee b_i) = 0\} = \vee\{a \otimes b | a \wedge b = 0\}$ (see end of Lemma [2.7.1]) and the last line follows since (i) $\Downarrow^{op} a \wedge \Uparrow^{op} b \leq a \wedge b$ and (ii) $\Uparrow^{op}, \Downarrow^{op}$ are both idempotent since the relation is a preorder.
As for the 'easier' way round, say we are given a, b with $\Downarrow^{op} a = a, \Uparrow^{op} b = b$ and $a \wedge b = 0$. Recall $a_{\leq} = (\Downarrow^{op} \otimes 1)(\#)$. (I could have chosen $a_{\leq} = (1 \otimes \Uparrow^{op})(\#)$ and followed an obvious parallel route.) So

$$\begin{aligned} a \otimes b &= (\Downarrow^{op} a) \otimes b \\ &= (\Downarrow^{op} a) \otimes 0 \wedge 0 \otimes b \\ &\leq (\Downarrow^{op} a) \otimes 0 \wedge (\Downarrow^{op} 0 \otimes b) \\ &= (\Downarrow^{op} \otimes 1)(a \otimes 0) \wedge (\Downarrow^{op} \otimes 1)(0 \otimes b) \\ &= (\Downarrow^{op} \otimes 1)(a \otimes b) \leq (\Downarrow^{op} \otimes 1)(\#) = a_{\leq} \qquad \square \end{aligned}$$

This last result can be stated as a 'preframe fact' as well: along the way we saw that

$$a_{\leq} = \bigvee^{\uparrow}\{\wedge_i(\Downarrow^{op} a_i \otimes \Uparrow^{op} b_i)| \wedge_i (a_i \vee b_i) = 0\}.$$

In fact the lemma can be stated and proved more easily as,

Lemma 5.1.3 *If $(X, \leq)$ is a compact Hausdorff locale with a closed preorder then:*

$$a_{\leq} = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes b_i)| \Downarrow^{op} a_i = a_i, \Uparrow^{op} b_i = b_i, \wedge_i(a_i \vee b_i) = 0\}$$

Notice that the proof to follow is a lot simpler than our last proof since we don't have to worry about translating the conclusion from its preframe form to its SUP-lattice form.
Proof:

$$\begin{aligned} a_{\leq} &= a_{\leq} * a_{\leq} \\ &= (\Downarrow^{op} \otimes 1)(\#) * (1 \otimes \Uparrow^{op})(\#) \\ &= (\Downarrow^{op} \otimes \Uparrow^{op})(\#) \text{ from Lemma [4.2.1]} \\ &= \vee^{\uparrow}\{\wedge_i(\Downarrow^{op} a_i \otimes \Uparrow^{op} b_i)| \wedge_i (a_i \vee b_i) = 0\} \\ &\leq \vee^{\uparrow}\{\wedge_i(\bar{a}_i \otimes \bar{b}_i)| \Downarrow^{op} \bar{a}_i = \bar{a}_i, \Uparrow^{op} \bar{b}_i = b_i, \wedge_i(\bar{a}_i \vee \bar{b}_i) = 0\} \end{aligned}$$

In the other direction say we have a finite collection $(a_i, b_i)_{i \in I}$ such that $\Downarrow^{op} a_i = a_i$ for all i, $\Uparrow^{op} b_i = b_i$ for all i and $\wedge_i(a_i \vee b_i) = 0$. Then

$$\begin{aligned} \wedge_i(a_i \otimes b_i) &= \wedge_i(\Downarrow^{op} a_i \otimes b_i) \\ &= (\Downarrow^{op} \otimes 1)(\wedge_i(a_i \otimes b_i)) \\ &\leq (\Downarrow^{op} \otimes 1)(\#) = a_{\leq} \qquad \square \end{aligned}$$

Theorem 5.1.2 (Nac65) *Assume the excluded middle. Let $(X, \leq)$ be a compact Hausdorff topological space with a closed partial order. Then the sets of the form $U \cap V$ where U is an open upper set and V is an open lower set, form a base for the topology on X.*

Proof: Say $W \subseteq X$ is an open subset of X. Then $\forall x \in W$ we need to find open sets U, V such that $x \in U \cap V \subseteq W$, $\uparrow U = U$ and $\downarrow V = V$. Say $y \notin W$ Then $x \neq y$ and so either $x \not\leq y$ or $y \not\leq x$.
If $x \not\leq y$ then there exists opens U_y, V_y such that U_y upper, V_y lower, $x \in U_y$, $y \in V_y$

and $U_y \cap V_y = \phi$.
If $y \not\leq x$ then there exists opens U_y, V_y such that U_y lower, V_y upper, $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$.
Thus $W^c \subseteq \bigcup_{y \in W^c} V_y$ and so, since W^c is closed and thus compact,

$$W^c \subseteq \bigcup_{i \in I} V_{y_i}$$

for some finite I. Hence $\bigcap_{i \in I} U_{y_i} \subseteq W$ and $x \in \bigcap_{i \in I} U_{y_i}$. $\Box$

The localic version of this theorem is an easy corollary to the work that has already been done. Its proof, unsurprisingly, requires the antisymmetry axiom on the order $\leq$.

Theorem 5.1.3 $(X, \leq)$ *is such that* X *is a compact Hausdorff locale and* $\leq$ *is a closed partial order (i.e.* $\Delta \leq (\leq), (\leq) \circ (\leq) \leq (\leq), (\leq) \vee (\geq) \leq \Delta$*) then every* $c \in \Omega X$ *is the join of elements of the form* $a \wedge b$ *where* $\neg a$ *is a lower closed closed sublocale of* X *and* $\neg b$ *is an upper closed closed sublocale of* X.

Proof: Notice that the problem is equivalent to checking that the subframe of ΩX generated by the set,

$$\{a | \Uparrow^{op} a = a\} \cup \{a | \Downarrow^{op} a = a\}$$

is the whole of ΩX.
$\leq$ is antisymmetric and reflexive. So $(\leq) \vee (\geq) = \Delta$. i.e. $a_{\leq} \vee a_{\geq} = \#$. But for any $a \in \Omega X$, $a = \# * a$ and so $a = (a_{\leq} \vee a_{\geq}) * a$. Now in the last lemma ([5.1.3]) we saw that if $\leq$ is a closed preorder on a compact Hausdorff X then

$$a_{\leq} = \bigvee^{\uparrow} \{\wedge_i (\Downarrow^{op} a_i \& \Uparrow^{op} b_i) | \wedge_i (a_i \vee b_i) = 0\}$$

Thus

$$a_{\geq} = \bigvee^{\uparrow} \{\wedge_i (\Uparrow^{op} b_i \& \Downarrow^{op} a_i) | \wedge_i (a_i \vee b_i) = 0\}$$

Hence $a_{\leq} \vee a_{\geq}$ is a directed join of meets of elements of the form

$$(\Downarrow^{op} a \vee \Uparrow^{op} b) \& (\Uparrow^{op} d \vee \Downarrow^{op} e)$$

and so $a = [(a_{\leq} \vee a_{\geq}) * a]$ is a directed join of meets of elements of the form:

$$(\Downarrow^{op} a \vee \Uparrow^{op} b) \vee \Omega!(1 \leq \Uparrow^{op} d \vee \Downarrow^{op} e \vee a)$$

Since 1 certainly belongs to $\{a | \Uparrow^{op} a = a\} \cup \{a | \Downarrow^{op} a = a\}$ and
$\Omega!(1 \leq \Uparrow^{op} d \vee \Downarrow^{op} e \vee a) = \bigvee \{1 | 1 \leq \Uparrow^{op} d \vee \Downarrow^{op} e \vee a\}$ we can now easily see that the frame generated by this set is the whole of ΩX. $\Box$

5.2 Compactness result

There is a technical lemma which will be needed later on. It bears a similarity to the result $(1 \otimes \Uparrow^{op})(\#) = (\Downarrow^{op} \otimes 1)(\#)$ that has proved useful so far.

Lemma 5.2.1 *Say* $R \hookrightarrow X \times Y$ *is a closed relation on compact Hausdorff* X, Y. *If* $\psi_R : \Omega Y \to \Omega X$ *is the preframe homomorphism corresponding to* R *and* $\phi_R : \Omega X \to \Omega Y$ *is the preframe homomorphism corresponding to* τR *then if* $b \in \Omega Y$ *and* $a \in \Omega X$ *we have*

$$1 \leq \psi_R(b) \vee a \quad \Leftrightarrow \quad 1 \leq b \vee \phi_R(a)$$

Proof: If $a_R = \bigvee_j^{\uparrow} \wedge_i a_i^j \& b_i^j$ then the LHS of the implication is:

$$\begin{aligned} & 1 \leq (\bigvee_j^{\uparrow} \wedge_i [a_i^j \vee \Omega!(1 \leq b \vee b_i^j)]) \vee a \\ \Leftrightarrow \quad & 1 \leq \bigvee_j^{\uparrow} \wedge_i [a \vee a_i^j \vee \Omega!(1 \leq b \vee b_i^j)] \\ \Leftrightarrow \quad & (\exists j)(\forall i)[1 \leq (a \vee a_i^j) \vee \Omega!(1 \leq b \vee b_i^j)] \end{aligned}$$

where the last line is by compactness and the definition of meet.
But for any compact locale Z with $\alpha, \beta \in \Omega Z$ we must have

$$1 \leq \alpha \vee \Omega!(1 \leq \beta) \quad \Leftrightarrow \quad 1 \leq \beta \vee \Omega!(1 \leq \alpha)$$

since $\alpha \vee \Omega!(1 \leq \beta) = \bigvee^{\uparrow}(\{\alpha\} \cup \{1 | 1 \leq \beta\})$.
So we conclude that $1 \leq \psi_R b \vee a \quad \Leftrightarrow \quad (\exists j)(\forall i)[1 \leq (b \vee b_i^j) \vee \Omega!(1 \leq a \vee a_i^j)]$
But $1 \leq b \vee \phi_R(a)$ is just the statement:

$$1 \leq [\bigvee_j^{\uparrow} \wedge_i (b_i^j \vee \Omega!(1 \leq a \vee a_i^j))] \vee b$$

which as above (via compactness of X) translates to,

$$(\exists j)(\forall i)[1 \leq (b \vee b_i^j) \vee \Omega!(1 \leq a \vee a_i^j)] \qquad \Box$$

As a corollary note that if R is a closed relation on some compact Hausdorff locale X and $b, a \in \Omega X$ then

$$1 \leq \Downarrow^{op} b \vee a \quad \Leftrightarrow \quad 1 \leq b \vee \Uparrow^{op} a.$$

5.3 Order preserving locale maps

We now turn to the definition of morphism between ordered locales. We find again that it is appropriate to define something by analogy to our spatial intuition. A map $f : X \to Y$ where X, Y are two ordered spaces is a morphism of the category of ordered spaces if and only if it is continuous and preserves order. An ordered locale is a locale with a sublocale of the product of the locale with itself. So if $(X, R_X), (Y, R_Y)$ are two ordered locales then a locale map $f : X \to Y$ is a morphism of the category of ordered locales if and only if there exists a locale map $n : R_X \to R_Y$ such that

$$\begin{tikzcd} R_X \arrow[r, "n"] \arrow[d, hook] & R_Y \arrow[d, hook] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd}$$

commutes.
For closed R_X, R_Y it is easy to check that the above diagram can be defined and commutes if and only if

$$\Omega(f \times f)(a_{R_Y}) \leq a_{R_X}$$

Of course we are not going to investigate things at this level of generality. We are only interested the case when the locales are compact Hausdorff and the relations are closed partial orders. We shall call such ordered locales compact Hausdorff posets. The notation $(X, \leq_X)$ will be used to denote such posets. Say $f : X \to Y$ is a locale map and $(X, \leq_X), (Y, \leq_Y)$ are two compact Hausdorff posets. Then f is a map in the category of ordered locales if and only if

$$\Omega(f \times f)(a_{\leq_Y}) \leq a_{\leq_X} \qquad (*)$$

We now translate this condition further.
Assume $(*)$ holds. Then if we a are given $a \in \Omega Y$ (and $a_{\leq_Y} = \bigvee^{\uparrow}_j(\wedge_i(a_i^j \otimes b_i^j))$ then

$$\begin{aligned}
\Omega f \Downarrow^{op} a &= \Omega f(a_{\leq_Y} * (a \otimes 0)) \\
&= \bigvee_j^{\uparrow} \wedge_i(\Omega f a_i^j \vee \Omega!(1 \leq b_i^j \vee a))
\end{aligned}$$

But $\Omega!(1 \leq b_i^j \vee a) \leq \Omega!(1 \leq \Omega f b_i^j \vee \Omega f a)$ and so

$$\begin{aligned}
\Omega f \Downarrow^{op} a &\leq \bigvee_j^{\uparrow} \wedge_i(\Omega f a_i^j \vee \Omega!(1 \leq \Omega f b_i^j \vee \Omega f a)) \\
&= [(\Omega f \otimes \Omega f)(a_{\leq_Y})] * (\Omega f a \otimes 0) \\
&\leq a_{\leq_X} * (\Omega f a \otimes 0) \\
&= \Downarrow^{op} \Omega f a
\end{aligned}$$

Hence $\Omega f \Downarrow^{op} a \leq \Downarrow^{op} \Omega f a$ for all $a \in \Omega Y$ if we assume $(*)$.
For the converse assume $\Omega f \Downarrow^{op} a \leq \Downarrow^{op} \Omega f a \quad \forall a \in \Omega Y$, and recall that since $(Y, \leq_Y)$ is a compact Hausdorff poset we know (Lemma [5.1.3]) that

$$a_{\leq_Y} = \bigvee^{\uparrow}\{\wedge_i a_i \otimes b_i | \wedge_i (a_i \vee b_i) = 0, \Downarrow^{op} a_i = a_i, \Uparrow^{op} b_i = b_i\}.$$
$$\text{So} \quad \Omega(f \times f)a_{\leq_Y} = \bigvee^{\uparrow}\{\wedge_i \Omega f a_i \otimes \Omega f b_i | \wedge_i (a_i \vee b_i) = 0, \Downarrow^{op} a_i = a_i, \Uparrow^{op} b_i = b_i\}$$

But for any finite collection of a_i, b_is satisfying $\wedge_i(a_i \vee b_i) = 0$ and $\Downarrow^{op} a_i = a_i$, $\Uparrow^{op} b_i = b_i$ we have

$$\Omega f a_i = \Omega f \Downarrow^{op} a_i \leq \Downarrow^{op} \Omega f a_i \leq \Omega f a_i$$

by reflexivity of $\leq_X$ and assumption. Similarly $\Omega f b_i = \Downarrow^{op} \Omega f b_i$.
Clearly $\wedge_i(a_i \vee b_i) = 0 \;\Rightarrow\; \wedge_i(\Omega f a_i \vee \Omega f b_i) = 0$, and so

$$\begin{aligned}
\Omega(f \times f)(a_{\leq_Y}) &\leq \bigvee^{\uparrow}\{\wedge_i a_i \otimes b_i | \wedge_i (a_i \vee b_i) = 0, \Downarrow^{op} a_i = a_i, \Uparrow^{op} b_i = b_i\} \\
&= a_{\leq_X}
\end{aligned}$$

So we have translated the condition $(*)$ to

$$\Omega f \circ \Downarrow^{op} \leq \Downarrow^{op} \circ \Omega f$$

Notice, incidentally, that exactly the same proof shows us that $(*)$ is equivalent to

$$\Omega f \circ \Uparrow^{op} \leq \Uparrow^{op} \circ \Omega f$$

We can now define the category **KHausPos**: its objects are compact Hausdorff posets and its maps are order preserving locale maps.

5.4 Compact Regular Biframes

The compact regular biframes were introduced by Banaschewski, Brümmer and Hardie in [BBH83]. Spatially they are the compact regular T_0 bispaces and have been related to the stably locally compact locales ([BB88]). We shall investigate this relation extensively in the last chapter. For the moment we prove a new result: the compact regular biframes are dually equivalent to the compact Hausdorff posets.

The objects of **KR2Frm** (the category of compact regular biframes) are triples (L_0, L_1, L_2) such that L_0 is a compact frame and L_1, L_2 are two subframes of L_0 which generate the whole of L_0 and are required to satisfy two regularity-like conditions:
(i) $\forall a \in L_1 \quad a = \bigvee^{\uparrow}\{c | c \in L_1 \quad c \prec_1 a\}$ where
$c \prec_1 a \quad \Leftrightarrow \quad \exists d \in L_2 \quad c \wedge d = 0 \quad a \vee d = 1$
(ii) $\forall a \in L_2 \quad a = \bigvee^{\uparrow}\{c | c \in L_2 \quad c \prec_2 a\}$ where
$c \prec_2 a \quad \Leftrightarrow \exists d \in L_1 \quad c \wedge d = 0 \quad a \vee d = 1$
It follows, since L_1, L_2 generate the whole of L_0, that if (L_0, L_1, L_2) is a compact regular biframe then L_0 is the frame of opens of a compact regular locale. So $L_0 = \Omega X$ for some compact Hausdorff locale X.
If $(L_0, L_1, L_2), (L_0{}', L_1{}', L_2{}')$ are two objects of **KR2Frm** then morphisms are given by frame homomorphisms $l : L_0 \to L_0{}'$ which satisfy:

$$\begin{array}{l} l(a_1) \in L_1{}' \quad \forall a_1 \in L_1 \\ l(a_2) \in L_2{}' \quad \forall a_2 \in L_2 \end{array}$$

Theorem 5.4.1 KR2Frm $\cong$ KHausPos

Proof: Although the proof is quite straightforward it is not short.

