Topos Investigations of the Extended Priestley Duality

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Abstract

This is a brief summary of the original results in the thesis.

First, there is a localic version of the correspondence between perfect and patch continuous monotone maps. To this end, Escardó's localic patch construction for a stably compact locale is used. Given a stably compact locale X, we define constructively the order in its patch locale. We also introduce localically the notion of a monotone patch continuous function in this context. The fact that lax pullbacks of perfect maps produce proper maps in **Loc** is proved. Vickers' preframe techniques are used throughout. Beck-Chevalley conditions for lax-coequalisers are also proved.

When working in $\mathfrak{Top}$, the 2-category of Grothendieck topoi and geometric morphisms, it is natural to consider functors between the (generalised) points of topoi. A 2-categorical criterion of an adjunction situation $F : X \rightleftharpoons Y : G$ in $\mathfrak{Top}$ is proved by constructing the classifying topoi of maps $Fx \to y$ and $x \to Gy$, where x, y are points of X, Y respectively and identifying them with inserters in $\mathfrak{Top}$.

Next, it is demonstrated that relative tidiness (in the sense of Moerdijk and Vermeulen) is the right topos-generalisation of perfectness. Vickers has shown that the exponential of topoi $[set]^X$, where X is a stably compact locale, classifies the geometric theory of B-sheaves which implies that a point of $[set]^X$ at stage Z is a B-sheaf in the sheaves over Z. For $f : X \to Y$ a perfect map between two stably compact locales, a description of the map $[set]^f : [set]^X \to [set]^Y$ is given and is shown to have a right adjoint. The definitions of the geometric morphisms are given by geometric constructions on the points of the exponential topoi, i.e., the B-sheaves. The geometricity of these constructions is guaranteed by the fact that we can represent perfect maps by strong homomorphisms between strong proximity lattices. The adjunction is proved by application of the 2-categorical criterion in the 2-category $\mathfrak{Top}$. The main result of this chapter is that for a map $f : X \to Y$ between stably compact locales, f is perfect if and only if f is relatively tidy.

Finally, there are investigations with a possible topos analogue of the patch construction. Some results are given on relatively tidy maps between structures that are examples of "stably compact topoi". It is argued by example, that "stably compact topoi" and relatively tidy maps should convey the notion of local partial ordering in the same sense that stably compact locales and perfect maps amount to (globally) partially ordered locales and monotone continuous maps.

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Chapter 1

Background

1.1 Our slice of topoi-On notation

Throughout this thesis, the term "topos" is used for a Grothendieck topos, or equivalently for a elementary topos that fulfills the assumptions of Giraud's theorem (see e.g. [Joh77], theorem 0.45). The 2-category of Grothendieck topoi, geometric morphisms and natural transformations is denoted as $\mathfrak{T}op$. We are thus usually suppressing the symbol of the base topos from its notation, except when there is ambiguity from the context. Also, sometimes, the symbol $\mathfrak{T}op$ will be using for the 1-category of Grothendieck topoi. We explicitly refer to (the sheaves of) our base topos as **Sets**, but we do not assume that these are the classical sets; we do not allow use of the law of excluded middle or the axiom of choice. This implies that the subobject classifier Ω of **Sets** (i.e. the power object of the terminal set) is not the Boolean **2** but it is still the terminal locale (initial frame). One thing that we assume is the existence of a natural numbers object. Indeed we rely on universal algebra for several of our results. Therefore, everywhere, **Sets** could be substituted with any Grothendieck topos B with a natural numbers object.

A frame is a poset with all joins and finite meets such that the arbitrary joins distribute over finite meets. A frame homomorphism is a function that preserves arbitrary joins and finite meets. We denote that the category of frames and frame homomorphisms as **Fr**. We define **Loc**, the category of locales to be the opposite of the category of frames. We use a single letter (say) X for the locale and ΩX for its defining frame. Locale theory offers the possibility to do point-free topology and thus constructive topology; a frame and therefore a locale is definable inside the sheaves of any topos.

We adopt the standpoint that contemplates Grothendieck topoi as generalised topological spaces or generalised locales. This manifests itself in the notation we use; an object of $\mathfrak{Top}$ is denoted with a Latin letter (e.g. X), whereas we reserve the symbol $\mathcal{S}X$ for the category of its sheaves, very much in parallel with the juxtaposition of a locale X with the frame of its opens ΩX. In this context, a geometric morphism will be denoted as $f : X \longrightarrow Y$, whereas, e.g. the inverse image functor of f as $f^* : \mathcal{S}Y \longrightarrow \mathcal{S}X$.

Let us for a while write B for the base topos. Let X be a locale inside $\mathcal{S}B$, corresponding to an $\mathcal{S}B$-internal frame ΩX. Taking sheaves over X (i.e. the category of functors $\Omega X \longrightarrow SB$ obeying the usual patching conditions) amounts to a functor

$$\mathcal{S} : \mathbf{Loc}(B) \longrightarrow \mathfrak{Top}/B$$

between the category of locales inside $\mathcal{S}B$ to topoi over B ([JT84]). We say that the image of a locale X inside $\mathcal{S}B$ under the functor $\mathcal{S}$ is the *locale over* B *that corresponds* to X.

Conversely, if $f : X \longrightarrow B$ is a topos over B, we choose to write ΩX for the frame

$$\Omega X := f_* \Omega_{\mathcal{S}X}$$

where $\Omega_{\mathcal{S}X}$ is the subobject classifier of $\mathcal{S}X$. Therefore, twisting the notation a little, we write $\mathbf{\Omega}$ for the functor

$$\mathbf{\Omega} : \mathfrak{Top}/B \longrightarrow \mathbf{Loc}(B)$$

that sends a topos X over B to the locale inside $\mathcal{S}B$ whose defining frame is $\Omega X := f_* \Omega_{\mathcal{S}X}$.

The functor $\mathcal{S}$ is a full embedding of the category $\mathbf{Loc}(B)$ of locales in $\mathcal{S}B$, into the category $\mathfrak{Top}/B$ of topoi over B.

Following Joyal and Tierney, we say that a topos $f : X \longrightarrow B$ is localic over B, iff there is an equivalence

$$\mathcal{S} \circ \Omega X \simeq \mathcal{S}X$$

Concluding, a locale X has opens ΩX and also sheaves ("generalised opens") $\mathcal{S}X$. Conversely, a topos X has sheaves $\mathcal{S}X$ and also opens ΩX.

We return to suppressing the base topos from the notation. If $\mathbf{C}$ is an (small) external category, and by that we mean a category on **sets**, we denote the topos whose category of sheaves is $[\mathbf{C}, \mathbf{Sets}]$ by the symbol $\hat{}\,\mathbf{C}$, i.e. $\mathcal{S}\,\hat{}\,\mathbf{C} \equiv [\mathbf{C}, \mathbf{Sets}]$. Furthermore, if $\mathfrak{I}$ is explicitly internal in the sheaves of a topos X, this is indicated by using the symbol $X\hat{}\,\mathfrak{I}$ for the topos whose sheaves are the $\mathcal{S}X$-internal (covariant) diagrams $\mathcal{S}X^{\mathfrak{I}}$, i.e.

$$\mathcal{S}(X\hat{}\,\mathfrak{I}) := \mathcal{S}X^{\mathfrak{I}}$$

When we consider finite products of topoi (that always exist), we can approach them via the geometric theories they classify. Nevertheless, it is useful to have a more concrete characterisation of them. If X and Y are two Grothendieck topoi, the categories of their sheaves $\mathcal{S}X$ and $\mathcal{S}X$ are cocomplete (and locally small) categories of set valued functors from a small category $\mathbf{C}$ that preserve a Grothendieck topology J on that category. It is therefore natural to seek their tensor product. Pitts [Pit85] indeed defines their tensor product $\mathcal{S}X \otimes \mathcal{S}Y$ in the 2-category **COCTS** of cocomplete locally small categories, small colimit preserving functors and natural transformations, in the usual way: it amounts to a functor $\otimes_{\mathcal{S}X,\mathcal{S}Y} : \mathcal{S}X \times \mathcal{S}Y \longrightarrow \mathcal{S}X \otimes \mathcal{S}Y$ in **COCTS** (i.e. a functor that preserves small colimits in both arguments), such that for any category $\mathbf{C}$ and any functor $F : \mathcal{S}X \times \mathcal{S}Y \longrightarrow \mathbf{C}$ in **COCTS**, there is a unique (up to isomorphism) functor $H : \mathcal{S}X \otimes \mathcal{S}Y \longrightarrow \mathbf{C}$ such that $H \circ \otimes_{\mathcal{S}X,\mathcal{S}Y} = F$.

Pitts proves the following ([Pit85], 2.3).

Theorem 1.1 *The category of sheaves over the product $X \times Y$ of two Grothendieck topoi is given by the tensor product $\mathcal{S}X \otimes \mathcal{S}Y$ in* **COCTS***. In particular, this implies that $\mathcal{S}X \otimes \mathcal{S}Y$ always exists.*

This is an other analogy with locale theory ($\Omega(X \times Y) = \Omega X \otimes \Omega Y$) and more precisely, as Pitts argues, with the fact that $\Omega(X \times Y)$ is the tensor product of ΩX and ΩY as sup-lattices ([JT84]).

Suppose that $\mathbf{C}$ is a category in **Sets** and X any topos. The product $X \times \hat{}\mathbf{C}$ in $\mathfrak{Top}$ is the trivial pullback

$$\begin{tikzcd} X \times \hat{}\mathbf{C} \arrow[r, "!^{\mathbf{C}}"] \arrow[d] & \hat{}\mathbf{C} \arrow[d] \\ X \arrow[r, "!"] & \mathbf{1} \end{tikzcd} \tag{1.1}$$

Therefore, it is a consequence of Diaconescu's theorem that $X \times \hat{}\mathbf{C}$ is the topos (that we denote) $X\hat{}!^*\mathbf{C}$, i.e. the topos whose sheaves are the internal diagrams

$$!^*(\mathbf{C}) \longrightarrow \mathcal{S}X$$

where $!^*(\mathbf{C})$ is the "internalised version" in $\mathcal{S}X$ of $\mathbf{C}$. Diaconescu's theorem is proved in [Joh77], 4.34 and the point we make about diagram 1.1 is a direct application of corollary [Joh77], 4.35, which can be viewed as a change of base law for presheaf topoi.

We finish with an observation which, although it seems innocuous, it is of great importance in the construction of geometric morphisms (see section 1.2). let $\mathbf{2}$ be the finite category $\bullet \longrightarrow \bullet$ (plus the two identities) in **Sets**. Then $\hat{}\mathbf{2}$ is the *Sierpiński* topos and we denote it by \$. An object of $\mathcal{S}\$$ is a pair of sets A_1, A_0 together with a function $f : A_1 \longrightarrow A_0$. The category $\mathbf{2}$ is finite and therefore it is indistinguishable from its internal version in $!^*(\mathbf{2})$ in $\mathcal{S}X$, for $! : X \longrightarrow \mathbf{1}$ (see example 2.39 in [Joh77]). Therefore, according to the discussion above, the category of sheaves over the topos

$$X \times \hat{} \mathbf{2} := X \times \$$$

is $[!^*(\mathbf{2}), \mathcal{S}X] \cong [\mathbf{2}, \mathcal{S}X]$. Moreover, the tensor product of $\mathbf{2}$ with any topos X exists ($\mathbf{2}$ is not a topos). More specifically, we have the following fact which is a special case of lemma B3.4.2 in [Joh02].

Lemma 1.2 *Any topos X has a tensor product with $\mathbf{2}$ and this is (the topos whose sheaves are) $[\mathbf{2}, \mathcal{S}X]$. This implies that for any topos X, we have the equivalence*

$$\mathfrak{Top}(X \times \$, Y) \simeq [\mathbf{2}, \mathfrak{Top}(X, Y)] \tag{1.2}$$

1.2 Constructivism and geometricity

As mentioned in section 1.1 this thesis dwells in a universe of mathematical discourse where the axiom of excluded middle and the axiom of choice do not exist. In other words all our results are constructive.

Adhering to constructive mathematics guarantees that constructions and facts are meaningful inside the sheaves of an arbitrary topos. Yet, this does not ensure that they are preserved when *transferred* between different topoi.

Definition 1.3 *We say that a construction is geometric when it is preserved by the inverse image of geometric morphisms.*

Geometric constructions include constructions that can be formalised in terms of finite limits, arbitrary colimits, epiness, monicness, (Kuratowski) finiteness, existential quantification, universal quantification over finite objects, free algebraic constructions (including free essentially algebraic constructions, see section 5.5), natural numbers and others, although we shall be basically using the ones mentioned.

A large part of this thesis purports to ascertain the existence of geometric counterparts for structures that are fundamentally non geometric, namely structures related to stably compact locales and perfect maps. In chapter 3 we represent perfect maps between stably compact locales using certain homomorphisms between strong proximity lattices. The latter are geometric being homomorphisms between geometric structures. In chapter 5 we demonstrate that sheaves over stably compact locales (non geometric structure) are equivalent with "$\mathcal{B}$-sheaves" over strong proximity lattices. Finally, in chapter 6 we essentially construct the geometric counterpart of a geometric morphism

$$f : Sh(X) \longrightarrow Sh(Y)$$

induced by a perfect map $f : X \longrightarrow Y$ between two stably compact locales.

Geometricity is obviously more restrictive than constructivism. On the other hand there is much to be gained if one abides its rigour. After all, one of the properties of this versatile entity called Grothendieck topos is that it classifies geometric theories. We proceed to outline how the construction of a geometric morphism between two topoi can be facilitated by geometricity.

By a *geometric transformation* F^γ between two geometric theories $\mathbb{T}_X$ and $\mathbb{T}_Y$ we mean a geometric construction that takes models of the theory $\mathbb{T}_X$ inside the sheaves of a topos Z to models of the theory $\mathbb{T}_Y$ in the sheaves of the same topos Z. In more detail, suppose that $\mathbb{T}_X$ contains sorts $\widetilde{X_i}$, relations $\widetilde{R_i}$, function symbols $\widetilde{f_i}$, constants $\widetilde{c_i}$ and axioms $\widetilde{A_i}$. A geometric transformation amounts to specifying a geometric process according to which, once a model of this theory in the sheaves of it any topos Z has been chosen, say

$$\mathcal{M}_X = \{\{X_i^{\mathcal{M}}\}, \{R_i^{\mathcal{M}}\}, \{f_i^{\mathcal{M}}\}, \{c_i^{\mathcal{M}}\}\}$$

(such that the axioms R_i are valid), a new set of objects and arrows

$$F^\gamma(\mathcal{M}_X) = \{\{Y_j^{F(\mathcal{M})}\}, \{S_j^{F(\mathcal{M})}\}, \{g_j^{(F\mathcal{M})}\}, \{d_j^{(F\mathcal{M})}\}\}$$

in $\mathcal{S}Z$ can be constructed which is a model of the theory $\mathbb{T}_Y$, i.e., the axioms B_j are valid.

We are going to demonstrate that a geometric transformation uniquely specifies a geometric morphism. We point out that the exposition in this section follows the one in [Vic98a], where the following exists.

Lemma 1.4 *Let $\mathbb{T}_X$ and $\mathbb{T}_Y$ be two geometric theories and $X = [\mathbb{T}_X]$, $Y = [\mathbb{T}_Y]$ their respective classifying topoi.*

A geometric morphism $F : [\mathbb{T}_X] \longrightarrow [\mathbb{T}_Y]$ can be uniquely specified (up to equivalence) by defining a geometric transformation F^γ between the geometric theories $\mathbb{T}_X$ and $\mathbb{T}_Y$.

II. As a consequence, functoriality of F is automatic

Proof. Indeed, the machinery is provided by the theory of classifying topoi. A geometric transformation F^γ can be applied to the universal model $\mathcal{U}_X$ of $\mathbb{T}_X$ in $\mathcal{S}X$. $F^\gamma(\mathcal{U}_X)$ is then an object in $\underline{Mod}(\mathcal{S}X, \mathbb{T}_Y)$ and (the object part of) the functor τ_X in

$$\underline{Mod}(\mathcal{S}X, \mathbb{T}_Y) \underset{\sigma_X}{\overset{\tau_X}{\rightleftarrows}} \mathfrak{T}op(X, Y) \qquad (1.3)$$

specifies uniquely (up to isomorphism) a geometric morphism $F : X \longrightarrow Y$. ■

Let us suppose now that we fix the base topos Z. We have stressed already that F^γ is a mere map between the objects of the categories $Mod(\mathcal{S}Z, \mathbb{T}_X)$ and $Mod(\mathcal{S}Z, \mathbb{T}_Y)$. But we can argue that geometricity also provides us with an arrow part so that F^γ becomes a functor between these categories. Indeed, if F^γ is applicable on the models of $\mathbb{T}_X$ in the sheaves of any topos, we can choose this topos to be $Z \times \$$, where $\$ \equiv \hat{\ }\mathbf{2}$ is the Sierpiński topos (see section 1.1). Hence, F^γ also specifies a map between the objects of $Mod(\mathcal{S}(Z \times \$), \mathbb{T}_X)$ and $Mod(\mathcal{S}(Z \times \$), \mathbb{T}_Y)$. The following lemma suffices to complete the argument.

Lemma 1.5 *Let $\mathbb{T}_X$ be a geometric theory and X its classifying topos. Let also Z be any Grothendieck topos. Then the following are equivalent.*

i Geometric morphisms $Z \longrightarrow [\mathbb{H}om\mathbb{T}_X]$.

ii 2-cells between geometric morphisms

$$\begin{tikzcd} Z \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow"] & X \end{tikzcd}$$

iii Geometric morphisms $Z \times \$ \longrightarrow X$.

iv Geometric morphisms $Z \longrightarrow X^{\$}$.

Proof. (i)⇔ (ii): This is the classifying topos property, i.e. the equivalence 1.3.

(iii)⇔ (iv): Because of the exponentiation adjunction.

(ii)⇔(iii): We look at the equivalence 1.2. The geometric morphisms of (iii) are the objects of the category in the L.H.S. of 1.2. The 2-cells of (ii) are objects of the R.H.S. category of 1.2. ■

Lemmas 1.4 and 1.5 basically express the fact that we can construct geometric morphisms between topoi by specifying their action on their points provided that this specification is geometric enough to make sense on points at any stage. This is an essentially topological aspect of topoi. We are going to augment this aspect in chapter 4, where we prove that in order to establish an adjunction $F \dashv G$ between *geometric morphisms*

$F : X \rightleftharpoons Y : G$, it suffices to show that there is a *bijection* of homomorphisms

$$F(x) \longrightarrow y \text{ and } x \longrightarrow Gy$$

where x, y are points of X and Y. There, naturality with respect to x and y comes for free by geometricity.

1.3 Change of base for locales

Locale theory is a constructive way of doing topology. Nevertheless, a frame is not a geometric structure; it is not preserved when pulled back along geometric morphisms. We give an account of how the change of base for locales is treated. It will transpire that, in some sense, a frame is not geometric but a locale is.

Let X be a locale inside the sheaves $\mathcal{S}Z$ of a topos. In section 1.1, we mentioned that there is a locale X' over Z that corresponds to X and this is obtained as the topos whose category of sheaves is $\mathcal{S}X$ (the image of X under the functor $\mathcal{S}$). Joyal and Tierney prove the following in [JT84].

Theorem 1.6 *If $X' \longrightarrow Z$ is a locale over Z, then for any topos $p : E \longrightarrow Z$, there is an equivalence between geometric morphisms $E \longrightarrow X'$ and frame homomorphisms $\Omega X' \longrightarrow \Omega E$ inside $\mathcal{S}Z$.*

The above theorem is equivalent with the statement that the functor $\mathbf{\Omega}$ is left adjoint to the functor $\mathcal{S}$. Since $\mathcal{S}$ is essentially an inclusion of locales into topoi, the adjunction $\mathbf{\Omega} \dashv \mathcal{S}$ is a reflection of $\mathfrak{T}op/Z$ in $\mathbf{Loc}(Z)$, usually referred to as the *localic reflection.*

Using theorem 1.6 we can easily prove the next lemma.

Lemma 1.7 *Suppose that Z is a topos and X is a locale inside $\mathcal{S}Z$. The locale $q : X' \longrightarrow Z$ over Z (i.e. a localic topos over Z) corresponds to X iff*

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "p"'] & & X' \arrow[dl, "q"] \\ & Z & \end{tikzcd} \tag{1.4}$$

for any topos $p : E \longrightarrow Z$ over Z, there is an equivalence between geometric morphisms $E \longrightarrow X'$ over Z and frame homomorphisms

$$\Omega X \longrightarrow \Omega E (= p_*(\Omega_{\mathcal{S}E})) \tag{1.5}$$

inside $\mathcal{S}Z$.

Proof. Let it be the case that X' corresponds to X. By theorem 1.6 we have that for any topos E over Z, $\mathfrak{Top}/Z(E, X') \simeq \mathbf{Fr}(Z)(\Omega X', \Omega E)$. But since X' is just the topos over Z whose sheaves are $\mathcal{S}X$, we get

$$\Omega X' \cong \Omega X \text{ in } \mathcal{S}Z$$

Conversely, let X'' be any locale over Z such that, for any topos E over Z, $\mathfrak{Top}/Z(E, X'') \simeq \mathbf{Fr}(Z)(\Omega X, \Omega E)$. Since $\Omega X' \cong \Omega X$, we also get $\mathfrak{Top}/Z(E, X') \simeq \mathbf{Fr}(Z)(\Omega X, \Omega E)$. Therefore, $\mathfrak{Top}/Z(E, X') \simeq \mathfrak{Top}/Z(E, X'')$. This gives the equivalence $X' \simeq X'$ over Z on the grounds of X' and X'' having equivalent points (=models) at any stage over Z. ∎

Now, following Vickers [Vic97], instead of a locale X inside $\mathcal{S}Z$ (which corresponds to a frame inside $\mathcal{S}Z$), we can consider a *frame presentation* $\mathbf{Fr}\langle G|R\rangle$ inside $\mathcal{S}Z$. Then, by the universal property of frame presentations, the frame homomorphisms of 1.5 are equivalent to maps $G \longrightarrow \Omega E$ that preserve the relations R. The advantage is that frame presentations, say in $\mathcal{S}Z$, are geometric in the sense that they are models in $\mathcal{S}Z$ of a geometric theory.

In short, a frame presentation is formally equivalent to an *essentially propositional* geometric theory, i.e. a geometric theory with no sorts and hence no function symbols. The generators of the frame presentation correspond to the relation symbols of the theory and the relations to the axioms. The presence of the adjective "essential" signifies that models of such a theory can contain objects corresponding to sorts that do not exist a priori, i.e. objects other than subobjects of **1**. These objects come into existence naturally by manipulating the frame presentation geometrically (see an example in section 3.3). For more on frame presentations as essentially propositional geometric theories, see [Vic99a].

Vickers (see [Vic97]) considers frame presentations brought into a particular form, the "GRD" form. Using that, for a given frame presentation inside, say $\mathcal{S}Z$, he constructs the locale over Z that corresponds to this presentation. The following is the centrepiece result pertaining to the change of base for locales ([Vic01], corollary 5.4).

Theorem 1.8 *Let $q : X \longrightarrow Z$ be a locale over Z that corresponds to a frame ΩX with a frame presentation P in $\mathcal{S}Z$. Let $f : E \longrightarrow Z$ be any geometric morphism. Then the locale over E that corresponds to the frame presentation $f^*(P)$ is given by the pullback in $\mathfrak{Top}$*

$$\begin{tikzcd} E \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ E \arrow[r, "f"] & Z \end{tikzcd} \tag{1.6}$$

Roughly speaking, the underlying idea in Vickers' construction is that a frame may not be geometric, but free constructions are. So instead of pulling back a frame in $\mathcal{S}Z$ along $f : E \longrightarrow Z$, we can first pull back its presentation and then obtain the frame that $f^*(P)$ generates in $\mathcal{S}E$.

Joyal and Tierney have investigated the change of base for locales in [JT84]. They demonstrated that locales can be regarded as certain types of monoids in the symmetric monoidal category of SUP-lattices (a SUP-lattice is a partial ordered set with all arbitrary joins). A geometric morphism $E \longrightarrow Z$ induces a functor $f^{\#}$ that takes SUP-lattices in $\mathcal{S}Z$ to SUP-lattices in $\mathcal{S}E$ ([JT84], proposition VI.1). Moreover, $f^{\#}$ preserves the symmetric monoidal structure of SUP-lattices and hence locales. Therefore, a locale $X \longrightarrow Z$ regarded as "a certain monoid" in the category of SUP-lattices inside $\mathcal{S}Z$ can be pulled back along a geometric morphism $f : E \longrightarrow Z$. They first proved that the locale over E that corresponds to $f^{\#}(X)$ is given by the pullback diagram 1.6.

1.4 Stably compact locales - the extended Priestley duality

Definition 1.9 *A map $f : X \longrightarrow Y$ between two locales is perfect iff f_* preserves directed joins, i.e. it is Scott continuous. In literature, perfect maps are more usually referred to as semi proper maps.*

Lemma 1.10 *A map $f : X \longrightarrow Y$ between two locales is perfect iff for any $b \in \Omega Y$ and any $a \in \Omega X$ with $b \ll f_*(a)$, there is $a' \ll a$ such that $b \leq f_*(a')$.*

Proof. ?? ■

Definition 1.11 *A locale X is stably locally compact iff the following conditions are satisfied.*

(i) X is locally compact, i.e. $a = \bigvee^{\uparrow}\{a_i | a_i \ll a\}$ for any $a \in \Omega X$.

(ii) The way below relation of the frame ΩX is meet-stable, i.e. $a \ll b_1, a \ll b_2 \Rightarrow a \ll b_1 \wedge b_2$ in ΩX.

Definition 1.12 *A locale X is stably compact iff, in addition to the conditions of definition 1.11, we have*

(iii) X is compact, i.e. $1 \ll 1$ in ΩX.

We are going to denote by **StKLoc** the category of stably compact locales and perfect maps.

The following lemma is proved by Banaschewski & Brümmer in [BB88].

Lemma 1.13 *If ΩX is the frame of a stably compact locale, the set of its Scott-open filters partially ordered by inclusion is the frame QX of a stably compact locale. Its directed joins are unions, its binary meets are intersections and its binary joins are given by*

$$K \vee L = \{a \in \Omega X | \exists a_1 \in K \text{ (and) } a_2 \in L : \ a \geq a_1 \wedge a_2\} \text{ for any } K, L \in QX$$

For any Scott-open filter $K \in QX$, $K = \bigcup^{\uparrow}\{\Uparrow a | a \in K\}$ and hence

$$K \ll L \text{ iff } \exists a \in QX \text{ such that } K \subseteq \Uparrow a \subseteq L$$

There is the following well known fact.

Lemma 1.14 *Let $f : X \longrightarrow Y$ be a continuous map between two stably compact locales. Then the following are equivalent.*

(i) f is perfect.

(ii) f^* preserves the way below relation $\ll$.

(iii) For any $L \in QY$, $\uparrow f^*[L] \in QX$.

Proof. (i)$\Rightarrow$ (ii): Let $b_1 \ll b_2$ and $f^*(b_2) \leq \bigvee^{\uparrow} S$. Then we have

$$\begin{aligned} b_2 &\leq f_*(\bigvee{}^{\uparrow} S) \text{ (because } f^* \dashv f_*) \\ \Leftrightarrow b_2 &\leq \bigvee{}^{\uparrow}\{f_*(s) | s \in S\} \text{ (by assumption)} \\ \Rightarrow \exists s \in S : \ b_1 &\leq f_*(s) \ \ (\Uparrow b_1 \text{ is Scott open}) \\ \Rightarrow f^*(b_1) &\leq s \text{ (because} f^* \dashv f_*) \end{aligned}$$

Therefore, $f^*(b_1) \ll f^*(b_2)$.

(ii)$\Rightarrow$ (i): We shall prove that, if f^* preserves the way below relation, then for any $a \in OmegaX$,

$$f_*(a) = \bigvee{}^{\uparrow}\{b \in \Omega Y | f^*(b) \ll a\} \tag{1.7}$$

It can be easily checked that given 1.7, f_* indeed preserves directed joins. To prove **??**, first we observe that $f^*(b) \ll a$ implies $f^*(b) \leq a$ and hence $b \leq f_*(a)$. So the R.H.S. of 1.7 is bounded by the L.H.S.. For the other direction, we observe that $f_*(a) = \bigvee^{\uparrow}\{b \in \Omega Y | b \ll f_*(a)\}$ because Y is locally compact. But $b \ll f_*(a)$ implies that $f^*(b) \ll f^* f_*(a)$ because F^* preserves the way below relation and so that $f^*(b) \ll a$ because $f^* \dashv f_*$.

(ii)$\Rightarrow$(iii): Trivially, $\uparrow f^*[L]$ is a filter. We show that $\uparrow f^*[L]$ is Scott open. Let $f^*(b) \leq \bigvee^{\uparrow} S$ for some b in ΩY. Because L is Scott open and ΩY is locally compact, there is $b \in L$ with $b' \ll b$. This and perfectness of f implies that $f^*(b') \ll f^*(b)$. So, by definition of the way below relation, there is $s \in S$ such that $f^*(b') \leq s$ which asserts that $s \in \uparrow f^*[L]$.

(iii)$\Rightarrow$ (ii): Let $b_1 \ll b_2$ in ΩY and $f^*(b_2) \leq \bigvee^{\uparrow} S$. We consider the set

$$M := \uparrow \{f^*(c) | b_1 \ll c\}$$

The set $\uparrow b_1$ is a Scott open filter, therefore by assumption, M is a Scott open filter and $\bigvee^{\uparrow} S \in M$. So there is $s \in S$ with $s \in M$, i.e. $s \geq f^*(c)$ for some c with $b_1 \ll c$. We deduce that $f^*(b_1) \leq s$ by the fact that $f^*(b_1) \leq f^*(c)$. ∎

An important class of stably compact locales is the compact regular locales. A locale X is regular iff for any $a \in \Omega X$, $a = \bigvee^{\uparrow}\{a_i \in \Omega X | a_i \lessdot a\}$. In any frame ΩX, we say that $b \lessdot a$ (b well inside a) iff there is $c \in \Omega X$ such that $b \wedge c = 0$ and $a \vee c = 1$. The fact that a compact regular locale is stably compact transpires from the following.

Lemma 1.15 *If X is a compact regular locale and $a, b \in \Omega X$, then*

$$b \lessdot a \Leftrightarrow b \ll a$$

Proof. Let $b \lessdot a$ and $a \leq \bigvee^{\uparrow} S$. Then we have that $1 \leq \neg b \vee a \leq \neg b \vee \bigvee^{\uparrow} S = \bigvee^{\uparrow}\{\neg b \vee s | s \in S\}$. But $1 \ll 1$ since X is compact, so there is $s \in S$ such that $1 \leq \neg b \vee s$ which implies $b \leq s$.

Conversely, let $b \ll a$. X is regular, so $a = \bigvee^{\uparrow}\{a_i | a_i \lessdot a\}$. Consequently, there is $a_0 \lessdot a$ with $b \leq a_0$. Obviously it holds that $b \wedge a_0 = 0$ and $a \vee a_0 = 1$, i.e. $b \lessdot a$. ∎

Definition 1.16 *A map $f : X \longrightarrow Y$ between two locales is proper iff it is perfect and in addition it satisfies the coFrobenius condition*

$$f_*(a \vee f^*(b)) = b \vee f_*(a) \ \textit{ for all } a \in \Omega X, b \in \Omega Y \tag{1.8}$$

Recall that a map $f : X \longrightarrow Y$ between two topological spaces is proper iff f^{-1} sends compact subsets to compact subsets, or equivalently, iff it is closed and has compact fibres.

Vermeulen's exposition [Ver94] on proper maps between locales contains the following results.

Theorem 1.17 *(i) Proper maps are pullback stable in* **Loc**, *i.e. in the following pull-*

back,

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "h"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{1.9}$$

if h is proper then p_1 is proper. Also the Beck-Chevalley condition holds, i.e.

$$p_{1*} \circ p_2^* = f^* \circ h_*$$

(ii) Proper surjections are coequalisers of their kernel pair.

(iii) A map $f : X \longrightarrow Y$, where X is compact and Y is regular is always proper.

In section 2.3 we show that items (i) and (ii) of the theorem 1.17 are true for perfect maps if we substitute pullbacks with lax pullbacks and coequalisers with lax coequalisers respectively.

