Topology via Constructive Logic

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Abstract

By working constructively in the sense of geometric logic, topology can be hidden. This applies also to toposes as generalized topological spaces.

1 Introduction

One aim of my book *Topology via Logic* [6] was to describe the use of toplogy in the denotational semantics of computer programming languages, explaining the topology through a logic of observations that describes a computer program by "what you can observe just by using it". Surprisingly, however, a mathematical structure introduced there, the so-called *topological systems*, found parallels in the work of situation theory, suggesting that the observational analysis has wider applicability. I say this merely to justify the presence of this article in this volume, for its content is more technical.

Specifically, I wish to show how, once one has accepted the desirability of topology and continuity, its use can be *simplified* by working within the constraints of constructive mathematics. Thus I am trying to sell constructivity not as a piece of dogma but for its practical usefulness.

There are various flavours or schools of constructivism, and the one I shall describe is the "geometric" (the name derives from historical roots in algebraic geometry rather than from any geometry evident in its use). Geometric logic is — essentially — described in [2] and [5] (amongst others), but the reader should beware of the terminology. Geometric logic is generally understood to include infinitary disjunctions, but Mac Lane and Moerdijk restrict their definition to the fragment in which all disjunctions are finitary. This is usually called *coherent*. Note that the classical completeness result (Corollary X.7.2 in Mac Lane and Moerdijk, an application of Deligne's theorem) holds only for coherent theories.

I intend to be brief, so I shall assume that the reader already has at least some aquaintance with the first few chapters of [6].

2 Observational Logic

Though it is easy to think of classical logic as a universal language of statements, in any given context it is reasonable to ask whether all its connectives are equally meaningful. Specifically, it is argued in [6] that if formulae represent (finite) observations, then the only reasonable connectives are conjunction and disjunction (and **true** and **false**); but that infinitary disjunctions are also reasonable.

The corresponding logic is (propositional) *geometric* logic. However, the definition of geometric theories is slightly surprising, for the extralogical axioms of a theory are more general in form than just formulae.

A propositional geometric theory is defined by —

- a set of propositional sysmbols
- a set of axioms of the form $\phi \vdash \psi$, where ϕ and ψ are formulae built up from the primitive symbols using the geometric connectives $\wedge$, **true**, $\bigvee$ and **false** (the big $\bigvee$ is intended to indicate arbitrary disjunctions, possibly infinitary).

(Note that negation can be expressed only to a limited extent — negated *formulae* appear as *axioms* $\phi \vdash$ **false**.) The observational intuition is that formulae represent observations, while axioms — how observations relate to each other — represent scientific hypotheses or background assumptions.

Although geometric theories are not mentioned as such in [6], it is evident that they are equivalent to presentations of frames by generators and relations: the propositional symbols are the generators, and the axioms the relations. The frame $\mathrm{Fr}\langle T\rangle$ corresponding to a theory T should be thought of as the "Lindenbaum algebra" for T — the algebra of formulae modulo equivalence.

Let us write $\Omega[T]$ for this frame, so that the corresponding locale is written [T]. What are its points? They are the frame homomorphisms from $\mathrm{Fr}\langle T\rangle$ to Ω (the frame of truth values — classically, $\Omega = \{\mathbf{false}, \mathbf{true}\}$). The universal property of "presenting by generators and relations" says that such a homomorphism is equivalent to a function assigning truth values to the propositional symbols of T, in such a way that the axioms are respected; but that is exactly a model of T: the points of $[T]$ are equivalent to models of T. We shall try to develop the idea that a locale is "the space of models" for a geometric theory.