We first construct a contravariant functor from **KR2Frm** to **KHausPos**. Let us assume we are given a compact regular biframe (L_0, L_1, L_2). We can define a couple of preframe endomorphisms on L_0: for $i = 1, 2$ set

$$\epsilon_i(a) = \bigvee^{\uparrow}\{c | c \in L_i \quad c \prec_i a\}$$

That ϵ_i preserves finite meets is straightforward. (Recall that L_1, L_2 are subframes of L_0, so certainly $\epsilon_i(1) = 1$ for $i = 1, 2$.) Compactness of L_0 shows that ϵ_1, ϵ_2 are preframe endomorphisms. The conditions (i) and (ii) in the definition of compact regular biframe given above tell us that the images of ϵ_1, ϵ_2 are exactly L_1, L_2 respectively. Notice $b \in L_i$ if and only if $\epsilon_i(b) = b$. It follows that ϵ_i is idempotent.

Bearing in mind the correspondence between preframe endomorphisms and closed relations, as worked out in Theorem [4.3.1], we define our compact Hausdorff poset $(X, \leq)$ from (L_0, L_1, L_2) as follows:

$$\begin{array}{rcl} \Omega X & = & L_0 \\ a_{\leq} & = & (\epsilon_1 \otimes 1)(\#) \end{array}$$

Reflexivity and transitivity of $\leq$ follows immediately since $\epsilon_1(b) \leq b \quad \forall b \in L_0$ and ϵ_1 is idempotent.
In fact
$(\alpha) \qquad a_{\leq} = (1 \otimes \epsilon_2)(\#)$
$(\beta) \qquad a_{\leq} \vee a_{\geq} \geq \#$
i.e. (α): we haven't 'lost' any information by picking ϵ_1 over ϵ_2 in our definition of $(X, \leq)$ and (β): $\leq$ is antisymmetric and therefore is a partial order.
Proof of (α) We want,

$$(\epsilon_1 \otimes 1)(\#) = (1 \otimes \epsilon_2)(\#).$$

We prove that

$$(\epsilon_1 \otimes 1)(\#) \leq (1 \otimes \epsilon_2)(\#)$$

and appeal to the symmetry between ϵ_1, ϵ_2 for the full result.
Now $\neg(1 \otimes \epsilon_2)(\#)$ is a closed sublocale of $X \times X$ and so gives rise to a unique preframe endomorphism of ΩX by:

$$a \longmapsto (1 \otimes \epsilon_2)(\#) * a$$

It follows that if we can prove

$$\epsilon_1(a) \leq (1 \otimes \epsilon_2)(\#) * a$$

for every $a \in \Omega X$ then we can conclude

$$(\epsilon_1 \otimes 1)(\#) \leq (1 \otimes \epsilon_2)(\#)$$

since $\neg(\epsilon_1 \otimes 1)(\#)$ is the closed sublocale corresponding to the preframe endomorphism ϵ_1.
But

$$(1 \otimes \epsilon_2)(\#) * a \quad = \quad \bigvee\nolimits^{\uparrow}\{\wedge_i[a_i \vee \Omega!(1 \leq (\epsilon_2(b_i) \vee a))] | \wedge_i (a_i \vee b_i) = 0\}$$

and

$$\epsilon_1(a) = \bigvee\{c | c \prec_1 a \quad c \in L_1\}.$$

Now if $c \prec_1 a$ then $\exists d \in L_2$ such that $c \wedge d = 0$ and $d \vee a = 1$. So $\epsilon_2(d) = d$. If we take $(a_1, b_1) = (c, 0)$ and $(a_2, b_2) = (0, d)$ and $I = \{1, 2\}$ then $\wedge_{i \in I}(a_i \vee b_i) = 0$. But for these a_is and b_is we see

$$\begin{aligned} & \wedge_i[a_i \vee \Omega!(1 \leq (\epsilon_2(b_i) \vee a))] \\ = \; & [c \vee \Omega!(1 \leq \epsilon_2(0) \vee a)] \wedge \Omega![1 \leq (\epsilon_2(d) \vee a)] \\ \geq \; & c \quad \text{since } \epsilon_2(d) = d \text{ and } d \vee a = 1. \end{aligned}$$

Hence

$$\epsilon_1(a) \leq (1 \otimes \epsilon_2)(\#) * a$$

and so we may conclude that ϵ_2 is the preframe homomorphism corresponding to upper closure as outlined above. $\square$
Proof of (β) Recall that L_1, L_2 are subframes of L_0 which generate the whole of L_0 (by the definition of compact regular biframe). We have observed that:

$$\begin{aligned} a \in L_1 \quad & \Leftrightarrow \quad \epsilon_1(a) = a \\ a \in L_2 \quad & \Leftrightarrow \quad \epsilon_2(a) = a \end{aligned}$$

(This is really just a restatement of the regularity-like conditions (i), (ii).) So the fact that L_1, L_2 generate L_0 lets us write:

$$a = \bigvee\nolimits^{\uparrow}\{b \wedge c | \epsilon_1(b) = b, \epsilon_2(c) = c, b \wedge c \leq a\}$$

for any $a \in L_0$.
But $a_{\leq} = (\epsilon_1 \otimes 1)(\#)$ (definition), and $a_{\leq} = (1 \otimes \epsilon_2)(\#)$ (α). And so by applying the twist isomorphism on $X \times X$ to the second of these we see that: $a_{\geq} = (\epsilon_2 \otimes 1)(\#)$.
Thus

$$\epsilon_1 = \Downarrow^{op} \text{ and } \epsilon_2 = \Uparrow^{op}.$$

So $\epsilon_1(b) = b \Leftrightarrow \Downarrow^{op} b = b \Leftrightarrow a_{\leq} * b = b$
and $\epsilon_2(c) = c \Leftrightarrow \Uparrow^{op} c = c \Leftrightarrow a_{\geq} * c = c.$
We want to prove $(a_{\leq} \vee a_{\geq}) \geq \#$. We know from our equivalence between closed sublocales on $X \times X$ and preframe endomorphisms of ΩX that it is sufficient to prove

$$\begin{array}{ll} \# * a \leq (a_{\leq} \vee a_{\geq}) * a & \forall a \in L_0 \\ \text{i.e.} \quad a \leq (a_{\leq} \vee a_{\geq}) * a & \forall a \in L_0 \end{array}$$

Now say b satisfies $\epsilon_1(b) = b$. Then

$$b = (a_{\leq} * b) \leq (a_{\leq} \vee a_{\geq}) * b$$

and if c satisfies $\epsilon_2(c) = c$ then

$$c = (a_{\geq} * c) \leq (a_{\leq} \vee a_{\geq}) * c.$$

Hence for any such b, c with $b \wedge c \leq a$ we have

$$\begin{array}{rcll} b \wedge c & \leq & ((a_{\leq} \vee a_{\geq}) * b) \wedge ((a_{\leq} \vee a_{\geq}) * c) & \\ & = & (a_{\leq} \vee a_{\geq}) * (b \wedge c) & \text{(because $*$ is a bipreframehomomorphism)} \\ & \leq & (a_{\leq} \vee a_{\geq}) * a. & \end{array}$$

But $a = \bigvee\{b \wedge c | \epsilon_1(b) = b, \quad \epsilon_2(c) = c, \quad b \wedge c \leq a\}$ since L_0 is generated by L_1, L_2 and so $a \leq (a_{\leq} \vee a_{\geq}) * a$ as required. $\Box$

Recall that

$$f : (X, \leq_X) \to (Y, \leq_Y)$$

is a morphism of **KHausPos** iff there exists a locale map $n : \leq_X \longrightarrow \leq_Y$ such that

$$\begin{tikzcd} \leq_X \arrow[r, "n"] \arrow[d, hook] & \leq_Y \arrow[d, hook] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd}$$

commutes. We saw in the last section that this condition is equivalent to:

$$\Omega f \circ {}^{Y}\!\Downarrow^{op} \leq {}^{X}\!\Downarrow^{op} \circ \Omega f$$

If l is a compact regular biframe map from (L_0, L_1, L_2) to $(L_0{}', L_1{}'L_2{}')$ certainly there exists

$$f : X \to Y$$

a locale map where $\Omega X = L_0{}', \quad \Omega Y = L_0$ and $\Omega f = l$. The order on X (as constructed above) corresponds to the preframe homomorphism $\epsilon_1^X : \Omega X \to \Omega X$. But

$$l\epsilon_1^Y(a) \leq \epsilon_1^X l(a)$$

since

$$c \prec_1 a \quad \Rightarrow \quad l(c) \prec_1 l(a)$$

as $l(c) \in L_1{}'$ if $c \in L_1$ and $l(d) \in L_2{}'$ if $d \in L_2$. So f is a map in the category **KHausPos** and we have defined a contravariant functor from **KR2Frm** to **KHausPos**.

Now on the other hand say we are given a Hausdorff poset $(X, \leq)$.
We know that

$$a_{\leq} = (1 \otimes \Uparrow^{op})(\#)$$
$$a_{\leq} = (\Downarrow^{op} \otimes 1)(\#)$$

where $\Downarrow^{op}, \Uparrow^{op}$ are the preframe endomorphism whose actions are the lower/upper closure of closed sublocales. Thus we have preframe endomorphisms of ΩX. Since $\leq$ is reflexive we know that $\Uparrow^{op} a \leq a \quad \forall a \in \Omega X$ and $\Downarrow^{op} a \leq a \quad \forall a \in \Omega X$, and so the sets

$$\begin{aligned} \{a \mid \Downarrow^{op} a = a\} &\subseteq \Omega X \\ \{a \mid \Uparrow^{op} a = a\} &\subseteq \Omega X \end{aligned}$$

are not only subpreframes but are subframes of the compact frame ΩX. Do they generate the whole of ΩX ? The answer is yes; we saw exactly this fact in the proof of Theorem [5.1.3].

So if we set $L_0 = \Omega X$ and $L_1 = \{a \mid \Downarrow^{op} a = a\}, L_2 = \{a \mid \Uparrow^{op} a = a\}$ then L_0 (is compact and) is generated by these two subframes.
We are now in a position to check the regularity-like condition (i) for (L_0, L_1, L_2) ((ii) will clearly follow by symmetry from this).

(i) states that if $a \in L_1 \equiv \{a \mid \Downarrow^{op} a = a\}$ then

$$a = \bigvee\{c \mid c \prec_1 a \quad \Downarrow^{op} c = c\}$$

where $c \prec_1 a \quad \Leftrightarrow \quad \exists d$ with $\Uparrow^{op} d = d, \quad d \wedge c = 0$ and $a \vee d = 1$. But we know by regularity of X that $\Downarrow^{op} a = a = \bigvee^{\uparrow}\{b \mid b \lhd a\}$ and by taking $\Downarrow^{op}$ of both sides we see $a = \bigvee^{\uparrow}\{\Downarrow^{op} b \mid b \lhd a\}$, and so to check (i) all we need do is check

$$b \lhd a \quad \Rightarrow \quad \Downarrow^{op} b \prec_1 a$$

Now if $b \lhd a$ then there exists d with $1 \leq a \vee d$ and $b \wedge d = 0$. But $a = \Downarrow^{op} a$ and so $\Downarrow^{op} a \vee d = 1$ letting us conclude $a \vee \Uparrow^{op} d = 1$ by the compactness result, Lemma [5.2.1].
Also $\Uparrow^{op} d \leq d$ and $\Downarrow^{op} b \leq b$ (reflexivity of $\leq$): thus $\Downarrow^{op} b \wedge \Uparrow^{op} d = 0$, and since $\Uparrow^{op} d \in L_2$ we may conclude $\Downarrow^{op} b \prec_1 a$.

Thus $(\Omega X, \{a \mid \Downarrow^{op} a = a\}, \{a \mid \Uparrow^{op} a = a\})$ is a compact regular biframe for any compact Hausdorff poset $(X, \leq)$.
As for morphisms, say $f : (X, \leq) \to (Y, \leq)$ is a map of **KHausPos** then as well as the condition

$$\Omega f \Downarrow^{op} \leq \Downarrow^{op} \Omega f$$

recall that we noted in the last section that the symmetric condition

$$\Omega f \Uparrow^{op} \leq \Uparrow^{op} \Omega f$$

is implied by (and implies) the assumption 'f is a **KHausPos** map'. Hence

$$\Omega f : (\Omega Y, \{b| \Downarrow^{op} b = b\}, \{b| \Uparrow^{op} b = b\}) \longrightarrow (\Omega X, \{a| \Downarrow^{op} a = a\}, \{a| \Uparrow^{op} a = a\})$$

is a map of **KR2Frm** and so we have a contravariant functor ($\mathcal{C}$) from compact Hausdorff posets to compact regular biframes.

Now say $(L_0, L_1, L_2) \equiv \mathcal{C}(X, \leq)$. Near the beginning of this proof we defined for any compact regular biframe a preframe endomorphism ϵ_i by

$$b \mapsto \bigvee\nolimits^{\uparrow}\{a|a \in L_i \quad a \prec_i b\}$$

I claim that since $L_1 = \{a| \Downarrow^{op} a = a\}$ then

$$\bigvee\nolimits^{\uparrow}\{a|a \in L_1 \quad a \prec_1 b\} = \Downarrow^{op} b$$

Certainly $\epsilon_1(b) \leq \Downarrow^{op} b$ for if $a \prec_1 b, \quad a \in L_1$ then $a \leq b$ and so

$$a = \Downarrow^{op} a \leq \Downarrow^{op} b.$$

In the other direction: $\forall b \in L_0 = \Omega X$

$$\Downarrow^{op} b = \bigvee\nolimits^{\uparrow}\{a|a \lhd \Downarrow^{op} b\}$$

and so by applying $\Downarrow^{op}$ to both sides we get

$$\Downarrow^{op} b = \bigvee\nolimits^{\uparrow}\{\Downarrow^{op} a|a \lhd \Downarrow^{op} b\}$$

and we know from above $a \lhd \Downarrow^{op} b$ implies $\Downarrow^{op} a \prec_1 \Downarrow^{op} b$. Thus $\Downarrow^{op} a \prec_1 b$ since $\Downarrow^{op} b \leq b$.

Hence $\epsilon_1 = \Downarrow^{op}$, and so mapping $(\Omega X, \{a| \Downarrow^{op} a = a\}, \{a| \Uparrow^{op} a = a\})$ to $(\bar{X}, \leq_{\bar{X}})$ where $\Omega\bar{X} = \Omega X$ and $\leq_{\bar{X}}$ is the closed sublocale corresponding to the preframe endomorphism ϵ_1 returns us to $(X, \leq)$.

Finally to check that **KR2Frm** and **KHausPos** are dually equivalent we need to check, given a compact regular biframe (L_0, L_1, L_2) that

$$(L_0, L_1, L_2) = (L_0, \{a| \Downarrow^{op} a = a\}, \{a| \Uparrow^{op} a = a\})$$

where $\Downarrow^{op}$ comes from the closed relation $\leq$ defined by

$$a_{\leq} = (\epsilon_1 \otimes 1)(\#).$$

Thus $\Downarrow^{op} = \epsilon_1$ and so $\{a| \Downarrow^{op} a = a\} = L_1$ as required. (Recall that $b \in L_1$ iff $\epsilon_1(b) = b$.)
But we saw

$$a_{\geq} = (\epsilon_2 \otimes 1)(\#) \qquad - (\alpha)$$

and so $\Uparrow^{op} = \epsilon_2$ and, just as with ϵ_1, the ϵ_2 fixed elements of L_0 are precisely the elements of L_2. $\square$

The classical version of this result was proved in Priestley's paper 'Ordered Topological Spaces and the Representation of Distributive Lattices' [Pri72]. Proposition 10 of that paper is (effectively): 'The compact order-Hausdorff topological spaces are equivalent to the compact regular T_0-bispaces'. It is shown in [BBH83] how to prove that the compact regular biframes are equivalent to the compact regular T_0-bispaces assuming the prime ideal theorem, and in fact it is clear that the proof can be repeated assuming the constructive prime ideal theorem. So in order to recover the classical result we need to make sure that our compact Hausdorff posets are classically equivalent to the compact order-Hausdorff topological spaces. We find that we only need to assume the constructive prime ideal theorem (CPIT). We've shown that compact Hausdorff locales are, given this assumption, spatial and so it is easy to check that they are then equivalent to the compact Hausdorff spaces (where in this constructive context it is easiest to define the compact Hausdorff spaces, **KHausSp**, as those topological spaces whose frame of opens are compact regular). To avoid the difficulties that come from constructively discussing the closed subsets of a topological space (such as the fact that arbitrary intersections of closeds are not closed via the usual proof since we cannot assume that arbitrary intersections distribute over finite unions), we use as motivation the classical result that the subspace of a compact Hausdorff space is closed if and only if it is compact. Hence we define the order-Hausdorff topological spaces to be those pairs $(X, \leq)$ such that X is a compact Hausdorff space and $\leq \subseteq X \times X$ is a compact partial order. Notice that if **KHausSp** $\cong$ **KHausLoc** then monomorphisms are going to correspond to injections of points i.e. to subspaces. In other words sublocales in **KHausLoc** correspond to compact subspaces in **KHausSp** assuming CPIT. But does the notion of relational composition of compact sets of points correspond to relational composition as we've defined it via a preframe homomorphism? To see that it does we need to check that pullbacks and image factorisations of compact Hausdorff topological spaces are (on points) constructed as in **Set**. We need

Lemma 5.4.1 *Assuming CPIT, the forgetful functor from* **KHausSp** *to* **Set** *creates pullbacks.*

Proof: If

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a pullback diagram in **KHausLoc** then $pt(X \times_Z Y)$ is isomorphic as a set to the set of pairs of points $p_1 : 1 \to X$, $p_2 : 1 \to Y$ such that $fp_1 = fp_2$. Hence

$$\begin{tikzcd} pt(X \times_Z Y) \arrow[r] \arrow[d] & pt(Y) \arrow[d, "pt(g)"] \\ pt(X) \arrow[r, "pt(f)"] & pt(Z) \end{tikzcd}$$

is a pullback diagram in **Set**. The result follows since we are assuming CPIT and so **KHausSp** $\cong$ **KHausLoc**. $\Box$

The forgetful functor also creates image factorisations. The proof of this is completely straightforward since if $f : X \to Y$ is a continuous map between compact Hausdorff spaces then $\{f(x)|x \in X\}$ can be endowed with a topology (the subspace topology from Y) which makes it into a compact Hausdorff topological space.