To put the scope of this thesis into a perspective, we now turn to the full subcategory of **StKLoc** comprising the coherent (or spectral) locales (see [Joh82], II.3) and perfect maps. We denote this category as **CohLoc**. By the term "Priestley Duality" we conventionally understand the equivalence

$$\mathbf{CohSp} \simeq \mathbf{OStoneSp} \tag{1.10}$$

where **CohSp** is the category of coherent spaces (the spatial counterpart of coherent locales) and perfect maps and **OStoneSp** is the category of ordered Stone spaces and monotone continuous maps. The equivalence 1.10 was effectively first proved by Priestley. She actually showed the duality $\mathbf{DLat}^{op} \simeq \mathbf{OStoneSp}$ which together with the generalised Stone representation $\mathbf{DLat}^{op} \simeq \mathbf{CohSp}$ yields 1.10, thus attaching the term "duality" to what is usually expressed as an equivalence. The localic version of the Priestley duality

$$\mathbf{CohLoc} \simeq \mathbf{OStoneLoc} \tag{1.11}$$

was proved by Townsend in [Tow97] (see also [Tow96]).

Banaschewski, Brümmer and Hardie introduced the notion of a compact regular biframe in [BBH83]. It is a triple (L, L_1, L_2), where L is a compact frame and L_1, L_2 are two subframes of L that generate L. L_1 and L_2 are equipped with two relations defined as follows.

$a_1 \preccurlyeq a_2$ in L_1 iff there is $b \in L_2$ such that $a_1 \wedge b = 0$ and $a_2 \vee b = 1$.

$b_1 \preccurlyeq b_2$ in L_2 iff there is $a \in L_1$ such that $b_1 \wedge a = 0$ and $b_2 \vee a = 1$.

The subframes L_1 and L_2 are also required to fulfill the conditions

$$\forall c \in L_i \;\; c = \bigvee^{\uparrow}\{d \in L_i | d \preccurlyeq c\}$$

for $i = 1, 2$. It follows that L is the frame of a compact regular locale. A biframe homomorphism $(L, L_1, L_2) \longrightarrow (L', L'_1, L'_2)$ is a frame homomorphism $l : L \longrightarrow L'$ that also satisfies $c \in L_i \Rightarrow l(c) \in L'_i$, for $i = 1, 2$. We denote by **KR2Fr** the category of compact regular biframes and biframe homomorphisms.

Banaschewski & Brümmer proved in [BB88] that there is the following duality

$$\mathbf{StKLoc} \simeq \mathbf{KR2Fr}^{op}$$

This duality is exhibited by assigning a biframe (L, L_1, L_2) to any stably compact locale X, where the frame L_1 is the frame of the closed nuclei on ΩX, L_2 is isomorphic to the frame of Scott-open filters of ΩX and L is stipulated to be the frame generated by L_1 and L_2. Townsend in his thesis [Tow96] gave a localic definition of compact regular ordered locales and proved the duality

$$\mathbf{KR2Fr} \simeq \mathbf{KRegPos}^{op}$$

where **KRegPos** is the category of compact regular ordered locales and monotone continuous maps. This establishes the equivalence

$$\mathbf{StKLoc} \simeq \mathbf{KRegPos}$$

The above equivalence will be referred to as the *extended localic Priestley duality.* Note that the classical extended Priestley duality, which says that the category of stably compact spaces and perfect maps is equivalent to the category of compact *pospaces* and monotone continuous functions, is usually attributed to Nachbin ([Nac50]). The term pospace is used here for a (necessarily) Hausdorff space equipped with a closed partial order.

1.5 Various compactness definitions for topoi

We have mentioned that proper maps between locales are defined by stipulating that they satisfy a lattice theoretic condition (definition 1.16). Johnstone has defined proper maps in [Joh79] (in this paper he actually calls them "perfect") by giving a much more topological characterisation.

Definition 1.18 *A map $f : X \longrightarrow Y$ between two locales is proper iff it renders X compact as a locale in the sheaves over Y.*

The two definitions are proven to be equivalent by Vermeulen in [Ver94]. A special case of this equivalence, is the well known fact that a locale X is compact iff the map

$$! : X \longrightarrow \mathbf{1}$$

is proper, i.e. X is compact in **Sets**. In fact, when the codomain of a map is the terminal locale $\mathbf{1}$, the coFrobenius condition is redundant in definition 1.16 as it is implied by Scott continuity of $!_*$. For a short constructive proof of this last fact see [Tow96], below theorem 3.2.2. Alternatively, it is easily observed that the statement that X is compact, i.e.

$$\bigvee^{\uparrow} S \geq 1 \Rightarrow \exists s \in S : \; s \geq 1$$

is logically equivalent with the statement that the global sections functor $!_*$ has the property

$$!_*(\bigvee{}^{\uparrow}\{s_i\}) = \bigvee{}^{\uparrow}\{!_*(s_i)\}$$

where $\{s_i\} \equiv S$ is a collection of subobjects of 1 in $Sh(X)$.

Definition 1.19 *More generally, a map $f : X \longrightarrow Y$ between two locales renders X compact inside the sheaves of Y when the direct image functor f_* preserves colimits of diagrams $\{s_i\}$ of subobjects of 1 in $Sh(X)$, indexed by $Sh(Y)$-internal filtered categories (see [Joh79]).*

Johnstone took a step further and defined a *proper geometric morphism* $f : X \longrightarrow Y$ between two arbitrary topoi to be one that renders $f_*(\Omega_{\mathcal{S}X})$ compact lattice object inside the sheaves $\mathcal{S}Y$. This amounts to the same condition as that of definition 1.19 with the notational difference of $\mathcal{S}X$ and $\mathcal{S}Y$ instead of $Sh(X)$ and $Sh(Y)$.

Tierney proposed a stricter condition for properness which was studied by Lindgren in his thesis [Lin]. Following Moerdijk and Vermeulen [MV97] we call the geometric morphisms that satisfy the Tierney-Lindgren condition *tidy*.

Definition 1.20 *A geometric morphism $f : X \longrightarrow Y$ is tidy iff*

$$f_*(colim_{i\in\Im}\{c_i\}) = colim_{i\in\Im}\{f_*(c_i)\}$$

where $\{c_i\}$ is a diagram of objects of $\mathcal{S}X$ indexed by a $\mathcal{S}Y$-internal filtered category.

Definition 1.21 *A topos X is strongly compact iff the direct image functor of the essentially unique geometric morphism $! : X \longrightarrow \mathbf{1}$ is tidy.*

Example 1.22 *([MV97], III.1.1) Any coherent topos or compact Hausdorff space is a strongly compact topos.*

With the expression "a diagram of objects of $\mathcal{S}X$ indexed by a $\mathcal{S}Y$-internal filtered category" we mean an object of the category $\mathcal{S}X^{f^*\mathfrak{I}}$, where $\mathfrak{I}$ is an internal category in $\mathcal{S}Y$. We refer the reader to [Joh77] for more on colimits of internal diagrams, or to section 6.8 where we briefly outline colimits of topos internal diagrams and give some more specific pointers in literature.

We mention that in her thesis K.Edwards first studied topoi with the property of definition 1.21 (strongly compact topoi).

Moerdijk and Vermeulen in their rather exhaustive monograph on compactness and regularity conditions for topoi [MV97], investigated the properties of proper and tidy maps providing (among other things) a different proof of a fact first established by Lindgren.

Theorem 1.23 *In the pullback in $\mathfrak{Top}$,*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "h"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{1.12}$$

if h is tidy then p_1 is also tidy. Furthermore, the induced natural transformation

$$f^* \circ h_* \Rightarrow p_{1*} \circ p_2^* \tag{1.13}$$

is an isomorphism, i.e. the Beck-Chevalley condition holds for pullbacks of tidy maps in $\mathfrak{Top}$.

Theorem 1.23 about tidy maps is the topos analogue of theorem 1.17 about proper maps. In [MV97], the authors prove that proper geometric morphisms are also pullback stable and the natural transformation of expression 1.13 is a monomorphism if h is proper.

It is reasonable to pose the question: what is a topos analogue of perfectness? By definition (1.9) perfect maps $f : X \longrightarrow Y$ between locales have defining frame homomorphisms whose right adjoints preserve joins of directed *subsets* of ΩY and directed subsets of ΩY are diagrams of elements of ΩY indexed by a filtered category in the sheaves of the base topos **Sets**[1] (i.e. where the frames ΩX and ΩY live). We point at a class of geometric morphisms introduced by Moerdijk and Vermeulen called *relatively tidy.*

Definition 1.24 *Let $X \longrightarrow B$ and $Y \longrightarrow B$ be two topoi in $\mathfrak{T}op/B$. Then a geometric morphism $f : X \longrightarrow Y$ is relatively tidy (relative to B) iff f_* preserves colimits of diagrams of objects in $\mathcal{S}X$ indexed by a $\mathcal{S}B$-internal category.*

Compared with tidy maps, relatively tidy maps possess good stability properties not under pullbacks but under *lax pullbacks* in $\mathfrak{T}op$. A lax pullback is an example of a 2-categorical limit and always exists in $\mathfrak{T}op$. For any two geometric morphisms with common codomain it consists of a square

$$\begin{tikzcd}
X \Rightarrow_Z Y \arrow[r, "\vartheta_2"] \arrow[d, "\vartheta_1"'] \arrow[dr, phantom, "\nearrow"] & Y \arrow[d, "h"] \\
X \arrow[r, "f"] & Z
\end{tikzcd} \tag{1.14}$$

with the universal property that it commutes up to a 2-cell $\tau : f \circ \vartheta_1 \Rightarrow h \circ \vartheta_2$. Lax pullbacks are unique up to equivalence. The 2-cell τ and the adjunctions $f^* \dashv f_*$ and $\vartheta_2^* \dashv \vartheta_{2*}$ also induce a natural transformation $f^* \circ h_* \Rightarrow \vartheta_{1*} \circ \vartheta_2^*$ in the following stages

$$\begin{array}{rcl}
1_{\mathcal{S}X} & \Rightarrow & \vartheta_{2*} \circ \vartheta_2^* \text{ (the unit of the adjunction)} \\
h_* & \Rightarrow & h_* \circ \vartheta_{2*} \circ \vartheta_2^* \\
h_* & \Rightarrow & f_* \circ \vartheta_{1*} \circ \vartheta_2^* \text{ (because exists } \tau'' : h_* \circ \vartheta_{2*} \Rightarrow f_* \circ \vartheta_{1*}) \\
f^* \circ h_* & \Rightarrow & \vartheta_{1*} \circ \vartheta_2^* \text{ (because } f^* \dashv f_*)
\end{array}$$

The following result proven by Moerdijk and Vermeulen ([MV97], theorem 5.1) was in part the motivation for chapter 2 of this thesis.

Theorem 1.25 *Suppose that diagram 1.14 is a lax pullback of topoi over a base B. If h is relatively tidy then ϑ_1 is tidy. Furthermore, the natural transformation $f^* \circ h_* \Rightarrow \vartheta_{1*} \circ \vartheta_2^*$ is an isomorphism, i.e. the Beck-Chevalley condition holds for lax pullbacks of relatively tidy geometric morphisms.*

In this thesis we manifest the view that relatively tidiness is the topos counterpart of perfectness with the following results. First, in complete analogy with relatively tidy

[1] Is it correct?

maps in $\mathfrak{T}op$, we demonstrate (chapter 2) that a lax pullback of a perfect map in **Loc** produces a proper map and that the Beck-Chevalley condition holds for this square. Also, a consequence of the results of chapter 6 is that a perfect map between stably compact locales is a relatively tidy geometric morphism between the corresponding topoi of sheaves over these locales. A corollary to this, which improves example 1.22, is that a stably compact locale is strongly compact as a topos.

We close this section with a historical note. Joyal and Tierney studied *open* maps of locales and topoi in [JT84] establishing a paradigm in which the study of proper maps of locales and topoi also fits. After all, openness and compactness/properness are dual notions in topology and topos theory.

1.6 Locally ordered topological spaces

A stably compact space is equivalent to a compact pospace, i.e. a compact Hausdorff space equipped with a closed partial order on its points, via the generalised Priestley duality (section 1.4). In this section briefly describe the notion of a *local pospace*, i.e. a compact Hausdorff space space equipped with a *local partial order*. This notion is due to Fajstrup, Goubault and Raussen in [FGR98]. Local pospaces were introduced by the authors in order to be used for a directed homotopy account of iterative concurrent systems in computer science. The definitions here are spatial.

Definition 1.26 *Let X be a Hausdorff space. A local partial order on X is a collection of pairs $\{(a_i, \leq_{a_i})|i \in I\}$, such that $\{a_i|i \in I\}$ is an open covering of X and $\leq_{a_i}$ is a partial order on the open set a_i for any $i \in I$, which in addition fulfills the following condition. For any pair a_i, a_j, $i, j \in I$ and any $x, y \in a_i \cap a_j$, it holds $x \leq_{a_i} y \Leftrightarrow x \leq_{a_j} y$.*

Let $\{(a_i, \leq_{a_i})|i \in I\}$ be a local partial order on a Hausdorff space X. Denote C the open covering $\{a_i|i \in I\}$. A refinement of this local partial order is a refinement D of the open covering C such that for any $b \subseteq a$, $b \in D$ and $a \in C$, it holds for any $x, y \in b$ that $x \leq_b y \Leftrightarrow x \leq_a y$. Two local partial orders on X are said to be equivalent iff they have a common refinement. It can be proved easily that this is in fact an equivalence relation.

Definition 1.27 *A Hausdorff space X together with an equivalence class of local partial orders is called a locally partially ordered space, or locally ordered space.*

A locally ordered space is a local pospace if there is a local partial order $\{(a_i, \leq_{a_i})|i \in I\}$ such that all pairs $(a_i, \leq_{a_i}$ are pospaces. We denote a local pospace as a pair (X, C), where C is one of the equivalent local orders on X.

Example 1.28 *Consider the circle* $\mathcal{S}^1$ *covered by three opens:* $a_1 = \{e^{i\vartheta} \in S^1 | \vartheta \in (-3\pi/4 \leq, \pi/2)\}$, $a_2 = \{e^{i\vartheta} | \vartheta \in (\pi/4, \pi)\}$ *and* $a_3 = \{e^{i\vartheta} | \vartheta \in (3\pi/4, 3\pi/2)\}$. *Each of these opens is ordered by increasing* ϑ. *The circle together with the equivalence class of* $\{(a_i, \leq_{a_i}) | i = 1, 2, 3\}$ *is a compact local pospace.*

Note that in order to define a local order on S^1, one has to cover it with at least three partially opens. In other words the intersection of any two opens in the local partial order must be connected.

We finally give an account of the morphisms between local pospaces.

Definition 1.29 *Let* (X, C) *and* (Y, D) *be two local pospaces. Then a continuous map* $f : X \longrightarrow Y$ *is called a directed map (or dimap) iff there are equivalent local orders* $C' \sim C$ *and* $D' \sim D$ *such that, for any* $a \in C'$ *and any* $b \in D'$ *and any* $x, y \in a \cap f^{-1}(b)$, *it holds*

$$x \leq_a y \Leftrightarrow f(x) \leq_b f(y)$$

Chapter 2

Perfect Maps

This chapter takes it cue from the end of section 1.4 in the introduction. In section 2.1 there is a revision of Escardó's patch construction for stably compact locales. In section 2.2 we prove that *lax pullbacks* of perfect maps produce proper maps in **Loc**. This is the localic analogue of the result by Moerdijk and Vermeulen ([MV97]) according to which lax pullbacks of relatively tidy geometric morphisms produce tidy maps (see section 1.5). Also in section 2.2 we show that perfect surjections always emerge as lax coequalisers of their kernel pair in **Loc**. Compare these two properties of perfect maps with the corresponding ones for proper maps (Vermeulen [Ver94]): proper maps are pullback stable and proper surjections always emerge as coequalisers of their kernel pair. Finally in section 2.4 we demonstrate the bijection between perfect maps between stably compact locales and patch continuous monotone functions. This is essentially generalised Priestley duality. Our proof benefits from the insight of Escardó's patch construction and is thus relatively short.

2.1 The patch construction

A major insight on constructing a biframe out of a stably compact locale was offered by Escardó in his paper [Esc01]. He proved that the "total" biframe L is exactly the frame of perfect nuclei on ΩX. By his definition, a nucleus is perfect iff it is Scott continuous. More generally, he defines the *patch* of a locale X to be the locale $PatchX$ whose frame is the frame of perfect nuclei on ΩX. Thus the patch locale of a stably compact locale is a compact regular locale. Furthermore, we have the following.

Lemma 2.1 *If X is a stably locally compact locale, the nuclei of the form*

$$\hat{a} \wedge \hat{K} \quad a \in \Omega X \text{ and } K \in QX \tag{2.1}$$

constitute a base of $\Omega(PatchX)$ *([Esc01], lemma 5.4).*

- *For* $a \in \Omega X$*, we denote by* $\hat{a}$ *the perfect closed nucleus defined by* $\hat{a}(b) := a \vee b$*, for any* $b\imath\Omega X$*.* L_1 *is exactly the frame of these nuclei. In the lattice of sublocales of* X*, nuclei of the form* $\hat{a}$ *correspond to open sublocales.*
- *For any Scott open filter* $K \in QX$*, we denote by* $\hat{K}$ *the perfect nucleus defined by* $\hat{K}(b) := \bigvee\{a \Rightarrow b | a \in K\}$ *for any* $b \in \Omega X$*, where* $\Rightarrow$ *is the Heyting implication in* ΩX*.* L_2 *is exactly the frame of these nuclei. In the lattice of sublocales of* X*, nuclei of the form* $\hat{K}$ *correspond to compact fitted sublocales. A sublocale is fitted iff it is an intersection of open sublocales.*

The following result exists in [Esc01b] (proposition 5.3) which is basically an other version of [Esc01].

Lemma 2.2 *If* X *is a stably locally compact locale*

$$\hat{K} = \bigvee\{\neg b | b \in K\}$$

This implies that nuclei of the form

$$\hat{a} \wedge \neg\hat{b} \quad a, b \in \Omega X \tag{2.2}$$

also constitute a base of $\Omega(PatchX)$*.*

Restricting to the case of stably compact locales, the patch construction is extended to a functor

$$\mathbf{Patch} : \mathbf{StKLoc} \longrightarrow \mathbf{KRegLoc}$$

determined by stipulating that for any map $g : X \longrightarrow Y$ between two stably compact locales

$$(\mathbf{Patch}g)^*(\hat{b}) := \widehat{g^*(b)} \text{ and } (\mathbf{Patch}g)^*(\hat{K}) := \widehat{\uparrow g^*[K]}$$

for any $b \in \Omega Y$ and $K \in QY$. Note that although $g^*[K]$ is not necessarily a filter,

$$\bigvee\{a \Rightarrow b | a \in g^*[K]\} = \bigvee\{a \Rightarrow b | a \in \uparrow g^*[K]\} \text{ for any } b \in \Omega X$$

Theorem 2.3 *The functor* **Patch** *is the right adjoint of the inclusion* $\mathbf{inc} : \mathbf{KRegLoc} \hookrightarrow \mathbf{StKLoc}$*, i.e.* **KRegLoc** *is a coreflective subcategory of* **StKLoc***. The counit of the adjunction has components* $\varepsilon_X : \mathbf{Patch}X \longrightarrow X$*, for any stably compact locale* X*, defined by*

$$\varepsilon_X^*(a) = \hat{a}$$

for $a \in \Omega X$. Moreover, ε_X is a perfect surjection.

Proof. [Esc01], definition 2.4, lemma 2.5 and corollary 5.9. ■

We close this section with briefly delving into spatial topology for the benefit of our intuition. A sober space X is stably compact iff it is locally compact and finite intersections of compact saturated subsets of X produce compact saturated subsets. A subset C is saturated if $\uparrow C = C$ in the specialisation order of X. The patch topology on X is given by considering the collection of opens in X and the complements of compact saturated subsets in X. $PatchX$ is a compact Hausdorff space and it can be naturally equipped with a closed partial order, namely the specialisation order of X. A compact Hausdorff space with a closed partial order is often called a compact *pospace*. Conversely, any compact pospace $(P, \leq)$ gives rise to a stably compact space Y, by stipulating that the opens of Y are subsets of P that are upper closed with respect to the order $\leq$.

A continuous function $g : X \longrightarrow Y$ between two stably compact spaces is perfect iff it reflects compact saturated sets (c.f. lemma 1.14, item (iii)). The same set theoretic function is a continuous monotone if viewed as a map $PatchX \longrightarrow PatchY$. Conversely, if $f : PatchX \longrightarrow PatchY$ is a monotone and continuous, the same set theoretic function is perfect if viewed as a map $X \longrightarrow Y$. The category of stably compact spaces and perfect maps is equivalent to the category of compact pospaces and monotone continuous functions. The spatial analogue of the map $\varepsilon_X : PatchX \longrightarrow X$ is the set theoretical identity (which is 1-1 and epi but not an isomorphism in **Top**). Therefore, diagrammatically, we have the trivial statement that

$$\begin{tikzcd} PatchX \arrow[r, "f"] \arrow[d, "\varepsilon_X"'] & PatchY \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{2.3}$$

for each perfect map g there is a unique monotone continuous function f that makes the above diagram commutative and the converse. In section 2.4 we prove localically this bijection between perfect and monotone (patch) continuous maps.

2.2 Lax pullbacks in Loc

The category of locales and continuous maps is *poset enriched* in the sense that for any pair of parallel continuous maps $f, g : X \rightrightarrows Y$,

$$f \leq g \;\text{ iff }\; f^*(b) \leq g^*(b)\ \forall b \in \Omega Y$$

By abusing the notation, for the rest of this chapter, we are going to denote by **Loc** the poset enriched category of locales and continuous maps.

We recall that in a poset enriched category $\mathbf{C}$, a lax pullback of two arrows $h : b \longrightarrow c$ and $f : c \longrightarrow c$ is an object $a \sqsubseteq_c b$ together with two arrows $p_1 : a \sqsubseteq_c b \longrightarrow a$ and $p_2 : a \sqsubseteq_c b \longrightarrow b$ such that

- $f \circ p_1 \leq h \circ p_2$ and
- For any object d and arrows $q_1 : d \longrightarrow a$, $q_2 : d \longrightarrow b$ that satisfy $f \circ q_1 \leq h \circ q_2$, there exists a unique arrow $e : d \longrightarrow a \sqsubseteq_c b$ such that $q_1 = p_1 \circ e$ and $q_2 = p_2 \circ e$.

In **Loc** lax pullbacks can be easily described by means of their frame presentations.

Lemma 2.4 *In* **Loc** *lax pullbacks exist. In particular, the frame of a lax pullback*

$$\begin{tikzcd}
X \sqsubseteq_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "h"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega(X \sqsubseteq_Z Y) \arrow[dr, phantom, "\leq"] & \Omega Y \arrow[l, "p_2^*"'] \\
\Omega X \arrow[u, "p_1^*"] & \Omega Z \arrow[l, "f^*"] \arrow[u, "h^*"']
\end{tikzcd}
\tag{2.4}$$

is presented as

$$\Omega(X \subseteq_Z Y) \cong \mathbf{Fr}\langle \Omega X \otimes \Omega Y \ \textit{(qua frame)} | f^*(a) \otimes 1 \leq 1 \otimes h^*(a)\rangle \tag{2.5}$$

and the frame homomorphisms p_1^* and p_2^* are given by

$$p_1^* = \bullet \otimes 1 \quad \textit{and} \quad p_2^* = 1 \bullet \otimes 1$$

Proof. We denote A the frame of the presentation. The maps $(\bullet \otimes 1) \circ f^* = f^*(\bullet) \otimes 1$ and $(1 \otimes \bullet) \circ h^* = 1 \otimes h^*(\bullet)$ are well defined frame homomorphisms $\Omega Z \longrightarrow A$ and trivially $f^*(a) \otimes 1 \leq 1 \otimes h^*(a)$, for any $a \in \Omega Z$.

Also for any frame ΩS and maps $q_1^* : \Omega X \longrightarrow \Omega S$ and $q_2^* : \Omega Y \longrightarrow \Omega S$ such that $q_1^* \circ f^*(a) \leq q_2^* \circ h^*(a)$ for any $a \in \Omega Z$, the frame homomorphism $e^* : A \longrightarrow \Omega S$ defined on the sub-basic elements of A by stipulating

$$e^* : b \otimes 1 \mapsto q_1^*(b) \text{ and } e^* : 1 \otimes c \longrightarrow q_2^*(c)$$

for any $b \in \Omega X$ and $c \in \Omega Y$, induces the equalities $q_1 = p_1 \circ e$ and $q_2 = p_2 \circ e$ and is clearly the unique such. ■

Consider now the case where $Y \equiv X$ and $f \equiv h$, i.e. the special case where the pullback 2.4 becomes the coequaliser

$$\begin{tikzcd} X \sqsubseteq_Z X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "h"] & Z \end{tikzcd} \tag{2.6}$$

and $X \sqsubseteq_Z X$ is the *lax kernel pair* of h. Let also X be a compact regular locale. We contemplate the monic

$$X \sqsubseteq_Z X \xhookrightarrow{< p_1, p_2 >} X \times X$$

as a relation on the locale X. Recall that in any regular category $\mathbf{C}$ (and **KRegLoc** is a regular category), a relation on an object X is a monic $< p_1, p_2 >: R \hookrightarrow X \times X$ (equivalently, a subobject $R \subseteq X \times X$). The relational composition $R \circ R$ is the relation

$$R \times_X R \xhookrightarrow{p_1 \circ a_1 \times p_2 \circ a_2} X \times X$$

where $R \times_X R$ is the following pullback in $\mathbf{C}$.

$$\begin{tikzcd} R \times_X R \arrow[r, "a_2"] \arrow[d, "a_1"'] & R \arrow[d, "p_1"] \\ R \arrow[r, "p_2"] & X \end{tikzcd}$$

In this context, we have the following definition.

Definition 2.5 *Let $R \hookrightarrow X \times X$ be a relation on X (in a regular category $\mathbf{C}$). R is a partial order if the following three axioms are satisfied.*

(reflexivity) The diagonal $\Delta : X \hookrightarrow X \times X$ factors through $< p_1, p_2 >$ (equivalently $\Delta \subseteq R$).

(transitivity) The map $p_1 \circ a_1 \times p_2 \circ a_2$ factors through $< p_1, p_2 >$ (equivalently, $R \circ R \subseteq R$).

(antisymmetry) If for an object Q, there are monics $Q \hookrightarrow R$ and $Q \hookrightarrow R^c$, where R^c is the relation

$< p_2, p_1 >: R \hookrightarrow X \times X$ (p_1 and p_2 are swapped), then there is a monic $Q \hookrightarrow \Delta$ (equivalently $R \cap R^c \subseteq \Delta$).

For more on relations in category theory the reader is referred to [FŠ90].

Using presentations of frames, the proof of the following lemma is straightforward and therefore omitted.

Lemma 2.6 *Let $h : X \longrightarrow Z$ be a continuous map and X be compact regular. Then $< p_1, p_2 >: X \sqsubseteq_Z X \hookrightarrow X \times X$ is a reflexive and transitive relation on X.*

If we make more specific demands on Y and h, we can further refine lemma 2.6.

Theorem 2.7 *Let $h : X \longrightarrow Z$ be a perfect map, X be compact regular and Y stably compact. Then $X \sqsubseteq_Z X$ is a closed reflexive and transitive relation on X, i.e. $X \sqsubseteq_Z X$ is a closed sublocale of $X \times X$.*

Proof. By lemma 2.4, we know that the frame of $X \sqsubseteq_Z X$ is given by the presentation

$$\Omega(X \sqsubseteq_Z X) \cong \mathbf{Fr}\langle \Omega X \otimes \Omega X \text{ (qua frame)} | h^*(a) \otimes 1 \leq 1 \otimes h^*(a),\ \forall a \in \Omega Z \rangle \tag{2.7}$$

To prove that $X \sqsubseteq_Z X$ is a closed sublocale of $X \times X$ it suffices to demonstrate that the relations in 2.7 are logically equivalent to the relation

$$s \leq 0 \quad \text{for some } s \in \Omega X \otimes \Omega X$$

Because then we shall know that $X \sqsubseteq_Z X$ corresponds to the complement of the open s. We claim that

$$s = \bigvee_{a \in \Omega Z} h^*(a) \otimes \neg h^*(a) \tag{2.8}$$

where $\neg$ is the Heyting complement in $\Omega X \otimes \Omega X$.

Suppose that $h^*(a) \otimes 1 \leq 1 \otimes h^*(a)$ for any $a \in \Omega Z$. Then

$$\begin{aligned}
s &= \bigvee_{a \in \Omega Z} h^*(a) \otimes \neg h^*(a) \\
&= \bigvee_{a \in \Omega Z} (h^*(a) \otimes 1 \wedge 1 \otimes \neg h^*(a)) \text{ (meets of tensor products are computed componentwise)} \\
&\leq \bigvee_{a \in \Omega Z} (1 \otimes h^*(a) \wedge 1 \otimes \neg h^*(a)) \text{ (by assumption)} \\
&= \bigvee_{a \in \Omega Z} (1 \otimes (h^*(a) \wedge \neg h^*(a))) \\
&= 1 \otimes 0
\end{aligned}$$

which is (isomorphic to) the bottom element 0 of $\Omega X \otimes \Omega X$.

Conversely, assume that $s \leq 0$. Let $a_i \ll a$ in ΩZ. Since h is a perfect map between two stably compact locales, $h^*(a_i) \ll h^*(a)$. Furthermore, in the frame of a compact regular locale, $\ll = \lessdot$ (lemma 1.15), so $a_i \ll a$ implies that $h^*(a_i) \lessdot h^*(a)$. By the definition of the well inside relation, the last implication becomes

$$a_i \ll a \ \in \Omega Z \ \Rightarrow h^*(a) \vee \neg h^*(a_i) \ \in \Omega X \tag{2.9}$$

Using this information we have:

$$\begin{aligned}
h^*(a) \otimes 1 &= h^*(\textstyle\bigvee^{\uparrow}\{a_i | a_i \ll a\}) \otimes 1 \ \ (Z \text{ is locally compact}) \\
&= \textstyle\bigvee^{\uparrow}\{h^*(a_i) | a_i \ll a\} \otimes 1 \ \ (h^* \text{ is frame homomorphism}) \\
&= \textstyle\bigvee^{\uparrow}\{(h^*(a_i) \otimes 1) | a_i \ll a\} \quad (\text{in general } \bigvee(a_i \otimes b) = \bigvee a_i \otimes b) \\
&= \textstyle\bigvee^{\uparrow}\{h^*(a_i) \otimes h^*(a) \vee \neg h^*(a_i) | a_i \ll a\} \quad (\text{because of 2.9}) \\
&= \textstyle\bigvee^{\uparrow}\{(h^*(a_i) \otimes h^*(a)) \vee (h^*(a_i) \otimes \neg h^*(a_i)) | a_i \ll a\} \\
&\leq \textstyle\bigvee^{\uparrow}\{(1 \otimes h^*(a)) \vee (h^*(a_i) \otimes \neg h^*(a_i)) | a_i \ll a\} \\
&= (1 \otimes h^*(a)) \vee \textstyle\bigvee^{\uparrow}\{h^*(a_i) \otimes \neg h^*(a_i) | a_i \ll a\} \\
&\leq (1 \otimes h^*(a)) \vee s \quad (\text{c.f. 2.8}) \\
&\leq 1 \otimes h^*(a) \quad (s \leq 0 \text{ by assumption})
\end{aligned}$$

■

Now, for Z a stably compact locale, let us consider the case of the map

$$\varepsilon_Z : \mathbf{Patch}(Z) \longrightarrow Z$$

where ε is the counit of the adjunction $\mathbf{inc} \dashv \mathbf{Patch}$. $\mathbf{Patch}(Z)$ is compact regular and ε_Z is perfect (and surjective), therefore by lemma 2.6 and theorem 2.7, $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(X)$ is a closed, reflexive and symmetric relation on Z. Additionally, we know (c.f. lemma 2.2) that any open $b \in \Omega\mathbf{Patch}(Z)$ can be written as join of elements of the form

$$\varepsilon^*(a_i) \wedge \neg\varepsilon^*(a_j) \quad \text{with } a, a' \in \Omega Z \tag{2.10}$$

Lemma 2.8 $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z)$ *is an antisymmetric relation on* Z.