3 An example

An example given in [6] is that of bitstreams — finite (unterminated) or infinite sequences of zeros and ones. A first observational theory Th_1 takes propositional symbols of the form $[s_n = x]$ where n is an element of the set $\mathbf{N}$ of natural numbers and x is an element of $2 = \{0, 1\}$. (This is a symbol *schema*, describing an $(\mathbf{N} \times 2)$-indexed family of symbols.) The axioms are given by schemas

$[s_n = 0] \wedge [s_n = 1] \vdash \mathbf{false}$

$[s_{n+1} = 0] \vee [s_{n+1} = 1] \vdash [s_n = 0] \vee [s_n = 1]$

Let us immediately analyse the models of this theory. A model interprets each propositional symbol as a truth value, and hence corresponds to a set

$s = \{(n, x) \in \mathbf{N} \times 2 : [s_n = x] \text{ is interpreted as } \mathbf{true}\}$

In addition, the axioms must be respected: so $(n, 0)$ and $(n, 1)$ cannot both be in s — this says just that s is a partial function from $\mathbf{N}$ to 2. Moreover, by the other axiom the domain of definition of s is an initial segment of $\mathbf{N}$ (if it contains n then it also contains all natural numbers less than n). So the points of $[Th_1]$ are equivalent to the partial functions from $\mathbf{N}$ to 2, with domain of definition an initial segment.

An equivalent formulation Th_2 was also given, with propositional symbols **starts** l for l in 2^* (i.e. l a *finite* sequence of elements of 2) and axioms

starts $l \vdash$ **starts** m if $m \sqsubseteq l$ (i.e. if m is a prefix of l)

starts $l \wedge$ **starts** $m \vdash$ **false** if neither l nor m prefixes the other

By a similar analysis, a model of this is equivalent to an *ideal* of 2^*, a subset of 2^* that is lower closed (under the prefix ordering $\sqsubseteq$) and in which any finite subset $S \subseteq_{\text{fin}} I$ has an upper bound in I. So the points of $[Th_2]$ are equivalent to the ideals of 2^*.

It was left as an exercise in [6] to show that the two theories are equivalent — they have isomorphic Lindenbaum frames; we shall look at part of this from a slightly different perspective.

Let us define a transformation F, transforming models of Th_2 (ideals) to models of Th_1 (partial functions on $\mathbf{N}$):

$F(I) = \{(n, x) \in \mathbf{N} \times 2 : \exists l \in 2^*.(\#l = n \wedge l{+}{+}[x] \in I)\}$

($\#$ is the length function on finite lists, $++$ is concatenation, $[-]$ constructs singleton lists. We are assuming that the natural numbers n start with 0.) It is easy enough to see that $F(I)$ is indeed a partial function whose domain of definition is an initial segment.

These ideals and partial functions were only convenient representations of more strictly defined models of propositional theories; let us examine how the construction works on these.

An ideal I corresponds to a 2^*-indexed family of truth values $[\![\mathbf{starts}\ l]\!]$ (i.e. the truth values of the formulae $l \in I$). In the corresponding model of Th_1, $[s_n = x]$ gets the value **true** iff $(n, x) \in F(I)$, and it follows that each $[s_n = x]$ gets the truth value

$\bigvee\{[\![\mathbf{starts}\ (l{+}{+}[x])]\!] : \#l = n\}$

This is a geometric combination of the given truth values, so we can say (at least at the propositional level) that we have a gometric construction of models of Th_1 out of models for Th_2.

Now let us look at something closer to [6], namely the inverse images under F. A proposition $[s_n = x]$ can be viewed as a collection of models of Th_1, namely those for which $[s_n = x]$ is interpreted as **true**. Consider its inverse image under F:

$I \in F^{-1}([s_n = x]) \Leftrightarrow F(I) \in [s_n = x] \Leftrightarrow (n, x) \in F(I)$

$\Leftrightarrow$ for some $l \in 2^*, \#l = n$ and $l{+}{+}[x] \in I$

$\Leftrightarrow I \in \bigvee\{\mathbf{starts}\ (l{+}{+}[x]) : \#l = n\}$

We thus see essentially the same formula used to calculate both inverse images of propositions and direct images of models.