Thus if we recall the definition of relational composition in terms of pullback and image factorization (as presented at the beginning of Chapter 4) then provided we have **KHausSp** $\cong$ **KHausLoc**, we know that set theoretic relational composition of compact subspaces is given by relational composition of closed sublocales. Hence, assuming CPIT, the order-Hausdorff topological spaces are equivalent to the compact Hausdorff posets.

Chapter 6

Localic Priestley Duality

6.1 Introduction

Priestley duality describes how the category of coherent spaces is equivalent to the category of ordered Stone spaces. We define ordered Stone locales (which classically are just the ordered Stone spaces) and present a new theorem that shows that the category of ordered Stone locales is equivalent to the category of coherent locales. Preframe techniques are used to prove this result.

6.2 Ordered Stone locales

A Stone space is a compact Hausdorff topological space which is also coherent. If we assume CPIT then we know that the category of Stone spaces is equivalent to the category of Stone locales i.e. compact Hausdorff locales which are also coherent. The frames of opens of such locales were seen (in Theorem [1.7.5]) to be exactly the ideal completions of Boolean algebras. From this we conclude that the category of Stone spaces is dual to the category of Boolean algebras. This is Stone's representation theorem [Sto 36],[Sto37].
The equivalence between Stone locales and Boolean algebras is trivial, it is when showing that Stone locales are equivalent to Stone spaces that we invoke a choice axiom.
Working in a classical context Priestley ([Pri70]) introduced ordered Stone spaces (also known as Priestley spaces). These are pairs $(X, \leq)$ where X is a compact space and $\leq$ is a partial order on X satisfying the requirement that for every $x, y \in X$ with $x \not\leq y$ there is a clopen upper set U containing x and not containing y. From this data it is a classical exercise to prove that an ordered Stone space is a Stone space. It is immediate that $\leq$ must be a closed subspace of $X \times X$, in fact the condition on $\leq$ above can be rewritten as the equation

$$\not\leq = \bigcup\{U \otimes U^c | U \text{ clopen } \uparrow U = U\}$$

where $\uparrow U$ is the upper closure of U with respect to the order $\leq$. Notice that we could use this condition to prove that $\leq$ is transitive. Also note that this condition can be written

$$\not\leq = \bigcup\{U \otimes U^c | U \text{ clopen } \downarrow U^c = U^c\}$$

since classically a subset is upper closed iff its complement is lower closed. Finally since we know that X is compact Hausdorff we may classically conclude that U is clopen if and only if it is a compact open subset of X and so, since X is coherent, $U \in \Omega X \cong Idl(K\Omega X)$ is in $K\Omega X$ if and only if it is clopen.

Given these classical observations it should be clear that the following is a reasonable definition of an *ordered Stone locale*
Definition: An ordered Stone locale is a pair $(X, \leq)$ where X is a Stone locale (i.e. $\Omega X \cong IdlB_X$ for some Boolean algebra B_X) and $\leq \rightarrowtail X \times X$ is a closed partial order satisfying

$$a_{\leq} = \bigvee\{a \otimes \neg a | a \in B_X, \Downarrow^{op} a = a\} \ (!)$$

where $\leq = \neg a_{\leq} \rightarrowtail X \times X$ and $\Downarrow^{op}: \Omega X \to \Omega X$ is the preframe endomorphism of ΩX corresponding to the action of taking the lower closure of closed sublocales.
Notation warning: We have a notation clash between Boolean algebra negation ($\neg$) and 'closed sublocale corresponding to the open a' ($\neg a \hookrightarrow X$). However context will eliminate any ambiguity.

The equation (!) is a SUP-lattice equation. It has a preframe equivalent which will be useful:

$$a_{\leq} = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i)| \wedge_{i \in I} (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in B_X \quad \Downarrow^{op} a_i = a_i, \Downarrow^{op} b_i = b_i, I \text{ finite } \}$$

Proving these two expressions to be the same requires the same manipulation (demonstrated in Lemma [2.7.1]) that proves that the closure of the diagonal of a locale can be expressed both as

$$\neg \bigvee\{a \otimes b | a \wedge b = 0\}$$

and

$$\neg \bigvee^{\uparrow}\{\wedge_i a_i \otimes b_i | \wedge_i (a_i \vee b_i) = 0\}$$

When it comes to the manipulations that follow we will find that the prefame version of the equation (!) will be the one to apply.
Our first manipulation comes with a proof that if we are given a pair (X, R) such that X is a Stone locale and R is a closed relation which satisfies (!) then R is transitive. To see this proof note that if $a \in \Omega X$ then $\Downarrow^{op} a$ is given by the formula

$$\bigvee^{\uparrow}\{\wedge_i(a_i \vee \Omega!(1 \leq \neg b_i \vee a))\}$$

where the directed join is over sets $\{a_i, b_i | i \in I\}$ such that I is finite, a_is and b_is are in the Boolean algebra of compact opens of X and $\Downarrow^{op} a_i = a_i$, $\Downarrow^{op} b_i = b_i$, $\wedge_i(a_i \vee \neg b_i) = 0$. So $\Downarrow^{op}\Downarrow^{op} a$ is equal to

$$\begin{aligned}\Downarrow^{op} \bigvee^{\uparrow} \wedge_i[\bigvee^{\uparrow}(\{a_i\} \cup \{1 | 1 \leq \neg b_i \vee a\})] &= \bigvee^{\uparrow} \wedge_i \bigvee^{\uparrow}(\{\Downarrow^{op} a_i\} \cup \{\Downarrow^{op} 1 | 1 \leq \neg b_i \vee a\}) \\ &= \bigvee^{\uparrow} \wedge_i \bigvee^{\uparrow}(\{a_i\} \cup \{1 | 1 \leq \neg b_i \vee a\}) = \Downarrow^{op} a.\end{aligned}$$

Idempotency of $\Downarrow^{op}$ is equivalent to idempotency of R with respect to relational composition. Idempotency of R is enough to prove that R is transitive. Notice that the condition (!) also implies that R is reflexive.

The morphisms between ordered Stone spaces are taken to be the continuous order preserving functions and so the category **OStoneSp** is defined. We take **OStoneLoc** to be the full subcategory of **KHausPos** whose objects are the ordered Stone locales. Recall from Section 5.3 that it follows that

$$f : (X \leq_X) \longrightarrow (Y, \leq_Y)$$

is a map of **OStoneLoc** if and only if $f : X \to Y$ is a locale map and $\forall a \in \Omega Y$

$$\Omega f \circ \Downarrow^{op} (a) \leq \Downarrow^{op} (a) \circ \Omega f$$

6.3 Priestley's Duality

Priestley's initial result was proved in [Pri70] (though see [Pri94] for some more recent thinking about the duality). It consisted of the statement $\mathbf{DLat}^{op} \cong \mathbf{OStoneSp}$; hence the term 'duality'. However we take the equivalence $\mathbf{DLat}^{op} \cong \mathbf{CohSp}$ (i.e. generalization of Stone representation) for granted since we are familiar with this result as essentially the assertion that coherent locales are spatial. ('Essentially' since we need to factor in the complication that the maps between coherent spaces are those whose inverse images preserve compact opens i.e. localically the semi-proper maps.) We view Priestley duality as the equivalence $\mathbf{CohSp} \cong \mathbf{OStoneSp}$. So the reader is warned that the word 'duality' is not entirely appropriate. This view of the duality is also taken in II 4 of [Joh82]. There the functor:

$$\begin{array}{rcl} \mathcal{B} : \mathbf{CohSp} & \longrightarrow & \mathbf{OStoneSp} \\ (X, \Omega) & \longmapsto & (X, \text{`patch'}, \leq) \end{array}$$

is defined. $\leq$ is the specialization order on (X, Ω) and a base for the patch topology is given by

$$\{U \cap V^c | U, V \text{ compact open}\}$$

In the other direction we have

$$\begin{array}{rcl} \mathcal{C} : \mathbf{OStoneSp} & \longrightarrow & \mathbf{CohSp} \\ (X, \Omega, \leq) & \longmapsto & (X, \{U | U \in \Omega, \uparrow U = U\}) \end{array}$$

Lemma 6.3.1 *Classically,* $\{U | U \in \Omega, \uparrow U = U\} = Idl\{U | U \in K\Omega, \uparrow U = U\}$*. i.e.* $\mathcal{C}(X, \Omega, \leq)$ *is coherent.* $\square$

Priestley proved in [Pri70] that, provided we are free to use the prime ideal theorem (PIT), these functors define an equivalence. We now use the remarks in the notes to Section II 4.9 of Stone Spaces [Joh82] to see how an assumption that $\mathcal{BC}$ defines an equivalence allows us to conclude the PIT:
Let us assume that $\mathcal{B}, \mathcal{C}$ define an equivalence. We see straight away that if a coherent space is T_1 (i.e. if the specialization order $\leq$ is equality) then it is Stone. But T_1ness can equivalently be defined as saying that all points are closed. For any distributive lattice A the points of the associated coherent space are the prime ideals and the closed points are the maximal ideals. Hence the statement of T_1ness is equivalent to the statement that the maximal and prime ideals coincide. So assuming $\mathcal{B}, \mathcal{C}$ define an equivalence we know that a coherent space is T_1 if and only if it is Stone. Hence:

Lemma 6.3.2 (Nac49) *A distributive lattice is Boolean if and only if all its prime ideals are maximal.* $\square$

It is not immediately obvious that this lemma implies PIT. It certainly proves that any non-Boolean distributive lattice has a prime ideal. But any non-trivial Boolean can be embedded into a non-trivial non-Boolean distributive lattice and so we have PIT. To see how to construct such an embedding consult Exercise I 4.8 of Stone Spaces ([Joh82]).

Of course it is unfortunate that the above proof relies on the excluded middle. The reason why we repeat this characterization of PIT is to make it clear that we cannot hope to prove Priestley's duality without some choice axioms. i.e. we *have* to move to something like locales if we want to have a constructive theory of spaces that admits a Priestley duality.

6.4 Localic Version

We define an equivalence of categories via the functors $\mathcal{B}, \mathcal{C}$:

$$\mathbf{CohLoc} \underset{\mathcal{C}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{OStoneLoc}$$

The idea behind the construction of $\mathcal{B}$ comes from the following classical reasoning: if $x \not\leq y$ where x, y are points of a coherent space X and $\leq$ is the specialization order then there exists a compact open U such that $x \in U$ and $y \notin U$. Thus $(x, y) \in U \otimes U^c$ and, as always, $(U \otimes U^c) \cap \Delta = \phi$. Now when one is defining the functors of the original Priestley duality we take a coherent space X and give it a new *patch* topology. A base for the patch topology is given by

$$\{U \cap V^c | U, V \text{ compact open}\}$$

and so we see that the specialization order, $\leq$, is closed as a subset of $X \times X$ when X is given the patch topology. Thus there is evidence to suggest that we can find a *closed* sublocale of the locale obtained when we move from a coherent locale to its 'patch topology' locale. This closed sublocale will come from (via pullback it turns out) the specialization order on the original coherent locale.
We stay with our spatial intuitions for one more classical lemma:

Lemma 6.4.1 *The set of compact opens of the patch topology on a coherent space X is the free Boolean algebra on the distributive lattice of compact opens of X.*

Proof: Certainly if U is a compact open of X it is a compact open of the patch topology.
If W is in the patch topology then $W = \bigcup_{i \in I} U_i \cap V_i^c$ for some indexing set I. But if W is compact in the patch topology then I can be taken to be finite. The set

$$\beta \equiv \{\bigcup_{i \in I} U_i \cap V_i^c | U_i, V_i \text{ compact open, } I \text{ finite}\} \subseteq PX$$

is a Boolean algebra. The complement of

$$\bigcup_{i \in I} U_i \cap V_i^c$$

is given by the subset

$$\bigcup[(\cap_{i \in J_1} U_i^c) \cap (\cap_{i \in J_2} V_i)]$$

where the union if taken over all pairs $J_1, J_2 \subseteq I$ such that J_1, J_2 are finite and $I \subseteq J_1 \cup J_2$. Clearly any element of β is compact open in the patch topology. $\Box$

Thus the definition of this 'patch topology' locale, (which will be the definition of the localic part of $\mathcal{B}$) is clear: given a coherent locale X we know $\Omega X = Idl(K\Omega X)$ for some distributive lattice $K\Omega X$. Define $\mathcal{B}X$ by $\Omega\mathcal{B}X = Idl(B_X)$ where B_X is the free Boolean algebra on $K\Omega X$.

The distributive lattice injection $K\Omega X \rightarrowtail B_X$ gives rise to a frame homomorphism from $Idl(K\Omega X)$ to $Idl(B_X)$ and hence to a locale map $\mathcal{B}X \to X$ which we shall call l_X. l_X is a surjection. In fact

Lemma 6.4.2 *l_X is monic.*

Proof: Say

$$\begin{tikzcd} Y \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \mathcal{B}X \arrow[r, "l_X"] & X \end{tikzcd}$$

is a diagram in **Loc** such that $l_X f_1 = l_X f_2$. Then for all $I \in \mathcal{B}X$

$$I = \bigvee^{\uparrow}\{\downarrow b | b \in I\}$$

since I is an ideal of B_X. So to prove $f_1 = f_2$ it is sufficient to prove

$$\Omega f_1(\downarrow b) = \Omega f_2(\downarrow b) \qquad \forall b \in B_X$$

But for all $b \in B_X$

$$b = \wedge_{i \in I}(\Omega l_X a_i \vee \neg \Omega l_X b_i)$$

for some finite I with $a_i, b_i \in K\Omega X$. And so the result follows since any frame homomorphism clearly preserves complements. □

One way to find a sublocale of $\mathcal{B}X \times \mathcal{B}X$ is to look at the pullback of the specialization order on $X \times X$ (viewed as a sublocale) along the map $l_X \times l_X$. i.e. look at the pullback diagram

$$\begin{tikzcd} \leq_{\mathcal{B}X} \arrow[r] \arrow[d] & \sqsubseteq \arrow[d] \\ \mathcal{B}X \times \mathcal{B}X \arrow[r, "l_X \times l_X"] & X \times X \end{tikzcd}$$

where $\Omega(\sqsubseteq) \equiv Fr < \Omega X \otimes \Omega X$ qua frame $| \quad a \otimes 0 \leq 0 \otimes a \quad \forall a \in \Omega X >$ (see Lemma [2.7.2]) and hope that $\leq_{\mathcal{B}X}$ is closed.

Lemma 6.4.3 *Given the data above*

$$\begin{tikzcd} \neg I \arrow[r, "l"] \arrow[d] & \sqsubseteq \arrow[d, "q"] \\ \mathcal{B}X \times \mathcal{B}X \arrow[r, "l_X \times l_X"] & X \times X \end{tikzcd}$$

is a pullback diagram where $I = \bigvee\{a \otimes \neg a | a \in K\Omega X\}$. *(We view* $K\Omega X \subseteq B_X$.*)*

The reason for the choice of I should be apparent from the spatial reasoning presented above.

Proof: We can translate I to a preframe equivalent:

$$I = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0, \quad a_i, b_i \in K\Omega X\}$$

Use the method of Lemma [2.7.1] to see this.