Proof. Let $\mathbf{Patch}(Z) \sqsupseteq_Z \mathbf{Patch}(Z)$ be the lax kernel pair of $\varepsilon_Z : \mathbf{Patch}(Z) \longrightarrow Z$

$$\begin{tikzcd} \mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z) \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "p_1"'] \arrow[r, phantom, "\rotatebox{90}{$\leq$}"] & \mathbf{Patch}(Z) \arrow[r, "\varepsilon_Z"] & Z \end{tikzcd} \tag{2.11}$$

In other words, if $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z)$ is R, $\mathbf{Patch}(Z) \sqsupseteq_Z \mathbf{Patch}(Z)$ is what we denoted as R^c (compare with diagram 2.6). The frame $\Omega\mathbf{Patch}(Z) \sqsupseteq_Z \mathbf{Patch}(Z)$ is has the "symmetric presentation" of 2.7, i.e. the relations are

$$1 \otimes \varepsilon^*(a) \leq \varepsilon^*(a) \otimes 1 \quad \text{for any } a \in \Omega Z \tag{2.12}$$

The above relation also implies that for any $a \in \Omega Z$

$$\neg(1 \otimes \varepsilon^*(a)) \geq \neg(\varepsilon^*(a) \otimes 1) \Leftrightarrow 1 \otimes \neg\varepsilon^*(a) \geq \neg\varepsilon^*(a) \otimes 1 \tag{2.13}$$

Now suppose that there is a sublocale $i : Y \hookrightarrow \mathbf{Patch}Z \times \mathbf{Patch}Z$ which is simultaneously a sublocale of $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z)$ and $\mathbf{Patch}(Z) \sqsupseteq_Z \mathbf{Patch}(Z)$, i.e. there are monics

$$l : Y \hookrightarrow \mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z) \quad \text{and} \quad m : Y \hookrightarrow \mathbf{Patch}(Z) \sqsupseteq_Z \mathbf{Patch}(Z)$$

It holds $i^*(b_1 \otimes b_2) = l^*(b_1 \otimes b_2) = m^*(b_1 \otimes b_2)$, for any $b_1, b_2 \in \mathbf{Patch}(Z)$ We denote by $q_1, q_2 : \mathbf{Patch}(Z) \times \mathbf{Patch}(Z) \rightrightarrows \mathbf{Patch}(Z)$ the two projections. Let b be any open in $\Omega\mathbf{Patch}Z$. Let $\{a_i | i \in I\}$ and $\{a_j | j \in J\}$ be opens in ΩZ such that $b = \bigvee_{i,j}(\varepsilon^*(a_i) \wedge \neg\varepsilon^*(a_j)$. We have

$$\begin{aligned}
i^* \circ q_2^*(b) &= i^*(b \otimes 1) \\
&= i^*(\bigvee_{i,j}(\varepsilon^*(a_i) \wedge \neg\varepsilon^*(a_j)) \otimes 1) \\
&= \bigvee_{i,j} i^*(\varepsilon^*(a_i) \otimes 1 \wedge \neg\varepsilon^*(a_j) \otimes 1) \\
&= \bigvee_{i,j}(i^*(\varepsilon^*(a_i) \otimes 1) \wedge i^*(\neg\varepsilon^*(a_j) \otimes 1)) \\
&= \bigvee_{i,j}(l^*(\varepsilon^*(a_i) \otimes 1) \wedge m^*(\neg\varepsilon^*(a_j) \otimes 1)) \\
&\leq \bigvee_{i,j}(l^*(1 \otimes \varepsilon^*(a_i)) \wedge m^*(1 \otimes \neg\varepsilon^*(a_j))) \\
&= \bigvee_{i,j}(i^*(1 \otimes \varepsilon^*(a_i)) \wedge i^*(1 \otimes \neg\varepsilon^*(a_j)))
\end{aligned}$$

$$
\begin{aligned}
&= i^*(1 \otimes \bigvee_{i,j} (\varepsilon^*(a_i) \wedge \neg\varepsilon^*(a_j))) \\
&= i^*(1 \otimes b) \\
&= i^* \circ q_1^*(b)
\end{aligned}
$$

Similarly we prove $i^* \circ q_1^* \leq i^* \circ q_2^*$ and so that $q_1 \circ i = q_2 \circ i$. Hence Y is a sublocale of the diagonal of $\mathbf{Patch}(Z) \times \mathbf{Patch}(Z)$. ∎

Corollary 2.9 *The lax kernel pair* $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z)$ *of* $\varepsilon_Z : \mathbf{Patch}(Z) \longrightarrow Z$ *is a closed partial order on* Z*.*

So we have proved:

Corollary 2.10 $(\sqsubseteq)$ *is a closed sublocale of* $\mathbf{Patch}A \times \mathbf{Patch}A$*. Its presentation is (equivalently) given by:*

$$\Omega(\sqsubseteq) = \langle \Omega\mathbf{Patch}A \otimes \Omega\mathbf{Patch}A \textit{ (qua frame)} | \bigvee_{a \in A} \neg\hat{a} \otimes \hat{a} \leq 0 \,\rangle$$

Hence, it becomes apparent that $\mathbf{Patch}(Z) \sqsubseteq_Z \mathbf{Patch}(Z)$ qualifies as the sublocale that corresponds to the closed partial order in $\mathbf{Patch}(Z)$ induced by the specialisation order of Z. What we do next is to define monotone homomorphisms between two locales $\mathbf{Patch}X$ and $\mathbf{Patch}Y$, where X, Y are stably compact locales.

Definition 2.11 *Let* $f : \mathbf{Patch}X \longrightarrow \mathbf{Patch}Y$ *be continuous. Then* f *is monotone iff there is a continuous map* $f' : \mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X \longrightarrow \mathbf{Patch}Y \sqsubseteq_Y \mathbf{Patch}Y$ *that makes the upper square in the following diagram commutative.*

$$
\begin{tikzcd}
\mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X \arrow[r, "f'"] \arrow[d, hook, "i_X"'] & \mathbf{Patch}Y \sqsubseteq_Y \mathbf{Patch}Y \arrow[d, hook, "i_Y"] \\
\mathbf{Patch}X \times \mathbf{Patch}X \arrow[r, "f \times f"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \mathbf{Patch}Y \times \mathbf{Patch}Y \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] \\
\mathbf{Patch}X \arrow[r, "f"] & \mathbf{Patch}Y
\end{tikzcd}
\tag{2.14}
$$

By the definition of the cartesian product the two lower squares in the diagram are always commutative, so an equivalent statement would be that f is monotone iff there is a continuous map $f' : \mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X \longrightarrow \mathbf{Patch}Y \sqsubseteq_Y \mathbf{Patch}Y$ such that $f \circ p_1 \circ i_A = q_1 \circ i_B \circ f'$ and $f \circ p_2 \circ i_A = q_2 \circ i_B \circ f'$.

2.3 Perfect surjections as lax coequalisers in Loc.

In this section we move away from thinking about the patch construction and we study the more general case of perfect maps between arbitrary locales. It can be observed that the above construction of $\Omega(\sqsubseteq)$ can be generalised for any perfect surjection $h : \Omega Y \longrightarrow Z$ and not just $\varepsilon_A\ \mathbf{Patch}A \longrightarrow A$.

We consider the case where $h : Y \longrightarrow Z$ is perfect and $f : X \longrightarrow Z$ is any continuous map. Let $X \sqsubseteq_Z Y$ be the *lax* pullback in **Loc** as depicted below using the lax pullback square.

$$\begin{tikzcd} X \sqsubseteq_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lightning"] & Y \arrow[d, "h"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \qquad \begin{tikzcd} \Omega(X \sqsubseteq_Z Y) \arrow[dr, phantom, "\lightning"] & \Omega Y \arrow[l, "1 \otimes \bullet"'] \\ \Omega X \arrow[u, "\bullet \otimes 1"] & \Omega Z \arrow[l, "f^*"] \arrow[u, "h^*"'] \end{tikzcd} \tag{2.15}$$

The frame $\Omega(X \sqsubseteq_Z Y)$ can be presented as

$$\Omega(X \sqsubseteq_Z Y) = \mathbf{Fr}\langle \Omega\mathbf{Patch}X \otimes \Omega\mathbf{Patch}Y \text{ (qua frame)} \mid f^*(a) \otimes 1 \leq 1 \otimes h^*(a), \text{ for } a \in \Omega Z \rangle \tag{2.16}$$

Let us note here that we cannot turn the presentation 2.16 into an equivalent presentation like that of corollary 2.10. $\Omega(X \sqsubseteq_Z Y)$ is not closed in general.

Next, we construct a *preframe presentation* of the frame 2.16. Preframes are posets having all binary meets and directed joins, with the binary meets distributing over directed joins. Johnstone and Vickers [JV91] have proved the preframe version of the coverage theorem which states that frames can be presented as preframes with relations of a specific form (playing the role of a preframe coverage):

$$\mathbf{PreFr}\langle P \text{(qua poset)} \mid C \rangle \cong \mathbf{Fr}\langle P \text{(qua join semilattice)} \mid C \rangle$$

where C are relations of the form $\wedge S \leq \bigvee_i^{\uparrow} \wedge S_i$ where S, S_i are finite subsets of P and C are join stable in the sense that if a relation of the above form is in C, then for any $x \in P$ the relation $\wedge\{x \vee y, y \in S\} \leq \bigvee_i^{\uparrow} \wedge\{x \vee y, y \in S_i\}$ is also contained in C. So next we write the frame presentation of $\Omega(X \sqsubseteq_Z Y)$ in an equivalent form so that the preframe coverage theorem can be readily applied.

$$\Omega(X \sqsubseteq_Z Y) = \mathbf{Fr}\langle \Omega X \otimes \Omega Y (\text{qua frame}) \mid f^*(a) \otimes 1 \leq 1 \otimes h^*(a),\ a \in \Omega Z\ \rangle \tag{2.17}$$

which is equivalent to

$$\begin{aligned}\mathbf{Fr}\langle j \otimes k,\ j \in \Omega X,\ k \in \Omega Y \quad \mid \quad & \wedge(j_i \otimes k_i) = \wedge j_i \otimes \wedge k_i, \\ & \bigvee_i (j_i \otimes k) = \bigvee_i j_i \otimes k, \\ & \bigvee_i (j \otimes k_i) = j \otimes \bigvee_i k_i, \\ & f^*(a) \otimes 1 \leq 1 \otimes h^*(a),\ a \in \Omega Z\ \rangle\end{aligned}$$

where $j \otimes k$ are formal symbols in this stage. By defining $j \parr k = j \otimes 1 \ \vee \ 1 \otimes k$ we can "accumulate" all the binary joins in the generators' part of the presentation. Using $\parr$, the equivalent presentation is:

$$\begin{aligned}\mathbf{Fr}\langle j \parr k,\ j \in \Omega X, k \in \Omega Y \text{ (qua join semilattice)} | \wedge_i j_i \parr k \quad &= \quad \wedge_i (j_i \parr k), \\ j \parr \wedge_i k_i \quad &= \quad \wedge_i (j \parr k_i), \\ \bigvee\nolimits^{\uparrow}_i j_i \parr k \quad &= \quad \bigvee\nolimits^{\uparrow}_i (j_i \parr k), \\ j \parr \bigvee\nolimits^{\uparrow}_i k_i \quad &= \quad \bigvee\nolimits^{\uparrow}_i (j \parr k_i) & (2.18) \\ \text{and } \forall a \in \Omega Z, \quad (f^*(a) \vee j) \parr k \quad &\leq \quad j \parr (h^*(a) \vee k) & (2.19)\end{aligned}$$

where the first four types of relations generate the first three types of relations in the previous presentation because they are join-stable. The fifth type of relations can be expressed as the join of the relations

$$h^*(a) \parr 0 \leq 0 \parr h^*(a) \ \text{ with } \ j \parr k \leq j \parr k$$

which is the least general join-stable expression equivalent with $f^*(a) \otimes 1 \leq 1 \otimes h^*(a)$. By a fairly standard procedure we can put the relations in the required form $\wedge S \leq \bigvee^{\uparrow}_i \wedge S_i$ where S, S_i are finite (so we do not need to do it explicitly), and therefore, by direct application of the Johnstone-Vickers preframe coverage theorem:

$$\Omega(X \sqsubseteq_Z Y) = \mathbf{PreFr}\langle j \parr k,\ j \in \Omega X, k \in \Omega Y \text{ (qua poset)} \mid \text{same relations as above} \rangle \tag{2.20}$$

Next, we seek an explicit form of the right adjoint of the map p_1^* in the case where X, Y

and Z are general locales and h is perfect. We define a function

$$p_1' : \Omega(X \sqsubseteq_Z Y) \longrightarrow \Omega X$$

$$\text{by} \quad p_1'(j \otimes k) := j \wedge f^* h_*(k) \tag{2.21}$$

We need to show that this function is well-defined on $\Omega(X \sqsubseteq_Z Y)$. To this end, we use preframe techniques. The action of p_1' on $j \invamp k$, for any $j \in \Omega X$ and $k \in \Omega Y$ is

$$p_1'(j \invamp k) = j \vee f^* h_*(k) \tag{2.22}$$

Lemma 2.12 *The map $p_1' : \Omega(X \sqsubseteq_Z Y) \longrightarrow \Omega X$ is well defined.*

Proof. Thinking $\Omega(X \sqsubseteq_Z Y)$ in terms of the preframe presentation, we have to show that p_1' is a preframe homomorphism. Expression 2.12 defines p_1' on the generators of the presentation 2.20 and we need to demonstrate that it respects its relations. It is easily checked that p_1' preserves the first four. In particular, relation 2.18 is preserved because h is perfect and therefore h_* commutes with directed joins. We need a little more work about the fifth one. We need to prove that

$$p_1'(j \vee f^*(a) \invamp k) \leq p_1'(j \invamp h^*(a) \vee k)$$

for any $j \in \Omega X$, $k \in \Omega Y$ and $a \in \Omega Z$ We have:

$$\begin{aligned}
h^* h_*(k) &\leq k \ (h_* \dashv h^*) \\
\Leftrightarrow h^*(a) \vee h^* h_*(k) &\leq h^*(a) \vee k \\
\Leftrightarrow h^*(a \vee h_*(k)) &\leq h^*(a) \vee k \ (h^* \text{ is a frame homomorphism}) \\
\Leftrightarrow a \vee h_*(k) &\leq h_*(h^*(a) \vee k) \ (h_* \dashv h^*) \\
\Leftrightarrow f^*(a \vee h_*(k)) &\leq f^* h_*(h^*(a) \vee k) \ (f^* \text{ is monotone}) \\
\Leftrightarrow f^* \vee f^*(a) h_*(k) &\leq f^* h_*(h^*(a) \vee k) \ (f^* \text{ is a frame homomorphism} \\
\Leftrightarrow j \vee f^*(a) \vee f^* h_*(k) &\leq j \vee f^* h_*(h^*(a) \vee k) \\
\Leftrightarrow p_1'((f^*(a) \vee j) \invamp k) &\leq p_1'(j \invamp (h^*(a) \vee k)) \text{ (by definition)}
\end{aligned}$$

and so p_1' preserves the fifth relation. So these two maps are well defined. ■

So now we are ready for the following.

Lemma 2.13 *The map p_1' defined by expression 2.21 or (equivalently) by 2.22 is the right adjoint of p_1. Furthermore, p_1 is automatically perfect.*

Proof. The great advantage of having defined p'_1 on a preframe presentation is that we have already established that p'_1 is a *preframe homomorphism*, i.e. it preserves finite meets and directed join. We demonstrate now that p'_1 is indeed the right adjoint of p^*_1.

- Thinking $\Omega(X \sqsubseteq_Z Y)$ as a preframe, the general element in it is written as $\bigvee^{\uparrow} \wedge j \parr k$ for $j \in \Omega X$ and $k \in \Omega Y$. Therefore, the action of the composite $p^*_1 p'_1$ on an arbitrary element is $p^*_1 p'_1(\bigvee^{\uparrow} \wedge j \parr k) = \bigvee^{\uparrow} \wedge p^*_1 p'_1(j \parr k)$ because p'_1 is preframe homomorphism and p^*_1 is a frame homomorphism. For this reason, in order to show that $p^*_1 p'_1(s) \leq s$, for any element $s \in \Omega(X \sqsubseteq_Z Y)$, it suffices to show that for any $j \parr k \in \Omega(X \sqsubseteq_Z Y)$, $p^*_1 p'_1(j \parr k) \leq j \parr k$. We do that as follows

$$\begin{aligned}
p^*_1 p'_1(j \parr k) &= p^*_1(j \vee f^* h_*(k)) \text{ (by 2.22)} \\
&= (j \vee f^* h_*(k)) \otimes 1 \\
&= (j \otimes 1) \vee (f^* h_*(k) \otimes 1) \text{ (by the tensor product property)} \\
&\leq (j \otimes 1) \vee (1 \otimes h^* h_*(k)) \text{ (by the relation of presentation 2.17)} \\
&\leq (j \otimes 1) \vee (1 \otimes k) \text{ (because } h^* \dashv h_*) \\
&= (j \parr k) \text{ (by the definition of } \parr)
\end{aligned}$$

- It is more straightforward to prove that $p'_1 p^*_1(j) \geq j$, for any $j \in \Omega Y$. Indeed we have
$$p'_1 p^*_1(j) = p_{1*}(j \otimes 1) \;=\; j \wedge f^* h_*(1) \;=\; j \wedge 1 \;=\; j \;\geq\; j$$
Where we used the fact that h_* is monotone and so $h_*(1) = 1$ and $f^* h_*(1) = 1$.

This completes the proof of $p^*_1 \dashv p'_1$. ■

Lemma 2.13 has an immediate important consequence.

Corollary 2.14 *(i) Consider the lax pullback in* **Loc**.

$$\begin{tikzcd}
X \sqsubseteq_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "h"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega(X \sqsubseteq_Z Y) & \Omega Y \arrow[l, "1 \otimes \bullet"'] \\
\Omega X \arrow[u, "\bullet \otimes 1"] & \Omega Z \arrow[l, "f^*"] \arrow[u, "h^*"']
\end{tikzcd}$$

If h is perfect then p_1 is proper.

(ii) The Beck-Chevalley condition holds in lax pullback squares of perfect maps in **Loc**,

i.e. with the notation of the lax pullback 2.15,

$$f^* \circ h_* = p_{1*} \circ p_2^* \tag{2.23}$$

Proof. (i) We have already shown in lemma 2.12 that p_1' is a preframe homomorphism and in lemma 2.13 that $p_1' = p_{1*}$. What remains to be verified is that p_1 satisfies the coFrobenius condition (definition 1.16). Since p_{1*} is a preframe homomorphism, it suffices to show that

$$p_{1*}((j \parr k) \vee p_1^*(j')) = p_{1*}(j \parr k) \vee j' \quad \forall j, j' \in \Omega X, k \in \Omega Y$$

We have

$$\begin{aligned} p_{1*}((j \parr k) \vee p_1^*(j')) &= p_{1*}((j \parr k) \vee (j' \parr 0)) \\ &= p_{1*}((j \vee j') \parr k) \\ &= (j \vee j') \vee f^* h_*(k) \quad \text{(by 2.22)} \\ &= j' \vee (j \vee f^* h_*(k)) \\ &= j' \vee p_{1*}(j \parr k) \end{aligned}$$

(ii) The Beck-Chevalley condition can be got out directly from the explicit form of p_{1*} (expression 2.21); for any $k \in \Omega Y$, we have $p_{1*} p_2^*(k) = p_{1*}(1 \otimes k) = 1 \wedge f^* h_*(k) = f^* h_*(k)$. ■

Consider now the special case of the lax pullback 2.15 in **Loc** with $X \equiv Y$ and $f \equiv h$ still a perfect map. Then the Beck-Chevalley condition becomes

$$p_{1*} \circ p_2^* = h^* \circ h_* \tag{2.24}$$

By relying just on the above equality we are able to prove the following theorem

Theorem 2.15 *Let $h : X \longrightarrow Z$ be a perfect surjection between two locales. Then h is the lax coequaliser of its kernel pair in* **Loc**.

$$\begin{tikzcd} X \sqsubseteq_Z X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "h"] \arrow[dr, "f'"'] & Z \arrow[d, dashed, "g"] \\ & & W \end{tikzcd} \qquad \begin{tikzcd} \Omega(X \sqsubseteq_Z Y) & \Omega X \arrow[l, shift right, "p_2^*"'] \arrow[l, shift left, "p_1^*"] & \Omega Z \arrow[l, "h^*"'] \\ & & \Omega W \arrow[ul, "f'^*"] \arrow[u, dashed, "g^*"'] \end{tikzcd}$$

Proof. We shall prove that a perfect surjection possesses the required universal property. Suppose there is continuous map $f : X \longrightarrow W$ such that $p_1^* f^*(b) \leq p_2^* f^*(b)$, for any $b \in \Omega W$. We need to demonstrate that there is a unique continuous map $g : Z \longrightarrow W$, such that $f^* = h^* \circ g^*$. First we observe that if such a map g exists then it is clearly unique since h^* is an injection by assumption.

To prove existence, we claim that

$$g := h_* \circ f^* \tag{2.25}$$

is such a map. At first glance $h_* \circ f^*$ is not necessarily a frame homomorphism because h_* does not in general preserve finite joins. However, we show that when h_* is restricted to the subframe A of ΩX with $j \in A$ iff $p_1^*(j) \leq p_2^*(j)$, then h_* is the inverse of h^* (and hence a frame homomorphism) and it obviously holds $A \supseteq Im(f^*)$. If $j \in A$ then we have

$$\begin{aligned}
h^* \circ h_*(j) &= p_{1*} \circ p_2^*(j) \text{ (by 2.24)} \\
&\geq p_{1*} \circ p_1^*(j) \text{ (by definition of } A) \\
&= p_{1*}(j \otimes 1) \\
&= j \wedge h^* \circ h_*(1) \text{ (c.f. 2.21)} \\
&= j \wedge 1 \;=\; j
\end{aligned}$$

For the other direction, we recall that the adjunction $h^* \dashv h_*$ implies $h^* \circ h_*(j) \leq j$. This shows that $h_* : A \longrightarrow \Omega Z$ is a right inverse of h^*. The fact that h^* is an injection guarantees that h_* is also a left inverse of h^*. This in particular implies that the choice of 2.25 is the correct one since

$$h^* \circ (h_* \circ f^*) = f^*$$

■

Remark 2.16 *The elements j of ΩX for which $p_1^*(j) \leq p_2^*(j)$ are referred to as the elements equipped with lax descent data. We denote $LDes(h)$ the poset of objects with such lax descent data. In the above proof we effectively showed that $LDes(h)$ is isomorphic to ΩZ. When this happens, we say that "h is of lax descent". We refer to [MV99] for a brief description of lax descent in the more general case of topoi.*

From the proof of theorem 2.15 we extract the following fact which will be evoked later.

Lemma 2.17 *If $h : X \longrightarrow Z$ is a perfect surjection between two locales, then $h_* \circ h^* = id_{\Omega Z}$*

2.4 The relation between perfect and patch continuous maps

In this section we return to our study of stably compact locales. We investigate the relation of perfect maps between stably compact locales and monotone continuous maps between their respective patch frames in order to reach a conclusion with the same content as the classical one.

The following theorem relies on the fact that $\varepsilon_X : \mathbf{Patch}X \longrightarrow X$ is a perfect surjection[1].

Theorem 2.18 *(i) Let X and Y be stably compact locales. If $f : \mathbf{Patch}X \longrightarrow \mathbf{Patch}Y$ is continuous and monotone (in the sense of definition 2.11) there is a unique continuous function $g : X \longrightarrow Y$ that makes the following diagram commutative.*

$$\begin{tikzcd}
\mathbf{Patch}X \arrow[r, "f"] \arrow[d, "\varepsilon_X"'] & \mathbf{Patch}Y \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, dashed, "g"] & Y
\end{tikzcd} \tag{2.26}$$

(ii) Any map g that makes the above diagram commutative is perfect.

Proof. (i) We exploit the lax coequaliser result of theorem 2.15. For this reason we construct the following diagram.

$$\begin{tikzcd}
\mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X \arrow[r, "f'"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \arrow[d, phantom, "\leq"] & \mathbf{Patch}Y \sqsubseteq_Y \mathbf{Patch}Y \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] \arrow[d, phantom, "\leq"] \\
\mathbf{Patch}X \arrow[r, "f"] \arrow[d, "\varepsilon_X"'] & \mathbf{Patch}Y \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, dashed, "g"] & Y
\end{tikzcd} \tag{2.27}$$

We sum up by saying that ε_X and ε_Y are perfect surjections so they are the coequalisers of their respective lax kernel pairs in **Loc**. The map f' is the unique continuous map that makes the two top squares commutative according to the definition of monotonicity

[1] Put reference.

(definition 2.11). We have

$$\begin{aligned} \varepsilon_Y \circ q_1 &\leq \varepsilon_Y \circ q_2 \text{ (by construction of } (\sqsubseteq)_Y) \\ \Rightarrow \varepsilon_Y \circ q_1 \circ f' &\leq \varepsilon_Y \circ q_2 \circ f' \\ \Rightarrow \varepsilon_Y \circ f \circ p_1 &\leq \varepsilon_Y \circ f \circ p_2 \text{ (definition 2.11)} \end{aligned}$$

This means that the function $\varepsilon_Y \circ f$ is also a coequaliser of p_1 and p_2. Hence, by the universal property of ε_X there must be a unique $g : X \longrightarrow Y$ that makes the lower diagram commutative which is the claimed.

(ii) Suppose that g is a map that makes the diagram 2.26 commutative with f continuous and monotone. By the uniqueness of the right adjoint, the same diagram involving the right adjoints of the defining frame homomorphism involved is also commutative. Starting with this equality, we have the following implications.

$$\begin{aligned} g_* \circ \varepsilon_{X*} &= \varepsilon_{Y*} \circ f_* \\ \Rightarrow g_* \circ \varepsilon_{X*} \circ \varepsilon_X^* &= \varepsilon_{Y*} \circ f_* \circ \varepsilon_X^* \\ \Rightarrow g_* &= \varepsilon_{Y*} \circ f_* \circ \varepsilon_X^* \text{ (lemma 2.17)} \end{aligned} \tag{2.28}$$

Now, in equation 2.28

ε_X^* is a frame homomorphism so it preserves directed joins.

ε_{X*} preserves directed joins because $\varepsilon_X$is perfect.

Patch$X$ and **Patch**Y are compact regular locales and it is known (first proved by Vermeulen in [Ver94]) that any map between compact regular locales is *proper*. Therefore, f is proper and this in particular entails that f_* preserves directed joins.

Hence the function g_* preserves directed joins as a composite of functions that do so. This completes the proof of perfectness of g. ∎

The converse of theorem 2.18 can also be established by using the adjunction **inc** $\dashv$ **Patch** that Escardó demonstrated in [Esc01].

Theorem 2.19 *If $g : X \longrightarrow Y$ is a perfect map between two stably compact locales, then there is a unique map f that makes the familiar diagram below commutative. Moreover,* *this unique map is monotone.*

$$\begin{tikzcd} \mathbf{Patch}X \arrow[r, dashed, "f"] \arrow[d, "\varepsilon_X"'] & \mathbf{Patch}Y \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{2.29}$$

Proof. All the ingredients of this proof are essentially provided by Escardó in [Esc01]. We recall that ε is the counit of the adjunction $\mathbf{inc} \dashv \mathbf{Patch}$. Therefore, for any stably compact locale Y

$$\varepsilon_Y : \mathbf{inc} \circ \mathbf{Patch}Y \longrightarrow Y$$

is universal from **inc** to Y. This means that, if we consider the map $g \circ \varepsilon_X$ (which is perfect because both g and ε_X are), there is indeed a unique continuous map $f : \mathbf{Patch}X \longrightarrow \mathbf{Patch}Y$ that makes the diagram 2.29 commutative.

This unique map is clearly $\mathbf{Patch}g$. This comes out of the definition of the arrow part of **Patch** (see discussion that precedes theorem 2.3) and the fact that $\varepsilon_X : \mathbf{Patch} \longrightarrow X$ is defined by the frame homomorphism

$$a \in \Omega X \mapsto \hat{a}$$

where $\hat{a}$ is the closed nucleus corresponding to the open a.

It remains to prove that $\mathbf{Patch}g$ is monotone which is trivial if we consider the frame presentation

$$\Omega(\mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X) \cong \mathbf{Fr}\langle \Omega\mathbf{Patch}X \otimes \Omega\mathbf{Patch}X \text{ (qua frame) } | \hat{a} \otimes 1 \leq 1 \otimes \hat{a}, \;\; \forall a \in \Omega X \rangle \tag{2.30}$$

and the corresponding for $\Omega(\mathbf{Patch}X \sqsubseteq_X \mathbf{Patch}X)$. The map

$$f' := \mathbf{Patch}g \times \mathbf{Patch}g$$

makes the two upper squares of diagram and it is well defined because for any $a \in \Omega Y$,

$$\begin{aligned} (\mathbf{Patch}g)^* \otimes (\mathbf{Patch})^* g(\hat{a} \otimes 1) &= ((\mathbf{Patch}g)^*(\hat{a}) \otimes (\mathbf{Patch}g(1))^*) \\ &= \widehat{g^*(a)} \otimes 1 \text{ (by the definition of } \mathbf{Patch}) \\ &\leq 1 \otimes \widehat{g^*(a)} \\ &= (\mathbf{Patch}g)^* \otimes (\mathbf{Patch}g)^*(1 \otimes a) \end{aligned}$$

■

Chapter 3

Finitary Representations of Stably Compact Locales

3.1 Introduction

We have mentioned in section 1.4, the generalised Stone representation

$$\mathbf{DLat}^{op} \simeq \mathbf{CohSp} \tag{3.1}$$

Suppose that one wants to obtain an analogous representation for stably compact spaces. The lattices of stably compact locales are in general "more manifestly continuous" so one would need an extra element on the distributive lattice structure that conveys this continuity. It turns out that this extra element is an additional binary relation (referred to as strong order), not necessarily reflexive, that imitates the way below relation of the frame of a stably compact locale. More specifically, Jung and Sünderhauf considered in [JS96] *strong proximity lattices* (see definition in section 3.2 below) and proved the equivalence of the category of stably compact spaces and continuous maps with the category of strong proximity lattices and certain *relations*. The essence of this equivalence is the fact that the lattice of opens of any stably compact locale is always the *rounded ideal* completion of some strong proximity lattice, where a rounded ideal is an ideal with respect to the additional binary relation.

The idea of representing continuous domains as rounded ideal completions of sets endowed with a "strong order" goes back to Smyth who introduced in [Smy77] the notion of the *R-structure*. An R-structure is a set A with a transitive binary relation ($\prec$) such that for any $a \in A$, $\downarrow a := \{a' \in A | a' \prec a\}$ is $\prec$-directed. It follows that any continuous dcpo D is the rounded ideal completion of some R-structure, namely the R-structure $(ptD, \ll)$

where ptD is the set of points of D and $\ll$ the way below relation of D.

The primary reason we are interested in representations of stably compact locales with strong proximity lattices is that the theory of strong proximity lattices is geometric and is thus preserved when pulled back by means of the inverse image functors of geometric morphisms. Moreover, we are interested in representing perfect maps (rather than general continuous maps) between stably compact locales as certain maps between the corresponding strong proximity lattices. To this end we introduce the notion of a *strong homomorphism* between two strong proximity lattices.

More analytically, this chapter is organised as follows. In section 3.2 we introduce strong proximity lattices and it is shown that if B is a strong proximity lattice then its *rounded ideal* completion $RIdl(B)$ is the frame of a stably compact locale. In section 3.3, we momentarily forget about $RIdl(B)$. Instead, given a strong proximity lattice B, we consider a particular presentation of a frame $\Omega RSpec(B)$. By treating this frame presentation as an *essentially propositional* geometric theory, we characterise the points of the locale $RSpec(B)$. Then we prove that $RIdl(B) \simeq RSpec(B)$. In section 3.4, we start from an arbitrary stably compact locale X and we construct a strong proximity lattice $\mathcal{B}X$. In section 3.5, we make the constructions $RSpec$ and $\mathcal{B}$ functorial: We introduce the notion of a *strong homomorphism* between two strong proximity lattices and we construct a perfect map $\mathbf{RSpec}(\mu) : \mathbf{RSpec}(B_1) \longrightarrow \mathbf{RSpec}(B_2)$ out of a strong homomorphism $\mu : B_2 \longrightarrow B_1$ and conversely, a strong homomorphism $\mathcal{B}f : \mathcal{B}X_2 \longrightarrow \mathcal{B}X_1$ out of a perfect map $f : X_1 \longrightarrow X_2$ between two stably compact locales. Finally we show that $\mathbf{RSpec} \circ \mathcal{B} = id_{\mathbf{StKLoc}}$. In section 3.6 we revisit the points of the locale $\mathbf{RSpec}(B)$ with the purpose to investigate how a strong homomorphism $\mu : B_2 \longrightarrow B_1$ acts on the points $pt \circ \mathbf{RSpec}(B_1) \longrightarrow \mathbf{RSpec}(B_2)$.

3.2 Strong proximity lattices

We introduce the notion of strong proximity lattices and their basic properties. All the definitions and results of this section were established by Jung & Sünderhauf in [JS96].

Definition 3.1 *A strong proximity lattice $(B, \wedge, \vee, \prec)$ is a distributive lattice with an additional relation $\prec$ which is transitive and interpolative and satisfies the following:*

(i) if $a \leq b \prec c \leq d$ then $a \prec d$.

(ii) if $a \prec b_i$ $(i \in I finite)$ then $a \prec \wedge_i b_i$.