Once we have constructed the inverse images of the primitive formulae $[s_n = x]$, we know those of the more general ones — for inverse image preserves unions and intersections. We find that we get (according to the techniques of [6]) a frame homomorphism from $\Omega[Th_1]$ to $\Omega[Th_2]$, in other words a continuous map from the locale $[Th_2]$ to $[Th_1]$. What this suggests is a connection between, on the one hand, continuous maps between locales (i.e. — by the usual definition — frame homomorphisms going backwards), and, on the other, geometric transformations of models. This is quite general.

Theorem 3.1 *Let T and U be two propositional geometric theories. Then the following are equivalent:*

1. *frame homomorphisms from $\Omega[U]$ to $\Omega[T]$*
2. *geometric transformations of models of T into models of U*

Proof (sketch) The key is that geometric constructions do not rely on classical logic (without negation, excluded middle is not even expressible, let alone assumed), and the frame $\Omega[T]$ can be viewed as a non-classical algebra of truth values. In such an algebra, we can seek non-classical models of propositional geometric theories. In particular, the symbols of T have their obvious interpretation in $\Omega[T]$, and this interpretation respects the axioms of T (it is forced to by the very construction of $\Omega[T]$): this gives a "generic" model of T in $\Omega[T]$, and everything else in $\Omega[T]$ is constructed geometrically from it. In these terms, a frame homomorphism from $\Omega[U]$ to $\Omega[T]$ is just a model of U in $\Omega[T]$, and so a model of U constructed geometrically from the generic model of T. But being a generic model means that it has no properties whatsoever other than those that follow from being a model of T. It follows that any construction on the generic model can be specialized to any specific model. Hence a geometric construction of a model of U from the generic model of T is equivalent to a geometric transformation of arbitrary models of T into models of U. □

Slogan: continuity = geometricity

4 Predicate geometric theories

The models of Th_1 and Th_2 were most naturally expressed as models of *predicate* theories (many-sorted, first order):

For Th_1, we use a binary predicate $s(n, x)$ ($n : \mathbf{N}$, $x : 2$) with axioms —

$s(n, x) \wedge s(n, y) \vdash x = y$

$s(n + 1, x) \vdash \exists y.s(n, y)$

For Th_2, we use a unary predicate $starts(l)$ ($l : 2^*$) with axioms —

$starts(l) \wedge m \sqsubseteq l \vdash starts(m)$

$\mathbf{true} \vdash starts(\epsilon)$ (ϵ here is the empty list)

$starts(l) \wedge starts(m) \vdash \exists n.(starts(n) \wedge l \sqsubseteq n \wedge m \sqsubseteq n)$

(We have directly formulated the ideal condition.)

The geometric theories as defined were propositional, but can the ideas be extended to predicate theories? In fact, predicate geometric theories are well known. The connectives for formulae include not only $\wedge$ and $\bigvee$, but also $=$ and $\exists$; then a predicate geometric theory comprises —

- sorts
- function and predicate symbols, each with declared arity (number and sorts of arguments and result)
- axioms of the form $\phi \vdash_S \psi$ where ϕ and ψ are formulae, constructed from the symbols using the geometric connectives, and whose free variables are all taken from the finite set S (see [4], p.245).

This looks insufficient for our predicate theories for Th_1 and Th_2 — there is much that appears extralogical, such as $\mathbf{N}$, $+$, 2^*, $\#$, $++$, etc. However, these can all be characterized uniquely up to isomorphism by geometric theory, so the theories as given can be augmented by extra vocabulary and axioms to define these symbols. For instance, for $\mathbf{N}$ and $+$, use $0 : \mathbf{N}, s : \mathbf{N} \to \mathbf{N}$ with axioms

$s(x) = 0 \vdash_{\{x\}} \mathbf{false}$

$s(x) = s(y) \vdash_{\{x,y\}} x = y$

$\mathbf{true} \vdash_{\{x\}} \bigvee_n x = s^n(0)$

(This last one has the air of cheating — it presumes an external $\mathbf{N}$ to index the infinite disjunction. However, there has to be some kind of trick at this point, for Gödel's theorem tells us that $\mathbf{N}$ can't be characterized in finitary first order logic. We shall return to this point later.) Then $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is characterized by