Define

$$\begin{aligned} \Omega l : \Omega(\sqsubseteq) &\longrightarrow \uparrow I \\ a \otimes b &\longmapsto I \vee (\Omega l_X a \otimes \Omega l_X b) \end{aligned}$$

This is seen to satisfy the 'qua frame' part of the definition of $\Omega(\sqsubseteq)$. To conclude that Ωl is well defined we need:

$$I \vee (\Omega l_X a \otimes 0) \leq I \vee (0 \otimes \Omega l_X a)$$

for all $a \in K\Omega X$. Notice that for any $a \in K\Omega X$ since $(a \vee 0) \wedge (0 \vee \neg a) = 0$ we have that $I = I \vee [(a \otimes 0) \wedge (0 \otimes \neg a)]$. But

$$\begin{array}{rcl} I \vee (a \otimes 0) & = & I \vee [(a \otimes 0) \wedge (0 \otimes 1)] \\ & = & I \vee [(a \otimes 0) \wedge (0 \otimes (\neg a \vee a))] \\ & = & I \vee [(a \otimes 0) \wedge [(0 \otimes \neg a) \vee (0 \otimes a)]] \\ & = & I \vee [(a \otimes 0) \wedge (0 \otimes \neg a)] \vee [(a \otimes 0) \wedge (0 \otimes a)] \\ & = & I \vee [(a \otimes 0) \wedge (0 \otimes a)] \\ & \leq & I \vee (0 \otimes a) \end{array}$$

Hence l is well defined, and the diagram in the statement of the lemma clearly commutes. Now say we are given Q, m, t such that

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "m"'] & \sqsubseteq \arrow[d, tail, "q"] \\ \mathcal{B}X \times \mathcal{B}X \arrow[r, "l_X \times l_X"] & X \times X \end{tikzcd}$$

commutes. Then the function

$$\begin{array}{rcl} \Omega z : \uparrow I & \longrightarrow & \Omega Q \\ J & \longmapsto & \Omega(m)J \end{array}$$

will (i) be well defined, (ii) make the appropriate triangles commutes and (iii) be a frame homomorphism, provided we can check that $\Omega(m)I = 0$.
But $\Omega m(I) = \bigvee^{\uparrow}\{\Omega m \wedge_i (a_i \otimes \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\}$ and so it is sufficient to prove

$$\Omega m \wedge_i (a_i \otimes \neg b_i) = 0$$

whenever $\wedge_i(a_i \vee \neg b_i) = 0$ for $a_i, b_i \in K\Omega X$. With such conditions we see that $(\Omega l_X \otimes \Omega l_X)(a_i \otimes 0) = a_i \otimes 0$, and so

$$\begin{array}{rcl} \Omega m(\wedge_i(a_i \otimes \neg b_i)) & = & \wedge_i \Omega m((a_i \otimes 0) \vee (0 \otimes \neg b_i)) \\ & = & \wedge_i[\Omega m((\Omega l_X \otimes \Omega l_X)(a_i \otimes 0)) \vee \Omega m(0 \otimes \neg b_i)] \\ & = & \wedge_i[\Omega t \Omega q(a_i \otimes 0) \vee \Omega m(0 \otimes \neg b_i)] \\ & \leq & \wedge_i[\Omega t \Omega q(0 \otimes a_i) \vee \Omega m(0 \otimes \neg b_i)] \\ & = & \wedge_i[\Omega m(0 \otimes a_i) \vee \Omega m(0 \otimes \neg b_i)] \\ & = & \Omega m[\wedge_i(0 \otimes (a_i \vee \neg b_i))] \\ & = & \Omega m(0 \otimes \wedge_i (a_i \vee \neg b_i)) \\ & = & \Omega m(0 \otimes 0) = 0. \qquad \square \end{array}$$

Now $I \leq \#$ so $\leq_{\mathcal{B}X}$ is certainly reflexive. It is shown in Lemma [2.7.3] that the specialization order is antisymmetric ($\sqsubseteq \wedge \sqsupseteq = \Delta$) and so $\leq_{\mathcal{B}X}$ will be antisymmetric since (i) the diagonal is preserved by pullback along a monic and (ii) pullback preserves finite meets of subobjects (as pullback is right adjoint to image factorization).

It is nice to know that the order on our ordered Stone locale can be found by pulling back the specialization order since then antisymmetry and reflexivity of the order follows from the fact that these two axioms hold for the specialization order. However we can prove that $\leq_{\mathcal{B}X}$ is antisymmetric directly:

Lemma 6.4.4 *$\leq_{\mathcal{B}X}$ is antisymmetric, where $\leq_{\mathcal{B}X}$ is given by*

$$a_{\leq_{\mathcal{B}X}} = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i)| \wedge_i (a_i \vee \neg b_i) = 0, \quad a_i, b_i \in K\Omega X\}.$$

Proof: We need to prove that $(\leq_{\mathcal{B}X}) \wedge (\geq_{\mathcal{B}X}) \overset{(p_1,p_2)}{\rightarrowtail} \mathcal{B}X \times \mathcal{B}X$ is the diagonal. We may conclude this provided we check that its right hand projection is equal to its left hand projection. i.e. $p_1 = p_2$. As a statement about frames this reads

$$\Omega(\pi_1)(I) \vee a_{\leq} \vee a_{\geq} = \Omega(\pi_2)(I) \vee a_{\leq} \vee a_{\geq} \quad \forall I \in IdlB_X$$

Note that we may restrict to the case that $I \in Idl(K\Omega X)$. This is because l_X is a monomorphism. In fact we only need worry about compact Is. i.e. we may assume $I = a \in K\Omega X$. In such a case $\Omega\pi_1 I = a \otimes 0, \quad \Omega\pi_2 I = 0 \otimes a$. Hence we need

$$a \otimes 0 \vee a_{\leq} \vee a_{\geq} = 0 \otimes a \vee a_{\leq} \vee a_{\geq} \quad \forall a \in K\Omega X.$$

Before proof note that for any $a \in K\Omega X$ since $(a \vee 0) \wedge (0 \vee \neg a) = 0$ we have that

$$\begin{aligned} a_{\leq} &= a_{\leq} \vee [(a \otimes 0) \wedge (0 \otimes \neg a)] \qquad (I) \\ a_{\geq} &= a_{\geq} \vee [(\neg a \otimes 0) \wedge (0 \otimes a)] \qquad (II) \end{aligned}$$

Hence for any $a \in K\Omega X$

$$\begin{aligned} a \otimes 0 \vee a_{\leq} \vee a_{\geq} &= a_{\leq} \vee [[a_{\geq} \vee (\neg a \otimes 0) \vee (a \otimes 0)] \wedge [a_{\geq} \vee (a \otimes a)]] \text{ by } (II) \\ &= a_{\leq} \vee a_{\geq} \vee (a \otimes a) \\ 0 \otimes a \vee a_{\leq} \vee a_{\geq} &= a_{\geq} \vee [[a_{\leq} \vee (a \otimes a)] \wedge [a_{\leq} \vee (0 \otimes \neg a) \vee (0 \otimes a)]] \text{ by } (I) \\ &= a_{\leq} \vee a_{\geq} \vee (a \otimes a). \qquad \Box \end{aligned}$$

So to be sure that $\mathcal{B}$ actually gives us an ordered Stone locale we need but check that

$$a_{\leq} = \bigvee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i)| \wedge_{i \in I} (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in B_X \quad \Downarrow^{op} a_i = a_i, \\ \Downarrow^{op} b_i = b_i, I \text{ finite } \}.$$

This will follow once we've shown that

Lemma 6.4.5 *If X is a coherent locale and $\Downarrow^{op}$ is the preframe endomorphism of ΩX that corresponds to the relation $\leq_{\mathcal{B}X}$ then for all $a \in B_X$,*

$$a \in K\Omega X \quad \Leftrightarrow \quad a = \Downarrow^{op} a$$

Proof: It is always the case that $\Downarrow^{op} a \leq a$ since $\leq_{\mathcal{B}X}$ is reflexive. Hence we need but prove

$$a \in K\Omega X \quad \Leftrightarrow \quad a \leq \Downarrow^{op} a.$$

We know that

$$a_{\leq_{\mathcal{B}X}} = \vee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i)| \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\}.$$

Assume we are given $a \in K\Omega X$. So

$$\Downarrow^{op} a = \vee^{\uparrow}\{\wedge_i[a_i \vee \Omega!(1 \leq \neg b_i \vee a)]| \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\}$$

Simply take $I = \{1, 2\}$

$$\begin{aligned} a_1 &= a \qquad & b_1 &= 0 \\ a_2 &= 0 \qquad & b_2 &= \neg a \end{aligned}$$

to see that $a \leq \Downarrow^{op} a$.

Conversely say $a \in B_X$ and $a \leq \Downarrow^{op} a$. Since '$a \in B_X$' means a is compact we see from our expression above for $\Downarrow^{op} a$ that

$$a \leq \wedge_{i\in I}[a_i \vee \Omega!(1 \leq \neg b_i \vee a)]$$

for some a_i, b_is in $K\Omega X$ with $\wedge_i(a_i \vee \neg b_i) = 0$. Hence

$$\begin{array}{rcl} a & \leq & \wedge_i(a_i \vee \Omega!(b_i \leq a)) \\ & = & \bigvee_{I=J_1\cup J_2} (\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2}\Omega!(b_i \leq a)) \\ & = & \bigvee_{I=J_1\cup J_2} (\wedge_{i\in J_1} a_i) \wedge (\Omega!(\vee_{i\in J_2} b_i \leq a)) \\ & = & \bigvee_{I=J_1\cup J_2} (\vee\{\wedge_{i\in J_1} a_i | \vee_{i\in J_2} b_i \leq a\}) \\ & = & \bigvee^{\uparrow}(\bigcup_{I=J_1\cup J_2} \{\wedge_{i\in J_1} a_i | \vee_{i\in J_2} b_i \leq a\}) \end{array}$$

The union is over all pairs $J_1, J_2 \subseteq I$ such that J_1, J_2 are finite and $I \subseteq J_1 \cup J_2$. The fact that this union is directed follows since if $(J_1, J_2), (\bar{J}_1, \bar{J}_2)$ are two pairs of the indexing set then $(J_1 \cap \bar{J}_1, J_2 \cup \bar{J}_2)$ is in the indexing set. Hence

$$a \leq \bigvee^{\uparrow}(\bigcup\{\wedge_{i\in J_1} a_i | \vee_{i\in J_2} b_i \leq a\})$$

So, by compactness of a, it is possible to find J_1, J_2 subsets of I such that $I \subseteq J_1 \cup J_2$ with the property that $a \leq \wedge_{i\in J_1} a_i$ and $\vee_{i\in J_2} b_i \leq a$. But the statement $\wedge_i(a_i \vee \neg b_i) = 0$ implies

$$\begin{array}{rl} & \vee_{I\subseteq J_1\cup J_2}[(\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2}\neg b_i)] = 0 \\ \Rightarrow & (\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2}\neg b_i) = 0 \\ \Rightarrow & \wedge_{i\in J_1} a_i \leq \neg(\wedge_{i\in J_2}\neg b_i) = \vee_{i\in J_2} b_i \end{array}$$

Hence $a = \wedge_{i\in J_1} a_i$ and since $a_i \in K\Omega X \quad \forall i$ we see that $a \in K\Omega X$. $\Box$

It is unfortunate that we have to rely on a distributivity law in the middle of the above proof. A more natural way to proceed would be to say: for every $i \in I$

$$\begin{array}{rcl} a & \leq & a_i \vee \Omega!(b_i \leq a) \\ & = & \bigvee^{\uparrow}(\{a_i\} \cup \{1 | b_i \leq a\}) \end{array}$$

and so if we define:

$$\begin{array}{c} J_1 \equiv \{i | a \leq a_i\} \\ J_2 \equiv \{i | b_i \leq a\} \end{array}$$

then compactness of a implies that $I \subseteq J_1 \cup J_2$. This is all very well but we now don't know for sure whether J_1, J_2 are finite.

As for the effect of $\mathcal{B}$ on morphisms, say we are given a semi-proper locale map $f : X \to Y$. So Ωf restricts to a distributive lattice homomorphism from $K\Omega Y$ to $K\Omega X$, and hence extends naturally to a distributive lattice homomorphism on the respective free Boolean algebras B_Y, B_X. This induces a locale map $\mathcal{B}f$ from $\mathcal{B}X$ to $\mathcal{B}Y$. We must check that this map is an ordered Stone locale map. i.e. that

$$\Omega\mathcal{B}f \circ \Downarrow^{op} a \leq \Downarrow^{op} \circ \Omega\mathcal{B}f(a)$$

for every $a \in \Omega\mathcal{B}X$
But

$$\begin{aligned} LHS &= \Omega\mathcal{B}f(\vee^{\uparrow}\{\wedge_i[a_i \vee \Omega!(1 \leq \neg b_i \vee a)] | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\}) \\ &= \vee^{\uparrow}\{\wedge_i[\Omega\mathcal{B}f(a_i) \vee \Omega!(1 \leq \neg b_i \vee a)] | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\} \\ &\leq \vee^{\uparrow}\{\wedge_i[\Omega\mathcal{B}f(a_i) \vee \Omega!(1 \leq \neg\Omega\mathcal{B}f b_i \vee \Omega\mathcal{B}f a)] | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X\} \\ &\leq \Downarrow^{op} \circ \Omega\mathcal{B}f(a) \end{aligned}$$

To comprehend the last two lines we need to remind ourselves that $\Omega\mathcal{B}f(a) = \Omega f(a) \in K\Omega X$ if $a \in K\Omega Y$, and that if $\bar{\Omega f}$ is the extension of $\Omega f : K\Omega Y \to K\Omega X$ to the Boolean completions then $\bar{\Omega f}(\neg b) = \neg(\Omega f b)$ for every $b \in K\Omega Y$. Thus $\mathcal{B}$ defines a functor from **CohLoc** to **OStoneLoc**.

Fortunately the construction of a functor $\mathcal{C}$ in the opposite direction is less involved than our construction of $\mathcal{B}$. Define $\mathcal{C}$ as follows

$$\begin{aligned} \mathcal{C} : \mathbf{OStoneLoc} &\longrightarrow \mathbf{CohLoc} \\ (X, \leq) &\longmapsto Idl(\{a \in K\Omega X | \Downarrow^{op} a = a\}) \end{aligned}$$

N.B. $\{a \in K\Omega X | \Downarrow^{op} a = a\}$ is a subdistributive lattice of $K\Omega X$. The only tricky bit in proving this is closure under finite joins. But $\Downarrow^{op} a \leq a \quad \forall a$, so (i) $0 \leq \Downarrow^{op} 0 \leq 0$ and (ii) if $a = \Downarrow^{op} a, b = \Downarrow^{op} b$ then $a \vee b = \Downarrow^{op} a \vee \Downarrow^{op} b \leq \Downarrow^{op} (a \vee b) \leq a \vee b$.
The definition of $\mathcal{C}$ on morphisms is also clear: if $f : (X, \leq_X) \to (Y, \leq_Y)$ is an ordered Stone locale map then it is proper and so is semi-proper; Ωf preserves compact opens. The fact that $\Omega f(\Downarrow^{op} (a)) \leq \Downarrow^{op} \Omega f(a) \quad \forall a \in \Omega Y$ means that Ωf restricts to a distributive lattice homomorphism from $\{a \in K\Omega Y | \Downarrow^{op} a = a\}$ to $\{a \in K\Omega X | \Downarrow^{op} a = a\}$. So f induces a semi-proper map $\mathcal{C}(f)$ from $\mathcal{C}(X, \leq_X)$ to $\mathcal{C}(Y, \leq_Y)$.
It is now clear that checking that

$$\mathcal{C}\mathcal{B}(X) \cong X \qquad \forall X \in Ob(\mathbf{CohLoc})$$

amount to showing that $\forall a \in B_X$

$$a \in K\Omega X \quad \Leftrightarrow \quad a = \Downarrow^{op} a$$

(where B_X is the free Boolean algebra over the distributive lattice $K\Omega X$). But we have shown this already in Lemma [6.4.5].
So all we need to do is ask: is $\mathcal{B}\mathcal{C}(Y) \cong Y$ for all $Y \in \mathbf{OStoneLoc}$?
Well we know that there is a distributive lattice inclusion,

$$\{a \in K\Omega Y | \Downarrow^{op} a = a\} \hookrightarrow K\Omega Y$$

but is it universal? *If* it is then the fact that we require

$$a_{\leq_Y} = \bigvee{}^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i) | \wedge_{i \in I} (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in B_X \quad \Downarrow^{op} a_i = a_i, \\ \Downarrow^{op} b_i = b_i, I \text{ finite } \}$$

for Y to be an ordered Stone locale means that

$$\leq_Y = \leq_{\mathcal{B}\mathcal{C}(Y)} .$$

Thus we will be finished provided we can check that the above inclusion is universal. Assume a diagram

$$\begin{tikzcd} K\mathcal{C}(Y) \arrow[r, hook] \arrow[dr, "f"'] & K\Omega Y \arrow[d, dotted, "\phi"] \\ & B \end{tikzcd}$$

where f is a distributive lattice homomorphism and B is a Boolean algebra.
Say $a \in K\Omega Y$ and we have found two finite sets of elements $\{a_i, b_i | i \in I\}$, $\{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ such that $\wedge_i(a_i \vee \neg b_i) = a = \wedge_{\bar{i}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}})$. (Where the $a_i, b_i, \bar{a}_{\bar{i}}, \bar{b}_{\bar{i}}$s are in $\{a \in K\Omega Y | \Downarrow^{op} a = a\}$.) We want to check,

Lemma 6.4.6 $\wedge_i(fa_i \vee \neg fb_i) = \wedge_{\bar{i}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$

(For then it will be 'safe' to define $\phi(a) = \wedge_i(fa_i \vee \neg fb_i)$ for any $\{a_i, b_i | i \in I\} \subseteq K\mathcal{C}(Y)$ such that $a = \wedge_i(a_i \vee \neg b_i)$.)

Proof: We have done this already really in Lemma [1.3.3]. To conclude that $\wedge_i(fa_i \vee \neg fb_i) \leq \wedge_{\bar{i}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$ we need to prove that for every $\bar{i}$ and for every pair $J_1, J_2 \subseteq I$ with $I \subseteq J_1 \cup J_2$ we have

$$(\wedge_{i \in J_1} fa_i) \wedge (\wedge_{i \in J_2} \neg fb_i) \leq (f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$$

This relies on the by now well known finite distributivity law being applied to the meet $\wedge_i(fa_i \vee \neg fb_i)$. But the last inequality can be manipulated to

$$f((\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i) \leq f((\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i))$$

and the fact that $(\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i \leq (\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i)$ follows from exactly the same manipulations applied to the assumption

$$\wedge_i(a_i \vee \neg b_i) \leq \wedge_{\bar{i}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}). \ \Box$$

Assumption: $\forall a \in K\Omega Y \quad \exists\{a_i, b_i | i \in I\} \subseteq K\mathcal{C}Y$ s.t. $\wedge_i(a_i \vee \neg b_i) = a$.