(iii) if $a_i \prec b$ $(i \in I finite)$ then $\vee_i a_i \prec b$.

(iv) *if* $a \prec \vee_i b_i$ $(i \in I finite)$ *then* $a \prec \vee_i b_i'$ *for some* $b_i' \prec b_i$.

(v) *if* $\wedge_i a_i \prec b$ $(i \in I finite)$ *then* $\vee_i a_i' \prec b$ *for some* $a_i' \succ a_i$.

Remark 3.2 *In general,* $a \prec b$ *does not imply* $a \leq b$.

Remark 3.3 *The theory of strong proximity lattices is a geometric theory.*

It can be readily noticed that strong proximity lattices are self-dual ($B \cong B^{op}$), because of the symmetry of its definition. We will be referring to the additional relation $\prec$ as the *strong order*. We define ideals and filters with respect to the strong order.

Definition 3.4 *Let* B *be a strong proximity lattice. A rounded ideal is a subset* I *of* B *such that*

(i) I *is lower closed with respect to* $\prec$: *if* $a \in I$ *and* $a' \prec a$, *then* $a' \in I$

(ii) I *is rounded: if* $a \in I$, *then there is* a' *with* $a \prec a'$ *and* $a' \in I$.

(iii) I *is* $\vee$*-closed: if* $a, b \in I$ *then* $a \vee b \in I$.

(iv) $\bot \in I$.

The definition of a rounded filter is the dual.

A prime rounded filter is a filter $F \subseteq B$ *which in addition possesses the property that for any pair* a, b *of elements of* B *with* $a \vee b \in F$, $a \in F$ *or* $b \in F$.

We shall always be denoting by $\downarrow$ and $\uparrow$ the lower closure and upper closure with respect to the strong order $\prec$. Note that properties (i) and (ii) of definition 3.4 can be expressed respectively as $\downarrow I \subseteq I$ and $\downarrow I \supseteq I$. Hence, (i) and (ii) can be substituted by the property

$$\downarrow I = I$$

Similarly, a filter of B is a subset of B such that $\uparrow F = F$ and F is $\wedge$-closed. It is easy to check that the subsets $\downarrow a$ and $\uparrow a$, $a \in B$ are rounded ideals and filters respectively (properties (iii) and (ii) respectively of definition 3.1).

Also we remark that if I is a rounded ideal then I is a weak rounded ideal, i.e. a lower closed subset of B with respect to the weak order $\leq$ that also satisfies properties (ii) and (iii) of definition 3.4. For if I is a rounded ideal and $a' \leq a$ with $a \in I$, then by roundedness there is $a'' \in I$ with $a \prec a''$ and so $a' \leq a \prec a''$ or $a' \prec a''$ (by definition 3.1(i)). Therefore $a' \in I$.

Lemma 3.5 *Let I be an ideal in a strong proximity lattice B. Then $I = \bigcup^{\uparrow}\{\downarrow b | b \in I\}$*

Proof. If $a \in I$, then by roundedness there is an element $b \in I$ with $a \prec b$. Therefore, $a \in \downarrow b$ and this shows that $I \subseteq \bigcup^{\uparrow}\{\downarrow b | b \in I\}$. If $a \prec b$ and $b \in I$, then $a \in I$ because $\downarrow I = I$ and this shows $I \supseteq \bigcup^{\uparrow}\{\downarrow b | b \in I\}$. The union is obviously directed, for if $a_1 \in I$ and $a_2 \in I$, then the principal rounded ideal $\downarrow (a_1 \vee a_2)$ is a superset of both $\downarrow a_1$ and $\downarrow a_2$. ■

It is well known that the ideal completion of a distributive lattice yields a a frame. We prove that the rounded ideal completion of a strong proximity lattice yields a frame of a stably compact locale.

Theorem 3.6 *Let B be a strong proximity lattice. Let $RIdl(B)$ be the set of rounded ideals of B partially ordered by set inclusion. Then $RIdl(B)$ is a frame of a stably compact locale. Its binary meets, directed joins and binary joins are computed as follows:*

(i) For $I, J \in RIdl(B)$, $I \wedge J = I \cap J$.

(ii) For $\{I_i \in RIdl(B)\}$ a directed family, $\bigvee^{\uparrow} I_i = \bigcup^{\uparrow} I_i$.

(iii) For $I_1, ..., I_n \in RIdl(B)$ a finite family, $\vee_{i=1}^{n} I_i = \{a \in B | \exists a_1 \in I_1, ..., a_n \in I_n, \text{ with } a \prec \vee_{i=1}^{n} a_i\}$

The way below relation of $RIdl(B)$ is given by

$$I \ll J \text{ iff } \exists a \in J \text{ such that } I \subseteq \downarrow a (\subseteq J)$$

Proof. Obviously $I_1 \wedge I_2$ is a rounded ideal if I_1 and I_2 are. It is easy to show that $\bigcup^{\uparrow} I_i$ is a rounded ideal if $\{I_i\}$ is a directed family of rounded ideals. Indeed, obviously $\downarrow \bigcup^{\uparrow} I_i = \bigcup^{\uparrow} I_i$. Let $a_1, a_2 \in \bigcup^{\uparrow} I_i$ and $\{I_i\}$ is a directed family of rounded ideals. Say without loss of generality that $a_1 \in I_1$ and $a_2 \in I_2$, where I_1, I_2 are two rounded ideals of the family. By directedness there is an ideal I_3 in the family such that $I_1 \subseteq I_3$ and $I_2 \subseteq I_3$. This implies that $a_1, a_2 \in I_3$ and I_3 is $\vee$-closed, hence $a_1 \vee a_2 \in I_3$ and so $a_1 \vee a_2 \in \bigcup^{\uparrow} I_i$. It is also easy to show that $I \vee J$ is a rounded ideal if I and J are. We check easily that $\downarrow (I \vee J) = I \vee J$ (using the transitivity and interpolation property of $\prec$). Also, if $a, b \in I \vee J$, then there are $a_1 \in I, a_2 \in J$ and $b_1 \in I$ and $b_2 \in J$ such that $a \prec a_1 \vee a_2$ and $b \prec b_1 \vee b_2$. This gives $a \prec a_1 \vee a_2 \vee b_1 \vee b_2$ and $b \prec a_1 \vee a_2 \vee b_1 \vee b_2$ (using property (i), definition 3.1). The last pair of inequalities yields $a \vee b \prec a_1 \vee a_2 \vee b_1 \vee b_2$ by applying property (iii) of definition 3.1. Now, $a_1 \vee b_1 \in I$ and $a_2 \vee b_2 \in J$ and therefore $a \vee b$ is an element of $I \vee J$ according to the definition.

We need to show that the binary join corresponds to the least upper bound in $RIdl(B)$. If I, J are two rounded ideals then clearly $I \vee J$ is an upper bound of I and J, for if (say) $a \in I$, then roundedness guarantees that there is $a' \in I$ with $a \prec a'$ and therefore $a \prec a'veeb$, where b is any element of J. Let K be any rounded ideal and simultaneously an upper bound of I and J. Then if a is an element of $I \vee J$, there are $a_1 \in I$ and $a_2 \in J$ such that $a \prec a_1 \vee a_2$. Now K contains all elements present in I and J, so $a_1, a_2 \in K$. Also K is $\vee$-closed and so $a_1 \vee a_2 \in K$. Finally $\downarrow K = K$, which proves that $a \in K$, i.e. that $I \vee J \subseteq K$.

For a directed family $\{J_i\}$ of rounded ideals and a rounded ideal I we have that

$$\begin{aligned} I \wedge \bigvee_i{}^{\uparrow} J_i &= I \cap \bigcup_i{}^{\uparrow} J_i = \\ \bigcup_i{}^{\uparrow}(I \cap J_i) &= \bigvee_i{}^{\uparrow}(I \wedge J_i) \end{aligned}$$

We will show that the binary joins also distribute with meets. Let I, J, K three rounded ideals. It can be easily checked that $(I \wedge J) \vee (I \wedge K) \subseteq I \wedge (J \vee K)$. For the opposite direction, let $a \in I \wedge (J \vee K)$. Then $a \in I$ and $a \prec a_1 \vee a_2$ for some $a_1 \in J$ and $a_2 \in K$. By roundedness of I, there is $a' \in I$ such that $a \prec a'$. By applying the property (ii) of definition 3.1 we get $a \prec a' \wedge (a_1 \vee a_2)$ or $a \prec (a' \wedge a_1) \vee (a' \wedge a_2)$ and this demonstrates that $a \in (I \wedge J) \vee (I \wedge K)$.

Up to this point in the proof we have effectively established that $RIdl(B)$ is indeed a frame. As always the finite joins are computed inductively out of the binary joins and arbitrary joins can be got out of the directed and finite ones. Also the above demonstration suffices to prove that the arbitrary joins distribute over (binary and hence) finite meets.

Now suppose that $I \ll J$ in $RIdl(B)$. By lemma 3.5 $J = \bigcup^{\uparrow}\{\downarrow b | b \in J\}$ and so there is $b \in J$ such that $I \subseteq \downarrow b$ (and obviously $\downarrow b \subseteq J$). Conversely, if there is $b \in J$ such that $I \subseteq \downarrow b \subseteq J$ and $\{I_i\}$ is a directed family of rounded ideal with $J \subseteq \bigcup_i^{\uparrow} I_i$. By assumption $b \in J$, which implies that there is index i_0 such that $b \in I_{i_0}$. I_{i_0} is rounded and so there is $b' \in I_{i_0}$ such that $b \prec b'$ and therefore $\downarrow b \subseteq I_{i_0}$. This together with the assumption $I \subseteq \downarrow b$ yield $I \subseteq I_{i_0}$.

Next we prove that $RIdl(B)$ is a continuous lattice, i.e. that for any rounded ideal I, $I = \bigcup^{\uparrow}\{I_i \in RIdl(B) | I_i \ll I\}$. We show that $I \subseteq \bigcup^{\uparrow}\{I_i \in RIdl(B) | I_i \ll I\}$. To that end, let $a \in I$. Evoking roundedness, there is an element $a_1 \in I$ such that $a \prec a_1$ and, subsequently, an other element $a_2 \in I$ such that $a_1 \prec a_2$. The element a is contained in the rounded ideal $\downarrow a_1$ and obviously $\downarrow a_1 \subseteq \downarrow a_2$, i.e. $\downarrow a_1 \ll I$. Therefore $a \in \bigcup^{\uparrow}\{I_i \in RIdl(B) | I_i \ll I\}$. The inclusion $I \supseteq \bigcup^{\uparrow}\{I_i \in RIdl(B) | I_i \ll I\}$ is obvious from the fact that $\ll$ entails $\leq$ in $RIdl(B)$.

The fact that $\ll$ is stable under meets is an immediate result of the axioms of strong proximity lattices. Suppose that $I \ll I_1$ and $I \ll I_2$ which we proved that is equivalent to the fact that there are $a \in I_1$ and $b \in I_2$ such that $I \subseteq\downarrow a$ and $I \subseteq\downarrow b$. Now $a \wedge b \leq a$ and $a \wedge b \leq b$ and by the comment just before lemma 3.5 this implies that $a \wedge b \in I_1 \cap I_2 = I_1 \wedge I_2$. Also for any element $c \in I$, $c \prec a$ and $c \prec b$ by assumption and therefore $c \prec a \wedge b$ by the property (ii) of the definition 3.1. Hence $I \subseteq\downarrow (a \wedge b)$ and so we establish that $I \ll I_1 \wedge I_2$.

Finally, we show that $RIdl(B)$ is the frame of a compact locale, i.e. that the top element of $RIdl(B)$ is compact. Let $\bot$ and $\top$ be the bottom and top elements of B as a distributive lattice. If we implement property (ii) of definition 3.1 with the index set I being empty we get that $a \prec \wedge\emptyset = \top$, for any $a \in B$. Similarly, using the property (iii) of definition 3.1, we get $\vee\emptyset = \bot \prec a$, for any $a \in B$. In particular, $\top \prec \top$ which implies that $\downarrow \top \subseteq \top \subseteq \top$ which can be written as

$$B \subseteq\downarrow \top \subseteq B$$

which amounts to $RIdl(B)$ being the frame of a compact locale. ■

The next lemma is almost trivial but we place it here for future reference.

Lemma 3.7 *Let B be a strong proximity lattice. Then in $RIdl(B)$,*

(i) $(\downarrow a) \wedge (\downarrow b) =\downarrow (a \wedge b)$.

(ii) $\downarrow a \vee \downarrow b =\downarrow (a \vee b)$.

Proof. (i) If $c \in (\downarrow a) \wedge (\downarrow b) = (\downarrow a) \cap (\downarrow b)$, then $c \prec a$ and $c \prec b$, so by the defining properties of strong proximity lattices, $c \prec a \wedge b$. If $c \prec a \wedge b$, then $c \prec a \wedge b \leq a$ and $c \prec a \wedge b \leq b$ and hence $c \in\downarrow (a \wedge b)$.

(ii) Obviously $\downarrow (a \vee b) \subseteq (\downarrow a) \vee (\downarrow b)$. Also if $c \in (\downarrow a) \vee (\downarrow b)$, then, by the definition of binary joins in $RIdl(B)$, there are $a' \prec a$ and $b' \prec b$ such that $c \prec a' \vee b'$. But $a' \prec b$ and $b' \prec b$ implies $a' \vee b' \prec a \vee b$ and hence $c \prec a \vee b$ which entails that $c \in\downarrow (a \vee b)$. ■

3.3 Presentation of frames of stably compact locales

The subject of this section is to write presentations of frames of stably compact locales using strong proximity lattices and then argue geometrically about the points of such locales.

We start with some generalities about frame presentations as essentially propositional geometric theories. The reader is also referred to section 1.3. Let $\mathbb{T}_X$ be an essentially propositional geometric theory, i.e. a geometric theory that has no (primitive) sorts and (hence) no function symbols, only nullary relation symbols $\widetilde{R_i}$ and axioms $\widetilde{A_i}$. Such a geometric theory, after a formal modification is equivalent to a frame presentation; the relation symbols are the formal generators of the presentation and, in the absence of sorts, the axioms on the relations can always be written in equational form $A_1 = A_2$, where each expression A_i is a join of finite meets of generators. Conversely, a frame presentation readily yields a propositional geometric theory where the generators are the relation symbols and the equational relations are the geometric axioms on those relation symbols.

Now let $\mathbb{T}_X$ be a propositional geometric theory that corresponds to a presentation of a frame ΩX. Then the standard construction of the syntactic site of a geometric theory $\mathbb{T}_X$ (e.g. the one in [MM92]) results to the site $(\Omega X, J)$, where J is the Grothendieck topology that assigns to each $a \in \Omega X$ the subfunctors of the representable presheaf $\mathbf{y}_{\Omega X}(a)$ that are the *open covering* sieves. The sheaves over the site $(\Omega X, J)$ is just $\mathcal{S}X$, the topos of sheaves over the frame ΩX. Adhering to the notation that displays topoi as generalised topological spaces, we can write that the classifying topos of the geometric theory $\mathbb{T}_X$ is the locale X.

Since such a geometric theory has no sorts then a model $\mathcal{M}_X^Z$ of $\mathbb{T}_X$ inside the sheaves of a topos Z is obtained by identifying its relation symbols $\widetilde{R_i}$ with subobjects of the empty product of objects in this topos, i.e., with subobjects of the terminal object $\mathbf{1}_{\mathcal{S}Z}$ in such a way so that the axioms are valid. It is important to stress that a model of the theory $\mathbb{T}_X$ in $\mathcal{S}Z$ can ultimately comprise not only these interpretations R_i of the primitive relation symbols $\widetilde{R_i}$ but also other objects that are constructed geometrically, i.e. by means of finite limits and arbitrary colimits. A model of the theory $\mathbb{T}_X$ inside $\mathcal{S}Z$ is equivalent to a geometric morphism $Z \longrightarrow X$ and the latter is a point of the locale X at stage Z. Therefore, the models of a propositional geometric theory are equivalent to the generalised points of the locale that this theory corresponds to. Specifying the points of a locale at any stage Z (Z being a Grothendieck topos) completely determines the locale itself.

Before we focus on frame presentations of stably compact locales, let us recall that given a distributive lattice D, its ideal completion is isomorphic to the frame presented as

$$Fr < D \ \text{(qua D.L.)} | \ > \tag{3.2}$$

("D.L." stands for distributive lattice). The points of the locale that this frame defines are *prime filters* of D and these are equivalent to lattice homomorphisms $D \longrightarrow \Omega$.

We define the following refinement of the presentation 3.2.

Definition 3.8 *Let B be a strong proximity lattice. We define $RSpec(B)$ (rounded spectrum of B) to be the locale whose frame is presented as*

$$\Omega RSpec(B) = Fr < B \text{ (qua D.L.) } \mid b = \bigvee{}^{\uparrow}\{a' | a' \prec a\} > \tag{3.3}$$

We turn the frame presentation 3.3 into the equivalent propositional geometric theory, which we denote $\mathbb{T}_{RSpec(B)}$. In accordance with the preceding discussion the relation symbols of $\mathbb{T}_{RSpec(B)}$ correspond to the elements of B and the axioms of $\mathbb{T}_{RSpec(B)}$ correspond to the properties of a distributive lattice plus the extra axiom in the presentation 3.3.

Definition 3.9 *Given a strong proximity lattice B, let $\mathbb{T}_{RSpec(B)}$ be the geometric theory presented as follows.*

relations: a relation symbol $\widetilde{F_b} \subseteq \mathbf{1}$ for each element $b \in B$

axioms:
1. $\top \Leftrightarrow \widetilde{F_\top}$.
2. $\widetilde{F}_{a \wedge b} \Leftrightarrow \widetilde{F}_a \wedge \widetilde{F}_b$.
3. $\perp \Leftrightarrow \widetilde{F}_\perp$.
4. $\widetilde{F}_{a \vee b} \Leftrightarrow \widetilde{F}_a \vee \widetilde{F}_b$.
5. $\widetilde{F}_b \Rightarrow \bigvee\{\widetilde{F}_{b'} | b' \prec b\}$.
6. $\widetilde{F}_b \Leftarrow \bigvee\{\widetilde{F}_{b'} | b' \prec b\}$.

Axioms 1-4 correspond to the (qua D.L.) part of the relations whereas axioms 5-6 correspond to the relation $b = \bigvee{}^{\uparrow}\{a' | a' \prec a\}$ in the frame presentation 3.3.

The locale $RSpec(B)$ of definition 3.8 is the same as the locale that classifies the geometric theory $\mathbb{T}_{RSpec(B)}$. The global points of this locale are equivalent to the geometric morphisms $\mathbf{1} \longrightarrow RIdl(B)$. Geometricity also allows us to speak about the points of $RIdl(B)$ at a stage Z, where Z is any Grothendieck topos. Such generalised points are the same (up to natural equivalence) as models of the geometric theory $\mathbb{T}_{RIdl(B)}$ inside $\mathcal{S}Z$. By following the standard recipe we can determine the models of $\mathbb{T}_{RIdl(B)}$ in $\mathcal{S}Z$.

A model of $\mathbb{T}_{RIdl(B)}$ inside $\mathcal{S}Z$ (equivalently a point of $RSpec(B)$ at stage Z) is determined by assigning a subobject of the terminal object $\mathbf{1}$ of $\mathcal{S}Z$ to each relation symbol $\widetilde{F}_a$, or in other words assigning a truth value in the subobject classifier $\Omega_{\mathcal{S}Z}$ to each $\widetilde{F}_a$ in such a way that the axioms hold.

Let us first consider the case where $Z = \mathbf{1}$, i.e. global points of $RSpec(B)$. Then $\mathcal{S}Z = \mathbf{Sets}$ and we can argue as follows. There are as many relations $\widetilde{F}_b$ as the elements of the strong proximity lattice B. Hence, assigning truth values to $\widetilde{F}_b$ is the same as fixing a characteristic function $\chi_F : B \longrightarrow \Omega$, i.e. a map $B \longrightarrow \Omega$ that respects the axioms of $\mathbb{T}_{RSpec(B)}$. Such an arrow uniquely specifies a subset F of B as the set with the property

$$b \in F \Leftrightarrow \chi_F(b) = \top$$

For such a set $F \subseteq B$, the axioms of $\mathbb{T}_{RIdl(B)}$ become

1. $\top \in F$.
2. $a \wedge b \in B \Leftrightarrow a \in B$ and $b \in B$.
3. $\perp \notin F$ (this is the intuitionistic negation).
4. $a \vee b \in F \Leftrightarrow a \in B$ or $b \in B$.
5. $b \in F \Rightarrow$ there is $b' \in F$ with $b' \prec b$.
6. If $b' \prec b$ and $b' \in F \Rightarrow b \in F$.

Axioms 1 and 2 say that F is a filter, axioms 3 and 4 that F is a prime filter. Axiom 5 says that F is rounded with respect to the strong order and axiom 6 that F is lower closed with respect to the strong order. So F is a rounded prime filter (with the definition of roundedness given in 3.4). Hence, the points of the locale $RSpec(B)$ are rounded prime filters of B, or in other words, the *topos* $RSpec(B)$ classifies rounded prime filters of B.

We remark that, although the geometric theory $\mathbb{T}_{RSpec(B)}$ has no sorts, the set B came into existence as a result of purely geometric manipulations. The analysis was hitherto carried out in **Sets** but geometricity of frame presentations guarantees that its results can be transferred inside the sheaves of any Grothendieck topos Z by means of the inverse image of the essentially unique geometric morphism $! : Z \longrightarrow \mathbf{1}$. The theory of strong proximity lattices is geometric, therefore in $\mathcal{S}Z$, the object $!^*(B)$ is a strong proximity lattice object and the subobjects $F \hookrightarrow !^*(B)$ corresponding to rounded prime filter of $!^*(B)$ in $\mathcal{S}Z$ can be constructed naturally as follows.

Using theorem 1.6, a generalised point

$$p : Z \longrightarrow RIdl(B)$$

corresponds to a frame homomorphism

$$\Omega(RIdl(B)) \longrightarrow !_*(\Omega_{\mathcal{S}Z})$$

We recall that the set $!_*(\Omega_{\mathcal{S}Z})$ is a frame (because $!_*$ preserves frames) and is actually the one that we usually write as ΩZ. By the universal property of frame presentations, frame homomorphisms $\Omega(RIdl(B)) \longrightarrow \Omega Z$ are in bijection with distributive lattice homomorphisms $f : B \longrightarrow \Omega Z$ that preserve the extra relation of the presentation 3.3. Furthermore, via the isomorphism

$$\mathbf{Sets}(B, \Omega Z) \xrightarrow{\cong} \mathcal{S}Z(!^*(B), \Omega_{\mathcal{S}Z})) \quad (\text{due to } !^* \dashv !_*)$$

the map p gives rise to a *characteristic* map $\chi :!^*(B) \longrightarrow \Omega_{\mathcal{S}Z}$ in $\mathcal{S}Z$. The rounded prime filters of $!^*(B)$ in $\mathcal{S}Z$ are realised as pullbacks

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, "!"'] & !^*(B) \arrow[d, "\chi"] \\ \mathbf{1} \arrow[r, hook, "true"] & \Omega_{\mathcal{S}Z} \end{tikzcd}$$

We have therefore demonstrated that the points at stage Z of the topos $RSpec(B)$, defined as the classifying (localic) topos of the geometric theory $\mathbb{T}_{RSpec(B)}$, are rounded prime filters of the strong proximity lattice object $!^*(B)$ inside $\mathcal{S}Z$.

Next we are going to show that the frame presented by 3.3 is exactly the frame of rounded ideals of B as in theorem 3.6. By juxtaposition with the presentation 3.2, the extra relation $b = \bigvee^{\uparrow}\{a'|a' \prec a\}$ has the effect of restricting to the frame of *rounded* ideals (instead of any ideals) and we know that this is the frame of a stably compact locale by theorem 3.6.

Lemma 3.10 *The presentation $Fr < B$ (qua D.L.) $|b = \bigvee^{\uparrow}\{b'|b' \prec b\} >$ presents the frame $RIdl(B)$, for any strong proximity lattice B.*

Proof. Let A be the frame $Fr < B$ (qua D.L.)$|b = \bigvee^{\uparrow}\{b'|b' \prec b\} >$. We are going to define two frame homomorphisms

$$A \underset{g}{\overset{f'}{\rightleftarrows}} RIdl(b)$$

By the universal property of the presentation, in order to define the map f', it suffices to define a distributive lattice homomorphism $f : B \longrightarrow RIdl(B)$ respecting the extra relation, because then there is a unique frame homomorphism f' that makes the following diagram commutative

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, hook, "i"'] & RIdl(B) \\ A \arrow[ur, "f'"'] & \end{tikzcd}$$

where i is the inclusion of generators. We define f to be the map with the action $f(b) :=\downarrow b$ which is a lattice homomorphism by lemma 3.7 and respects the extra relation by lemma 3.5.

We define g to be the map with action $g(I) := \bigvee^{\uparrow}\{i(b)|b \in I\}$, where $\bigvee^{\uparrow}$ is the operation of calculating directed joins in A. We show that this is indeed a frame homomorphism. Let $I_1, I_2 \in RIdl(B)$. Then (dropping the inclusion symbol i) $g(I_1 \wedge I_2) = g(I_1 \cap I_2) = \bigvee^{\uparrow}\{a_i|a_i \in I_1 \text{ and } a_i \in I_2\}$. On the other hand $g(I_1) \wedge g(I_2) = (\bigvee^{\uparrow}\{a_i \in I_1\}) \wedge (\bigvee^{\uparrow}\{a_j \in I_2\}) = \bigvee^{\uparrow}\{a_i \wedge a_j|a_i \in I_1, a_j \in I_2\}$. Obviously $\{a_i|a_i \in I_1 \text{ and } a_i \in I_2\} \subseteq \{a_i \wedge a_j|a_i \in I_1, a_j \in I_2\}$. Also the assumption that I_1, I_2 are rounded ideals of $\mathcal{B}X$ implies that they are (weak) ideals. So if $a_i \in I_1$ and $a_j \in I_2$, then $a_i \wedge a_2 \in I_1$ and $a_1 \wedge a_2 \in I_2$. Therefore, $\{a_i|a_i \in I_1 \text{ and } a_i \in I_2\} = \{a_i \wedge a_j|a_i \in I_1, a_j \in I_2\}$.

To prove that g preserves directed joins in $RIdl(B)$, let $\{I_k\}_k$ is a directed collection of rounded ideals. Then $g(\bigvee^{\uparrow}_k I_k) = g(\bigcup^{\uparrow}_k I_k) = \bigvee^{\uparrow}\{a \in \bigcup^{\uparrow}_k I_k\} = \bigvee^{\uparrow}_k \bigvee^{\uparrow}\{a \in I_k\} = \bigvee^{\uparrow}_k g(I_k)$.

Finally, we prove that g preserves finite joins. Trivially it preserves nullary joins. Also, for I_1, I_2 two rounded ideals, we have

$$\begin{aligned} g(I_1) \vee g(I_2) &= \bigvee{}^{\uparrow}\{a \in I_1\} \vee \bigvee{}^{\uparrow}\{a \in I_2\} \\ &= \bigvee{}^{\uparrow}\{a_1 \vee a_2|a_1 \in I_1, a_2 \in I_2\} \end{aligned}$$

The relation in the presentation asserts that $a_1 \vee a_2 = \bigvee^{\uparrow}\{a'|a' \prec a_1 \vee a_2\}$. Therefore, iterating the above equalities,

$$\begin{aligned} g(I_1) \vee g(I_2) &= \bigvee{}^{\uparrow}\{\bigvee{}^{\uparrow}\{a' \in B|a' \prec a_1 \vee a_2\}|a_1 \in I_1, a_2 \in I_2\} \\ &= \bigvee{}^{\uparrow}\{a' \in B|a' \prec a_1 \vee a_2, a_1 \in I_1, a_2 \in I_2\} \\ &= g(I_1 \vee I_2) \end{aligned}$$

To complete the proof of the isomorphism, it suffices to prove that for any rounded ideal $I \in RIdl(B)$, $f' \circ g(I) = I$ and that for any *formal generator* $i(b), b \in B$ of A, it holds

$g \circ f'(i(b)) = i(b)$. For any rounded ideal I, we have

$$\begin{aligned} f' \circ g(I) &= f'(\bigvee{}^{\uparrow}\{i(b)|b \in I\}) \\ &= \bigvee{}^{\uparrow}\{f'(i(b))|b \in I\} \\ &= \bigvee{}^{\uparrow}\{f(b)|b \in I\} \;\; \text{(by the definition of } f'\text{)} \\ &= \bigvee{}^{\uparrow}\{\downarrow b|b \in I\} \\ &= I \text{ (because of lemma 3.5)} \end{aligned}$$

Also, for any $b \in B$, $g \circ f'(i(b)) = g(\downarrow b) = \bigvee^{\uparrow}\{b'|b' \prec b\} = b$ by the relation of the presentation. This completes the proof. ■

3.4 Derivation of strong proximity lattices from stably compact locales

In sections 3.2 and 3.3 we demonstrated how any strong proximity lattice gives rise to a stably compact locale. In this section we demonstrate the inverse construction. Associated with a stably compact locale X is not only the frame of its opens ΩX but also the frame of its Scott-open filters QX that classically corresponds to its compact saturated sets. The typical construction that assigns a strong proximity lattice $\mathcal{B}X$ to the stably compact locale X takes both ΩX and QX into account. Jung & Sünderhauf first gave a classical account of this construction in [JS96]. The constructive one developed in this section is due to Vickers [Vic98b].

We start by recalling a simple fact.

Lemma 3.11 *Scott open subsets of frames of locally compact locales are rounded.*

Proof. The claim of the lemma means that if ΩX is a continuous frame and $a \in K$, where K is a Scott open filter, then there is $a' \in K$ with $a' \ll a$. Take the Scott open set $S := \twoheaddownarrow a$. By continuity of ΩX, $\bigvee^{\uparrow} S = a \in K$ and since K is Scott open there is $s \in S$, i.e. $a' \ll a$, such that $a' \in K$. ■

Lemma 3.12 *Let X be a stably compact locale. Let $b_i \in \Omega X$ $(i \in I)$ be a finite number of opens and $K \subseteq \Omega X$ a Scott-open filter such that $\vee_i b_i \in K$. Then there are Scott open filters $K_i \subseteq \Omega X$ such that $b_i \in K_i$ for all $i \in I$ and $\cap_i K_i = K$.*

Proof. If the index set I is empty, the assumption $(\perp =) \vee \emptyset \in K$ automatically implies that $K = \Omega X = \wedge\emptyset$.

We cover the case $I = \{1, 2\}$. Let $b_1, b_2 \in \Omega X$ with $b_1 \vee b_2 \in K$. We consider the subset of ΩX

$$L := \{a \in \Omega X | a \vee b_2 \in K\} \tag{3.4}$$

It is easy to prove that L is a Scott-open filter. Let $a \in L$ and $a' \geq a$. Then $a' \vee b_2 \geq a \vee b_2 \in K$ and so $a' \in L$ because K is upper closed. If $a, a' \in L$, then $(a \wedge a') \vee b_2 = (a \vee b_2) \wedge (a' \vee b_2) \in K$ because K is $\wedge$-closed. Finally, $\bigvee_j^{\uparrow} a_i \in L$ implies that $(\bigvee_j^{\uparrow} a_i) \vee b_2 \in K$ or that $\bigvee_j^{\uparrow}(a_i \vee b_2) \in K$. K is Scott-open and so there is index j_0 such that $a_{j_0} \vee b_2 \in K$ and this forces $a_{j_0} \in L$.

Now $b_1 \in L$ and L is a rounded with respect to the relation $\ll$ (lemma 3.11), so there is $b_1' \in L$ with $b_1' \ll b_1$. We define the following Scott-open filter

$$K_1 := \Uparrow b_1' \tag{3.5}$$

Obviously $b_1 \in K_1$. We also define the Scott-open filter

$$K_2 := \{a \in \Omega X | b_1' \vee a \in K\} \tag{3.6}$$

It holds that $b_1' \vee b_2 \in K$ and so $b_2 \in K_2$. We observe that

$$(K \vee K_1) \wedge (K \vee K_2) = K \vee (K_1 \wedge K_2) \tag{3.7}$$

We also observe that if $c \in K_1 \wedge K_2 = K_1 \cap K_2$, then $c \gg b_1' \Rightarrow c \geq b_1'$ and at the same time $b_1' \vee c \in K$. Therefore, $c = b_1' \vee c \in K$ and thus we conclude that $K_1 \wedge K_2 \subseteq K$. This insight and equation 3.7 yield $(K \vee K_1) \wedge (K \vee K_2) = K$ which proves the claim of the lemma for the binary case.