$\mathbf{true} \vdash_{\{y\}} 0 + y = y$

$\mathbf{true} \vdash_{\{x,y\}} s(x) + y = s(x + y)$

The effect is to give a collection of geometric constructions that can be used within the theories. These include —

- Cartesian product
- Disjoint union
- Equalizers and coequalizers (quotients)
- Free algebra constructions (e.g. $\mathbf{N}$ and list types X^*)
- Recursively defined functions
- Finite powersets $\mathcal{F}X$ (isomorphic to free semilattices)
- Universal quantification bounded over finite sets: $\forall x \in S.P(x)$, where $S : \mathcal{F}X$

However, they do not include exponentiation X^Y or full power sets $\mathcal{P}X$ (the logic is weak second order). Technically, the "geometric constructions" are those that are preserved by the inverse image functors of geometric morphisms between toposes.

It now turns out that the transformation F of models (Th_2 to Th_1) of the *predicate* theories can be expressed geometrically. So also can the corresponding inverse transformation G from Th_1 to Th_2, and the isomorphisms $s \cong F(G(s))$, $I \cong G(F(I))$ needed to show the theories equivalent.

Hence: we argued that continuity was geometricity on models for propositional geometric theories; but in practice it is more convenient to work with equivalent predicate theories.

Of course, there is a non-trivial technical claim here: that the propositional and predicate notions of "geometric transformation" agree. This comes out of the machinery of sheaf theory.

5 Observational intuition

Expanding on [6]'s observational intuition for the propositional geometric logic, one can also [7] give an observational account of the predicate logic. The idea is to describe a set not as a fully comprehended collection of elements, but as instructions for dealing with such elements as you might encounter:

1. how to know when you've "apprehended" an element of the set;
2. how to know when you've observed two apprehended elements to be equal.

(cf. Bishop's [see 1] definition of a set as comprising a stock of representations of elements, and a defined equality relation on them.)

Though this is informal, the intuition fits well with the constructions listed as geometric. For instance for a free algebra, you know how to recognize terms and how to check proofs of equality between terms. (Notice how for algebraic theories with undecidable word problem, *in*equality between terms is not algorithmically checkable.)

In practice, these intuitions provide a good bench mark for testing the geometric validity of arguments. For example, we claimed that the finite powerset construction $\mathcal{F}$ was geometric. If X is described observationally as above, then so is $\mathcal{F}X$: a finite subset of X is apprehended by apprehending all its elements and listing them (though because *in*equality is not necessarily observable, you can't guarantee that all the elements of the list are distinct). To observe that $\{x_1, \cdots, x_m\} = \{y_1, \cdots, y_n\}$, you observe that each x_i is equal to some y_j and vice versa.

Suppose now that $S = \{x_1, \cdots, x_m\}$ is a finite set, and $\phi(x)$ is an observable property. Is $\{x \in S : \phi(x)\}$ finite? Not in general! To list *all* its elements, you'd have to know that the unlisted elements don't have property ϕ, and the problem is that $\neg\phi$ is not necessarily observable. It turns out that this and similar unexpected behaviour is already known to topos theorists; the observational account gives a rough and ready way of anticipating it.

6 Generalized topological spaces (toposes)

The predicate versions of Th_1 and Th_2 were equivalent to propositional theories. (Technically, this follows from the fact that all the sorts — $\mathbf{N}$, 2^*, etc. — were geometrically derivable out of nothing: "propositional" means no *essentially new* sorts.) However, a truly predicate theory can also be thought of in this spatial way. A propositional theory was thought of as describing a "locale", its "space" of models (both classical and non-classical); but technically it was represented as a frame, the topology. A continuous map between locales is really a geometric transformation of models into models, technically representable in reverse by the inverse image function, a frame homomorphism.