If this assumption is true then ϕ will be a (necessarily unique) Boolean homomorphism extending f. [For if $a = \wedge_{i \in I}(a_i \vee \neg b_i)$ and $\bar{a} = \wedge_{i \in \bar{I}}(a_i \vee \neg b_i) \quad \Rightarrow \quad a \wedge \bar{a} = \wedge_{I \cup \bar{I}}(a_i \vee \neg b_i)$. So

$$\begin{aligned} \phi(a \wedge \bar{a}) &= \wedge_{I \cup \bar{I}}(fa_i \vee \neg fb_i) \\ &= [\wedge_{i \in I}(fa_i \vee \neg fb_i)] \wedge [\wedge_{i \in \bar{I}}(fa_i \vee \neg fb_i)] \\ &= \phi(a) \wedge \phi(\bar{a}) \end{aligned}$$

Similarly for $\vee$.]
We also have the following Boolean algebra lemma:

Lemma 6.4.7 *If $I, \bar{I}$ are finite sets and $\{a_i, b_i | i \in I\}$ and $\{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ are sets of elements of some Boolean algebra B, and $\wedge_i(a_i \vee \neg b_i) = 0, \wedge_{\bar{i}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) = 0$. Then for any $J_1, J_2 \subseteq I \times \bar{I}$, finite subsets, such that $I \times \bar{I} \subseteq J_1 \cup J_2$ we have*

$$\wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}}) \leq \vee_{(i,\bar{i}) \in J_2}(\neg \bar{a}_{\bar{i}} \wedge b_i)$$

Proof: The conditions imply:

$$\begin{array}{rl} & [\wedge(a_i \vee \neg b_i)] \vee [\wedge(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}})] = 0 \\ \Rightarrow & \wedge_{(i,\bar{i}) \in I \times \bar{I}}[a_i \vee \neg b_i \vee \bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}] = 0 \\ \Rightarrow & \vee_{I \times \bar{I} \subseteq J_1 \cup J_2}[(\wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}})) \wedge (\wedge_{(i,\bar{i}) \in J_2}(\bar{a}_{\bar{i}} \vee \neg b_i))] = 0 \\ \Rightarrow & (\wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}})) \wedge (\wedge_{(i,\bar{i}) \in J_2}(\bar{a}_{\bar{i}} \vee \neg b_i)) = 0 \end{array}$$

The result follows since

$$\neg(\wedge(\bar{a}_{\bar{i}} \vee \neg b_i)) = \vee(\neg \bar{a}_{\bar{i}} \wedge b_i). \ \Box$$

We can now prove our assumption:

Theorem 6.4.1 *If $(Y, \leq)$ is an ordered Stone locale and $a \in K\Omega Y$ then $a = \wedge_{i \in I}(a_i \vee \neg b_i)$ for some finite I with $a_i, b_i \in K\Omega Y$ and $\Downarrow^{op} a_i = a_i, \Downarrow^{op} b_i = b_i$.*

Proof: Clearly the antisymmetry axiom must now come into play. This axiom states that

$$(\leq) \wedge (\geq) \leq_{Sub(X \times X)} \Delta$$

which as a statement about the opens of $\Omega(X \times X)$ reads:

$$a_{\leq} \vee a_{\geq} \geq \#$$

But $a = \# * a$ since $\#$ corresponds to the identity of relational composition. Thus

$$a \leq (a_{\leq} \vee a_{\geq}) * a \qquad \text{(I)}$$

From our axioms used to define 'ordered Stone locale' we know,

$$a_{\leq} = \vee^{\uparrow}\{\wedge_i(a_i \otimes \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega Y \quad \Downarrow^{op} a_i = a_i \quad \Downarrow^{op} b_i = b_i\}$$

symmetrically

$$a_{\geq} = \vee^{\uparrow}\{\wedge_{\bar{i}}(\neg \bar{b}_{\bar{i}} \otimes \bar{a}_{\bar{i}}) | \wedge_{\bar{i}} (\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) = 0 \quad \bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} \in K\Omega Y \quad \Downarrow^{op} \bar{a}_{\bar{i}} = \bar{a}_{\bar{i}} \quad \Downarrow^{op} \bar{b}_{\bar{i}} = \bar{b}_{\bar{i}}\}.$$

Thus $a_{\leq} \vee a_{\geq}$ is a directed union of elements of the form

$$\begin{array}{rl} & [\wedge_i(a_i \otimes \neg b_i)] \vee [\wedge_{\bar{i}}(\neg \bar{b}_{\bar{i}} \otimes \bar{a}_{\bar{i}})] \\ = & \wedge_{(i,\bar{i}) \in I \times \bar{I}}[(a_i \otimes \neg b_i) \vee (\neg \bar{b}_{\bar{i}} \otimes \bar{a}_{\bar{i}})] \\ = & \wedge_{(i,\bar{i}) \in I \times \bar{I}}[(a_i \vee \neg \bar{b}_{\bar{i}}) \otimes (\neg b_i \vee \bar{a}_{\bar{i}})] \end{array}$$

Since a is compact and $(\_) * a$ preserves directed joins and finite meets we see from (I) that

$$a \leq \wedge_{(i,\bar{i}) \in I \times \bar{I}}([(a_i \vee \neg \bar{b}_{\bar{i}}) \otimes (\neg b_i \vee \bar{a}_{\bar{i}})] * a)$$

for some $\{a_i, b_i | i \in I\}, \{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ such that $\wedge_i(a_i \vee \neg b_i) = 0, \wedge_{\bar{i}}(\neg \bar{b}_{\bar{i}} \vee \bar{a}_{\bar{i}}) = 0$ and $\Downarrow^{op} a_i = a_i, \Downarrow^{op} b_i = b_i, \Downarrow^{op} \bar{a}_{\bar{i}} = \bar{a}_{\bar{i}}, \Downarrow^{op} \bar{b}_{\bar{i}} = \bar{b}_{\bar{i}}$. Now

$$\begin{array}{rl} & [(a_i \vee \neg \bar{b}_{\bar{i}}) \otimes (\neg b_i \vee \bar{a}_{\bar{i}})] * a \\ = & (a_i \vee \neg \bar{b}_{\bar{i}}) \vee \Omega!(1 \leq \neg b_i \vee \bar{a}_{\bar{i}} \vee a) \\ = & \vee^{\uparrow}[\{a_i \vee \neg \bar{b}_{\bar{i}}\} \cup \{1 | b_i \wedge \neg \bar{a}_{\bar{i}} \leq a\}] \end{array}$$

And so, similarly to Lemma [6.4.5], via compactness of a we can find finite subsets $J_1, J_2 \subseteq I \times I$ with the properties:

$$\begin{array}{rcl} a & \leq & a_i \vee \neg \bar{b}_{\bar{i}} \quad \forall (i, \bar{i}) \in J_1 \\ b_i \wedge \neg \bar{a}_{\bar{i}} & \leq & a \quad \forall (i, \bar{i}) \in J_2 \\ I \times \bar{I} & \subseteq & J_1 \cup J_2 \end{array}$$

Clearly (by definition of J_1, J_2)

$$\begin{array}{rl} & a \leq \bigwedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}}) \\ \text{and} & \bigvee_{(i,\bar{i}) \in J_2} (\neg \bar{a}_{\bar{i}} \wedge b_i) \leq a. \end{array}$$

But by the last lemma

$$\bigwedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}}) \leq \bigvee_{(i,\bar{i}) \in J_2} (\neg \bar{a}_{\bar{i}} \wedge b_i)$$

and so $a = \bigwedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}})$. $\square$

6.5 Notes

In his thesis 'The Structure of (free) Heyting Algebras' ([Pre93]) Pretorius proves a constructive version of Priestley's duality. He shows that the the coherent locales are dual to a particular subcategory of the category of pairs of frames (where the second element of the pair is a subframe of the first and morphisms of this category are frame homomorphisms that preserve the subframe). This particular subcategory is seen, assuming PIT, to be equivalent to the ordered Stone spaces and so Priestley's original duality is recovered. It is not clear how, from its definition, to view this particular subcategory localically; although given the results of this chapter we now know that it is constructively equivalent to the ordered Stone locales.

The methods of Pretorius' proof are very different from ours. He makes much use of the frame of congruences on a distributive lattice. His observation that 'the set of compact congruences on a distributive lattice is the free Boolean algebra on that lattice' has helped us in two important ways. Firstly it shows us how to construct the free Boolean algebra on a distributive lattice (see Section 1.3). This is not a trivial problem as the usual method, via finitary universal algebra, is not allowed in our context since it depends on the natural numbers. Secondly the fact that the compact congruences form the free Boolean algebra means that we have a much simpler proof of Banaschewski and Brümmer's result that the stably locally compact locales correspond to the compact regular biframes [BB88]. The consequences of this correspondence forms the content of our last chapter.

Chapter 7

Hausdorff Systems

7.1 Introduction

Given a poset $(X, \leq)$ we can construct $Idl(X)$, its ideal completion. $Idl(X)$ is an algebraic dcpo. For any algebraic dcpo, A, we can construct KA, the set of compact elements of A. These constructions are inverse to each other. However we cannot conclude that the category of posets is equivalent to the category of algebraic dcpos. This is because not all dcpo maps preserve compact opens. But if we extend the morphisms between posets to relations (satisfying suitable conditions) then a categorical equivalence can be established. This is the idea behind Scott's information systems (see [Sco82]). One of the reasons for presenting algebraic dcpos as posets (=information systems) is that it becomes possible to use the presentation to solve domain equations. Domains are special types of algebraic dcpos and the problem of solving domain equations is important in theoretical computer science. See [Vic89] for background on domains and [LW84] for details about how domain equations can be solved using information systems. The problem of extending this equivalence to the retracts of the algebraic dcpos (i.e. the continuous posets) is dealt with in [Vic93]. In [Vic93] Vickers introduces the category of continuous information systems (**InfoSys**). These are pairs (X, R) where X is a set and R is a relation on X which is idempotent with respect to relational composition. There are many morphisms possible between continuous information systems. The most general are relations:

$$R : (X, R_X) \to (Y, R_Y)$$

$R \subseteq X \times Y$ such that $R = R_Y \circ R \circ R_X$ where $\circ$ is relational composition. These are called the lower approximable semimappings.
We define *Hausdorff systems* to be the proper parallel to continuous information systems. So a Hausdorff system is a pair (X, R) where X is a compact Hausdorff locale and R is a closed relation such that $R \circ R = R$. Upper approximable semimappings between Hausdorff systems are closed relations $R \hookrightarrow X \times Y$,

$$R : (X, R_X) \to (Y, R_Y),$$

such that $R = R_Y \circ R \circ R_X$ where $\circ$ is compact Hausdorff relational composition. We have defined the category

$$\mathbf{HausSys}_U$$

If (X, R) is an infosys then we know (Chapter 4) that there is a SUP-lattice homomorphism $\downarrow^R: PX \to PX$ corresponding to R. $\downarrow^R$ is idempotent since R is. The set

$$\{T|T \in PX \quad \downarrow^R T = T\}$$

can then easily be seen to be a constructively completely distributive lattice. The essence of [Vic93] is a proof that all constructively completely distributive lattices arise in this way.
Given a Hausdorff system (X,R) we know that there is a preframe morphism $\Downarrow^{op}: \Omega X \to \Omega X$ corresponding to R (Chapter 4). Hence

$$\{a|a \in \Omega X \quad \Downarrow^{op} a = a\}$$

is a subpreframe of ΩX. It also has finite joins: $\Downarrow^{op} 0$ is least and the join of a,b is given by $\Downarrow^{op} (a \vee b)$. Further,

Lemma 7.1.1 $\Omega\bar{X} \equiv \{a|a \in \Omega X \quad \Downarrow^{op} a = a\}$ *is the frame of opens of a stably locally compact locale.*

Proof: First we check that the frame is continuous, i.e. that $\forall a \in \Omega\bar{X}$

$$a = \bigvee^{\uparrow}\{b|b \ll_{\Omega\bar{X}} a\} \quad (*)$$

Since ΩX is compact regular we know that $(\forall a, b \in \Omega X)$

$$a \lhd b \quad \Leftrightarrow \quad a \ll b$$

Hence to conclude $(*)$ all we need do is check that

$$b \ll a \quad \Rightarrow \quad \Downarrow^{op} b \ll_{\Omega\bar{X}} a$$

if $a \in \Omega\bar{X}$. Say $b \ll a$ and $a \leq \bigvee^{\uparrow} S \quad S \subseteq^{\uparrow} \Omega\bar{X}$ then $\exists s \in S \quad b \leq s \quad \Rightarrow$ $\Downarrow^{op} b \leq \Downarrow^{op} s = s$.
As for stability we need to check that $1 \ll_{\Omega\bar{X}} 1$ (trivial by compactness of ΩX) and $a \ll_{\Omega\bar{X}} b_1, b_2$ implies $a \ll_{\Omega\bar{X}} b_1 \wedge b_2$. Since $b_i \in \Omega\bar{X}$, ΩX is regular and $\Downarrow^{op}$ is a preframe homomorphism we know that

$$b_i = \bigvee^{\uparrow}\{\Downarrow^{op} c|c \lhd b_i\}$$

Hence $a \leq \Downarrow^{op} c_i$ for some c_1, c_2 with $c_i \lhd b_i$. Hence $a \leq \Downarrow^{op} (c_1 \wedge c_2)$. But $c_1 \wedge c_2 \lhd b_1 \wedge b_2$ and so $c_1 \wedge c_2 \ll b_1 \wedge b_2$. Hence $a \ll_{\Omega\bar{X}} b_1 \wedge b_2$. $\Box$

The next section is devoted to proving that every stably locally compact locale arises in this way. From then our program is to check that this equivalence can be made categorical by restricting the class of relations that are allowed to be Hausdorff system maps. The program is the proper parallel to the contents of [Vic93].

7.2 Stably locally compact locales

Let $\mathbf{StLocKLoc}_U$ be the category whose objects are stably locally compact locales and whose morphisms are formally reversed preframe maps. Bearing in mind the correspondence between preframe homomorphisms on the frame of opens of compact Hausdorff locales and closed relations on these locales (as captured by Theorem [4.3.1]) it should be clear that there is a functor:

$$\begin{array}{rcl} \mathcal{C}_U : \mathbf{HausSys}_U & \to & \mathbf{StLocKLoc}_U \\ (X,R) & \mapsto & \bar{X} \end{array}$$

where $\Omega\bar{X} = \{a \in \Omega X| \Downarrow^{op} a = a\}$.
If $R : (X, R_X) \to (Y, R_Y)$ is an upper approximable semimapping (i.e. if $R_Y \circ R \circ R_X = R$) then it is clear that ψ_R (the preframe homomorphism from ΩY to ΩX corresponding to R) is going to satisfy:

$$\psi_R = {}^X\Downarrow^{op} \circ \psi_R \circ {}^Y\Downarrow^{op}$$

From this it follows that ψ_R will restrict to a preframe homomorphism from $\Omega\bar{Y}$ to $\Omega\bar{X}$. $\mathcal{C}_U$ is functorial.

Lemma 7.2.1 *The map*

$$\begin{array}{rcl} \mathbf{HausSys}_U((X,R_X),(Y,R_Y)) & \longrightarrow & \mathbf{PreFrm}(\Omega\bar{Y},\Omega\bar{X}) \\ R & \longmapsto & (\psi_R)|_{\Omega\bar{X}} \end{array}$$

is a bijection. i.e. $\mathcal{C}_U$ *is full and faithful.*

Proof: Send a preframe map $\bar{\psi} : \Omega\bar{Y} \to \Omega\bar{X}$ to the relation corresponding to the preframe homomorphism

$${}^X\Downarrow^{op} \circ \bar{\psi} \circ {}^Y\Downarrow^{op} : \Omega Y \longrightarrow \Omega X \ \Box$$

We want to define

$$\mathcal{B}_U : \mathbf{StLocKLoc}_U \to \mathbf{HausSys}_U$$

Fix, for the rest of the section, X, a stably locally compact locale. Define $\Lambda\Omega X$ to be the set of Scott open filters of ΩX. So $F \in \Lambda\Omega X$ iff

$$\begin{array}{ll} (i) & F \text{ is upper} \\ (ii) & a,b \in F \quad \Rightarrow \quad a \wedge b \in F \\ (iii) & 1 \in F \\ (iv) & a \in F \quad \Rightarrow \quad \exists b \in F \quad b \ll a. \end{array}$$

The following lemma is in [BB88],

Lemma 7.2.2 $\Lambda\Omega X$ *is the frame of opens of a stably locally compact locale.*

Proof: If F_1, F_2 are two Scott open filters then

$$F_1 \vee F_2 = \uparrow \{a_1 \wedge a_2 | a_1 \in F_1, \quad a_2 \in F_2\}$$

Directed joins are given by union. $F_1 \wedge F_2 = F_1 \cap F_2$, finite distributivity is an easy manipulation. If G is a Scott open filter then

$$G = \bigcup\nolimits^{\uparrow} \{\Uparrow b | b \in G\}$$

Hence $F \ll G$ if and only if there is a $b \in \Omega X$ such that $F \subseteq \uparrow b \subseteq G$. $\Box$

Since X is stably locally compact we know that there is a frame injection $\downarrow\!\!\downarrow : \Omega X \to Idl\Omega X$. Now define $B_{\Omega X}$ to be the free Boolean algebra on ΩX qua distributive lattice. There is a frame injection of $Idl\Omega X$ into $Idl B_{\Omega X}$ which we will denote by Ωl. So if we compose this injection with $\downarrow\!\!\downarrow$ we find that ΩX can be embedded in $Idl B_{\Omega X}$. Notice that if $\bar{a} \ll a$ then $\downarrow \bar{a} \subseteq \Omega l \downarrow\!\!\downarrow a$.