The case where the index set I is any finite cardinal can be proved from the binary case by obvious induction on the cardinality of I. ∎

Theorem 3.13 *Let X be a stably compact locale. Let $\mathcal{B}X$ be the poset of those pairs (a, K) in*

$$\Omega X \times QX^{op} \quad \textit{qua distributive lattice}$$

that fulfill the property that a is a lower bound of K.

We endow $\mathcal{B}X$ with a relation $\prec$ by stipulating that $(a, K) \prec (b, L)$ iff $b \in K$ (where $a, b \in \Omega X$ and $K, L \in QX$). Then $\mathcal{B}X$ is a strong proximity lattice.

Proof. First, we verify that $\mathcal{B}X$ is a distributive lattice. We show that it is closed under meets. Let $(a, K), (b, L) \in \mathcal{B}X$. Then $(a, K) \wedge (b, L) = (a \wedge b, K \vee L)$ in $\Omega X \times QX^{op}$. Let

$c \in K \vee L$. Then by the definition of the joins in QX^{op}, there are $c_1 \in K$ and $c_2 \in L$ such that $c \geq c_1 \wedge c_2$. Moreover, $a \leq c_1$ and $b \leq c_2$, as a, b are lower bounds of K, L respectively. Therefore $a \wedge b \leq c_1 \wedge c_2$ and so $a \wedge b \leq c$.

$\mathcal{B}X$ is also closed under joins. The join of two pairs in $\mathcal{B}X$ is $(a, K) \vee (b, L) = (a \vee b, K \wedge L) = (a \vee K \cap L)$. Trivially, if $c \in K \cap L$, then $c \geq a$ and $c \geq b$ and so $c \geq a \vee b$. We thus establish that $\mathcal{B}X$ is a distributive lattice. Note that (by definition)

$$(a, K) \leq (b, L) \Rightarrow a \leq b \ \& \ K \supseteq L$$

Now we verify that the relation $\prec$ satisfies the properties of definition 3.1. To show that $\prec$ is interpolative, let $(a, K) \prec (b, L)$ in $\mathcal{B}X$. By definition $b \in K$ and the fact that K is a Scott-open filter of a continuous lattice implies that it is rounded with respect to $\ll$. Hence, there is an element $c \ll b$ with $c \in K$. The pair $(c, \Uparrow c)$ is an element of $\mathcal{B}X$ and it satisfies $(a, K) \prec (c, \Uparrow c) \prec (b, L)$.

The relation $\prec$ is transitive. For let $(a_1, K_1) \prec (a_2, K_2) \prec (a_3, K_3)$. This means that $a_3 \in K_2$ and the fact that a_2 is a lower bound of K_2 shows that $a_2 \leq a_3$. This last inequality together with the fact $a_2 \in K_1$ yield that $a_3 \in K_1$ (because K_1 is a filter). So $(a_1, K) \prec (a_3, K_3)$.

We verify the validity of definition 3.1(i). We consider four pairs in $\mathcal{B}X$ related as below

$$(a_1, K_1) \leq (a_2, K_2) \prec (a_3, K_3) \leq (a_4, K_4)$$

That means that $a_3 \in K_2$ and hence $a_3 \in K_1$ because $K_2 \subseteq K_1$. Also it holds $a_3 \leq a_4$ and we recall that K_1, being a filter, is upper closed and so $a_4 \in K_1$.

Now we come to the property (ii) of definition 3.1. Let $(b_i, L_i) \in \mathcal{B}X$ be a finite family and $(a, K) \in \mathcal{B}X$ such that $(a, K) \prec (b_i, L_i)$ for all i. Then $b_i \in K$ and so $\wedge_i b_i \in K$ because K is a filter. So $(a, K) \prec (\wedge_i b_i, \vee_i L_i) = \wedge_i (b_i, L_i)$.

The property (iii) of definition 3.1 is likewise trivial to verify. Let $(a_i, K_i) \in \mathcal{B}X$ be a finite family and $(b, L) \in \mathcal{B}X$ such that $(a_i, K_i) \prec (b, L)$ for all i. This translates as $b \in K_i$ for all i and so $b \in \cap_i K_i = \wedge_i K_i$. So $\vee_i (a_i, K_i) = (\vee_i a_i, \wedge_i K_i) \prec (b, L)$.

The property (iv) is the most non-trivial to verify. Let $(b_i, L_i) \in \mathcal{B}X$ be a finite family and $(a, K) \in \mathcal{B}X$ such that $(a, K) \prec \vee_i (b_i, L_i) = (\vee_i b_i, \wedge_i L_i)$. Then $\vee_i b_i \in K$. By applying lemma 3.12, there are Scott-open filters K_i such that $b_i \in K_i$ and $\wedge_i K_i = K$. By roundedness of the Scott-open filters K_i (with respect to $\ll$), there are elements $b'_i \in K_i$ such that $b'_i \ll b_i$ for all i. We consider the elements $(b'_i, \Uparrow b'_i)$. By construction it obviously holds that $(b'_i, \Uparrow b'_i) \prec (b_i, L_i)$ and $(a, K) \prec (\vee_i b'_i, \wedge_i \Uparrow b'_i)$.

Finally we verify the property (v) in definition 3.1. Let $(a_i, K_i) \in \mathcal{B}X$ be a finite family and $(b, L) \in \mathcal{B}X$ satisfying $\wedge_i(a_i, K_i) = (\wedge_i a_i, \vee_i K_i) \prec (b, L)$. This implies that $b \in \vee_i K_i$ and by the definition of joins in QX, it is the same with the assertion that there are $b_i \in K_i$, for all i, with $b \geq \wedge_i b_i$. Again, we evoke roundedness of the Scott-open filters K_i and we find elements $b'_i \in K_i$ such that $b'_i \ll b_i$, for all i. We consider the pairs $(b'_i, \Uparrow b'_i)$. It holds that $(a_i, K_i) \prec (b'_i, \Uparrow b'_i)$ because $b'_i \in K_i$ and

$$\wedge_i(b'_i, \Uparrow b'_i) = (\wedge_i b'_i, \vee_i \Uparrow b'_i)$$

because there are indeed elements $b''_i \gg b'_i$, namely $b''_i := b_i$, such that $b \geq \wedge_i b''_i$. ∎

Remark 3.14 *Clearly, in $\mathcal{B}X$ it holds that*

$$(a, K) \prec (b, L) \Rightarrow (a, K) \leq (b, L)$$

whereas this is not true for a general strong proximity lattice.

3.5 Perfect maps versus strong homomorphisms

In this section we shall extend the constructions of a stably compact locale $RSpec(B)$ out of a strong proximity lattice B and of a strong proximity lattice $\mathcal{B}X$ out of a stably compact locale X to functors

$$\mathbf{StKLoc} \underset{\mathbf{RSpec}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{StPrLat}^{op}$$

where **StKLoc** is the category of stably compact locales and perfect maps and **StrPrLat** is the category of strong proximity lattices and a special class of lattice homomorphisms that we first introduce here and christen *strong homomorphisms*. The main result of this section is that **RSpec** is a retraction (left inverse) of $\mathcal{B}$.

We point out that in [JS96] Jung and Sünderhauf consider the category of strong proximity lattices and certain relations and subsequently prove the *equivalence* of this category with the category of stably compact spaces and *continuous* maps. On the other hand, we are interested in the category **StKLoc** where the morphisms are perfect maps instead of general continuous functions. The mentioned result

$$\mathbf{RSpec} \circ \mathcal{B} = id_{\mathbf{StKLoc}} \tag{3.8}$$

falls short of an equivalence between **StKLoc** and **StrPrLat** but it nevertheless states that for any perfect map between stably compact locales there is an equivalent strong homomorphism between the corresponding strong proximity lattices, the latter regarded as the geometric counterpart of the former. We shall be relying on the fact of expression 3.8 in chapter 6 where we investigate the geometric counterpart of geometric morphisms induced by perfect maps between stably compact locales.

We briefly compare our result 3.8 with the duality

$$\mathbf{CohLoc} \simeq \mathbf{DLat}^{op}$$

where **CohLoc** is the category of coherent locales and perfect maps (see also section 1.4).

We start by introducing strong homomorphisms.

Definition 3.15 *We call a lattice homomorphism $\mu : B_1 \longrightarrow B_2$ between two strong proximity lattices strong iff in addition it fulfills the following three properties*

(i) If $a_1 \prec a_2$ in B_1 then $\mu(a_1) \prec \mu(a_2)$ in B_2.

(ii) For any two elements $a \in B_1$ and $b \in B_2$ with such that $b \prec \mu(a)$, there is $a' \in B_1$ with $a' \prec a$ such that $b \prec \mu(a')$.

(iii) For any two elements $a \in B_1$ and $b \in B_2$ with such that $b \succ \mu(a)$, there is $a' \in B_1$ with $a' \succ a$ such that $b \succ \mu(a')$.

First we define the functor **RSpec**. The following lemma is an immediate consequence of the the property (ii) of strongness.

Lemma 3.16 *Let $\mu : B_2 \longrightarrow B_1$ be a strong lattice homomorphism. Then for any $b \in B_2$*

$$\downarrow \mu[\downarrow b] = \downarrow \mu(b)$$

where $\downarrow \mu[\downarrow b] \equiv \{a \in B_1 | \exists b' \prec b \text{ with } a \prec \mu(b')\}$.

Proof. Clearly $\downarrow \mu[\downarrow b] \subseteq \downarrow \mu(b)$. Conversely, let $a \prec \mu(b)$. Then by the property (ii) of definition 3.15 there is $b' \prec b$ with $a \prec \mu(b')$. This means that $a \in \{a_i \in B_1 | \exists b' \prec b \text{ with } a = \mu(b')\}$ or that $a \in \downarrow \mu[\downarrow b]$. ■

Next we define the arrow part of the functor **RSpec**.

Theorem 3.17 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism. Then the function*

$$RSpec(\mu)^* : RIdl(B_2) \longrightarrow RIdl(B_1)$$

defined for any rounded ideal $I \subseteq B_2$ by

$$(RSpec\mu)^*(I) := \downarrow \{a \in B_1, \text{ such that} \, \exists b \in I \text{ with} \, a = \mu(b)\} = \downarrow \mu(I)$$

is the defining frame homomorphism of a perfect map $RSpec(B_1) \longrightarrow RSpec(B_2)$.

(The down closure $\downarrow$ *symbol appearing in the definition of* $RSpec(\mu)$ *is with respect to the strong order* $\prec$ *in* B_1*).*

Proof. For I a round ideal in B_2, $(RSpec\mu)^*(I)$ is easily checked to be a rounded ideal in B_1. It is lower closed by definition. Also if $a_1, a_2 \in (RSpec\mu)^*(I)$, then $a_1 \prec \mu(b_1)$ and $a_2 \prec \mu(b_2)$ for some $b_1, b_2 \in B_2$. So $a_1 \vee a_2 \prec \mu(b_1) \vee \mu(b_2) = \mu(b_1 \vee b_2)$. But $b_1 \vee b_2 \in I$ because I is an ideal, so $a_1 \vee a_2 \in (RSpec \circ \mu)^*(I)$.

The finite meets and directed joins in $RIdlB_1$ are intersections and unions respectively, which makes easy to check that $(RSpec\mu)^*$ preserves them. The finite joins involve a little more subtlety, so we demonstrate the fact that $(RSpec\mu)^*$ preserves the binary joins. Let I and J be two rounded ideals in B_2. Then the definition of binary joins in $RIdl(B_2)$ yields.

$$\begin{aligned}
(RSpec\mu)^*(I) \vee (RSpec\mu)^*(J) &= \{a \in B_1 | \exists a_1, a_2 \in B_1,\ \exists b_1 \in I,\ \exists b_2 \in J \\
&\quad \text{with } a_1 \prec \mu(b_1) \ \&\ a_2 \prec \mu(b_2) \ \&\ a \prec a_1 \vee a_2\} \\
(RSpec\mu)^*(I \vee J) &= \downarrow \{a \in B_1 | \exists b_1 \in I,\ \exists b_2 \in J,\ \exists b_{12} \in B_2 \\
&\quad \text{with } b_{12} \prec b_1 \vee b_2 \ \&\ a = \mu(b_{12})\}
\end{aligned}$$

Let $a' \in (RSpec\mu)^*(I \vee J)$. Then there are $b_1 \in I, b_2 \in J$ and $b_{12} \in B_2$ with $a' \prec \mu(b_{12})$ and $b_{12} \prec b_1 \vee b_2$. Since μ is strong, $b_{12} \prec b_1 \vee b_2$ implies $\mu(b_{12}) \prec \mu(b_1 \vee b_2) = \mu(b_1) \vee \mu(b_2)$. Evoking the property of strong proximity lattices, there are $a_1, a_2 \in B_1$ with $a_1 \prec \mu(b_1)$ and $a_2 \prec \mu(b_2)$ and $\mu(b_{12}) \prec a_1 \vee a_2$. Hence, we have $a' \prec \mu(b_{12}) \prec a_1 \vee a_2$ with $a_1 \prec \mu(b_1)$ and $a_2 \prec \mu(b_2)$ which implies that $a' \in (RSpec\mu)^*(I) \vee (RSpec\mu)^*(J)$. Conversely, let $a \in (RSpec\mu)^*(I) \vee (RSpec\mu)^*(J)$. Then there are $a_1, a_2 \in B_1$ and $b_1 \in I, b_2 \in J$ with $a_1 \prec \mu(b_1)$, $a_2 \prec \mu(b_2)$ and $a \prec a_1 \vee a_2$. This implies that $a_1 \vee a_2 \prec \mu(b_1 \vee b_2)$. Because μ is strong by assumption, there is $b_{12} \in B_2$ with $b_{12} \prec b_1 \vee b_2$ and $a_1 \vee a_2 \prec \mu(b_{12})$. Hence $a \in (RSpec\mu)^*(I \vee J)$.

Finally we prove that $RSpec\mu$ is perfect. Let $I \ll J$ in $RIdl(B_2)$. By theorem 3.6 this means that there is $c \in B_2$ such that $I \subseteq \downarrow c \subseteq J$. Since $(RSpec\mu)^*$ is monotone, it preserves these inclusions. So

$$(RSpec\mu)^*(I) \subseteq (RSpec\mu)^*(\downarrow c) \subseteq (RSpec\mu)^*(J)$$

All that remains to be proved is that $(RSpec\mu)^*(\downarrow c)$ is a principal rounded ideal in B_1 and this fact is demonstrated in lemma 3.16. Therefore, $(RSpec\mu)^*(I) \ll (RSpec\mu)^*(J)$. ∎

We recapitulate in the next definition.

Definition 3.18 *Let* **RSpec** : **StPrLat** $\longrightarrow$ **StKLoc**^{op} *be the functor whose object part is defined by definition 3.8 (c.f. lemma 3.10) and arrow part by theorem 3.17.*

Now we turn to the arrow part of the functor $\mathcal{B}$: **StKLoc** $\longrightarrow$ **StPrLat**^{op}. Given a perfect map $f : X \longrightarrow Y$ we seek a strong homomorphism $\mathcal{B}f : \mathcal{B}Y \longrightarrow \mathcal{B}X$. Note that $\mathcal{B}f$ is in the same direction as the defining frame homomorphism of f. We observe that if L is a Scott-open filter then its image $f^*[L]$ along f^* is not necessarily upper closed but it generates a Scott-open filter, namely the subset $\Uparrow f^*[L]$.

Theorem 3.19 *Let $f : X \longrightarrow Y$ be a perfect map between two stably compact locales. Then the function $\mathcal{B}f : \mathcal{B}Y \longrightarrow \mathcal{B}X$ defined as*

$$\mathcal{B}f : (b, L) \mapsto (f^*(b), \Uparrow f^*[L]) \text{ (where } f^*[L] := \{a \in \Omega X | \exists c \in L : a = f^*(c)\})$$

for a pair $(b, L) \in \mathcal{B}Y$ is a strong homomorphism.

Proof. First we prove that $\mathcal{B}f$ is a lattice homomorphism.

$\mathcal{B}f$ preserves $\wedge$: For any $(a, K), (b, L) \in \mathcal{B}Y$, by definition we have

$$\begin{aligned} \mathcal{B}f(a, K) \wedge \mathcal{B}f(b, L) &= (f^*(a) \wedge f^*(b), \Uparrow f^*[K] \vee \Uparrow f^*[L]) \quad \text{and} \\ \mathcal{B}f((a, K) \wedge (b, L)) &= \mathcal{B}f(a \wedge b, K \vee L) = \\ &= (f^*(a \wedge b), \Uparrow f^*[K \vee L]) \end{aligned}$$

Since f^* preserves meets, we have to show that

$$\Uparrow f^*[K] \vee \Uparrow f^*[L] = \Uparrow f^*[K \vee L]$$

Let $a \in \Uparrow f^*[K] \vee \Uparrow f^*[L]$. Then there is $c \gg f^*(c')$ and $d \gg f^*(d')$, for $c' \in K$ and $d' \in L$, such that $a \geq c \wedge d$. The fact that $c \gg f^*(c')$ and $d \gg f^*(d')$ implies that $c \wedge d \gg f^*(c') \wedge f^*(d') = f^*(c' \wedge d')$. So $a \gg f^*(c' \wedge d')$ and $c' \wedge d' \in K \vee L$ which implies that $a \in \Uparrow f^*[K \vee L]$. Conversely, let $b \in \Uparrow f^*[K \vee L]$. Then $b \gg f^(c)$ for some $c \in K \vee L$. Since $c \in K \vee L$, there are $c_1 \in K$ and $c_2 \in L$ such that $c \geq c_1 \wedge c_2$. This in turn implies that $f^*(c) \geq f^*(c_1 \wedge c_2) = f^*(c_1) \wedge f^*(c_2)$. Hence we have that $b \gg f^*(c_1) \wedge f^*(c_2)$ which implies that

$$b \geq f^*(c_1) \wedge f^*(c_2) \tag{3.9}$$

Now, by roundedness of K and L there are $c_1' \in K$ with $c_1' \ll c_1$ and $c_2' \in L$ with $c_2' \ll c_2$. This also implies that $f^*(c_1) \gg f^*(c_1')$ and $f^*(c_2) \gg f^*(c_2')$. The last two inequalities assert that $f^{(}c_1) \in \Uparrow f^*[K]$ and $f^*(c_2) \in \Uparrow f^*[L]$ and so by the inequality 3.9, $b \in \Uparrow f^*[K] \vee \Uparrow f^*[L]$.

$\mathcal{B}f$ preserves $\vee$: Because f^* preserves joins, we just need to show that

$$\Uparrow f^*[K] \cap \Uparrow f^*[L] = \Uparrow f^*[K \cap L] \tag{3.10}$$

Let $a \in \Uparrow f^*[K] \cap \Uparrow f^*[L]$. Then there are $a_1 \in K$ and $a_2 \in L$ with $a \gg f^*(a_1)$ and $a \gg f^*(a_2)$. This yields that $a \gg f^*(a_1) \vee f^*(a_2) = f^*(a_1 \vee a_2)$. The open $a_1 \vee a_2$ belongs both in K and L, therefore a is an open of the R.H.S. set of 3.10. The proof of $\Uparrow f^{[}K \cap L] \subseteq \Uparrow f^*[K] \cap \Uparrow f^*[L]$ is obvious.

Now we prove that $\mathcal{B}f$ is strong, i.e. it obeys the three axioms of definition 3.15.

Proof of (i): Let $(a, K) \prec (b, L)$ in $\mathcal{B}Y$. Then by definition $b \in K$. By roundedness of K, there is an open $c \in K$ with $c \ll b$. Since f is perfect, $f^*(c) \ll f^*(b)$ or equivalently $f^*(c) \in \Uparrow f^*(b)$. This immediately yields that

$$(f^*(a), \Uparrow f^*[K]) \prec (f^*(b), \Uparrow f^*[L])$$

Proof of (ii): First we assume that $(a, K) \prec \mathcal{B}f(b, L)$ in $\mathcal{B}X$. By the definition of $\mathcal{B}f$ this assumption reads

$$(a, K) \prec (f^*(b), \Uparrow f^*[L])$$

which by definition implies $f^*(b) \in K$. Now, since $f^*(b) \in K$ and K is rounded as a Scott-open filter, there is $a' \in K$ with $a' \ll f^*(b)$. Given that f is perfect, there is $b' \ll b$ with $a' \ll f^*(b')$. K is upper closed, so $f^*(b') \in K$.

We consider the pair $(b', \Uparrow b')$. ΩY is stably continuous and therefore $\Uparrow b'$ is a Scott-open filter and trivially b' is a lower strong bound of $\Uparrow b'$. So (theorem 3.13), $(b', \Uparrow b')$ is an element of $\mathcal{B}Y$. Since $b' \ll b$ or equivalently $b \in \Uparrow b'$, we have that $(b', \Uparrow b') \prec (b, L)$ in $\mathcal{B}Y$. Finally, the fact that $f^*(b') \in K$ guarantees that $(a, K) \prec \mathcal{B}f(b', \Uparrow b') = (f^*(b'), \Uparrow f^*[\Uparrow b'])$.

Proof of (iii): Now assume that $(a, K) \succ \mathcal{B}f(b, L)$, i.e., $(a, K) \succ (f^*(b), \Uparrow f^*[L])$. By definition, this now implies that $a \in \Uparrow f^*[L]$, or that there is $cb'' \in L$ such that $a \gg f^*(b'')$. Since L is rounded, $b'' \in L$ implies that there is $b' \ll c$ with $b' \in L$.

Now we consider the pair $(b', \Uparrow b') \in \mathcal{B}Y$. The fact that $b' \in L$ implies that $(b, L) \prec (b', \Uparrow b')$. Also, the fact that $b'' \succ b''$ and $a \succ f^*(b'')$ yields that $a \in \Uparrow f^*[\Uparrow b'']$. Therefore, $(a, K) \succ (f^*(b'), \Uparrow f^*[\Uparrow b']) = \mathcal{B}f(b', \Uparrow b')$ by the definitions of $\mathcal{B}Y$ and $\mathcal{B}f$. ■

Definition 3.20 *We define* $\mathcal{B} : \mathbf{StKLoc} \longrightarrow \mathbf{StPrLat}$ *to be the functor whose object part is given by theorem 3.13 and arrow part by theorem 3.19.*

The rest of this section is devoted to proving that **RSpec** is a retraction of $\mathcal{B}$. First two simple facts.

Lemma 3.21 *If $K \subseteq \Omega X$ is a Scott-open filter and b is a lower bound of K, then $a \in K$ implies $b \ll a$.*

Proof. Out of roundedness of Scott open filters (

Lemma 3.22 *3.11). If b is a lower bound of a Scott open filter K then for any $a \in K$, there is $a' \ll a$ such that $a' \in K$. So $b \leq a' \ll a$ or $b \ll a$.* ■

Lemma 3.23 *For X stably compact locale, $(b, L) \prec (a, K)$ implies that $K \subseteq L$.*

Proof. Trivial from the definition. ■

Theorem 3.24 *Let X be a stably compact locale. Then the following isomorphism holds*

$$\Omega X \cong \Omega(\mathbf{RSpec} \circ \mathcal{B} \circ X) \tag{3.11}$$

Proof. Given a stably compact locale X, $\mathcal{B}X$ consists of pairs (a, K) where $a \in \Omega X$, $K \subseteq \Omega X$ is a Scott-open filter and a is a lower bound of K. Therefore, $\Omega(\mathbf{RSpec} \circ \mathcal{B} \circ X)$ consists of rounded ideals $I = \{(a_i, K_i)\}$ of such pairs. We define a map

$$\beta : \Omega(\mathbf{RSpec} \circ \mathcal{B} \circ X) \longrightarrow \Omega X \quad \text{by } \beta(I) := \bigvee{}^{\uparrow} a_i \tag{3.12}$$

We start by observing that β is monotone. We prove that β is a frame homomorphism. Let $I_1, I_2 \in \Omega(\mathbf{RSpec} \circ \mathcal{B} \circ X)$. Then $\beta(I_1 \wedge I_2) = \beta(I_1 \cap I_2) = \bigvee^{\uparrow}\{a_i | a_i \in I_1 \text{ and } a_i \in I_2\}$. On the other hand $\beta(I_1) \wedge \beta(I_2) = (\bigvee^{\uparrow}\{a_i \in I_1\}) \wedge (\bigvee^{\uparrow}\{a_j \in I_2\}) = \bigvee^{\uparrow}\{a_i \wedge a_j | a_i \in I_1, a_j \in I_2\}$ and we have shown in the proof of lemma 3.10 that this is the same as $\bigvee^{\uparrow}\{a_i | a_i \in I_1 \text{ and } a_i \in I_2\}$.

Directed joins in $\Omega(\mathbf{RSpec}(\mathcal{B}X))$ are unions of rounded ideals of $\mathcal{B}X$, so for a collection of ideals $\{I_k\} = \{(a_{ki}, K_{ki})\}$, we have $\beta(\bigvee^{\uparrow}_k I_k) = \beta(\bigcup^{\uparrow}_k I_k) = \bigvee^{\uparrow}_{ik}\{a_{ik}\} = \bigvee^{\uparrow}_k \bigvee^{\uparrow}_i \{a_{ki}\} = \bigvee^{\uparrow}_k \beta(I_k)$.

The binary joins need a little more attention (theorem 3.6). Let I_1, J_2 two rounded ideals of $\mathcal{B}X$. Then

$$\begin{aligned} \beta(I_1 \vee I_2) &= \beta(\{(a_i, K_i) \in \mathcal{B}X | \exists (a_1, K_1) \in I_1, \exists (a_2, K_2) \in I_2 \\ &\quad\ \text{and } (a_1, K_1) \prec (a_1, K_1) \vee (a_2, K_2)\}) \\ &= \bigvee{}^{\uparrow}\{a_i\} \end{aligned}$$

On the other hand

$$\begin{aligned} \beta(I_1) \vee \beta(I_2) &= \bigvee{}^{\uparrow}\{a_j | \exists (a_j, K_j) \in I_1\} \vee \bigvee{}^{\uparrow}\{a_k | \exists (a_k, K_k) \in I_2\} \\ &= \bigvee{}^{\uparrow}\{a_j \vee a_k | \exists (a_j, K_j) \in I_1 \text{ and } \exists (a_k, K_k) \in I_2\} \end{aligned}$$

By lemma 3.21, $(a_i, K_i) \prec (a_1, K_1) \vee (a_2, K_2) = (a_1 \vee a_2, K \cap K_2)$ implies that $a_i \ll a_1 \vee a_2$, so obviously $\beta(I_1 \vee I_2) \leq \beta(I_1) \vee \beta(I_2)$. Also, by the roundedness of I_1, I_2, for $(a_j, K_j) \in I_1$ and $(a_k, K_k) \in I_2$ there are $(a'_j, K'_j) \in I_1$ and $(a'_k, K'_k) \in I_2$ with $(a_j, K_j) \prec (a'_j, K'_j)$ and $(a_k, K_k) \prec (a'_k, K'_k)$. This implies that $(a_j, K_j) \vee (a_k, K_k) \prec (a'_j, K'_j) \vee (a'_k, K'_k)$ and by the interpolation property, there is $(a_i, K_i) \in \mathcal{B}X$ with $(a_j, K_j) \vee (a_k, K_k) \prec (a_i, K_i) \prec (a'_j, K'_j) \vee (a'_k, K'_k)$. Therefore, $(a_i, K_i) \in I_1 \vee I_2$ and $a_j \vee a_k \ll a_i \ll a'_j \vee a'_k$. This proves that $\beta(I_1) \vee \beta(I_2) \subseteq \beta(I_1 \vee I_2)$.

Now we prove that β is perfect. Suppose that $I_1 = \{(a_i, K_i)\}$ and $I_2 = \{(a_j, K_j)\}$ are two rounded ideals of elements of $\mathcal{B}X$ and $I_1 \ll I_2$ in $\Omega(\mathbf{RSpec} \circ \mathcal{B}X)$. By theorem (3.6), there is a pair (a, K) in I_2 with

$$\{(a_i, K_i)\} \subseteq \downarrow (a, K) \subseteq \{(a_j, K_j)\} \tag{3.13}$$

It is easy to show that in general the following sets are identical

$$\{a_k | \exists K_k \in QX : \ (a_k, K_k) \prec (a, K)\} = \twoheaddownarrow a$$

Indeed, if $(a_k, K_k) \prec (a, K)$ then by lemma 3.21, $a_k \ll a$. Conversely, if $a_k \ll a$ then $(a_k, \Uparrow a_k) \prec (a, K)$. Therefore, $\beta(\downarrow (a, K)) = \bigvee^{\uparrow}\{\twoheaddownarrow a\} = a$ because ΩX is a continuous lattice. By monotonicity, $\beta(I_1) \leq \beta(\downarrow (a, K)) \leq \beta(I_2)$ which with the exact labelling of expression 3.13 gives $\bigvee^{\uparrow}\{a_i\} \leq a \leq \bigvee^{\uparrow}\{a_j\}$. By expression 3.13 above, $(a_i, K_i) \in I_1$ implies that $(a_i, K_i) \prec (a, K)$ which in turn implies that $a_i \ll a$. Therefore, $\bigvee^{\uparrow}\{a_i\} \ll a$. This proves that $\beta(I_1) \ll a \leq \beta(I_2)$ which gives $\beta(I_1) \ll \beta(I_2)$.

Now we define a map on the other direction as follows.

$$\gamma : \Omega X \longrightarrow \Omega(\mathbf{RSpec} \circ \mathcal{B} \circ X) \quad \text{by } \gamma(a) := \{(a_i, K_i) | a \in K_i\} \tag{3.14}$$

It can be readily checked (lemma 3.23) that for any $a \in \Omega X$, $\gamma(a)$ is a rounded ideal in $\mathcal{B}X$. It is also straightforward to prove that γ preserves finite meets and directed joins.

We will prove now that it preserves binary joins. By laying out the definition we have that

$$\gamma(a \vee b) = \{(a_i, K_i) | a \vee b \in K_i\} \quad \text{and} \tag{3.15}$$

$$\begin{aligned}\gamma(a) \vee \gamma(b) &= \{(a_l, K_l) | \exists (a_j, K_j),\ a \in K_j \ \&\ \exists (a_k, K_k),\ b \in K_k \\ &\quad \&\ (a_l, K_l) \prec (a_j, K_j) \vee (a_k, K_k)\} \\ &= \{\downarrow (a, \top)\} \ \text{(where } \top \text{ is the top element in } \Omega X)\end{aligned}$$

It is trivial to verify that γ preserves finite meets and directed joins. It is only marginally more difficult to prove that γ preserves finite joins. We have that $\gamma(a_1) \vee \gamma(a_2) = \{(a_i, K_i) | \exists (a_1', K_1') \prec (a_1, \top) \ \&\ \exists (a_2', K_2') \prec (a_2, \top) \ : (a_i, K_i) \prec (a_1', K_1') \vee (a_2', K_2')\}$ and $\gamma(a_1 \vee a_2) = \{\downarrow (a_1 \vee a_2, \top)\}$. If $(a_i, K_i) \in \gamma(a_1) \vee \gamma(a_2)$ then there are $(a_1', K_1'), (a_2', K_2')$ such that $(a_1', K_1') \prec (a_1, \top)$ and $(a_2', K_2') \prec (a_2, \top)$ with $(a_i, K_i) \prec (a_1', K_1') \vee (a_2', K_2')$. This implies that $(a_i, K_i) \prec (a_1', K_1') \vee (a_2', K_2') \prec (a_1, \top) \vee (a_2, \top)$, i.e. that $(a_i, K_i) \in \gamma(a_1 \vee a_2)$. Conversely, if $(a_i, K_i) \in \gamma(a_1 \vee a_2)$, then $(a_i, K_i) \prec (a_1 \vee a_2, \top) = (a_1, \top) \vee (a_2, \top)$. But since $\mathcal{B}X$ is a strong proximity lattice, there are $(a_1', K_1') \prec (a_1, \top)$ and $(a_2', K_2') \prec (a_2, \top)$ such that $(a_i, K_i) \prec (a_1', K_1') \vee (a_2', K_2')$. Therefore, $(a_i, K_i) \in \gamma(a_1 \vee a_2)$ since $(a_1', K_1') \vee (a_2', K_2') \prec (a_1, \top) \vee (a_2, \top) = (a_1 \vee a_2, \top)$.

We also prove that γ preserves the way below relation in ΩX. Let $a \ll b$ in ΩX. Then there is $c \in \Omega X$ with $a \ll c \ll b$. Then $\downarrow (a, \top) \subseteq \downarrow (c, \top) \subseteq \downarrow (b, \top)$ or $\gamma(a) \subseteq \downarrow (c, \top) \subseteq \gamma(b)$ which by theorem 3.6 implies $\gamma(a) \ll \gamma(b)$ in $\Omega(\mathbf{RSpec} \circ \mathcal{B}X)$.