Similary, a predicate theory can be thought of as describing a "generalized space" of its models, and this is a *topos* in the sense of Grothendieck's dictum, "a topos is a generalized topological space". However, the technical definition, the "generalized topology", is more complicated. The Lindenbaum frame of propositions (formulae without free variables) is no longer adequate for predicate theories, and has to be extended to a "Lindenbaum category of sets", the category you see constructed in — for instance — [5] as the classifying topos of the theory. (This is "topos as generalized category of sets". Note that this "generalized topology" is different from the "Grothendieck topology" that is used at a certain stage in the construction. The category and Grothendieck topology that comprise a "site" are more analogous to the generators and relations of a presentation, something that can be seen more clearly in Johnstone's [3] sites for frames — the Grothendieck topology is the analogue of the coverage.) The "continuous maps" — the geometric transformations of models — now appear as geometric morphisms between toposes.

Hence our reinterpretation of continuity as geometricity has also cast light on the notion of topos as generalized space, a notion of which it is easy to lose sight in the standard accounts of toposes.

7 Arithmetic Universes

What follows is more speculative, though existing results lend support to the broad argument suggested.

A crucial feature of geometric logic is the arbitrary disjunctions, that is to say disjunctions of arbitrary sets of formulae. This was used to justify inductive and recursive constructions as geometric, and suggest a "geometric mathematics" that is algorithmic in flavour.

However, there is a gap here. The arbitrary set-indexed disjunctions of geometric logic encompass far more than the recursively indexed ones, and in fact the extent of geometric tranformations (as continuous maps) depends on your underlying idea of what sets are — for this determines what disjunctions you can form. Thus geometric logic is not absolute in itself, but relative to the chosen set theory. In fact, this leads to certain anomalies in the observational interpretation.

A simple one raised by Mike Smyth (and mentioned in [6]) concerns the discrete topology on the natural numbers $\mathbf{N}$. For sure, every singleton $\{n\}$ represents a finitely observable property of natural numbers, so any disjunction of singletons — i.e. any subset of $\mathbf{N}$ — should also be finitely observable. Consider then, for an algorithm A,

$\bigvee\{\{n\} : A(n) \text{ does not terminate}\}$

Can this really be "finitely observable"? That would seem to imply a solution to the halting problem, which of course is impossible. The catch is that we have used classical set theory to comprehend the disjuncts, and this has smuggled in inobservable features. Thus geometric logic based on classical set theory does not exactly capture the observational ideas.

A likely-looking way out is to restrict the infinities to effective ones, by taking a primitive collection of "geometric constructions" (such as those listed in Section 4) as directly defining our notion of set theory. The logical disjunctions come out of set-theoretic disjoint unions (taking images to obtain non-disjoint unions). These would be of only finitely many sets (properly, geometric logic countenances disjoint unions of infinitely many sets), but this would be partially compensated for by the inductive constructions. It is conjectured that this can be precisely formalized in category theory, by using Joyal's *arithmetic universes* [unfortunately unpublished]: that the categorical structure postulated in an arithmetic universe models the "effective geometric constructions". It is hoped that by thus restricting the notion of geometric construction (and hence the corresponding notion of continuous map), a fragment of topology can be found that genuinely matches the observational ideas.

To return to the "cheating" characterization of **N** given in Section 4, **N** would be characterized not logically, with an externally indexed countable disjunction, but categorically, as a free algebra with constant 0 and unary operator s.

8 Conclusions

By keeping one's mathematics constructive, one can make a lot of topology implicit: if the points are described as models of a geometric theory, then the topology is defined implicitly, and if transformations are defined geometrically then continuity is automatic. This works not only for ordinary topology, but also for Grothendieck's "generalized topological spaces" (toposes), generalizing from propositional to predicate theories.

On a practical level, constructive reasoning can thus can lighten the burden of topological discussion; at a deeper level it is hoped that the approach can bring a true reconcilliation between topology and effective mathematics.

Further exposition of these ideas can be found in [8] and [9].

9 Bibliography

Some of my own papers are also available electronically — see Web page

http://theory.doc.ic.ac.uk:80/people/Vickers/

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