Lemma 7.2.3 $\Lambda\Omega X$ *can be embedded into* $Idl B_{\Omega X}$.

Proof: Send F to $\bigvee^{\uparrow}_{b \in F} \downarrow \neg b$. It is routine to check that this is a frame injection. $\Box$

Define: ΩY=the subframe of $Idl B_{\Omega X}$ generated by the image of the above two embeddings.

Theorem 7.2.1 *Y is a compact Hausdorff locale.*

Proof: Compactness is immediate since ΩY is a subframe of the compact frame $IdlB_{\Omega X}$. As for regularity it is clearly sufficient to check that

$$\Omega l{\Downarrow}a = \bigvee\nolimits^{\uparrow}\{I|I \lhd \Omega l{\Downarrow}a\}$$

for every $a \in \Omega X$ and

$$\bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b = \bigvee\nolimits^{\uparrow}\{I|I \lhd \bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b\}$$

$\forall F \in \Lambda\Omega X$.
However $a = \bigvee^{\uparrow}\{x|x \ll a\}$ and $F = \bigvee^{\uparrow}\{G|G \ll F\}$ since both ΩX and $\Lambda\Omega X$ are continuous posets. So it is sufficient to prove that

$$\begin{array}{rcl} x \ll a & \Rightarrow & \Omega l{\Downarrow}x \lhd \Omega l{\Downarrow}a \text{ (I)} \\ G \ll F & \Rightarrow & \bigvee\limits^{\uparrow}_{b\in G} \downarrow \neg b \lhd \bigvee\limits^{\uparrow}_{b\in F} \downarrow \neg b \text{ (II).} \end{array}$$

(I): Say $x \ll a$. Set $F = \Uparrow x$ (a Scott open filter). Then $\bigvee^{\uparrow}_{b\in F} \downarrow \neg b \in \Omega Y$. But clearly

$$\Omega l{\Downarrow}x \wedge \bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b = 0$$

Further $x \ll a \quad \Rightarrow \quad \exists \bar{a} \quad x \ll \bar{a} \ll a$. Hence

$$\begin{array}{rcl} \Omega l{\Downarrow}a \vee \bigvee\limits^{\uparrow}_{b\in F} \downarrow \neg b & \geq & \Omega l{\Downarrow}a \vee \downarrow \neg \bar{a} \\ & \geq & \downarrow \bar{a} \vee \downarrow \neg \bar{a} = 1 \end{array}$$

Hence $\Omega l{\Downarrow}x \lhd \Omega l{\Downarrow}a$.

(II): Say $G \ll F$. So $\exists x \in F \quad G \subseteq \Uparrow x \subseteq F$ (since $F = \bigvee^{\uparrow}\{\Uparrow x|x \in F\}$). Then

$$\bigvee\nolimits^{\uparrow}_{b\in G} \downarrow \neg b \wedge \Omega l{\Downarrow}x = 0$$

Now $x \in F \quad \Rightarrow \quad \exists \bar{x} \in F \quad \bar{x} \ll x$ and so

$$\Omega l{\Downarrow}x \vee \bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b \geq \downarrow \bar{x} \vee \downarrow \neg \bar{x} = 1 \qquad \square$$

We want a closed idempotent relation on Y and so we need to find a preframe endomorphism $\Downarrow^{op}: \Omega Y \to \Omega Y$ such that $(\Downarrow^{op})^2 = \Downarrow^{op}$. If $I, J \in \Omega Y$ we write $I \prec_1 J$ if and only if $\exists F \in \Lambda\Omega X$ such that

$$\begin{array}{l} I \wedge \bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b = 0 \\ J \vee \bigvee\nolimits^{\uparrow}_{b\in F} \downarrow \neg b = 1 \end{array}$$

Clearly $\prec_1 \subseteq \lhd$ and the last proof has shown us that $x \ll a$ implies $\Omega l{\Downarrow}x \prec_1 \Omega l{\Downarrow}a$. Define

$$\begin{array}{rcl} \Downarrow^{op}: \Omega Y & \longrightarrow & \Omega Y \\ J & \longmapsto & \bigvee\limits^{\uparrow}\{I|I = \Omega l{\Downarrow}a \text{ for some } a, I \prec_1 J\}. \end{array}$$

<u>Facts about $\Downarrow^{op}$:</u>
- $\star$ $\forall J, \quad \Downarrow^{op}(J) = \Omega l{\Downarrow}a$ for some $a \in \Omega X$
- $\star$ $\Downarrow^{op}(\Omega l{\Downarrow}a) = \Omega l{\Downarrow}a \quad \forall a$
- $\star$ $(\Downarrow^{op})^2 = \Downarrow^{op}$

$\star$ $\Downarrow^{op}$ is a preframe homomorphism.
Hence define $\mathcal{B}_U : \mathbf{StLocKLoc}_U \to \mathbf{HausSys}_U$ by $\mathcal{B}(X) = (Y, R)$, where R is the closed relation corresponding to $\Downarrow^{op}$.

The above definition did not simply jump out of a hat. Although presented in a very different way it is essentially just a restructuring of Banaschewski and Brümmer's construction of a compact regular biframe from a stably locally compact locale. In their paper [BB88] they embedded ΩX and $\Lambda\Omega X$ into the frame of frame congruences via exactly the same functions; regularity of the frame generated follows the same path. Compactness in their proof is not immediate. They embed the frame generated into the frame of frame congruences of the ideal completion of ΩX, pointing out that this embedding will be contained within the frame generated by congruences of the form

$$(\downarrow a \hookrightarrow Z) \wedge (\neg \downarrow b \hookrightarrow Z)$$

where $\Omega Z = Idl\Omega X$, $a, b \in \Omega X$. Another lemma verifies that the frame generated by these congruences is compact. But it can be seen that the frame generated by these congruences is just the ideal completion of the compact distributive lattice congruences on ΩX. Pretorius [Pre93] tells us that the set of such compact congruences is the Boolean completion of the distributive lattice ΩX and so we see that we can embed into the ideal completion of the Boolean completion of ΩX; see Section 1.3. This is exactly what is done above.

How is $\mathcal{B}_U$ defined on morphisms? Say $f : X_1 \to X_2$ is a morphism of $\mathbf{StLocKLoc}_U$ (so $\Omega f : \Omega X_2 \to \Omega X_1$ is a preframe homomorphism). From the starred 'facts about $\Downarrow^{op}$' above we see that the set of $\Downarrow^{op}$-fixed opens of $\mathcal{B}_U(X)$ is just the image of the inclusion $\Omega l_{\Downarrow} : \Omega X \to \Omega\mathcal{B}_U(X)$. Hence ΩX is isomorphic to $\Omega\mathcal{C}_U\mathcal{B}_U(X)$. So we can find a unique $\bar{f}$ such that

$$\begin{tikzcd}
\mathcal{C}_U\mathcal{B}_U X_1 \arrow[r, "\bar{f}"] \arrow[d, "\cong"'] & \mathcal{C}_U\mathcal{B}_U X_2 \\
X_1 \arrow[r, "f"] & X_2 \arrow[u, "\cong"']
\end{tikzcd}$$

commutes. But $\mathcal{C}_U$ is full and faithful. So there is a unique $\mathcal{B}_U f : \mathcal{B}_U X_1 \to \mathcal{B}_U X_2$ such that $\bar{f} = \mathcal{C}_U\mathcal{B}_U f$.

Lemma 7.2.4 $(X, R) \cong (Y, S)$ *in* $\mathbf{HausSys}_U$ *if and only if* $\Omega\bar{X} \cong \Omega\bar{Y}$ *as posets.*

Proof: Say $(X, R) \cong (Y, S)$ in $\mathbf{HausSys}_U$. It follows that there are upper approximable mappings

$$\begin{aligned} T : (X, R) &\longrightarrow (Y, S) \\ Q : (Y, S) &\longrightarrow (X, R) \end{aligned}$$

such that $T \circ Q = S$ and $Q \circ T = R$, where $\circ$ is relational composition. To see this notice that $R : (X, R) \to (X, R)$ is the identity on the Hausdorff system (X, R). If ψ_T, ψ_Q are the preframe homomorphisms corresponding to T, Q then $\psi_T \circ \psi_Q = {}^R\Downarrow^{op}$ and $\psi_Q \circ \psi_T = {}^S\Downarrow^{op}$. From which it follows

$$\begin{aligned} \psi_T|_{\Omega\bar{Y}} : \Omega\bar{Y} &\longrightarrow \Omega\bar{X} \\ \psi_Q|_{\Omega\bar{X}} : \Omega\bar{X} &\longrightarrow \Omega\bar{Y} \end{aligned}$$

are (order preserving) bijections. Conversely , say

$$\begin{tikzcd} \Omega\bar{X} \arrow[r, shift left, "\bar{\psi}"] & \Omega\bar{Y} \arrow[l, shift left, "\bar{\phi}"] \end{tikzcd}$$

are order preserving bijections. Then $\bar{\psi}$ and $\bar{\phi}$ are preframe homomorphisms. So if ϕ is defined so as to make

$$\begin{tikzcd} \Omega Y \arrow[r, "\phi"] \arrow[d, "{{}^{Y}\Downarrow^{op}}"'] & \Omega X \\ \Omega\bar{Y} \arrow[r, "\bar{\phi}"] & \Omega\bar{X} \arrow[u, hook] \end{tikzcd}$$

commute and ψ is defined to make

$$\begin{tikzcd} \Omega X \arrow[r, "\psi"] \arrow[d, "{{}^{X}\Downarrow^{op}}"'] & \Omega Y \\ \Omega\bar{X} \arrow[r, "\bar{\psi}"] & \Omega\bar{Y} \arrow[u, hook] \end{tikzcd}$$

commute we see that ϕ, ψ are preframe homomorphisms. If T, Q are the relations corresponding to ψ, ϕ respectively then clearly T, Q are upper approximable semimappings which are inverse to each other in $\mathbf{HausSys}_U$. $\Box$

Theorem 7.2.2 $\mathbf{HausSys}_U \cong \mathbf{StLocKLoc}_U$

Proof: We need to check $\mathcal{B}_U\mathcal{C}_U(X,R) \cong (X,R)$ in $\mathbf{HausSys}_U$, for every Hausdorff system (X,R). This is immediate from the preceding lemma since we know $\mathcal{C}_U\mathcal{B}_U\mathcal{C}_U(X,R) \cong \Omega\bar{X} = \mathcal{C}_U(X,R)$. $\Box$

7.3 Approximable Mappings

In the paper [Vic93] various different types of morphisms between continuous information systems are introduced. So far we have only examined the proper parallel to $\mathbf{InfoSys}_L$. i.e. to the case where the morphisms are relations $R : (X, R_X) \to (Y, R_Y)$ such that $R_Y \circ R \circ R_X = R$. On the 'open' side we see (Theorem 3.7 of [Vic93]) that

$$\mathbf{InfoSys}_L \cong \mathbf{CCDLoc}_L$$

where $\mathbf{CCDLoc}_L$ is the category whose objects are constructively completely distributive locales and whose morphisms are formally reversed SUP-lattice homomorphisms. On the proper side:

$$\mathbf{HausSys}_U \cong \mathbf{StLocKLoc}_U$$

In [Vic93] we see that the equivalence can be refined:

$$\mathbf{InfoSys} \cong \mathbf{CCDLoc}$$

CCDLoc has been introduced in Section 1.6. **InfoSys** has as objects all continuous information systems just as before. The morphisms are now the *approximable mappings*. Say $R : (X, R_X) \to (Y, R_Y)$ is a lower approximable semimapping. Then it is an approximable mapping provided it also satisfies:

$$\begin{array}{ll} (i) & s' R_X s \quad \Rightarrow \quad \exists t' \in Y \quad s' R t' \\ (ii) & s' R_X s \quad s R t_1 \quad s R t_2 \quad \Rightarrow \quad \exists t' \in Y \quad s' R t' \quad t' R_Y t_1 \quad t' R_Y t_2 \end{array}$$

For a justification of these axioms notice that if R_X, R_Y are partial orders then $(i), (ii)$ are saying that for every $s \in X$, $\{t | s R t\}$ is an ideal of Y.
It is quite easy to see that these two conditions can be expressed as:

$$\begin{array}{ll} (i) & \downarrow^X (X) \subseteq Y \circ R \\ (ii) & \downarrow^X (A_1 \circ R \cap A_2 \circ R) \subseteq (\downarrow^Y A_1 \cap \downarrow^Y A_2) \circ R \end{array}$$

where A_1, A_2 range over all subsets of Y. i.e. they range over all open sublocales of Y (viewed as a a discrete locale). Hence it should be clear what an approximable mapping between Hausdorff systems should be:

$$R : (X, R_X) \to (Y, R_Y)$$

is an approximable mapping of Hausdorff systems if and only if $R = R_X \circ R \circ R_Y$ and

$$\begin{array}{ll} (i) & \Downarrow^X (X) \leq_{Sub(X)} Y \circ R \\ (ii) & \Downarrow^X (F \circ R \wedge G \circ R) \leq_{Sub(X)} (\Downarrow^Y F \wedge \Downarrow^Y G) \circ R \end{array}$$

for all closed sublocales F, G of Y. Say $\psi_R : \Omega Y \to \Omega X$ is the preframe homomorphism corresponding to R. Then these equations are equivalent to the requirements:

$$\begin{array}{ll} (i) & \psi_R(0) \leq {}^X\!\Downarrow^{op}(0) \\ (ii) & \psi_R({}^Y\!\Downarrow^{op} a \vee {}^Y\!\Downarrow^{op} b) \leq {}^X\!\Downarrow^{op}(\psi_R(a) \vee \psi_R(b)). \end{array}$$

It is easy, from these definitions, to check that $R : (X, R) \to (X, R)$ is always an approximable mapping and that approximable mappings are closed under composition. Let **HausSys** be the category of Hausdorff systems with approximable mappings. It should now be clear that we have a functor:

$$\mathcal{C} : \mathbf{HausSys} \to \mathbf{StLocKLoc}$$

where **StLocKLoc** is the full subcategory of **Loc** consisting of the stably locally compact locales. The only difficulty is checking that the approximable mappings give rise to frame homomorphisms. Say $R : (X, R_X) \to (Y, R_Y)$ is an approximable mapping. Then, as in the $\mathbf{HausSys}_U$ case, we know that ψ_R restricts to a preframe homomorphism from $\Omega\bar{Y}$ ($\equiv \{a \in \Omega Y | \Downarrow^{op} a = a\}$) to $\Omega\bar{X}$. For every a and b in $\Omega\bar{Y}$

$$\begin{array}{lll} \psi_R(a \vee_{\Omega\bar{Y}} b) & = & \psi_R({}^Y\!\Downarrow^{op}(a \vee b)) \\ & = & \psi_R(a \vee b) \qquad (\psi_R = \psi_R \circ {}^Y\!\Downarrow^{op}) \\ & \leq & {}^X\!\Downarrow^{op}(\psi_R(a) \vee \psi_R(b)) \qquad (a, b \in \Omega\bar{Y}) \\ & = & \psi_R(a) \vee_{\Omega\bar{X}} \psi_R(b). \end{array}$$

And

$$\begin{array}{lll} \psi_R(0_{\Omega\bar{Y}}) & = & \psi_R(\Downarrow^{op} 0) \\ & = & \psi_R(0) \leq {}^X\!\Downarrow^{op} 0 \\ & = & 0_{\Omega\bar{X}}. \end{array}$$

So ψ_R restricts to a frame homomorphism from $\Omega\bar{Y}$ to $\Omega\bar{X}$. On the other hand it is easy to follow the definitions and prove that every frame homomorphism from $\Omega\bar{Y}$ to $\Omega\bar{X}$ gives rise to an approximable mapping from (X, R_X) to (Y, R_Y) just as in Lemma [7.2.1]. In fact the conclusion of that lemma is easily seen to hold here: $\mathcal{C}$ is full and faithful.

The next task is to check that the construction $\mathcal{B}_U$ gives rise to a well defined functor:

$$\mathcal{B} : \mathbf{StLocKLoc} \to \mathbf{HausSys}$$

This amounts to checking that if $f : X_1 \to X_2$ is a locale map between two stably locally compact locales then $\mathcal{B}_U f : \mathcal{B}_U(X_1) \to \mathcal{B}_U(X_2)$ is an approximable mapping. By reexamining the construction of $\mathcal{B}_U f$ we see that this fact follows from our observation that $\mathcal{C}$ is full and faithful.

Notice that Lemma [7.2.4] can now be repeated with **HausSys** in place of $\mathbf{HausSys}_U$ and we may conclude:

Theorem 7.3.1 **HausSys** $\cong$ **StLocKLoc**. $\Box$

7.4 Hoffmann-Lawson Duality

We use the blanket term Hoffmann-Lawson duality to cover dualities induced by the action of taking Scott open filters. Hoffmann and Lawson initially proved such a duality for continuous posets in [Hof79],[Hof81] and [Law79]. In [Vic93] we see how to make the duality constructive: the Hoffmann-Lawson dual of a continuous poset is found by taking the opposite of the corresponding continuous information system.