Finally we prove that β and γ are inverse to each other. For any $a \in \Omega X$ it holds that

$$\begin{aligned}\beta \circ \gamma(a) &= \bigvee{}^{\uparrow} \{a_i | \exists (a_i, K_i) \prec (a, \top)\} \\ &= \bigvee{}^{\uparrow} \{a_i | a_i \ll a\}\end{aligned}$$

Indeed, $(a_i, K_i) \prec (a, \top)$ implies $a_i \ll a$ by lemma 3.21 and if $a_i \ll a$ then $(a_i, \Uparrow a_i) \prec (a_\top)$. But since X is locally compact, $\bigvee^{\uparrow} \{a_i | a_i \ll a\} = a$, so $\beta \circ \gamma(a) = a$.

Furthermore, for any rounded ideal $I \subseteq \mathcal{B}X$, we have that

$$\gamma \circ \beta(I) = \{\downarrow (\bigvee{}^{\uparrow} \{a_i | \exists (a_i, K_i) \in I\}, \top)\}$$

Let $(a_j, K_j) \in \gamma \circ \beta(I)$. Then $(a_j, K_j) \prec (\bigvee^{\uparrow} \{a_i | \exists (a_i, K_i) \in I\}, \top)$, which implies $\bigvee^{\uparrow} \{a_i | \exists (a_i, K_i) \in I\} \in K_j$. But since K_j is Scott-open, there is $(a_0, K_0) \in I$ with $a_0 \in K_j$ and this implies that $(a_j, K_j) \prec (a_0, K_0)$. Since I is lower closed, this entails that $(a_j, K_j) \in I$. Conversely, let $(a_i, K_i) \in I$. Because I is rounded, there is $(a_0, K_0) \in I$ with $(a_i, K_i) \prec (a_0, K_0)$. This implies that $a_0 \in K_j$ and there is Scott-open filter K_0 with $(a_0, K_0) \in I$. Hence, $\bigvee^{\uparrow}\{a_i | \exists (a_i, K_i) \in I\} \in K_i$ because K_i is upper closed. This completes the proof. ■

3.6 The points of strong proximity lattices (revisited)

We have shown in section 3.3 that the global points of the localic topos $\mathbf{RSpec}(B)$ that classifies (the geometric theory of) the frame presentation 3.8 are prime rounded filters of the strong proximity lattice B. The points of the topos $\mathbf{RSpec}(B)$ are in equivalence with the points of the locale carrying the same symbol. This is easily seen by recalling theorem 1.6 in the introduction. Putting $E \equiv \mathbf{1}$, the terminal topos, that theorem says that there is an isomorphism

$$\mathfrak{T}op(\mathbf{1}, \mathbf{RSpec}(B)) \cong \mathbf{Fr}(\Omega(\mathbf{RSpec}(B)), \Omega) = \mathbf{Loc}(\mathbf{1}, \mathbf{RSpec}(B)) \tag{3.16}$$

with $\mathbf{1}$ in the far right hand side being the terminal locale inside the sheaves of the base topos (which we denote as **Sets**). The elements of the far right hand side category are the global points of the locale $\mathbf{RSpec}(\mathbf{B})$. Let us recall a few things about the global points of a locale (see [Joh82]).

Suppose that we have a global point of an arbitrary locale X, i.e. continuous map $x : \mathbf{1} \longrightarrow X$. The defining frame homomorphism of such a point is $x^* : \Omega X \longrightarrow \Omega$, where Ω is the initial frame in **Sets** which coincides with the subobject classifier of **Sets**. It follows that such a frame homomorphism gives rise to a unique completely prime filter $H \subseteq \Omega X$, which is its true kernel

$$H = \{a \in \Omega X | x^*(a) = \top\}$$

Conversely, any completely prime filter is a true kernel of a unique frame homomorphism that defines a point of X. The set of points ptX of the given locale is partially ordered (specialisation order) by set inclusion of the corresponding completely prime filters. Furthermore, it has all directed joins, i.e. it is a dcpo. If $f : X \longrightarrow Y$ is a continuous map between two locales, there is an obvious point transformation (Scott-continuous map) $ptf : ptX \longrightarrow ptY$ obtained by post-composition with f in **Loc**: $ptf(x) = f \circ x$, or equivalently, by pre-composition with f^* in **Fr**. Since a point is the same as the completely prime filter of its true values, a point transformation ptf is equivalent to a map that sends completely prime filters H of ΩX to completely prime filters of ΩY defined by

$$pt f : H \mapsto \{a \in \Omega Y | f^*(a) \in H\} \tag{3.17}$$

Now we consider the case where $X = \mathbf{RSpec}(B)$. In section 3.3 we showed that the points of the topos $\mathbf{RSpec}(B)$ are rounded prime filters of B, so the poset of completely prime filters of rounded ideals of B, partially ordered by set inclusion of rounded ideals, must be equivalent to the dcpo of rounded prime filters of B.

We are going to give an other more tangible proof of this fact here.

Lemma 3.25 *Let B be a strong proximity lattice. We denote by $RPFilt(B)$ the dcpo of its rounded prime filters (it is easy to show it is a dcpo) and by $pt\mathbf{RSpec}(B)$ the dcpo of the completely prime filters of the frame $\Omega(\mathbf{RSpec}(B))$. Then the following isomorphism holds*

$$RPFilt(B) \cong pt \circ \mathbf{RSpec}(B) \tag{3.18}$$

Proof. It is trivial to verify that the set of prime rounded filters of a strong proximity lattice partially ordered by set inclusion has all directed joins (they are directed unions).

For $F \subseteq B$ any rounded prime filter, we define

$$\Xi(F) = \{I \in \Omega(\mathbf{RSpec}(B)) |\; I \cap F \neq \emptyset\} \tag{3.19}$$

It is easy to check that Xi is monotone and preserves directed unions. Conversely, for a completely prime filter $H \subseteq \Omega(\mathbf{RSpec}(B))$, we define

$$\Sigma(H) = \{b \in B| \downarrow b \in H\} \tag{3.20}$$

It is routine to check that $\Xi(F)$ upper closed and that $\Xi(F)$ is inaccessible by directed joins (they are unions). Let $I_1, I_2 \in RIdl(B)$ with $I_1 \cap F \neq \emptyset$ and $I_2 \cap F \neq \emptyset$ and in particular, let $a_1 \in I_1 \;\&\; a_1 \in F$ and $a_2 \in I_2 \;\&\; a_2 \in F$. Then $a_1 \wedge a_2 \in I_1 \;\&\; a_1 \wedge a_2 \in I_2$ but also $a_1 \wedge a_2 \in F$ because it is $\wedge$-closed. Hence $I_1 \cap I_2 \cap F \neq \emptyset$, i.e. $\Xi(F)$ is $\wedge$-closed. Finally, let $I_1 \vee I_2 \neq \emptyset$. Then there is $c \in I_1 \vee I_2 \;\&\; c \in F$. By the definition of the binary joins in $\Omega(\mathbf{RSpec}(B))$, there are $a_1 \in I_1$ and $a_2 \in I_2$ with $c \prec a_1 \vee a_2$. Since F is upper closed, $a_1 \vee a_2 \in F$ and since F is prime, either $a_1 \in I_1$ or $a_2 \in I_2$ which implies that either $I_1 \cap F \neq \emptyset$ or $I_2 \cap F \neq \emptyset$. Therefore $\Xi(F)$ is prime.

It is also routine to check that $\Sigma(H)$ is upper closed and $\wedge$-closed. Let $b_1 \vee b_2 \in \Sigma(H)$. Then $\downarrow (b_1 \vee b_2) \in H$. From lemma 3.7, $\downarrow (b_1 \vee b_2) = (\downarrow b_1) \vee (\downarrow b_2) \in H$ and since H is prime either $\downarrow b_1$ or $\downarrow b_2$ is an element of H which implies that either b_1 or b_2 is an element of $\Sigma(H)$. Therefore $\Sigma(H)$ is prime.

The two assignments are inverse to each other. Indeed it is easy to verify that for any rounded prime filter of B we have

$$\Sigma \circ \Xi(F) = \{b \in B| \downarrow b \cap F \neq \emptyset\} = F \tag{3.21}$$

On the other hand, for any completely prime filter H of rounded ideals of B we have

$$\Xi \circ \Sigma(H) = \{I \in \Omega(\mathbf{RSpec}(B))| \, \exists b \in B \text{ with } \downarrow b \in H \,\&\, b \in I\} \tag{3.22}$$

Let $I \in \Xi \circ \Sigma(H)$. Then there is $b \in B$ with $\downarrow b \in H$ and $b \in I$. From the latter fact we deduce that $\downarrow b \subseteq I \Leftrightarrow \downarrow b \leq I$. So I must be an element of H because H is upper closed. Conversely, let I be an element of H. Then $I = \bigvee^{\uparrow}\{\downarrow b | b \in B\}$ and because H is inaccessible by directed joins, there is $b \in I$ with $\downarrow b \in H$ which implies that I is an element of $\Xi \circ \Sigma(H)$. ■

Corollary 3.26 *Let QX be the set of Scott-open filters of $\Omega(\mathbf{RSpec}(B))$. Then $QX \cong \Omega(\mathbf{RSpec}(B^{op}))$.*

Proof.

■

Now let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices. It induces point transformations between

$$pt \circ \mathbf{RSpec}(B_1) \longrightarrow pt \circ \mathbf{RSpec}(B_2) \text{ or equivalently between} \tag{3.23}$$

$$RPFilt(B_1) \longrightarrow RPFilt(B_2) \tag{3.24}$$

In the rest of this section, we give an account of the above maps. We readily know that we can obtain the transformation of expression 3.23 by putting $f^* := (\mathbf{RSpec}\mu)^*$ in the expression 3.17, i.e

$$pt \circ \mathbf{RSpec}(\mu) : H \mapsto \{a \in \Omega\mathbf{RSpec}(B_2)|(\mathbf{RSpec}\mu)^*(a) \in F\} \tag{3.25}$$

where H is a completely prime filter of rounded ideals of B_1.

The following lemma describes the transformations of the expression 3.24.

Lemma 3.27 *Let $\Xi_B : RPFilt(B) \rightleftharpoons pt \circ \mathbf{RSpec}(B) : \Sigma_B$ be the isomorphisms as in the* *proof of lemma 3.25. Then the following two square diagrams are commutative.*

$$\begin{tikzcd} pt \circ \mathbf{RSpec}(B_1) \arrow[r, "(\downarrow \mu)^{-1}"] \arrow[d, shift left, "\Sigma_{B_1}"] & pt \circ \mathbf{RSpec}(B_2) \arrow[d, shift right, "\Sigma_{B_2}"'] \\ RPFilt(B_1) \arrow[r, "(\mu)^{-1}"] \arrow[u, shift left, "\Xi_{B_1}"] & RPFilt(B_2) \arrow[u, shift right, "\Xi_{B_2}"'] \end{tikzcd} \tag{3.26}$$

Proof. It suffices to show that $(\downarrow \mu)^{-1} = \Xi_{B_2} \circ (\mu)^{-1} \circ \Sigma_{B_1}$. We are going to prove that for any $H \in pt \circ \mathbf{RSpec}(B_1)$, we have that $(\downarrow \mu)^{-1}(H) = \Xi_{B_2} \circ (\mu)^{-1} \circ \Sigma_{B_1}(H)$. The definitions yield that

$$\begin{aligned} (\downarrow \mu)^{-1}(H) &= \{I \in \Omega(\mathbf{RSpec}B) | \exists J \in H \text{ with } (\downarrow \mu)(I) = J\} \\ \Xi_{B_2} \circ (\mu)^{-1} \circ \Sigma_{B_1}(H) &= \{I \in \Omega(\mathbf{RSpec}B) | \exists a \in B_2 \text{ with } \downarrow \mu(a) \in H \ \& \ a \in I\} \end{aligned}$$

Let $I \in (\downarrow \mu)^{-1}(H)$. Then there is $J \in H$ such that $(\downarrow \mu)(I) = J \in H$. But $I = \bigvee^{\uparrow}\{\downarrow a | a \in I\}$, so we have that

$$\begin{aligned} J = (\downarrow \mu)(I) &= (\downarrow \mu)(\bigvee{}^{\uparrow}\{\downarrow a | a \in I\}) \\ &= \bigvee{}^{\uparrow}\{(\downarrow \mu)(\downarrow a) | a \in I\} \end{aligned}$$

because $(\downarrow \mu) := (\mathbf{RSpec}\mu)^*$ preserves directed joins as a frame homomorphism. The fact that H is inaccessible by directed joins implies that there is $a_0 \in I$ such that $(\downarrow \mu)(\downarrow a_0) \in H$. Using lemma 3.16, this fact becomes $\downarrow \mu(a_0) \in H$ and by looking at its defining expression above, this means that $I \in \Xi_{B_2} \circ (\mu)^{-1} \circ \Sigma_{B_1}(H)$.

Now let $I \in \Xi_{B_2} \circ (\mu)^{-1} \circ \Sigma_{B_1}(H)$. Then there is $a \in B_2$ such that $\downarrow \mu(a) \in H$ and $a \in I$. But using again lemma 3.16, $\downarrow \mu(a) = (\downarrow \mu)(\downarrow a) \in H$, hence $\downarrow a \in (\downarrow \mu)^{-1}(H)$. Also the fact that $a \in I$ implies $\downarrow a \subseteq I$ and $(\downarrow \mu)^{-1}(H)$ is upper closed being a filter, therefore, $I \in (\downarrow \mu)^{-1}(H)$. ■

Corollary 3.28 *(i) By stipulating that*

$$\mathbf{RPFilt}(\mu)(F) := (\mu)^{-1}[F]$$

for any strong homomorphism $\mu : B_2 \longrightarrow B_1$ and any rounded prime filter of B_1, *the construction RPFilt is extended to a functor*

$$\mathbf{RPFilt} : \mathbf{StrPrLat} \longrightarrow \mathbf{dcpo}$$

and isomorphism 3.18 is extended to isomorphism between functors

$$\mathbf{RPFilt} \cong pt \circ \mathbf{RSpec} \tag{3.27}$$

(ii) Let X be a stably compact locale. Then we have the isomorphism

$$ptX \cong \mathbf{RPFilt} \circ \mathcal{B}X \quad \textit{(qua dcpo)}$$

Proof. (i) Obvious from lemma 3.27.

(ii) We put $B = \mathcal{B}X$ in 3.27 and we get the claimed isomorphism using theorem 3.24.

Note that if x is a point, i.e. a completely prime filter of ΩX, then the prime rounded filter of $\mathcal{B}X$ that corresponds to x is

$$\Sigma(x) = \{(a, \top) | a \in x\} \tag{3.28}$$

This is deduced from expression 3.20 after setting $H = x$ in its L.H.S. and $H = \gamma(x)$ in its R.H.S., where γ is as in the proof of theorem 3.24. ■

Chapter 4

Adjunctions in $\mathfrak{T}op$

4.1 Introduction

Suppose that $\mathbf{F} : \mathbf{X} \rightleftharpoons \mathbf{Y} : \mathbf{G}$ are functors between two small categories. Of the several equivalent definitions of the adjoint situation $F \dashv G$, we cite the following two (see [Mac71], IV, theorem 2)

Definition 4.1 $\mathbf{G}$ *is right adjoint to* $\mathbf{F}$ *iff, for any object* x *in* $\mathbf{X}$ *and any object* y *in* $\mathbf{Y}$ *there is a bijection* $\phi_{x,y}$

$$\mathbf{Y}(\mathbf{F}x, y) \xrightarrow[\phi_{x,y}]{\cong} \mathbf{X}(x, \mathbf{G}y) \tag{4.1}$$

which is natural both in x *and* y*.*

Definition 4.2 $\mathbf{G}$ *is right adjoint to* $\mathbf{F}$ *iff there are natural transformations*

$$\eta : \mathbf{id}_X \Rightarrow \mathbf{G} \circ \mathbf{F} \quad \text{and} \quad \varepsilon : \mathbf{G} \circ \mathbf{F} \longrightarrow \mathbf{id}_Y$$

such that the composites

$$\begin{array}{l} \mathbf{G} \xrightarrow{\eta_{\mathbf{G}}} \mathbf{G} \circ \mathbf{F} \circ \mathbf{G} \xrightarrow{\mathbf{G} \bullet \varepsilon} \mathbf{G} \qquad \text{and} \\ \mathbf{F} \xrightarrow{\mathbf{F} \bullet \eta} \mathbf{F} \circ \mathbf{G} \circ \mathbf{F} \xrightarrow{\varepsilon_{\mathbf{F}}} \mathbf{F} \end{array} \tag{4.2}$$

are the identities of $\mathbf{G}$ *and* $\mathbf{F}$ *respectively. This condition is usually referred to as the triangle identities. (*$\bullet$ *is the horizontal composition of 2-cells.)*

Of the two definitions, the second one can be interpreted in any 2-category. In particular, it can be interpreted in the 2-category $\mathfrak{T}op$ of Grothendieck topoi. We rewrite this definition, adapting it for the context of $\mathfrak{Top}$ plus a small change: we require the triangle identities to hold only up to isomorphism.

Definition 4.3 *Let $F : X \longrightarrow Y$ and $G : Y \longrightarrow X$ be two geometric morphisms between two Grothendieck topoi. We say that G is the right adjoint of F ($F \dashv G$) iff there are 2-cells*

$$\eta : id_X \Rightarrow G \circ F \quad \text{and} \quad \varepsilon : G \circ F \Rightarrow id_Y$$

such that the composites

$$\begin{array}{c} G \xrightarrow{\eta_G} G \circ F \circ G \xrightarrow{G \bullet \varepsilon} G \qquad \text{and} \\ F \xrightarrow{F \bullet \eta} F \circ G \circ F \xrightarrow{\varepsilon_F} F \end{array} \tag{4.3}$$

are both isomorphisms.

The aim of this chapter is to recover a sufficient condition for adjoint situations in $\mathfrak{Top}$ which is analogous to definition 4.1. The condition we establish is indeed analogous to definition 4.1, only easier; naturality comes for free.

We shall demonstrate that the analogue of 4.1 in $\mathfrak{Top}$ involves points x, y rather than objects of X and Y and the Hom-sets appearing in 4.1 are promoted to categories whose objects are 2-cells $F \circ x \Rightarrow y$ and $x \Rightarrow G \circ y$ respectively.

More specifically, given geometric morphisms $F : X \longrightarrow Y$ and $G : Y \longrightarrow X$, we define the following two topoi.

- The topos $\mathcal{E} := [\mathbb{T}_{\mathcal{E}}]$ that classifies (theories whose models are) triples

  $$(x, y; Fx \Rightarrow y)$$

  where $x : Z \longrightarrow X$ and $y : Z \longrightarrow Y$ are arbitrary points (at any stage Z)

- The topos $\mathcal{F} := [\mathbb{T}_{\mathcal{F}}]$ that classifies (theories whose models are) triples

  $$(x', y'; x' \Rightarrow Gy')$$

  where x', y' are also points of X and Y at any stage.

$\mathcal{E}$ and $\mathcal{F}$ are topoi *over* $X \times Y$. This is because one can geometrically construct maps $i_{\mathcal{E}} : \mathcal{E} \longrightarrow X \times Y$ and $i_F : \mathcal{F} \longrightarrow X \times Y$ that simply forget the 2-cells $Fx \Rightarrow y$ and $x' \Rightarrow Gy'$ from the ingredients of the theories $\mathbb{T}_{\mathcal{E}}$ and $\mathbb{T}_{\mathcal{F}}$ respectively (model reduction functors). In words, the topos $i_{\mathcal{E}} : \mathcal{E} \longrightarrow X \times Y$ over $X \times Y$ classifies 2-cells $Fx \Rightarrow y$ *given* two points x and y of X and Y respectively.

Now consider the situation that $i_{\mathcal{E}}$ and $i_{\mathcal{F}}$ are equivalent in $\mathfrak{T}op/(X \times Y)$, i.e. $\mathcal{E} \simeq \mathcal{F}$ and the following diagram commutes in $\mathfrak{T}op$

$$\begin{tikzcd} \mathcal{E} \arrow[rr, shift left] \arrow[rr, phantom, "\simeq"] \arrow[dr, "i_{\mathcal{E}}"'] & & \mathcal{F} \arrow[ll, shift left] \arrow[dl, "i_{\mathcal{F}}"] \\ & X \times Y & \end{tikzcd} \tag{4.4}$$

Let us see roughly when such situation occurs. Commutativity of diagram 4.4 means that the two pairs of points coincide $x = x'$ and $y = y'$. Equivalence of $\mathcal{E}$ and $\mathcal{F}$ in $\mathfrak{T}op$ means that models $(x, y; Fx \Rightarrow y)$ (say in $\mathcal{S}Z$) of $\mathbb{T}_{\mathcal{E}}$ are equivalent to models (in the same $\mathcal{S}Z$) $(x, y; x \Rightarrow Gy)$ and since the point parts have been identified already, the situation in discussion happens when models of the two kinds of 2-cells are equivalent (in $\mathcal{S}Z$).

$$(Fx \Rightarrow y) \simeq (x \Rightarrow Gy) \tag{4.5}$$

which is indeed analogous to the 1-categorical bijection 4.1.

We shall be referring to the topoi $\mathcal{E}$ and $\mathcal{F}$ the *geometric inserters* of the diagrams

$$X \times Y \underset{p_2}{\overset{F \circ p_1}{\rightrightarrows}} Y \qquad\qquad X \times Y \underset{G \circ p_2}{\overset{p_1}{\rightrightarrows}} X \tag{4.6}$$

respectively, where p_1 and p_2 are the first and second projections. The goal of this chapter is to prove the following theorem.

Theorem 4.4 *If the geometric inserters $\mathcal{E}$ and $\mathcal{F}$ are equivalent over $X \times Y$, then $F \dashv G$.*

4.2 Outline of the chapter

This section is intended to be a guide for the rest of the sections of this chapter. Pointers to literature are not provided as all the notions mentioned here are discussed in detail in the sections that follow.

Consider the diagram in $\mathfrak{T}op$

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y \tag{4.7}$$

that generalises the diagrams 4.6. The task in sections 4.3-4.5 is to write a geometric theory

$\mathbb{T}_{\mathcal{E}}$ whose models are pairs $(x; f)$ where x is a point of X and f is a 2-cell $F \circ x \Rightarrow G \circ x$. Such a theory $\mathbb{T}_{\mathcal{E}}$ must contain the ingredients (sorts, function symbols, relation symbols, constants and axioms) of the theory $\mathbb{T}_X$ that X classifies plus new function symbols (say $\widetilde{k}$) that correspond to the 2-cells $F \circ x \Rightarrow G \circ x$. We also need to incorporate in $\mathbb{T}_{\mathcal{E}}$ certain new naturality axioms for k.

The subtlety lies in how one formally defines the "new function symbols that correspond to the 2-cells $F \circ x \Rightarrow G \circ x$". F and G correspond to functors between the sheaves of the topoi X and Y and not homomorphisms between the geometric theories $\mathbb{T}_X$ and $\mathbb{T}_Y$. Our method is to construct *site morphisms* F^σ and G^σ out of the geometric morphisms F and G in the following sense.

Suppose that $(\mathbf{C}, J)$ and $(\mathbf{D}, K)$ are the syntactic sites for the topoi X and Y respectively. Then we can consider a subcategory $\mathbf{C}'$ of $\mathcal{S}(\mathbf{C}, J)$ such that together with some Grothendieck topology J' on $\mathbf{C}'$

- $\mathbf{C}'$ contains the image of $\mathbf{D}$ under both F^* and G^* and
- $\mathcal{S}(\mathbf{C}, J) \simeq (\mathbf{C}', J')$.

The existence of such a topology J' comes directly out of the Giraud's theorem but we prove it independently of that theorem in section 4.3.

Given such a site $(\mathbf{C}', J')$, precomposing the inverse image functors F^* and G^* with the Yoneda embedding $\mathbf{y_D}$ (the topology of the syntactic site is subcanonical) we obtain a pair of site morphisms

$$(\mathbf{C}', J') \underset{G^\sigma}{\overset{F^\sigma}{\leftleftarrows}} (\mathbf{D}, K) \tag{4.8}$$

Now we exploit the fact that X and Y classify the geometric theories (say $\mathbb{T}'_X$ and $\mathbb{T}_Y$) of *flat and continuous* covariant functors over the sites $(\mathbf{C}', J')$ and $(\mathbf{D}, K)$ respectively. Such geometric theories have languages with sorts and function symbols formally equivalent to the objects and arrows of the categories $\mathbf{C}$ and $\mathbf{D}$ respectively. In this context, F^σ and G^σ map sorts and function symbols of the geometric theory that Y classifies to the geometric theory that X classifies. This means that it is perfectly legitimate now to define the aforementioned new function symbols $\widetilde{k}$ as function symbols between sorts of the form

$$F^\sigma(d) \longrightarrow G^\sigma(d) \quad \text{for any object } d \text{ in } \mathbf{D}$$

The additional axioms needed are essentially the ones that guarantee that $\widetilde{k}$ are actually the components of a natural transformation between the functors F^σ and G^σ (naturality axioms).

Diagrams of the form 4.7 in any 2-category admit a particular well known 2-categorical (weighted) limit called the *inserter*. Section 4.6 offers a quick review of weighted limits and inserters. In section 4.7 we prove that, for a given pair of parallel geometric morphisms in $\mathfrak{T}op$, its inserter is equivalent with the geometric inserter.

Next we focus on the diagrams of the special form 4.6. Being able to interchange between the inserter and the geometric inserter facilitates the proof of the main theorem 4.4. This is done in section 4.8.

We have already discussed in section 4.1 how the equivalence of the categories of homomorphisms in expression 4.5 entail that $\mathcal{E} \simeq \mathcal{F}$ over $X \times Y$. In section 4.9 we explain how such an equivalence entails from a mere "set"-theoretic bijection between the objects of these categories.

4.3 Enhancing sites

Let X be a Grothendieck topos. Let $\mathbf{C}$ be a small category with pullbacks and J a topology on C such that the category of sheaves on the site $(\mathbf{C}, J)$ is equivalent to the category $\mathcal{S}X$.

The aim of this section is to prove that if $\mathbf{C}'$ is a subcategory of $\mathcal{S}X$ that contains the image of $\mathbf{C}$ under $\mathbf{ay_C}$, then there is a Grothendieck topology J on $\mathbf{C}'$ such that

$$\mathcal{S}X \simeq \mathcal{S}(\mathbf{C}', J')$$

We start with some well known facts. We denote by $\mathbf{ay_C} : \mathbf{C} \longrightarrow \mathcal{S}X$ the composite $\mathbf{y_C} : \mathbf{C} \longrightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$ followed by $\mathbf{a} : \mathbf{Sets}^{\mathbf{C}^{op}} \longrightarrow \mathcal{S}(\mathbf{C}, J)$, where $\mathbf{y_C}$ is the Yoneda embedding and $\mathbf{a}$ is left adjoint of the inclusion functor $\mathbf{i} : \mathcal{S}(\mathbf{C}, J) \longrightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$, i.e. the associated sheaf (or sheavification) functor. The Yoneda embedding functor $\mathbf{y_C}$ preserves whatever limits there are in $\mathbf{C}$. The associated sheaf functor $\mathbf{a}$ preserves finite limits ([MM92],III.5) which means that the composite $\mathbf{ay_C}$ preserves whatever finite limits there are in $\mathbf{C}$. In particular, our assumption is that $\mathbf{C}$ has pullbacks, so the following is always a pullback square in $\mathbf{C}$

$$\begin{tikzcd}
\mathbf{ay_C}(c_i \times_c c_j) \arrow[r, "\pi^2_{ij}"] \arrow[d, "\pi^1_{ij}"'] & \mathbf{ay_C}(c_j) \arrow[d, "\mathbf{ay}(f_j)"] \\
\mathbf{ay_C}(c_i) \arrow[r, "\mathbf{ay}(f_i)"] & \mathbf{ay_C}(c)
\end{tikzcd}$$

for any arrows $f_i : c_i \longrightarrow c$ and $f_j : c_j \longrightarrow c$. It is well-known that any object x of $\mathcal{S}X$ i.e. a sheaf on the site $(\mathbf{C}, J)$ has the form $x = colim_k\{\mathbf{ay_C}(c_k)\}$ for an appropriate diagram of objects $\{c_k\}$ in $\mathbf{C}$. This in particular implies that the objects of $\mathbf{ay_C}(\mathbf{C})$ *generate* the category $\mathcal{S}X$ in the sense that for any object x of $\mathcal{S}X$ the set of arrows

$$\{\mathbf{ay_C}(c_i) \longrightarrow x | c_i \text{ object in } \mathbf{C}\}$$

is a jointly epimorphic family. Since the objects of the category $\mathbf{ay_C}(\mathbf{C})$ generate the category $\mathcal{S}X$, obviously the objects of any subcategory $\mathbf{C}'$ of $\mathcal{S}X$ that contains $\mathbf{ay_C}(\mathbf{C})$ also generate $\mathcal{S}X$. Therefore, we know by Giraud's theorem that $\mathcal{S}X$ can be written as (equivalent to) $\mathcal{S}(\mathbf{C}', J')$ for some appropriate Grothendieck topology J' on $\mathbf{C}'$. What this section offers is a "shortcut" to the proof of Giraud's theorem in the much simplified case where $\mathcal{S}X$ is already the category of the sheaves on a site $\mathcal{S}(\mathbf{C}, J)$.

We fix the assumption that $\mathbf{C}'$ is any subcategory of $\mathcal{S}X$ that contains the image of $\mathbf{C}$ under the functor $\mathbf{ay_C}$.

The following lemma provides a hint about the choice of the Grothendieck topology J' on $\mathbf{C}'$.

Lemma 4.5 *Let $\mathcal{S}X = \mathcal{S}(\mathbf{C}, J)$. A family $\{f_i : c_i \longrightarrow c\}$ is a sieve in $J(c)$ iff the map*

$$\coprod_i \mathbf{ay_C}(c_i) \longrightarrow \mathbf{ay_C}(c)$$

is epi in $\mathcal{E}$.

Proof. [MM92] ■

We define the topology J' as follows. A sieve S on an object u in $\mathbf{C}'$ is included in $J'(u)$ iff S is a set of jointly epimorphic arrows in $\mathcal{S}X$. A set $\{h_i : u_i \longrightarrow u\}$ is a set of jointly epimorphic arrows in $\mathcal{E}$ exactly when $\coprod_i u_i \longrightarrow u$ is an epi arrow in $\mathcal{S}X$. This in turn is equivalent to the fact that for any pair of distinct arrows $f, g : u \longrightarrow v$ in $\mathcal{S}X$, there is an arrow h_i in S such that $f \circ h_i \neq g \circ h$.

Lemma 4.6 *The above collection of sieves on objects of $\mathbf{C}'$ is indeed a topology on $\mathbf{C}'$*

Proof. (i) The fact that $\mathbf{C}'$ contains $\mathbf{ay_C}(\mathbf{C})$ guarantees that for any object u in $\mathbf{C}'$ there is at least one epimorphism $h : \mathbf{ay_C}(c) \longrightarrow u$, for some object c in $\mathbf{C}$. Hence, $J'(u)$ includes the maximal sieve.

(ii) Let S be a sieve in $J'(u)$, i.e. S is a family of jointly epimorphic arrows $f_i : u_i \longrightarrow u$. Let also $h : v \longrightarrow u$ be any arrow in $\mathbf{C}'$. Then the family of the pullback arrows

$f_i^* : u_i \times_u v \longrightarrow v$ in $\mathcal{E}$ is also epimorphic.

$$\begin{tikzcd}
\mathbf{ay}_{\mathbf{C}}(c_{ij}) \arrow[r, "k_{ij}"] & u_i \times_u v \arrow[r, "f_i^*"] \arrow[d] & v \arrow[d, "h"] \\
& u_i \arrow[r, "f_i"] & u
\end{tikzcd}$$

Now, the fact that objects of the form $\mathbf{ac}(c)$, with c an object in $\mathbf{C}$ generate $\mathcal{S}X$ means that for each object $u_i \times_u v$ there is an epimorphic family $k_{ij} : \mathbf{ay}_{\mathbf{C}}(c_{ij}) \longrightarrow u_i \times_u v$, where c_{ij} are all objects in $\mathbf{C}$. Therefore, the composite arrows $f_i^* \circ k_{ij}$ constitute an epimorphic family whose codomain is v. Moreover, by the defining property of the pullback, $h \circ f_{ij}^*$ is an element of S which forces $h \circ f_i^* \circ k_{ij}$ to be an element of S. So, the epimorphic family $\{f_i^* \circ k_{ij}\}$ is a family of arrows in $\mathbf{C}'$ and is contained in the sieve $h^*(S)$. That means that $h^*(S)$ is an epimorphic family and so $h^*(S) \in J'(v)$. This proves stability.