By analogy, for a Hausdorff system (X, R) there is a duality (on objects) which takes (X, R) to $(X, \tau R)$ where τR is the composite

$$R \hookrightarrow X \times X \xrightarrow{\tau} X \times X$$

(τ is the twist isomorphism). It is not immediately clear how to make this duality functorial. i.e. how to define a functor

$$\tau : \mathbf{HausSys} \longrightarrow \mathbf{HausSys}^{op}$$

Notice that if we reexamine $(\mathbf{HausSys})_U$ then

$$\tau : \mathbf{HausSys}_U \longrightarrow \mathbf{HausSys}_U^{op}$$

clearly is well defined. This is because

$$R_Y \circ R \circ R_X = R \quad \Leftrightarrow \quad \tau R_X \circ \tau R \circ \tau R_Y = \tau R$$

and so we get our first duality:

$$(\mathbf{HausSys})_U \cong (\mathbf{HausSys})_U^{op}$$

We have also (by implication) just checked that

$$(\mathbf{StLocKLoc})_U \cong (\mathbf{StLocKLoc})_U^{op}$$

On the open side there is the result

$$\mathbf{CCDLoc}_U \cong \mathbf{CCDLoc}_U^{op}$$

where the U indicates that the morphisms are formally reversed SUP-lattice homomorphisms. Notice that in our constructive context we cannot just take the opposite of a constructively completely distributive lattice in order to get its dual; if we could then the opposite of a constructively completely distributive lattice would be constructively completely distributive and, following our discussion in 1.6, this would imply that the excluded middle is true. The easiest constructive way of describing this duality is by looking at the points. We know that a CCD locale is uniquely determined by its continuous poset of points. [Vic93] shows how the above duality corresponds to taking the Scott open filters of these points in order to get the points of the dual. i.e. we are looking at a Hoffmann-Lawson duality.
What is the dual of a stably locally compact locale? Given that we are looking for a Hoffmann-Lawson duality and we have observed already that $\Lambda\Omega\bar{X}$ is the frame of opens of a stably locally compact locale if $\bar{X}$ is stably locally compact, it is clearly desirable to prove,

Theorem 7.4.1 *If (X, R) is a Hausdorff system then*

$$\{a \in \Omega X | \Uparrow^{op} a = a\} \cong \Lambda\{b \in \Omega X | \Downarrow^{op} b = b\}.$$

Proof: Recall from Chapter 5 that if (X, R) is a Hausdorff system (i.e. $R^2 = R$) then

$$a_R = \bigvee^{\uparrow}\{\wedge_i(\Downarrow^{op} a_i \otimes \Uparrow^{op} b_i) | \wedge_{i\in I} (a_i \vee b_i) = 0 \ I \text{ finite}\}.$$

(We see this result contained within the first few lines of the proof of Lemma [5.1.3].) It follows that

$$\Uparrow^{op} a = \bigvee^{\uparrow}\{\wedge_i(\Uparrow^{op} a_i \vee \Omega!(1 \leq a \vee \Downarrow^{op} b_i)) | \wedge_i (a_i \vee b_i) = 0\} \ (*)$$

Define a function:

$$\begin{array}{rcl} \phi : \{a \in \Omega X | \Uparrow^{op} a = a\} & \longrightarrow & \Lambda\{b \in \Omega X | \Downarrow^{op} b = b\} \\ a & \longmapsto & \{\Downarrow^{op} b | 1 \leq a \vee \Downarrow^{op} b\} \end{array}$$

Clearly $\phi(a)$ is a filter on $\{b | \Downarrow^{op} b = b\} \equiv \Omega\bar{X}$. Say $\Downarrow^{op} b \in \phi(a)$. We know

$$\Downarrow^{op} b = \bigvee^{\uparrow}\{\Downarrow^{op} \bar{b} | \bar{b} \ll_{\Omega X} \Downarrow^{op} b\}$$

since $\Downarrow^{op} b = \bigvee^{\uparrow}\{\bar{b} | \bar{b} \ll_{\Omega X} \Downarrow^{op} b\}$. Thus by compactness of ΩX since $1 \leq a \vee \Downarrow^{op} b$ we know $\exists \bar{b} \ll_{\Omega X} \Downarrow^{op} b$ with $1 \leq a \vee \Downarrow^{op} \bar{b}$. Hence $\Downarrow^{op} \bar{b} \in \phi(a)$. But

$$\bar{b} \ll_{\Omega X} \Downarrow^{op} b \quad \Rightarrow \quad \Downarrow^{op} \bar{b} \ll_{\Omega\bar{X}} \Downarrow^{op} b \qquad [7.1.1]$$

and so $\phi(a)$ is a Scott open filter. i.e. ϕ is well defined.
Further note that ϕ reflects order: say we are given $a, \bar{a} \in \{a | \Uparrow^{op} a = a\}$ with $\{\Downarrow^{op} b | 1 \leq a \vee \Downarrow^{op} b\} \subseteq \{\Downarrow^{op} b | 1 \leq \bar{a} \vee \Downarrow^{op} b\}$ then $\forall b$

$$1 \leq \bar{a} \vee \Downarrow^{op} b \quad \Rightarrow \quad 1 \leq a \vee \Downarrow^{op} b$$

and so the fact that $\Uparrow^{op} \bar{a} \leq \Uparrow^{op} a$ can be read off from $(*)$.
In the other direction define

$$\begin{array}{rcl} \psi : \Lambda\Omega\bar{X} & \longrightarrow & \{a | a \in \Omega X \quad \Uparrow^{op} a = a\} \\ F & \longmapsto & \bigvee^{\uparrow}\{\Uparrow^{op} a | a \in \Omega X \text{ such that } \exists b \in \Omega X \text{ with } a \wedge b = 0 \quad \Downarrow^{op} b \in F\} \end{array}$$

We need to show that $\forall F \in \Lambda\Omega\bar{X}$

$$F = \{\Downarrow^{op} b | 1 \leq \psi(F) \vee \Downarrow^{op} b\}$$

Proof of this: Say $b \in F$ then $b = \Downarrow^{op} b$. Since F is a Scott open filter we know that $\exists \bar{b} \in F$ such that

$$\bar{b} \ll_{\Omega \bar{X}} b.$$

The dual of $(*)$ is

$$\Downarrow^{op} c = \bigvee^{\uparrow} \{\wedge_i (\Downarrow^{op} b_i \vee \Omega!(1 \leq c \vee \Uparrow^{op} a_i)) | \wedge_i (a_i \vee b_i) = 0\}.$$

But every $(\Downarrow^{op} b_i \vee \Omega!(1 \leq c \vee \Uparrow^{op} a_i))$ is in $\Omega \bar{X}$ since it can be expressed as a directed join of elements of $\Omega \bar{X}$. Hence

$$b = \Downarrow^{op} b = \bigvee^{\uparrow}_{\Omega \bar{X}} \{\wedge_i (\Downarrow^{op} b_i \vee \Omega!(1 \leq b \vee \Uparrow^{op} a_i)) | \wedge_i (a_i \vee b_i) = 0\}$$

$\bar{b} \ll_{\Omega \bar{X}} b \quad \Rightarrow \exists \bar{\bar{b}} \in \Omega \bar{X} \quad \bar{b} \ll_{\Omega \bar{X}} \bar{\bar{b}} \ll_{\Omega \bar{X}} b.$
Hence there exists a finite collection $(a_i, b_i)_{i \in I}$ with $\wedge_i (a_i \vee b_i) = 0$ such that

$$\bar{\bar{b}} \leq \wedge_i (\Downarrow^{op} b_i \vee \Omega!(1 \leq b \vee \Uparrow^{op} a_i))$$

Hence (see Lemma [6.4.5]) there exists $J_1, J_2 \subseteq I$ finite such that $I = J_1 \cup J_2$ and

$$\bar{b} \leq \wedge_{i \in J_1} (\Downarrow^{op} b_i) \qquad 1 \leq b \vee \wedge_{i \in J_2} \Uparrow^{op} a_i$$

Hence $\bar{b} \leq \Downarrow^{op} (\wedge_{i \in J_1} b_i)$ and so $\Downarrow^{op} (\wedge_{i \in J_1} b_i)$ is in F. Now by the familiar finite distributivity law we know that

$$\wedge_{i \in I} (a_i \vee b_i) = \bigvee_{I = J_1 \cup J_2} ((\wedge_{i \in J_1} a_i) \wedge (\wedge_{i \in J_2} b_i))$$

and so since $(\wedge_{i \in J_1} b_i) \wedge (\wedge_{i \in J_2} a_i) = 0$ we get that $\Uparrow^{op} (\wedge_{i \in J_2} a_i) \leq \psi(F)$. So $1 \leq \psi(F) \vee b$.

On the other hand say $1 \leq \psi(F) \vee b$ for some b with $\Downarrow^{op} b = b$. By the compactness of ΩX (and the definition of ψ) we know that

$$1 \leq \Uparrow^{op} a \vee b$$

for some $a \in \Omega X$ such that $\exists \bar{b} \in \Omega X$ with the properties that $a \wedge \bar{b} = 0$ and $\Downarrow^{op} \bar{b} \in F$. However recall Lemma [5.2.1]. This stated that for any $a, b \in \Omega X$ we have that

$$1 \leq \Uparrow^{op} a \vee b \quad \Leftrightarrow \quad 1 \leq a \vee \Downarrow^{op} b.$$

Hence $\bar{b} \leq \Downarrow^{op} b$. This implies $\Downarrow^{op} \bar{b} \leq \Downarrow^{op} b = b$. It follows that $b \in F$ since $\Downarrow^{op} \bar{b} \in F$. $\Box$

There is no natural way of finding a contravariant functor from **HausSys** to **HausSys** since if R is an approximable mapping then we cannot hope that τR is also an approximable mapping. Just as in the open parallel we symmetrize the definition of approximable mapping in order to define a new class of functions between Hausdorff systems which will give rise to a contravariant functor. Clearly the parts of the definition which need to be symmetrized are the conditions:

$$\begin{array}{ll} (i) & \Downarrow^X (X) \leq_{Sub(X)} Y \circ R \\ (ii) & \Downarrow^X (F \circ R \wedge G \circ R) \leq_{Sub(X)} (\Downarrow^Y F \wedge \Downarrow^Y G) \circ R \end{array}$$

Define a *Lawson approximable mapping* to be an approximable mapping which also satisfies

$$\begin{array}{ll} (i) & \Uparrow^Y (Y) \leq_{Sub(Y)} X \circ \tau R \\ (ii) & \Uparrow^Y (F \circ \tau R \wedge G \circ \tau R) \leq_{Sub(Y)} (\Uparrow^X F \wedge \Uparrow^X G) \circ \tau R \end{array}$$

where F, G are arbitrary closed sublocales of X. Hence define the category

$$(\mathbf{HausSys})_\Lambda$$

whose morphisms are the Lawson approximable mappings. It should be clear that if $R : (X, R_X) \to (Y, R_Y)$ is a Lawson approximable mapping then there are two frame homomorphisms:

$$\psi_R : \{b \in \Omega Y | \,^Y\Downarrow^{op} b = b\} \to \{a \in \Omega X | \,^X\Downarrow^{op} a = a\}$$
$$\psi_{\tau R} : \{a \in \Omega X | \,^X\Uparrow^{op} a = a\} \to \{b \in \Omega Y | \,^Y\Uparrow^{op} b = b\}.$$

We would like to define the class of Lawson maps between stably locally compact locales and so define a category $(\mathbf{StLocKLoc})_\Lambda$ with the property

$$(\mathbf{HausSys})_\Lambda \cong (\mathbf{StLocKLoc})_\Lambda$$

The nature of the duality induced by τ should then be clear. We will say that $f : \bar{X} \to \bar{Y}$ (a locale map) between stably locally compact locales is *Lawson* iff

$$(\Omega f)^{-1} : \Lambda\Omega\bar{X} \longrightarrow \Lambda\Omega\bar{Y}$$

preserves finite joins. That this is a sensible guess can be seen straightaway by noting that $\psi_{\tau R}$ is a frame homomorphism from $\Lambda\Omega\bar{X}$ to $\Lambda\Omega\bar{Y}$ for any Lawson approximable mapping R. This follows from the last theorem.

Theorem 7.4.2 $(\mathbf{HausSys})_\Lambda \cong (\mathbf{StLocKLoc})_\Lambda$

Proof: Although the proof is slightly trickier it is still essentially a variation of the proof of $\mathbf{HausSys}_U \cong \mathbf{StLocKLoc}_U$. As a first step we check the fact that the set of Lawson approximable maps from (X, R_X) to (Y, R_Y) corresponds to the set of Lawson maps from $\bar{X}$ to $\bar{Y}$ via the usual transformation (i.e. $R \mapsto \psi_R|_{\Omega\bar{Y}}$). Say we are given a Lawson approximable map $R : (X, R_X) \to (Y, R_Y)$. Then we will know that $\psi_R|_{\Omega\bar{Y}}$ is the frame homomorphism corresponding to a Lawson map form $\bar{X}$ to $\bar{Y}$ provided we can check my claim that the composite

$$\Lambda\Omega\bar{X} \xrightarrow{\cong} \{a | \,^X\Uparrow^{op} a = a\} \xrightarrow{\psi_{\tau R}} \{b | \,^Y\Uparrow^{op} b = b\} \xrightarrow{\cong} \Lambda\Omega\bar{Y}$$

is given by $(\psi_R)^{-1}$. (For then we know $(\psi_R)^{-1}$ preserves finite joins since $\psi_{\tau R}$ does.) Recalling the proof of the last theorem we see that the above composite takes $F (\in \Lambda\Omega\bar{X})$ to

$$G \equiv \{\Downarrow^{op} b | 1 \leq \bigvee\nolimits^{\uparrow}\{\Uparrow^{op} \psi_{\tau R}(a) | \exists \bar{a} \quad a \wedge \bar{a} = 0 \quad \Downarrow^{op} a \in F\} \vee \Downarrow^{op} b\}$$

We want

$$\Downarrow^{op} b \in G \quad \Leftrightarrow \quad \psi_R(\Downarrow^{op} b) \in F$$

Now $F = \{\Downarrow^{op} a | 1 \leq \bigvee^{\uparrow}\{\Uparrow^{op} a_0 | \exists \bar{a} \quad a_0 \wedge \bar{a} = 0 \quad \Downarrow^{op} \bar{a} \in F\} \vee \Downarrow^{op} a\}$.
So $\Downarrow^{op} b \in G$ if and only if $\exists \bar{a}, a_0 \quad a_0 \wedge \bar{a} = 0 \quad \Downarrow^{op} \bar{a} \in F$ such that

$$1 \leq \Uparrow^{op} \psi_{\tau R}(a_0) \vee \Downarrow^{op} b$$
$$\Leftrightarrow \quad 1 \leq \psi_{\tau R}(\Uparrow^{op} a_0) \vee \Downarrow^{op} b$$

and $\psi_R(\Downarrow^{op} b) \in F \quad \Leftrightarrow \quad \exists \bar{a}, a_0 \quad a_0 \wedge \bar{a} = 0 \quad \Downarrow^{op} \bar{a} \in F$,

$$1 \leq \Uparrow^{op} a_0 \vee \psi_R(\Downarrow^{op} b)$$

But we have seen that for every $a \in \Omega X, b \in \Omega Y$

$$(1 \leq \psi_{\tau R}(a) \vee b) \quad \Leftrightarrow \quad (1 \leq a \vee \psi_R(b))$$

(Lemma [5.2.1]), and the composition gives $(\psi_R)^{-1}$ as required.

On the other hand say we are given $f : X_1 \to X_2$ a Lawson map between stably locally compact locales. Set $(X, R_X) = \mathcal{B}(X_1)$, $\quad (Y, R_Y) = \mathcal{B}(X_2)$ and $R = \mathcal{B}f$. So

$$R : (X, R_X) \longrightarrow (Y, R_Y)$$

is an approximable mapping. We check that it is Lawson. As usual $\psi_R : \Omega Y \to \Omega X$ is the preframe homomorphism corresponding to R. Clearly

$$\begin{tikzcd}
\Omega\bar{Y} \arrow[r, "\psi_R|_{\Omega\bar{Y}}"] \arrow[d, "\cong"'] & \Omega\bar{X} \\
\Omega X_2 \arrow[r, "\Omega f"] & \Omega X_1 \arrow[u, "\cong"']
\end{tikzcd}$$

commutes (where $\cong$ is as in the verification that $\mathcal{CB}(X_i) \cong X_i$), and so $(\psi_R)^{-1} : \Lambda\Omega\bar{X} \to \Lambda\Omega\bar{Y}$ preserves joins since $(\Omega f)^{-1} : \Lambda\Omega X_1 \to \Lambda\Omega X_2$ does. But we have just shown that $(\psi_R)^{-1} : \Lambda\Omega\bar{X} \to \Lambda\Omega\bar{Y}$ is given by the composite

$$\Lambda\Omega\bar{X} \xrightarrow{\cong} \{a | \, {}^X\Uparrow^{op} a = a\} \xrightarrow{\psi_{\tau R}} \{b | \, {}^Y\Uparrow^{op} b = b\} \xrightarrow{\cong} \Lambda\Omega\bar{Y}$$

and so $\psi_{\tau R}|_{\{a| \, {}^X\Uparrow^{op} a = a\}}$ preserves joins which is sufficient to prove that

$$\tau R : (Y, \tau R_Y) \to (X, \tau R_X)$$

is an approximable mapping. i.e. R is Lawson. $\Box$

7.5 Products

Lemma 7.5.1 $(1,1)$ *is the terminal object of* **HausSys**. *If* $(X, R), (Y, S)$ *are two Hausdorff systems then*

$$(X, R) \times (Y, S) = (X \times Y, i(R \times S))$$

where $i : (X \times X) \times (Y \times Y) \to (X \times Y) \times (X \times Y)$ *is the twist isomorphism.*

Proof: Clearly $(1,1)$ is terminal. This follows since for any Hausdorff system (X, R) we know that approximable mappings from (X, R) to $(1,1)$ correspond to locale maps from $\bar{X}$ to 1.
If ${}^R\Downarrow^{op}$, ${}^S\Downarrow^{op}$ are the preframe homomorphisms corresponding to R, S then

$${}^R\Downarrow^{op} \otimes {}^S\Downarrow^{op} : \Omega X \otimes \Omega Y \longrightarrow \Omega X \otimes \Omega Y$$

is the preframe homomorphism corresponding to $i(R \times S)$. We need projection relations:

$$\begin{aligned} P_1 : (X \times Y, i(R \times S)) &\to (X, R) \\ P_2 : (X \times Y, i(R \times S)) &\to (Y, S) \end{aligned}$$

Define P_1 to be the pullback of R along

$$X \times Y \times X \xrightarrow{\pi_{13}} X \times X$$

and P_2 to be the pullback of S along

$$X \times Y \times Y \xrightarrow{\pi_{23}} Y \times Y$$

Hence the opens corresponding to P_1, P_2 are

$$\begin{aligned} a_{P_1} &= \Omega\pi_{13}(a_R) \\ a_{P_2} &= \Omega\pi_{23}(a_S) \end{aligned}$$

and the preframe homomorphisms corresponding to P_1, P_2 are

$$\begin{aligned} &\Omega\pi_1 \circ {}^{R}\Downarrow^{op} \\ &\Omega\pi_2 \circ {}^{S}\Downarrow^{op} \end{aligned}$$

where $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ are the usual projections. The best way of demonstrating this last claim is to look at the cases $a_R = a_1 \parr a_2, a_S = b_1 \parr b_2$. From this it is clear that P_1, P_2 are approximable mappings.
We need to check that if $Q_1 : (Z,T) \to (X,R)$ and $Q_2 : (Z,T) \to (Y,S)$ are two approximable mappings, then there exists a unique approximable map

$$L : (Z,T) \longrightarrow (X \times Y, i(R \times S))$$

such that $P_i L = Q_i$ for $i = 1, 2$.
Assume such an L exists. Say $\psi_{P_i}, \psi_L, \psi_{Q_i}$ are the corresponding preframe maps. Then since ψ_L is an approximable mapping it must satisfy $\psi_L \circ ({}^{R}\Downarrow^{op} \otimes {}^{S}\Downarrow^{op}) = \psi_L$. Hence for every $a \parr b \in \Omega X \otimes \Omega Y$ we must have

$$\begin{aligned} \psi_L(a \parr b) &= \psi_L({}^{R}\Downarrow^{op} a \parr {}^{S}\Downarrow^{op} b) \\ &= \psi_L(\psi_{P_1}(a) \vee \psi_{P_2}(b)) \\ &= \psi_L(({}^{R}\Downarrow^{op} \otimes {}^{S}\Downarrow^{op})(\psi_{P_1}(a)) \vee ({}^{R}\Downarrow^{op} \otimes {}^{S}\Downarrow^{op})(\psi_{P_2}(b))) \\ &= {}^{T}\Downarrow^{op}(\psi_L \psi_{P_1}(a) \vee \psi_L \psi_{P_2}(b)) \\ &= {}^{T}\Downarrow^{op}(\psi_{Q_1}(a) \vee \psi_{Q_2}(b)) \end{aligned}$$

The penultimate line is by the fact that ψ_L is an approximable map. Thus L is uniquely determined and it is clear from the above what formula should be assigned to ψ_L in order to define L such that $P_i L = Q_i$. $\Box$

7.6 Semi-Proper Maps

In Banaschewski and Brümmer's paper "Stably Continuous Frames" ([BB88]) there is a proof that the category of stably continuous frames and 'proper' maps is equivalent to the category of compact regular biframes. Their 'proper' maps are '$\ll$' preserving frame homomorphisms. We refer (see Section 1.7.3) to '$\ll$' preserving maps between stably locally compact locales as semi-proper maps. This is a good expression since it was shown (Lemma [3.2.1]) that a locale map $f : X \to Y$ between stably locally compact locales is semi-proper if and only if Ωf has a right adjoint that is a preframe homomorphism.

$$(\mathbf{StLocKLoc})_{SP}$$

is the category whose objects are stably locally compact locales and whose morphisms are semi-proper locale maps. Banaschewski and Brümmer's result is

$$(\mathbf{KR2Frm})^{op} \cong (\mathbf{StLocKLoc})_{SP}$$

But we saw in Section 5.4 that

$$(\mathbf{KR2Frm})^{op} \cong \mathbf{KHausPos}$$

So

$$\mathbf{KHausPos} \cong (\mathbf{StLocKLoc})_{SP}$$

The main purpose of this section is to outline a proof of this fact and to show how this equivalence can be viewed as an extension of localic Priestley duality. Interestingly, on objects, the proof uses exactly the same constructions as the proof that Hausdorff systems correspond to stably locally compact locales. For:

Lemma 7.6.1 *If X is a stably locally compact locales and (Y,R) is the Hausdorff system given by $\mathcal{B}X$ (as in the functor $\mathcal{B}$: $\mathbf{StLocKLoc} \to \mathbf{HausSys}$ of Section 7.2) then (Y,R) is a compact Hausdorff poset. i.e. R is a partial order.*

Proof: Recall the construction of $\mathcal{B}X$. $\Downarrow^{op}(J) \leq J \quad \forall J$ so R is reflexive and the $\Downarrow^{op}$-fixed ideals form a subframe of ΩY which is isomorphic to ΩX.

Further define $\epsilon_2 : \Omega Y \to \Omega Y$ by mapping any ideal J to

$$\bigvee^{\uparrow}\{I | I = \bigvee^{\uparrow}_{b \in F} \downarrow \neg b \text{ some Scott open filter } F, \quad I \prec_2 J\}$$

where

$$I \prec_2 J \quad \Leftrightarrow \quad \exists a \in \Omega X \qquad \begin{array}{l} I \wedge \Omega l {\downarrow\!\!\!\downarrow} a = 0 \\ J \vee \Omega l {\downarrow\!\!\!\downarrow} a = 1. \end{array}$$

Again ϵ_2 is a preframe homomorphism and $\epsilon_2(J) \leq J \quad \forall J$ and so the ϵ_2-fixed elements form a subframe isomorphic to $\Lambda\Omega X$. ΩY is generated by these subframes and from the definitions it is easy to check the regularity-like conditions for

$$(\Omega Y, \Downarrow^{op} - \text{fixed ideals}, \epsilon_2 - \text{fixed ideals})$$

Consequently this last object is a compact regular biframe and so corresponds to an object of **KHausPos**. □

We have a lemma which can be read as a justification for our choice of examining the semi-proper maps:

Lemma 7.6.2 *Say $f : X_1 \to X_2$ is a map between stably locally compact locales. Then f is semi-proper iff the mapping*

$$\begin{array}{rcl} (\Omega f)_\# : P\Omega X_2 & \longrightarrow & P\Omega X_1 \\ F & \longmapsto & \uparrow \{\Omega f(a) | a \in F\} \end{array}$$

takes Scott open filters to Scott open filters.

Proof: Say $(\Omega f)_\#$ maps Scott open filters to Scott open filters and $a \ll b$ where $a, b \in \Omega X_2$. Then the set

$$F \equiv \uparrow \{\Omega f(\bar{b}) | a \ll \bar{b}\}$$

is a Scott open filter. If $\Omega f(b) \leq \bigvee^{\uparrow} S$ for some $S \subseteq^{\uparrow} \Omega X_1$ then $\bigvee^{\uparrow} S \in F$. But F is a Scott open filter and so there exists $s \in S$ such that $s \in F$. Thus $\Omega f(a) \leq s$. The converse is trivial. □

From this (and the fact that $\Omega\mathcal{B}(X)$ is generated by an image of ΩX unioned with an image of $\Lambda\Omega X$) it should be clear how to define a functor:

$$\mathcal{B}_{SP} : (\mathbf{StLocKLoc})_{SP} \longrightarrow (\mathbf{KR2Frm})^{op} \cong \mathbf{KHausPos}$$

In the other direction we want:

$$\mathcal{C}_{SP} : (\mathbf{KR2Frm})^{op} \longrightarrow (\mathbf{StLocKLoc})_{SP}$$

This is given on objects by taking the second member of the triple ($(L_0, L_1, L_2) \mapsto L_1$) and is given on morphisms by restriction. The easiest way to see that this restriction corresponds to a semi-proper locale map is by noting that for $a, b \in L_1$ we have

$$a \ll_{L_1} b \quad \Leftrightarrow \quad a \prec_1 b$$

and that $\prec_1$ is preserved by any compact regular biframe map.

Clearly $\mathcal{C}_{SP}\mathcal{B}_{SP}(X) \cong X$.

In the other direction say (L_0, L_1, L_2) is a compact regular biframe. We know (Theorem [7.4.1]) that $L_2 \cong \Lambda L_1$ and so if $IdlB_{L_1}$ is the ideal completion of the free Boolean algebra qua distributive lattice on L_1 then there is an embedding of L_0 into $IdlB_{L_1}$.
L_0 (viewed as a subframe of $IdlB_{L_1}$) is the frame generated by the union of the images of the embeddings of L_1 and ΛL_1. So

$$(L_0, L_1, L_2) \cong \mathcal{B}_{SP}\mathcal{C}_{SP}(L_0, L_1, L_2)$$

and we have recaptured Banaschewski and Brümmer's result that

$$(\mathbf{KR2Frm})^{op} \cong (\mathbf{StLocKLoc})_{SP}$$

Consequently:

$$\mathbf{KHausPos} \cong (\mathbf{StLocKLoc})_{SP} \qquad \text{(a)}$$

It was pointed out at the end of Chapter 5 that the classical correspondence between compact regular biframes and compact Hausdorff posets was shown in Priestley's paper [Pri72]. As for the classical equivalence between stably locally compact spaces and compact regular T_0 bispaces we find that this appears in [Sal84]. Oswald Wyler's paper 'Compact ordered spaces and prime Wallman compactifications' ([Wyl84]) classically covers both equivalences: the stably locally compact locales correspond to the algebras of the prime Wallman compactification functor, a fact that is also in [Sim82].

We now make a set of observations which will allow us to conclude that result (a) above is an extension of localic Priestley duality. The category of coherent locales has as morphisms the semi-proper maps between coherent locales, **CohLoc** is a full subcategory of $(\mathbf{StLocKLoc})_{SP}$. It is certainly clear from the definition of the category of ordered Stone locales that it is a full subcategory of the compact Hausdorff posets. So it is natural to check whether the equivalence just checked (i.e. (a)) is an extension of the equivalence between ordered Stone locales and coherent locales as outlined in the previous chapter.

Recall that we defined

$$\mathcal{C} : \mathbf{OStoneLoc} \longrightarrow \mathbf{CohLoc}$$

by $\Omega\mathcal{C}(X, \leq) = Idl(\{a \in K\Omega X | \Downarrow^{op} a = a\})$. If we can show that:

$$Idl(\{a \in K\Omega X | \Downarrow^{op} a = a\}) \cong \{a | \Downarrow^{op} a = a\}$$

then it will be clear that the equivalence $\mathbf{KHausPos} \xrightarrow{\cong} \mathbf{StLocKLoc}$ is an extension of $\mathcal{C} : \mathbf{StoneLoc} \to \mathbf{CohLoc}$. Certainly we can define a frame homomorphism:

$$\alpha : Idl\{a \in K\Omega X | \Downarrow^{op} a = a\} \longrightarrow \{a | \Downarrow^{op} a = a\}$$

as the unique extension of the distributive lattice inclusion

$$\{a \in K\Omega X | \Downarrow^{op} a = a\} \hookrightarrow \{a | \Downarrow^{op} a = a\}$$

and injectivety of this map clearly lifts to α.
So is α surjective? Recall that the definition of an ordered Stone locale $(X, \leq)$ required:

$$a_{\leq} = \bigvee^{\uparrow}\{\wedge_i(a_i \& \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0,\ a_i, b_i \in K\Omega X,\ \Downarrow^{op} a_i = a_i,\ \Downarrow^{op} b_i = b_i\}$$

Say $a = \Downarrow^{op} a$, then a is a directed join of elements of the form

$$\wedge_i(a_i \vee \Omega!(1 \leq \neg b_i \vee a))$$

where $a_i, b_i \in K\Omega X$ and $\Downarrow^{op} a_i = a_i, \Downarrow^{op} b_i = b_i$. These elements are all intersections of the directed joins:

$$\bigvee^{\uparrow}(\{a_i\} \cup \{1 | 1 \leq \neg b_i \vee a\})$$

But $a_i, 1 \in \{a \in K\Omega X | \Downarrow^{op} a = a\}$ and so α is surjective.
This tells us that if **KHausPos** $\xrightarrow{\cong}$ **StLocKLoc** is applied to an ordered Stone locale then the result is a coherent locale which is isomorphic to the coherent locale given by the Priestley duality functor $\mathcal{C}$.

Similarly to our work on Priestley's duality we find

Lemma 7.6.3 *If (Y, R) is $\mathcal{B}(X)$ for some stably locally compact locale X then there is a pullback diagram:*

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook] & \sqsubseteq \arrow[d, hook] \\
Y \times Y \arrow[r, "k \times k"] & X \times X
\end{tikzcd}$$

where $\sqsubseteq$ is the specialization sublocale and $\Omega k = \Omega l_{\Downarrow}$.

Compare this lemma with Lemma [6.4.3].
Proof: It will be useful to have a formula for the open corresponding to R. I claim that

$$a_R = \bigvee^{\uparrow}\{\wedge_i(a_i \parr \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in \Omega X\}$$

(where we are taking $\Omega X \subseteq \Omega Y$ since Ωk is an injection). Notice that if this claim is true then the result follows by a proof identical to the proof of Lemma [6.4.3].

We translate the claim into its SUP-lattice form. This reads

$$a_R = \bigvee\{a \otimes \neg a | a \in \Omega X\}$$

Define $\aleph = \bigvee\{a \otimes \neg a | a \in \Omega X\}$.
Now $a_R = (\Downarrow^{op} \otimes 1)(\#)$ and so

$$a_R = \bigvee^{\uparrow}\{\wedge_i(\Downarrow^{op} N_i \parr M_i) | \wedge_{i \in I} (N_i \vee M_i) = 0 \quad N_i, M_i \in \Omega Y \quad I \text{ finite}\}$$

Say $\wedge_i(N_i \vee M_i) = 0$. Then

$$\wedge_i(\Downarrow^{op} N_i \parr M_i) = \bigvee_{I = J_1 \cup J_2}(\wedge_{i \in J_1} \Downarrow^{op} N_i) \otimes (\wedge_{i \in J_2} M_i)$$

and so we may conclude $a_R \leq \aleph$ by noting that for every pair J_1, J_2

$$(1) \quad \Downarrow^{op} (\wedge_{i \in J_1} N_i) \in \Omega X$$
$$(2) \quad \wedge_{i \in J_2} M_i \leq \neg \Downarrow^{op} (\wedge_{i \in J_1} N_i)$$

where the latter is by the fact that $(\wedge_{i \in J_1} N_i) \wedge (\wedge_{i \in J_2} M_i) = 0$ and $\Downarrow^{op} \leq Id$. Conversely notice that if $a \in \Omega X$, taking $N_1 = a, M_1 = 0, N_2 = 0, M_2 = \neg a$ proves $a \otimes \neg a \leq a_R$. $\square$

So the antisymmetry of R can be recaptured by noting that k is a monomorphism. Thus we don't have to use biframes in order to prove Lemma [7.6.1].

How does Priestley duality fit into out parallel? We could define 'Priestley Systems' as the images under $\mathcal{B}$ of the coherent locales. It is not quite clear whether these are the proper parallel to the simplest information systems (namely posets with certain relations as morphisms). Surely the proper parallel to a poset is a compact Hausdorff poset? But the posets correspond to the algebraic dcpos and the compact Hausdorff posets, we have seen, correspond to the stably locally compact locales. However the open parallel to the stably locally compact locales are the continuous posets (or CCD locales) rather than the algebraic dcpos (or Alexandrov locales). Perhaps the compact Hausdorff posets treated as Hausdorff systems (i.e. maps are approximable mappings) correspond to the coherent locales? Priestley duality would then show us that every compact Hausdorff poset is isomorphic (as a Hausdorff system) to an ordered Stone locale. This is quickly seen to be false since the equivalences of this chapter clearly prove that **HausSys** is equivalent to the full subcategory of compact Hausdorff posets and so a hypothesis of this kind would lead to the contradiction that the coherent locales are equivalent to the stably locally compact locales. The author's conclusion is that we are not looking at a left right symmetry. Recall the cube drawn at the end of Chapter 2. Algebraic dcpos are contained within the **dcpo** node and coherent locales are in the **Frm** node; the symmetry for these nodes is perpendicular to the preframe/SUP-lattice symmetry that has been the subject of this thesis.

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