(iii) Finally, transitivity is easily verified for J'. ■

We denote by X' the Grothendieck topos defined by $\mathcal{S}X' := \mathcal{S}(\mathbf{C}', J')$. We shall prove that $X \simeq X'$ in $\mathfrak{T}op$.

First we define a functor

$$\Psi : \mathcal{S}(\mathbf{C}', J') \longrightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$$

$$\Psi(P) = P \circ \mathbf{ay}_{\mathbf{C}}$$

for any object P in $\mathcal{S}(\mathbf{C}', J')$. We prove that this functor actually sends a J'-sheaf over $\mathbf{C}'$ to a J-sheaf over $\mathbf{C}$.

Lemma 4.7 *Let P be any object in $\mathcal{S}(\mathbf{C}', J')$. Then the presheaf $\Psi(P) = P \circ \mathbf{ay}_{\mathbf{C}}$ is a sheaf over $\mathbf{C}$ with respect to the J'-topology.*

Proof. Let $S = \{f_i : c_i \longrightarrow c\} \in J(c)$ be a sieve that covers an object c in $\mathbf{C}$. Suppose that $f_i \mapsto x_{f_i} \in P \circ \mathbf{ay}_{\mathbf{C}}(c_i)$ is any matching family for this sieve. Lemma (4.5) guarantees that the family of arrows $\mathbf{ay}_{\mathbf{C}}(f_i)$ is jointly epimorphic in $\mathbf{C}'$. It is not necessarily a sieve though, but it *generates* a sieve in $J'(\mathbf{ay}_{\mathbf{C}}(c))$. Indeed, we denote by S' the family of arrows in $\mathbf{ay}_{\mathbf{C}}(C')$ defined as

$$S' = \{G : b \longrightarrow c \mid G = \mathbf{ay}_{\mathbf{C}}(f_i) \circ F\} :=\uparrow \mathbf{ay}(f_i)$$

where F is any arrow in $\mathbf{ay}_{\mathbf{C}}(C')$ with codomain c_i for some index i. This family is obviously a sieve on c in $\mathbf{C}'$. Moreover, it is a sieve in $J'(c)$ because it is still a jointly epimorphic family of arrows with codomain c because for any c_i the maps $\mathbf{ay_C}(f_i) \circ id_{c_i}$ are included in S' which means that $S' \supseteq \{\mathbf{ay_C}(f_i)\}$.

Now, the matching family for the presheaf $P \circ \mathbf{ay_C}$ and the sieve S naturally induces a matching family for the sheaf P and the sieve S' as follows.

For an arrow of the form $G = \mathbf{ay_C}(f_i) \circ F$ in S' define a matching family given by

$$x_G = P(H)(x_{f_i})$$

where $\{x_{f_i}\}$ is a matching family for the presheaf $P \circ \mathbf{ay_C}$ and H is *any* arrow in $\mathbf{C}'$ such that $G = \mathbf{ay_C}(f_i) \circ H$. This definition is in fact independent of the choice of H and f_i as demonstrated below. Let it be the case that $G = \mathbf{ay_C}(f_i) \circ H = \mathbf{ay_C}(f_i) \circ K$.

$$\begin{tikzcd}
 & b \arrow[d, dashed, "I"'] \arrow[ddl, "H"'] \arrow[ddr, "K"] & \\
 & \mathbf{ay_C}(c_i \times_c c_j) \arrow[dl, "\Pi^1_{ij}"] \arrow[dr, "\Pi^2_{ij}"'] & \\
\mathbf{ay_C}(c_i) \arrow[dr, "\mathbf{ay_C}(f_i)"'] & & \mathbf{ay_C}(c_j) \arrow[dl, "\mathbf{ay_C}(f_j)"] \\
 & \mathbf{ay_C}(c) &
\end{tikzcd}$$

In the above diagram $\Pi^1_{ij} = \mathbf{ay_C}(\pi^1_{ij})$ and $\Pi^2_{ij} = \mathbf{ay_C}(\pi^2_{ij})$ as in diagram (4.3). By definition

$$\begin{aligned}
x_{\mathbf{ay_C}(f_i)} &= P(H)(x_{f_i}) \\
&= P(\Pi^1_{ij} \circ I)(x_{f_i}) \\
&= P(I) \circ P(\Pi^1_{ij})(x_{f_i}) \\
&= P(I) \circ \left(P \circ \mathbf{ay_C}(\pi^1_{ij})\right)(x_{f_i})
\end{aligned}$$

But $f_i \circ \pi^1_{ij}$ is an arrow in S and $\{x_{f_i}\}$ is a matching family for $P \circ \mathbf{ay_C}$, so $P(I) \circ \left(P \circ \mathbf{ay_C}(\pi^1_{ij})\right)(x_{f_i}) = P(I)(x_{f_i \circ \pi^1_{ij}}) = P(I)(x_{f_j \circ \pi^2_{ij}})$ (because of the property of the pullback). Hence,

$$\begin{aligned}
x_{\mathbf{ay_C}(f_i)} &= P(I)(x_{f_j \circ \pi^2_{ij}}) \\
&= P(I) \circ \left(P \circ \mathbf{ay_C}(\pi^2_{ij})\right)(x_{f_j}) \\
&= P(I) \circ P(\Pi^2_{ij})(x_{f_j}) \\
&= P(\Pi^2_{ij} \circ I)(x_{f_j}) \\
&= P(K)(x_{f_j}) \\
&= x_{\mathbf{ay_C}(f_j) \circ K}
\end{aligned}$$

The above proves that we have well-defined a matching family for a sieve $S' \in J'(\mathbf{ay_C}(c))$. The assumption is that P is a sheaf for the J'-topology, which leads to the conclusion that this matching family has a unique amalgamation, i.e. a unique element $x \in \mathbf{ay_C}(c)$ such that $x_G = P(G)(x)$ for all $G \in S'$. This element is clearly also an amalgamation for the matching family $f_i \mapsto x_{f_i}$ for the sieve S and sheaf $P \circ \mathbf{ay_C}$. It is also unique for this matching family. For if there was an other element $x' \in \mathbf{ay_C}(c)$ such that $x_{f_i} = P \circ \mathbf{ay_C}(f_i)(x')$, then for the arbitrary arrow $G = \mathbf{ay_C}(f_i) \circ F \in S'$,

$$\begin{aligned}
P(G)(x') &= P(\mathbf{ay_C}(f_i) \circ F)(x') = P(F) \circ P(\mathbf{ay_C}(f_i))(x') \\
&= P(F)(x_{f_i}) = x_{\mathbf{ay_C}(f_i) \circ F} \\
&= x_G
\end{aligned}$$

which proves x' also an amalgamation for $S' \in J'(\mathbf{ay_C}(c))$. So $x' = x$. ■

On the other direction, we can define a functor

$$\Phi : \mathcal{S}X = \mathcal{S}(\mathbf{C}, J) \longrightarrow \mathbf{Sets}^{\mathbf{C}'^{op}}$$

taking an object u in $\mathcal{S}X$ to the presheaf $\mathcal{S}X(-, u) : \mathbf{C}'^{op} \longrightarrow \mathbf{Sets}$ and an arrow $f : u_1 \longrightarrow u_2$ in $\mathcal{S}X$ to the morphism f_*, defined by

$$x \in \mathcal{S}X(c', u_1) \text{ then } f_{*_{c'}}(x) = x \circ f$$

The refer to ([MM92],Appendix) for the proof of the following lemma.

Lemma 4.8 *Let $\mathbf{C}'$ be a full subcategory of $\mathcal{S}X$ whose objects generate $\mathcal{S}X$. Let also J' be a topology on $\mathbf{C}'$ consisting of sieves of jointly epimorphic arrows for each object of $\mathbf{C}'$. Then $\mathcal{S}X(-, u)$ is a J'-sheaf for any object u of $\mathcal{S}X$.*

This lemma guarantees that the functor Φ sends objects of $\mathcal{S}X$ to objects of $\mathcal{S}(\mathbf{C}', J')$.

Indeed, the above lemma is applicable in our case because $\mathbf{C}$ contains the category $\mathbf{ay_C}(\mathbf{C})$, which in turn generates $\mathcal{S}X$. This lemma has an immediate corollary.

Corollary 4.9 *The Yoneda embedding map* $\mathbf{y_{C'}} : \mathbf{C'} \longrightarrow \mathbf{Sets}^{\mathbf{C'^{op}}}$ *takes objects of* $\mathbf{C'}$ *to* J'*-sheaves over* $\mathbf{C'}$*. In other words,* J' *is a subcanonical topology on* $\mathbf{C'}$*.*

Proof. Since $\mathbf{C'}$ is a full subcategory of $\mathcal{S}X$, we have the following equality of sets.

$$\mathcal{S}X(u', u) = \mathbf{C'}(u', u)$$

for any objects u, u' in $\mathbf{C'}$. So $\mathbf{y_{C'}}(u) = \mathcal{S}X(-, u)$ when u is an object of $\mathbf{C'}$. ■

We are going to evoke this corollary later as indeed the following lemma also proved by Mac Lane & Moerdijk in [MM92],Appendix.3, lemma 4.

Lemma 4.10 *Let* $\mathbf{in} : \mathbf{C'} \hookrightarrow \mathcal{S}X$ *be the inclusion functor Then the functor*

$$\mathcal{S}X(\mathbf{in}, -) : \mathcal{S}X \longrightarrow \mathcal{S}(\mathbf{C'}, J')$$

$$u \mapsto \mathcal{S}X(\mathbf{in}(-), u)$$

preserves colimits.

Finally, we will demonstrate that the pair of functors Φ, Ψ is an equivalence of categories.

Theorem 4.11 *The categories* $\mathcal{S}(\mathbf{C}, J)$ *and* $\mathcal{S}(\mathbf{C'}, J')$ *are equivalent.*

Proof. We need to show that $\Psi \circ \Phi(u)$ is naturally isomorphic to the identity u, for any sheaf u in $\mathcal{S}X$ and that $\Phi \circ \Psi(P)$ is naturally isomorphic to the identity P, for any sheaf in $\mathcal{S}X(\mathbf{C'}, J')$. For the former isomorphism is effectively demonstrated in [MM92]; it manifests itself in two stages. Let $\mathbf{i}$ be the inclusion $\mathcal{S}X \hookrightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$, then

$$\mathcal{S}X(\mathbf{ay_C}, u) \xrightarrow[\phi_{\mathbf{y_C},u}]{\cong} \mathbf{Sets}^{\mathbf{C}^{op}}(\mathbf{y_C}, \mathbf{i}(u)) \xrightarrow[\mathbf{Y}_u]{\cong} u \tag{4.9}$$

The map $\phi_{\mathbf{y_C},u}$ is the isomorphism (natural in u) of Hom-sets stemming from the adjunction $\mathbf{a} \dashv \mathbf{i}$ and $\phi_{\mathbf{y_C},u}$ is the isomorphism of the Yoneda lemma which is also natural in u (see e.g. [Bor91(1)], theorem 1.3.3).

To prove that $\Phi \circ \Psi(P) \cong P$, for P any J'-sheaf over $\mathbf{C'}$, we demonstrate that for any object c' in $\mathbf{C'}$ we have an isomorphism

$$\mathcal{S}X(c', P \circ \mathbf{ay_C}) \cong P(c')$$

We first prove the above isomorphism for any representable sheaf $\mathbf{y}_{\mathbf{C}'}(c'_i)$ in $\mathcal{S}(\mathbf{C}', J')$, for c'_i an object in $\mathbf{C}'$. (Recall that the Yoneda embedding $\mathbf{y}_{\mathbf{C}'} : \mathbf{C}' \longrightarrow \mathbf{Sets}^{\mathbf{C}'^{op}}$ in fact takes objects of $\mathbf{C}'$ to sheaves (corollary 4.9).)

It holds that $\mathbf{y}_{\mathbf{C}'}(c'_i) = \mathcal{S}X(-, c'_i)$, as discussed in corollary 4.9 and so

$$\begin{aligned} \mathbf{y}_{\mathbf{C}'}(c'_i) \circ \mathbf{ay}_{\mathbf{C}} &= \mathcal{S}X(\mathbf{ay}_{\mathbf{C}}(-), c'_i) && (4.10)\\ &\cong c'_i \text{ (the same natural isomorphism as in 4.9)} && (4.11) \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{S}X(c', \mathbf{y}_{\mathbf{C}'}(c'_i) \circ \mathbf{ay}_{\mathbf{C}}) &\cong \mathcal{S}X(c', c'_i) && (4.12)\\ &= \mathbf{y}_{\mathbf{C}'}(c'_i)(c') && (4.13) \end{aligned}$$

Now, any object P in $\mathcal{S}(\mathbf{C}', J')$ can be written as a colimit $P \cong colim_i \mathbf{y}_{\mathbf{C}'}(c'_i)$ for an appropriate diagram $\{c'_i\}$. So we have

$$\begin{aligned} \mathcal{S}X(c', P \circ \mathbf{ay}_{\mathbf{C}}) &\cong \mathcal{S}X(c', (colim_i \mathbf{y}_{\mathbf{C}'}(c'_i)) \circ \mathbf{ay}_{\mathbf{C}})\\ &\cong (c', colim_i\ (\mathbf{y}_{\mathbf{C}'}(c'_i) \circ \mathbf{ay}_{\mathbf{C}}) \text{ (colimits of sheaves are computed pointwise)}\\ &\cong colim_i\ \ \mathcal{S}X(c', \mathbf{y}_{\mathbf{C}'}(c'_i) \circ \mathbf{ay}_{\mathbf{C}}) \text{ (because of lemma 4.10)}\\ &\cong colim_i\ (\mathbf{y}_{\mathbf{C}'}(c'_i)(c')) \text{ (because of 4.12)}\\ &\cong (colim_i \mathbf{y}_{\mathbf{C}'}(c'_i))(c')\\ &\cong P(c') \end{aligned}$$

Hence, the functors P and $\mathcal{S}X(-, P \circ \mathbf{ay}_{\mathbf{C}})$ are isomorphic.

It remains to prove that this last isomorphism is natural in P. We argue as follows. The inclusion functor $\mathbf{in} : \mathbf{C}' \hookrightarrow \mathcal{S} = \mathcal{S}(\mathbf{C}, J)$ is in fact flat and continuous with respect to the J'-topology. It is flat because, trivially, it preserves finite limits (see theorem 4.13 below). Continuity of $\mathbf{in}$ with respect to the J' topology is also trivial (check with definition 4.14 below). Therefore, $\mathbf{in}$ gives rise to a geometric morphism $\mathcal{S}X \longrightarrow \mathcal{S}(\mathbf{C}', J')$ +++++++

■

The pair Φ, Ψ defines an equivalence of the categories $\mathcal{S}(\mathbf{C}, J)$ and this implies that $\Phi \dashv \Psi$ and also $\Phi \vdash \Psi$ ([Mac71],IV.4, theorem 1). We introduce two *geometric morphisms*

$$X \underset{e'}{\overset{e}{\rightleftarrows}} X' \qquad (4.14)$$

by stipulating that e is formally defined by the pair $\Psi \dashv \Phi$ and e' by the pair $\Phi \dashv \Psi$. This is possible because e and e' have both a left and a right adjoint, so they preserve all limits and colimits and in particular finite limits.

4.4 2-categorical aspects of $\mathfrak{T}op$ with respect to sites

Let $F : X \longrightarrow Y$ be a geometric morphism between two Grothendieck topoi as before. We can assume that the topoi are isomorphic to sheaves over sites $\mathcal{S}X = \mathcal{S}(\mathbf{C}, J)$ and $\mathcal{S}Y = \mathcal{S}(\mathbf{D}, K)$. Starting with the geometric theories $\mathbb{T}_X, \mathbb{T}_Y$ that are classified by the two topoi, the typical construction of $\mathcal{S}X, \mathcal{S}Y$ identifies $\mathbf{C}$ and$\mathbf{D}$ as the *syntactic categories* of $\mathbb{T}_X$ and $\mathbb{T}_Y$ respectively. In this construction the Grothendieck topologies are subcanonical, i.e. the Yoneda embedding functors $\mathbf{y}$ send objects of the syntactic categories to sheaves.

It is well known (e.g. [MM92], VII.7) that, for any Grothendieck topos X and any site site $(\mathbf{D}, K)$ there is an equivalence of categories

$$\mathfrak{T}op(X, \mathcal{S}(\mathbf{D}, K)) \underset{\phi}{\overset{\theta}{\rightleftarrows}} \underline{FlatCont}((\mathbf{D}, K), \mathcal{S}X) \tag{4.15}$$

where the R.H.S. of the equivalence 4.33 is the category with

- objects the functors $\mathbf{D} \longrightarrow \mathcal{S}X$ that are *flat* and *continuous*.
- arrows the natural transformations between such flat and continuous functors

We are not going to include the general definition of a flat functor here. In the case where a functor $F : \mathbf{D} \longrightarrow \mathcal{S}X$ targets a Grothendieck topos, it holds that F is flat iff it is filtering ([MM92], VII.9, Theorem 1). We outline what a filtering functor targeting a Grothendieck topos is. We start with the case where $\mathcal{S}X \equiv \mathbf{Sets}$.

Definition 4.12 *Let $\mathbf{C}$ be a small category and $F : \mathbf{C} \longrightarrow \mathbf{Sets}$ a functor. Then F is filtering iff it fulfills the following conditions.*

(i) F is nonempty, i.e. there is object c of $\mathbf{C}$ such that $F(c) \neq \emptyset$.

(ii) For any two elements $x \in F(c)$ and $y \in F(d)$, there exists an object a of $\mathbf{C}$, morphisms $f : a \longrightarrow c$, $g : a \longrightarrow d$ and an element $z \in F(a)$ such that $f(z) = x$ and $g(z) = y$.

(iii) For any two parallel arrows $f, g : c \longrightarrow d$ in $\mathbf{C}$ and an element $x \in F(c)$ with $f(x) = g(y)$, there is an arrow $h : a \longrightarrow c$ in $\mathbf{C}$ and an element $z \in F(a)$ such that $f \circ h = g \circ h$ and $h(z) = x$.

Definition 4.12 may assume that F targets **Sets**, but it transpires from it that the theory of filtering functors is in fact geometric and therefore makes sense in any topos. All that one has to do is to rewrite definition 4.12 in a diagrammatic form. If we now demand that the domain of a functor F has all finite limits we get the following theorem.

Theorem 4.13 ([MM92],VII.9) *If $\mathbf{D}$ has all finite limits, then a functor $\mathbf{D} \longrightarrow \mathcal{S}X$ is flat iff it preserves finite limits.*

Next we recall the definition of a continuous functor.

Definition 4.14 *Let $(\mathbf{D}, K)$ be a site and X a Grothendieck topos. Then a flat functor $\mathbf{D} \longrightarrow \mathcal{S}X$ is continuous with respect to to the topology K iff it sends K-sieves to jointly epimorphic families in $\mathcal{S}X$.*

Remark 4.15 *The Yoneda embedding is a flat and continuous functor.*

Proof. Obvious from lemma 4.5. ■

We conclude that, provided that the category $\mathbf{D}$ has all finite limits, the objects of the R.H.S. of the equivalence 4.33 are K-continuous left exact functors. Also we mention that, if K is a subcanonical topology, then the geometric morphism

$$F : \mathcal{S}X \longrightarrow \mathcal{S}(\mathbf{D}, K)$$

corresponds under the equivalence 4.33 to the flat continuous functor

$$F^{\#} : (\mathbf{D}, K) \longrightarrow \mathcal{S}X \quad \text{where} \quad F^{\#} := F^{*} \circ \mathbf{y_D} \tag{4.16}$$

Indeed $F^{\#}$ is obviously left exact because both F^{*} and $\mathbf{y_D}$ are left exact. For a proof of the continuity of $F^{\#}$ we point at [MM92], VII.7. We shall not need the explicit construction of the opposite direction of the equivalence 4.33. The reader is again referred to [MM92].

One direction of the equivalence 4.33 can be taken a stage further. Let $\mathbf{D}$ have all finite limits as before and let $\mathcal{S}X = \mathcal{S}(\mathbf{C}, J)$, where $\mathbf{C}$ also has all finite limits. We have the following definition.

Definition 4.16 *Let $(\mathbf{D}, K)$ and $(\mathbf{C}, J)$ be two sites where $\mathbf{C}$ and $\mathbf{D}$ are small categories with all finite limits. Then a site morphism $F^{\sigma} : (\mathbf{D}, K) \longrightarrow (\mathbf{C}, J)$ is a functor $F^{\sigma} : \mathbf{D} \longrightarrow \mathbf{C}$ that preserves finite limits and also preserves the Grothendieck topologies in the sense that if R is a sieve in $K(d)$ on an object d in $\mathbf{D}$, then the sieve $\uparrow F^{\sigma}(R)$ generated by $F^{\sigma}(R)$ is a sieve in $J(F^{\sigma}(d))$.*

Given such a site morphism $F^\sigma : (\mathbf{D}, K) \longrightarrow (\mathbf{C}, J)$ and assuming again that J and K are subcanonical topologies, there is an obvious induced geometric morphism $F : (\mathbf{C}, J) \longrightarrow \mathcal{S}(\mathbf{D}, K)$ ([MM92],VII.10). Indeed the composite

$$\mathbf{y_C} \circ F^\sigma : \mathcal{S}X \leftarrow (\mathbf{D}, K)$$

is flat and continuous. For, in view of theorem 4.13, $\mathbf{y_C} \circ F^\sigma$ is left exact because F^σ and $\mathbf{y_C}$ are left exact. Also, F^σ sends K-sieves to families that generate J'-sieves (definition 4.16) and $\mathbf{y_C}$ being continuous (remark 4.15) sends J-sieves to jointly epimorphic families in $\mathcal{S}X$. So, for a K-sieve $\{g_i\}$, $\mathbf{y_C}(\uparrow F^\sigma(\{g_i\}))$ is a jointly epimorphic family in $\mathcal{S}X$ and thus $\mathbf{y_C}(F^\sigma(\{g_i\}))$ is also a jointly epimorphic family in $\mathcal{S}X$. This amounts to $\mathbf{y_C} \circ F^\sigma$ being continuous. In fact we have just proved

Lemma 4.17 *Let $F^\sigma : (\mathbf{D}, K) \longrightarrow (\mathbf{C}, J)$ be a site morphism and $a^\# : (\mathbf{C}, J) \longrightarrow \mathcal{S}Z$ be flat and continuous. Then $a^\# \circ F^\sigma$ is flat and continuous.*

For the rest of this section we fix a pair of geometric morphisms between two Grothendieck topoi

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y$$

where $\mathcal{S}X = \mathcal{S}(\mathbf{C}, J)$ and $\mathcal{S}Y = \mathcal{S}(\mathbf{D}, K)$, with $\mathbf{C}$ and $\mathbf{D}$ small categories having all finite limits and J and K being subcanonical.

Definition 4.18 *Let $\mathbf{C}'$ be the closure under finite limits of a small full subcategory $\mathbf{C}''$ of $\mathcal{S}X$ that contains the images of the following three functors.*

- $\mathbf{ay} : \mathbf{C} \longrightarrow \mathcal{S}X$.
- $F^\# : \mathbf{D} \longrightarrow \mathcal{S}X$, *where* $F^\# = F^* \circ \mathbf{y_D}$.
- $G^\# : \mathbf{D} \longrightarrow \mathcal{S}X$, *where* $G^\# = G^* \circ \mathbf{y_D}$.

Let also J' be the topology on $\mathbf{C}'$ as in section 4.3, i.e. such that $\mathcal{S}(\mathbf{C}', J') \simeq \mathcal{S}X$.

Remark 4.19 *The category $\mathbf{C}'$ is still small.*

This enhancement of the generating category $\mathbf{C}$ provides the possibility to have a description of F and G by means of site morphisms.

Lemma 4.20 *The functors $F^\#$ and $G^\#$ factor uniquely through site morphisms $F^\sigma, G^\sigma : (\mathbf{D}, K) \longrightarrow (\mathbf{C}', J')$.*

Proof. By construction of $\mathbf{C}'$, the images of the functors $F^{\#}$ and $G^{\#}$ are confined inside $\mathbf{C}'$. We rename them as F^{σ} and G^{σ}. They are left exact and hence flat ($\mathbf{C}'$ has all finite limits adjoined so theorem 4.13 is valid). Also continuity of $F^{\#}$ and $G^{\#}$ means that they send covering K-sieves to jointly epimorphic families of arrows in $\mathcal{S}X$. So, by the definition of J' in section 4.3, if $S \in K(d)$, for an object d in $\mathbf{D}$, then $F^{\#}(S) \in J'(F^{\sigma}(d))$ and similarly for $G^{\#}$. So F^{σ} and G^{σ} are site morphisms. ■

Let now $\eta : F \Rightarrow G$ be a 2-cell in $\mathfrak{T}op$. The enhancement of the generating category $\mathbf{C}$ also provides the means to define a natural transformation $\eta^{\sigma} : F^{\sigma} \longrightarrow G^{\sigma}$ out of η. We recall that a 2-cell $\eta : F \Rightarrow G$ between two geometric morphisms $F, G : X \longrightarrow Y$ is generally defined by a natural transformation $\eta' : F^* \longrightarrow G^*$ (note that η' has the same direction as η'). For the sake of completeness we note here that, for any such natural transformation η', there is always a unique natural transformation $\eta'' : G_* \longrightarrow F_*$ and conversely.

The components of $\eta'_y : F^*(y) \longrightarrow G^*(y)$ for any object y of $\mathcal{S}Y$ satisfy the usual naturality square. By restricting the arguments of F^*, G^* to objects of the form $\mathbf{y_D}(d)$, with d object in $\mathbf{D}$, η' induces a natural transformation $\eta^{\sigma} : G^{\sigma} \longrightarrow F^{\sigma}$ by stipulating that its components are $\eta^{\sigma}_d = \eta_{\mathbf{y}(d)}$. In other words, the following diagram is commutative for any arrow $g : d \longrightarrow d'$ in $\mathbf{D}$.

$$\begin{tikzcd}
G^{\sigma}(d) \arrow[r, "\eta^{\sigma}_d"] \arrow[d, "G^{\sigma}(g)"'] & F^{\sigma}(d) \arrow[d, "F^{\sigma}(g)"] \\
G^{\sigma}(d') \arrow[r, "\eta^{\sigma}_d"] & F^{\sigma}(d')
\end{tikzcd}$$

We close this section by mentioning that geometricity also provides us with an internal picture of 2-cells in $\mathfrak{T}op$. A pair of parallel geometric morphisms $F, G : X \longrightarrow Y$ can also be construed as two (generalised) points of Y at stage X. A 2-cell between F and G then carries the meaning of a “transformation between points”. More concretely, if Y classifies a geometric theory $\mathbb{T}_Y$ then the inverse image functors F^* and G^* send the universal model $\mathcal{U}_Y$ of $\mathbb{T}_Y$ in $\mathcal{S}Y$ to two models $F^*(\mathcal{U}_Y)$ and $G^*(\mathcal{U}_Y)$ of $\mathbb{T}_Y$ in $\mathcal{S}X$. A 2-cell $\eta : F \Rightarrow G$ is a natural transformation that corresponds to a homomorphism between the models $F^*(\mathcal{U}_Y) \longrightarrow G^*(\mathcal{U}_Y)$ in $\mathcal{S}X$.

Remark 4.21 *Obviously, we can generalise the above result for any (finite) number of parallel geometric morphisms $F_i : X \longrightarrow Y$: We can always choose a small full subcategory with finite limits $\mathbf{C}'$ of $\mathcal{S}X$ and a Grothendieck topology J' on $\mathbf{C}'$, such that $\mathcal{S}X = \mathcal{S}(\mathbf{C}', J')$*

and the functors $F_i^{\#}$ factor through site morphisms $F_i^{\sigma} : (\mathbf{D}, K) \longrightarrow (\mathbf{C}', J')$.

4.5 The geometric inserter

Now, let us argue on the fact $\mathcal{S}X \simeq \mathcal{S}(\mathbf{C}', J')$. We have started with the assumption that X classifies a geometric theory $\mathbb{T}_X$. At the same time we have managed to write $\mathcal{S}X$ as the topos of sheaves over the site $(\mathbf{C}', J')$ (up to equivalence). We know that such a construction invariably yields that X (also) classifies the geometric theory (say $\mathbb{T}'_X$) of flat continuous covariant functors over $\mathbf{C}'$. A model of this theory in the sheaves of an arbitrary topos Z *corresponds to* a flat continuous functor $\mathbf{C}' \longrightarrow \mathcal{S}Z$. Equivalently, there is also the *internal* way of describing flat continuous functors, according to which, a model of $\mathbb{T}'_X$ in $\mathcal{S}Z$ is an internal flat continuous functor on $!^*(\mathbf{C}')$ inside $\mathcal{S}Z$. $!^*(\mathbf{C}')$ stands for the internalised version of this category in $\mathcal{S}Z$, i.e. the pullback of $\mathbf{C}'$ along the essential unique geometric morphism $! : Z \longrightarrow \mathbf{1}$.

We now present the geometric theory $\mathbb{T}_{X'}$ of flat continuous (covariant) functors over $(\mathbf{C}', J')$ (external version).

sorts:		a sort $\tilde{X}_i$ for each object X_i of $\mathbf{C}'$
functions:	F1.	a function symbol $\tilde{f} : \tilde{X}_1 \longrightarrow \tilde{X}_2$ for each arrow $f : X_1 \longrightarrow X_2$ in $\mathbf{C}'$
axioms:	A1.	$\forall x{\in}\tilde{X} \left(\top \Rightarrow \tilde{id}_{\tilde{X}_i}(x) = x\right)$
	A2.	$\forall x{\in}\tilde{X} \left(\top \Rightarrow \tilde{f}(\tilde{g}(x)) = \widetilde{(f \circ g)}(x)\right)$, where $g : X \to Y$, $f : Y \to Z$ in $\mathbf{C}'$
	A3.	$\top \Rightarrow \bigvee_{X_i \in \mathbf{C}'} \left\{\exists x \in \tilde{X}_i (x = x)\right\}$
	A4.	$\forall x{\in}\tilde{X}\ \forall y{\in}\tilde{Y} (\top \Rightarrow \bigvee_{Z \in \mathbf{C}'} \{\exists z{\in}\tilde{Z}\ (x = \tilde{f}(x) \wedge y = \tilde{g}(x),$ for $f : Z \longrightarrow X,\ g : Z \longrightarrow Y$in $\mathbf{C}'$)})
	A5.	$\forall x{\in}\tilde{X}\ ((\tilde{f}(x) = \tilde{g}(x)) \Rightarrow \bigvee_{Z \in \mathbf{C}'} \{\exists z{\in}\tilde{Z}\ (x = \tilde{h}(z),$ for $\tilde{h} : \tilde{Z} \longrightarrow \tilde{X},\ \tilde{f} \circ \tilde{h} = \tilde{g} \circ \tilde{h})\}),$ where $f, g : X \longrightarrow Y$ in $\mathbf{C}'$
	A6.	$\forall x{\in}\tilde{X}\ (\bigvee_{X_i} \{\exists y{\in}\tilde{X}_i (x = \tilde{f}_i(y))\}),$ where $\{f_i : X_i \longrightarrow X\}$ is a J'-sieve on X

Axioms (A1,A2) are covariant functoriality, axioms (A3,A4,A5) are flatness (c.f. definition 4.12) and axiom (A6) is J'-continuity (c.f. definition 4.14). The last one actually dictates that in a model (M) of $\mathbb{T}_{X'}$ in any topos, the arrows $\tilde{f}_i^{(M)}$ are jointly epimorphic.

Now we return to the pair of site morphisms $F^\sigma, G^\sigma : (\mathbf{D}, K) \longrightarrow (\mathbf{C}', J')$ of lemma 4.20. Let d be any object in $\mathbf{D}$. In accordance with our nomenclature, we denote by $\widetilde{F^\sigma(d)}$ and $\widetilde{G^\sigma(d)}$ the sorts of the geometric theory $\mathbb{T}_{X'}$ that correspond to the objects $F^\sigma(d)$ and $G^\sigma(d)$ of $\mathbf{C}'$ respectively. Also, for g an arrow in $\mathbf{D}$, we denote by $\widetilde{F^\sigma(g)}$ and $\widetilde{G^\sigma(g)}$ the function symbols of $\mathbb{T}_{X'}$ that correspond to the arrows $F^\sigma(g)$ and $G^\sigma(g)$ respectively.

It is worth remarking that the category $\mathbf{D}$ can be viewed as a "set" of indices for part of the sorts and functions of the geometric theory $\mathbb{T}_{X'}$. The two ways of assigning sorts and functions of $\mathbb{T}_{X'}$ to the indices are provided by the applications of the two functors F^σ and G^σ on the objects and arrows of $\mathbf{D}$.

We enhance the geometric theory [4.5], by adjoining the following ingredients to its presentation.

functions: F2. a function symbol $\widetilde{k_d} : \widetilde{F^\sigma(d)} \longrightarrow \widetilde{G^\sigma(d)}$ for each object d of $\mathbf{D}$, (*i.e.* for each pair of objects $F^\sigma(d)$,$G^\sigma(d)$ of $\mathbf{C}'$, for d object in $\mathbf{D}$)

axioms: A7. $\forall x \in \widetilde{F^\sigma(d_1)}\ (\widetilde{G^\sigma(g)}(\widetilde{k_{d_1}}(x)) = \widetilde{k_{d_2}}(\widetilde{F^\sigma(g)}(x)))$, where $\widetilde{k_{d_1}} : \widetilde{F^\sigma(d_1)} \longrightarrow \widetilde{G^\sigma(d_1)}$, $\widetilde{k_{d_2}} : \widetilde{F^\sigma(d_2)} \longrightarrow \widetilde{G^\sigma(d_2)}$ and $g : d_1 \longrightarrow d_2$ in $\mathbf{D}$

Axiom (7) can be categorically formulated by demanding the following naturality square to commute.

$$\begin{tikzcd}
\widetilde{F^\sigma(d_1)} \arrow[r, "\widetilde{k_{d_1}}"] \arrow[d, "\widetilde{F^\sigma(g)}"'] & \widetilde{G^\sigma(d_1)} \arrow[d, "\widetilde{G^\sigma(g)}"] \\
\widetilde{F^\sigma(d_2)} \arrow[r, "\widetilde{k_{d_2}}"] & \widetilde{G^\sigma(d_2)}
\end{tikzcd}$$

We denote the geometric theory presented by all the above ingredients by $\mathbb{T}_{\mathcal{E}}$. We also denote by $\mathcal{E}$ the classifying topos of this geometric theory.

The fact that $\mathbb{T}_{\mathcal{E}} \supseteq \mathbb{T}_X$ implies that, in any topos Z, a model of the geometric theory $\mathbb{T}_{\mathcal{E}}$ is also a model of the geometric theory $\mathbb{T}_X$. We define a "model reduction" functor $\mathbf{mr}_Z$

$$\mathbf{mr}_Z : Mod(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \longrightarrow Mod(\mathcal{S}Z, \mathbb{T}_X)$$

by stipulating that $\mathbf{mr}_Z$ sends a model of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$ to itself and a homomorphism between two models of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$ to itself.

We evoke the fact that when $\mathbb{T}_R$ is a geometric theory and R its classifying topos then the following pair of maps is an adjoint equivalence of categories

$$\begin{tikzcd} Mod(\mathcal{S}Z, \mathbb{T}_R) \arrow[r, shift left, "\tau_Z"] \arrow[r, phantom, "\cong"] & \mathfrak{Top}(Z, R) \arrow[l, shift left, "\sigma_Z"] \end{tikzcd} \tag{4.17}$$

So, we can define the composite functor

$$\begin{tikzcd} Mod(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \arrow[r, "\mathbf{mr}_Z"] & Mod(\mathcal{S}Z, \mathbb{T}_X) \arrow[d, "\tau_Z"] \\ \mathfrak{Top}(Z, \mathcal{E}) \arrow[u, "\sigma_Z"] \arrow[r, dashed, "\mathfrak{Top}(Z, i_{\mathcal{E}})"] & \mathfrak{Top}(Z, X) \end{tikzcd} \tag{4.18}$$

By putting Z for $\mathcal{E}$ in [4.18], we get a functor

$$\begin{tikzcd} Mod(\mathcal{S}\mathcal{E}, \mathbb{T}_{\mathcal{E}}) \arrow[r, "\mathbf{mr}_{\mathcal{E}}"] & Mod(\mathcal{S}\mathcal{E}, \mathbb{T}_X) \arrow[d, "\tau_{\mathcal{E}}"] \\ \mathfrak{Top}(\mathcal{E}, \mathcal{E}) \arrow[u, "\sigma_{\mathcal{E}}"] \arrow[r, dashed, "\mathfrak{Top}(\mathcal{E}, i_{\mathcal{E}})"] & \mathfrak{Top}(\mathcal{E}, X) \end{tikzcd} \tag{4.19}$$

We denote by $i_{\mathcal{E}}$ the geometric morphism that is the image of the identity $id_{\mathcal{E}}$ in $\mathfrak{Top}(\mathcal{E}, \mathcal{E})$ along the functor $\tau_{\mathcal{E}} \circ \mathbf{mr}_{\mathcal{E}} \circ \sigma_{\mathcal{E}}$. Using the terminology of section 1.2, the defining geometric construction $\mathbf{mr}_Z$ of $i_{\mathcal{E}}$ is a *geometric transformation* and corresponds to the postcomposition with $i_{\mathcal{E}}$.

The geometric construction of $i_{\mathcal{E}}$ exhibits the following obvious fact which is important to bear in mind in what follows.

Corollary 4.22 *Let $\tilde{X}$ be a sort in $\mathbb{T}_{X'}$ and X an object in the universal model of $\mathbb{T}_{X'}$ in $\mathcal{S}(\mathbf{C}', J')$ (denoted $\mathcal{U}_{X'}$). Then $i^*_{\mathcal{E}}$ takes X to an object X' in $\mathcal{S}\mathcal{E}$ which is the object of the universal model of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}\mathcal{E}$ (denoted $\mathcal{U}_{\mathcal{E}}$) that corresponds to the same sort $\tilde{X}$ in $\mathbb{T}_{X'} \subseteq \mathbb{T}_{\mathcal{E}}$.*

Proof. We read diagram [4.19]. The functor $\tau_{\mathcal{E}}$ by definition constructs a geometric morphism $i_{\mathcal{E}}$ such that

$$i^*_{\mathcal{E}}(\mathcal{U}_{X'}) = \mathbf{mr}_{\mathcal{E}}(\mathcal{U}_{\mathcal{E}})$$

where the R.H.S is the "reduct" of $\mathcal{U}_{\mathcal{E}}$.

Now, let $\mathbb{T}_R$ be any geometric theory, $(\mathbf{B}(\mathbb{T}_R), L)$ its *syntactic site* and (hence) $\mathcal{S}R = \mathcal{S}(\mathbf{B}(\mathbb{T}_R), L)$ the sheaves over its classifying topos. It is known (e.g. [MM92]) that the universal model $\mathcal{U}_R$ in $\mathcal{S}R$ is the same (up to equivalence) with the image of the Yoneda embedding $\mathbf{y}_{\mathbf{B}(\mathbb{T}_R)} : \mathbf{B}(\mathbb{T}_R) \longrightarrow \mathcal{S}R$, i.e.

$$\mathcal{U}_R = \mathbf{y}_{\mathbf{B}(\mathbb{T}_R)}(\mathbf{B}(\mathbb{T}_R))$$

(the topology on the syntactic site is subcanonical). If in particular $\mathbb{T}_R$ is the geometric theory of flat continuous covariant functors on a category $\mathbf{B}$ with respect to a topology L, then its syntactic site is $(\mathbf{B}, L)$ itself.

Hence, if $\widetilde{X_{c'}}$ is a sort of $\mathbb{T}_{X'}$ corresponding to an object c' in $\mathbf{C}'$, then $\mathbf{y}_{\mathbf{C}'}(c')$ is the object in $\mathcal{U}_{X'}$ that corresponds to $\widetilde{X_{c'}}$ and $i^*_{\mathcal{E}} \circ \mathbf{y}_{\mathbf{C}'}(c')$ is the object in $\mathcal{U}_{\mathcal{E}}$ that corresponds to $\widetilde{X'_c}$. ■

The geometric morphism $i_{\mathcal{E}}$ is a universal 1-cell associated with the classifying topos $\mathcal{E}$. There is also a universal 2-cell. For any object d in $\mathbf{D}$, let $[k_d]$ be the arrow corresponding to the function symbol of type [F2] $\widetilde{k_d} : \widetilde{F^\sigma(d)} \longrightarrow \widetilde{G^\sigma(d)}$ in the universal model $\mathcal{U}_{\mathcal{E}}$ of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}\mathcal{E}$. Axiom [A7] dictates that for any arrow $g : d \longrightarrow d'$ in $\mathbf{D}$, the following is commutative

$$\begin{tikzcd}
{[F^\sigma(d)]} \arrow[r, "k_d"] \arrow[d, "{[F^\sigma(g)]}"'] & {[G^\sigma d]} \arrow[d, "{[G^\sigma(g)]}"] \\
{[F^\sigma(d')]} \arrow[r, "k_{d'}"] & {[G^\sigma(d')]}
\end{tikzcd} \tag{4.20}$$

Using the insight of corollary 4.22, the object e.g. $[F^\sigma(d)]$ in $\mathcal{U}_{\mathcal{E}}$ is obtained as

$$[F^\sigma(d)] = i^{\#}_{\mathcal{E}}(F^\sigma(d)) := i^*_{\mathcal{E}} \circ \mathbf{y}_{\mathbf{C}'}(F^\sigma(d))$$

Therefore the square becomes for any object d in $\mathbf{D}$

$$\begin{tikzcd}
i^{\#}_{\mathcal{E}}(F^\sigma(d)) \arrow[r, "k_d"] \arrow[d, "i^{\#}_{\mathcal{E}}(F^\sigma(g))"'] & i^{\#}_{\mathcal{E}}(G^\sigma(d)) \arrow[d, "i^{\#}_{\mathcal{E}}(G^\sigma(g))"] \\
i^{\#}_{\mathcal{E}}(F^\sigma(d')) \arrow[r, "k_{d'}"] & i^{\#}_{\mathcal{E}}(G^\sigma(d'))
\end{tikzcd} \tag{4.21}$$

Hence the arrows k_d are the components of a natural transformation $k : i^{\#}_{\mathcal{E}} \circ F^\sigma \Rightarrow i^{\#}_{\mathcal{E}} \circ G^\sigma$ and by lemma 4.17, the functors $i^{\#}_{\mathcal{E}} \circ F^\sigma$ and $i^{\#}_{\mathcal{E}} \circ G^\sigma$ are flat and continuous. The image of

k along the arrow part of the functor ϕ is a natural transformation (say) $\epsilon : F \circ i_{\mathcal{E}} \Rightarrow G \circ i_{\mathcal{E}}$.

Definition 4.23 *We call the triple $(\mathcal{E}, i_{\mathcal{E}}, \epsilon)$ the geometric inserter of the diagram*

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y \tag{4.22}$$

in $\mathfrak{T}op$. The choice of the adjective "geometric" reflects the fact that both the topos $\mathcal{E}$ and the geometric morphism $i_{\mathcal{E}}$ have been defined entirely geometrically.

4.6 The 2-categorical inserter

In this section we introduce the notion of the inserter in the 2-category $\mathfrak{T}op$.

An inserter is an example of a *weighted limit*. A thorough study of weighted limits is in [Kel89] and [PR91]. Let $\mathbf{D}$ and $\mathbf{K}$ be two 2-categories and let $\mathbf{CAT}$ be the 2-category of all categories. We consider 2-functors $P : \mathbf{D} \longrightarrow \mathbf{K}$ and $F : \mathbf{D} \longrightarrow \mathbf{CAT}$. P can be interpreted as a diagram in $\mathbf{K}$ (in the appropriate 2-categorical sense) and F as a weight on $\mathbf{D}$.

Definition 4.24 *The F-weighted limit of the diagram P is an object $Lim_F P$ in $\mathbf{K}$ such that for any object Z in $\mathbf{K}$, we have the equivalence of categories*

$$\mathbf{K}(Z, Lim_F P) \simeq [\mathbf{D}, \mathbf{CAT}] \left(F, \mathbf{K}\left(Z, P\left(-\right)\right)\right) \tag{4.23}$$

which is natural in Z. The objects of the R.H.S. 2-category above are the natural transformations from F to $\mathbf{K}(Z, P(-))$ and can be viewed as F-weighted cones over P with vertex Z. Its arrows are the modifications of these F-weighted cones.

The above definition is a weaker version of the standard one appearing e.g. in [Kel89] and [PR91] in that it demands 4.23 to be natural equivalence instead of natural isomorphism. We point out that, if it exists, a weighted limit of a given weighted diagram is unique up to equivalence.

An inserter is the F-weighted limit for the special case where $\mathbf{D}$ is the category

$$d_0 \underset{f_2}{\overset{f_1}{\rightrightarrows}} d_1$$

and the weight F sends d_0 to the category $\mathbf{1}$ and d_1 to the category $\mathbf{2}$, the category of two objects $\perp$ and $\top$ and one arrow between them $\uparrow : \perp \longrightarrow \top$ apart from the identities. It also sends f_1 to the arrow that maps $\cdot$ (the single object of $\mathbf{1}$) to $\perp$ and f_2 to the arrow that maps $\cdot$ to $\top$. We fix the notation of the diagram $P(\mathbf{D})$ as below

$$X = P(d_0) \underset{G=P(f_2)}{\overset{F=P(f_1)}{\rightrightarrows}} P(d_1) = Y$$

We are interested in giving a concise description of the category $[\mathbf{D}, \mathbf{CAT}]\,(F, \mathbf{K}(Z, P(-)))$ for the special case of the defining diagram and weight of the inserter. The following lemma gives an equivalent definition of this category after discarding the redundant data of the above definition.

Lemma 4.25 *The category* $[\mathbf{D}, \mathbf{CAT}](F, \mathbf{K}(Z, G(-)))$ *for the weight and the diagram*

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y \tag{4.24}$$

of the inserter in a 2-category $\mathbf{K}$ *is (isomorphic to) the category whose objects and arrows are as below*

- *objects are pairs* (a, α) *where* $a : Z \longrightarrow X$ *is an arrow in* $\mathbf{K}$ *and* $\alpha : F \circ a \Rightarrow G \circ a$ *is a 2-cell in* $\mathbf{K}$.
- *Let* (a, α) *and* (b, β) *be two objects in* $[\mathbf{D}, \mathbf{CAT}](F, \mathbf{K}(Z, P(-)))$, *with* $a, b : Z \longrightarrow X$, $\alpha : F \circ a \Rightarrow G \circ a$ *and* $\beta : F \circ b \Rightarrow G \circ b$ *. Then an arrow* $(a, \alpha) \longrightarrow (b, \beta)$ *is a 2-cell* $\eta : a \Rightarrow b$ *such that the following diagram commutes*

$$\begin{tikzcd} F \circ a \arrow[r, "F \bullet \eta"] \arrow[d, "\alpha"'] & F \circ b \arrow[d, "\beta"] \\ G \circ a \arrow[r, "G \bullet \eta"] & G \circ b \end{tikzcd} \tag{4.25}$$

(The symbol $\bullet$ *stands for "horizontal" composition of 2-cells.)*

Now, we we consider the special case where $\mathbf{K}$ is the 2-category $\mathfrak{Top}$ of Grothendieck topoi and geometric morphisms. For brevity we denote

$$\mathbf{I}(Z) := [\mathbf{D}, \mathbf{CAT}]\,(F, \mathfrak{Top}\,(Z, P\,(-)))$$

and we preserve all the notation of the above lemma. In the next section we are going to prove that there is an adjoint equivalence between the categories $\mathbf{I}(Z)$ and $\mathfrak{Top}(Z, \mathcal{E})$, where

$\mathcal{E}$ is the geometric inserter of F, G (section 4.5), and that this equivalence is natural in Z. Such a natural equivalence implies that the Grothendieck topos $\mathcal{E}$ is the (2-categorical) inserter of a diagram of two parallel geometric morphisms $F, G : X \longrightarrow Y$ in the sense of definition 4.24.

4.7 Equivalence of inserter and geometric inserter

In this section we are going to demonstrate that there is a natural equivalence

$$\mathbf{I}(Z) \simeq \mathfrak{T}op(Z, \mathcal{E})$$

for any topos Z. We shall be using the construction of section 4.3 and relying on the natural equivalence

$$Mod(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathfrak{T}op(Z, \mathcal{E}) \tag{4.26}$$

as well as the natural equivalence 4.33 of section 4.4.

We are going to describe the construction of a pair of functors $\Lambda_Z : \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \rightleftharpoons \mathbf{I}(Z) : \Sigma_Z$ with great detail so that the fact that they are inverse to each other becomes obvious.

We first define a functor

$$\Lambda_Z : Mod(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \longrightarrow \mathbf{I}(Z) \tag{4.27}$$

Let $\mathcal{M}^Z_{\mathcal{E}}$ be a model of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$. The model reduction functor

$$\mathbf{mr}_Z : Mod(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \longrightarrow Mod(\mathcal{S}Z, \mathbb{T}_{X'}) \tag{4.28}$$

defines via the equivalence 4.26 a geometric morphism $a : Z \longrightarrow X'$ corresponding to the model $\mathbf{mr}_Z(\mathcal{M}^Z_{\mathcal{E}})$.

$$\begin{tikzcd} Z \arrow[r, "a"] & X' \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & Y \end{tikzcd} \tag{4.29}$$

$$\begin{tikzcd} \mathcal{S}Z & \mathcal{S}X' \arrow[l, "a^*"'] & (\mathbf{C}', J') \arrow[l, "\mathbf{y_{C'}}"'] & (\mathbf{D}, K) \arrow[l, shift right, "F^\sigma"'] \arrow[l, shift left, "G^\sigma"] \end{tikzcd} \tag{4.30}$$

Now, the fact that $\mathcal{M}^Z_{\mathcal{E}}$ is a model of $\mathbb{T}_{\mathcal{E}}$ implies that for any object d in $\mathbf{D}$, $\mathcal{M}^Z_{\mathcal{E}}$ fixes an arrow

$$[k_d] : [F^\sigma(d)] \longrightarrow [G^\sigma(d)]$$

in $\mathcal{S}Z$ that corresponds to the [F2]-function $\widetilde{K_d} : \widetilde{F^\sigma(d)} \longrightarrow \widetilde{G^\sigma(d)}$ of the language of $\mathbb{T}_\mathcal{E}$. By the way a is defined we immediately know that

$$[F^\sigma(d)] = a^\#(F^\sigma(d)) \text{ and } [G^\sigma(d)] = a^\#(G^\sigma(d))$$

because $\mathbf{y_{C'}}(F^\sigma(d))$ and $\mathbf{y_{C'}}(G^\sigma(d))$ are the objects corresponding to the sorts $F^\sigma(d)$ and $G^\sigma(d)$ in the universal model of $\mathbb{T}_{X'}$ in $\mathcal{S}X'$ (see corollary 4.22).

The axiom [A7] in $\mathcal{M}^Z_\mathcal{E}$ asserts that for any arrow $h : d \longrightarrow d'$ in $\mathbf{D}$ the following naturality square commutes

$$\begin{tikzcd} {[F^\sigma(d)]} \arrow[r, "{[k_d]}"] \arrow[d, "{[F^\sigma(h)]}"'] & {[G^\sigma(d)]} \arrow[d, "{[G^\sigma(h)]}"] \\ {[F^\sigma(d')]} \arrow[r, "{[k_{d'}]}"] & {[G^\sigma(d')]} \end{tikzcd} \tag{4.31}$$

Obviously we also have that $[F^\sigma(h)] = a^\#(F^\sigma(h))$ and $[G^\sigma(h)] = a^\#(G^\sigma(h))$, so if, in addition, we denote

$$\alpha^\sigma_d := [k_d]$$

the square 4.31 becomes

$$\begin{tikzcd} a^\# \circ F^\sigma(d) \arrow[r, "\alpha^\sigma_d"] \arrow[d, "a^\# \circ F^\sigma(h)"'] & a^\# \circ G^\sigma(d) \arrow[d, "a^\# \circ G^\sigma(h)"] \\ a^\# \circ F^\sigma(d') \arrow[r, "\alpha^\sigma_{d'}"] & a^\# \circ G^\sigma(d') \end{tikzcd} \tag{4.32}$$

This demonstrates that α^σ_d are the components of a natural transformation $\alpha^\sigma : a^\# \circ F^\sigma \Rightarrow a^\# \circ G^\sigma$. The functor $a^\#$ is just $a^* \circ \mathbf{y_{C'}}$ and hence it is flat and continuous. By the lemma 4.17 we conclude that the functors $a^\# \circ F^\sigma$ and $a^\# \circ G^\sigma$ are flat and continuous.

Hence α^σ induces a natural transformation $F \circ a \Rightarrow G \circ a$ via the equivalence of categories

$$\mathfrak{Top}(Z, Y) \underset{\phi}{\overset{\theta}{\rightleftarrows}} ConFlat((\mathbf{D}, K), \mathcal{S}Z) \tag{4.33}$$

The natural transformation α^σ is an arrow between objects $a^* \circ \mathbf{y_{C'}} \circ G^\sigma$ and $a^* \circ \mathbf{y_{C'}} \circ F^\sigma$ in the R.H.S category in 4.33. These objects are actually $\theta(G \circ a)$ and $\theta(F \circ a)$. Hence

$\phi(\alpha^{\sigma})$ provides as with a 2-cell $\alpha : \phi \circ \theta(F \circ a) \Rightarrow \phi \circ \theta(G \circ a)$. $\phi \circ \theta$ is isomorphic to the identity on the L.H.S. category in 4.33, hence we have constructed a 2-cell

$$\alpha : F \circ a \Rightarrow G \circ a \tag{4.34}$$

We have therefore proved the following lemma.

Lemma 4.26 *The pair (a, α) defined above is a well-defined object in $\mathbf{I}(Z)$.*

We know turn to the arrow part of the functor Λ_Z. Let $\mathcal{M}^Z_{\mathcal{E}}$ and $\mathcal{N}^Z_{\mathcal{E}}$ be two models of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$, corresponding to objects (a, α) and (b, β) in $\mathbf{I}(Z)$. Let H be a homomorphism between the two models. The sorts of $\mathbb{T}_{\mathcal{E}}$ are indexed with respect to the objects of $\mathbf{C}'$, so the components of H are arrows $H_{c'} : [c']_1 \longrightarrow [c']_2$ for any object c' of $\mathbf{C}'$, where $[c']_1$ is the object in $\mathcal{S}Z$ that corresponds to the object c' in $\mathbf{C}'$ as fixed by the first model and $[c']_2$ the object in $\mathcal{S}Z$ that corresponds to the same object in $\mathbf{C}'$ as fixed by the second model. Such components $H_{c'}$ are, by definition, subject to the property that the following diagram is commutative for any arrow $f : c'_1 \longrightarrow c'_2$ in $\mathbf{C}'$, equivalently for any function symbol $\tilde{f} : \widetilde{X_{c'_1}} \longrightarrow \widetilde{X_{c'_2}}$, where $\widetilde{X_{c'_i}}$ is the sort in $\mathbb{T}_{\mathcal{E}}$ that is indexed by c'_i.

$$\begin{tikzcd}
{[c'_1]_1} \arrow[r, "H_{c'_1}"] \arrow[d, "{[f]_1}"'] & {[c'_1]_2} \arrow[d, "{[f]_2}"] \\
{[c'_2]_1} \arrow[r, "\varepsilon_{c'_2}"] & {[c'_2]_2}
\end{tikzcd} \tag{4.35}$$

By assumption, $[c'_i]_1 = a^* \circ \mathbf{y}_{\mathbf{C}'}(c'_i)$ and $[c'_i]_2 = b^* \circ \mathbf{y}_{\mathbf{C}'}(c'_i)$ (i=1,2). We first *choose* $[f]$ to correspond to the function symbols [F1]. Then the above commutative square becomes

$$\begin{tikzcd}
a^* \circ \mathbf{y}_{\mathbf{C}'}(c'_1) \arrow[r, "H_{c'_1}"] \arrow[d, "a^* \circ \mathbf{y}_{\mathbf{C}'}(f)"'] & b^* \circ \mathbf{y}_{\mathbf{C}'}(c'_1) \arrow[d, "b^* \circ \mathbf{y}_{\mathbf{C}'}(f)"] \\
a^* \circ \mathbf{y}_{\mathbf{C}'}(c'_2) \arrow[r, "H_{c'_2}"] & b^* \circ \mathbf{y}_{\mathbf{C}'}(c'_2)
\end{tikzcd} \tag{4.36}$$

Repeating a previous argument, $a^* \circ \mathbf{y}_{\mathbf{C}'}$ and $b^* \circ \mathbf{y}_{\mathbf{C}'}$ are flat and continuous functors and the above diagram asserts that H is a natural transformation between them. Via the equivalence

$$\mathfrak{T}op(Z, X') \underset{\phi}{\overset{\theta}{\rightleftarrows}} \underline{FlatCont}((\mathbf{C}', J'), \mathcal{S}Z) \tag{4.37}$$

it corresponds to a 2-cell $\eta : a \Rightarrow b$. Before we claim that this natural transformation is a morphism of $\mathbf{I}(Z)$, we have to prove that η also makes the diagram 4.25 commutative.

Lemma 4.27 *The natural transformation is an arrow in* $\mathbf{I}(Z)$.

Proof. We go back to the diagram 4.35. $\widetilde{X_{c'_1}}$ and $\widetilde{X_{c'_2}}$ can be any sorts in $\mathbb{T}_{\mathcal{E}}$ and $\tilde{f}$ any function symbol connecting them. We set $c'_1 = F^\sigma(d)$, $c'_2 = G^\sigma(d)$ and $\underline{f = [k_d]}$ for any d object in $\mathbf{D}$ which corresponds by definition to a function symbol $\tilde{f} : \widetilde{F^\sigma(d)} \longrightarrow \widetilde{G^\sigma(d)}$. Then diagram 4.36 becomes

$$\begin{tikzcd}
a^{\#}(F^\sigma(d)) \arrow[r, "H_{F^\sigma(d)}"] \arrow[d, "\alpha^\sigma_d := {[k_d]}_1"'] & b^{\#}(F^\sigma(d)) \arrow[d, "\beta^\sigma_d := {[k_d]}_2"] \\
a^{\#}(G^\sigma(d)) \arrow[r, "H_{G^\sigma(d)}"] & b^{\#}(G^\sigma(d))
\end{tikzcd}$$

which corresponds exactly to the desired diagram 4.25. (e.g. the horizontal composite $F \bullet \eta$ is η_F.) ■

Now we define a functor on the opposite direction

$$\Sigma : \mathbf{I}(Z) \longrightarrow \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \tag{4.38}$$

Let (a, α) be an object in $\mathbf{I}(Z)$, where a is a geometric morphism $a : Z \longrightarrow X$ and α is a natural transformation $F \circ a \longrightarrow G \circ a$. Let a' be the composite geometric morphism

$$Z \xrightarrow{\quad a \quad} X \xrightarrow{\quad e \quad} X'$$

where e is the geometric morphism defined at the end of section 4.3.

It is intuitively clear that a is going to specify a model of $\mathbb{T}_{X'}$ in $\mathcal{S}Z$ and α will adjoin to that the arrows in $\mathcal{S}Z$ that correspond to the function symbols [F2] subject to axioms [A7] so that they "add up" to a model of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$.

The functor a'^* readily provides a model of $\mathbb{T}_{X'}$ in $\mathcal{S}Z$. Namely, the model

$$\mathcal{M}^Z_{X'} = a'^*(\mathcal{U}_{X'}) := a^* \circ e^*(\mathcal{U}_{X'}) \tag{4.39}$$

with $\mathcal{U}_{X'}$ being the universal model of $\mathbb{T}_{X'}$ in $\mathcal{S}X'$. Note that (section 4.3)

The natural transformation α is defined by a natural transformation $\alpha' : a^* \circ F^* \longrightarrow a^* \circ G^*$, i.e. by arrows $\alpha'_y : a^* \circ F^*(y) \longrightarrow a^* \circ G^*(y)$ for any object y in $\mathcal{S}Y$, subject to the commutativity of the naturality square

$$\begin{tikzcd} a^* \circ F^*(y_1) \arrow[r, "\alpha'_{y_1}"] \arrow[d, "a^* \circ F^*(h)"'] & a^* \circ G^*(y_1) \arrow[d, "a^* \circ G^*(h)"] \\ a^* \circ F^*(y_2) \arrow[r, "\alpha'_{y_2}"] & a^* \circ G^*(y_2) \end{tikzcd} \tag{4.40}$$

for any arrow $h : y_1 \longrightarrow y_2$ in $\mathcal{S}Y$. In particular, by choosing y to be any representable sheaf $y = \mathbf{y_D}$, we get an arrow $\alpha'_{\mathbf{y_D}(d)} : a^* \circ F^*(\mathbf{y_D}(d)) \longrightarrow a^* \circ G^*(\mathbf{y_D}(d))$ for any object in the category $\mathbf{D}$. Note that one of the fixed assumptions of this chapter is that the topology K on $\mathbf{D}$ is subcanonical (e.g. by choosing $(\mathbf{D}, K)$ to be the syntactic site of $\mathbb{T}_Y$), so that $\mathbf{y_D}(d)$ is a sheaf in $\mathcal{S}Y$. The square [4.40] now becomes

$$\begin{tikzcd} a^* \circ F^*(\mathbf{y_D}(d_1)) \arrow[r, "\alpha'_{\mathbf{y_D}(d_1)}"] \arrow[d, "a^* \circ F^*(\mathbf{y_D}(g))"'] & a^* \circ G^*(\mathbf{y_D}(d_1)) \arrow[d, "a^* \circ G^*(\mathbf{y_D}(g))"] \\ a^* \circ F^*(\mathbf{y_D}(d_2)) \arrow[r, "\alpha'_{\mathbf{y_D}(d_2)}"] & a^* \circ G^*(\mathbf{y_D}(d_2)) \end{tikzcd} \tag{4.41}$$

where $g : d_1 \longrightarrow d_2$ is any arrow in $\mathbf{D}$ and the functoriality of $\mathbf{y_D}$ was used. By definition $F^* \circ \mathbf{y_D} = F^\sigma$ and $G^* \circ \mathbf{y_D} = G^\sigma$ (remark 4.21). We also denote

$$[k_d] = \alpha'_{\mathbf{y_D}(d)} \tag{4.42}$$

For any object d in $\mathbf{D}$ the above is an arrow between objects $a^* \circ F^\sigma(d)$ and $a^* \circ G^\sigma(d)$ in $\mathcal{S}Z$ that makes the following square commutes for any arrow $g : d_1 \longrightarrow d_2$ in $\mathbf{D}$

$$\begin{tikzcd} a'^* \circ \mathbf{y_{C'}} \circ F^\sigma(d_1) \arrow[r, "{[k_{d_1}]}"] \arrow[d, "a'^* \circ \mathbf{y_{C'}} \circ F^\sigma(g)"'] & a'^* \circ \mathbf{y_{C'}} \circ G^\sigma(d_1) \arrow[d, "a'^* \circ \mathbf{y_{C'}} \circ G^\sigma(g)"] \\ a'^* \circ \mathbf{y_{C'}} \circ F^\sigma(d_2) \arrow[r, "{[k_{d_2}]}"] & a'^* \circ \mathbf{y_{C'}} \circ G^\sigma(d_2) \end{tikzcd} \tag{4.43}$$

where we used the identity

$$e^* \circ e_* \equiv \Psi \circ \Phi = id_{\mathcal{S}X}$$

proved in theorem 4.11, together with the fact that Φ acts on $\mathbf{C}'$ as the Yoneda embedding $\mathbf{y}_{\mathbf{C}'}$ and that $a'^* = a^* \circ \Psi$. The definition (4.42) gives the right arrow with respect to definition (4.39) because objects in $\mathcal{M}^Z_{X'}$, corresponding to sorts of types $\widetilde{F^\sigma(d)}$ and $\widetilde{G^\sigma(d)}$, are (up to isomorphism) $a'^*(\mathbf{y}_{\mathbf{C}'}(F^\sigma(d)))$ and $a'^*(\mathbf{y}_{\mathbf{C}'}(G^\sigma(d)))$ respectively (corollary (4.22)). So the arrows $[k_d]$ have the correct sources and targets in $\mathcal{M}^Z_{X'}$. The validity of the argument is "up to isomorphism" because of theorem (4.11) again.

Therefore, we stipulate that the object part of the functor Σ is defined by (4.39) and (4.42) in the sense that $\Sigma(a, \alpha)$ is the model $a'^*(\mathcal{U}_{X'})$ together with the arrows $[k_d]$ of (4.42). We denote

$$\Sigma(a, \alpha) := \mathcal{M}^{Z,(a)}_{\mathcal{E}}$$

We now define the arrow part of Σ. Let (a, α) and (b, β) be two objects in $\mathbf{I}(Z)$, such that $\Sigma(a, \alpha)$ and $\Sigma(b, \beta)$ are two respective objects $\mathcal{M}^Z_{\mathcal{E}}$ and $\mathcal{N}^Z_{\mathcal{E}}$ in $\underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}})$. Let also η an arrow between them in $\mathbf{I}(Z)$, i.e., $\eta : a \Rightarrow b$ is a natural transformation rendering the diagram (4.25) commutative. Such an η amounts to component arrows $\eta'_x : b^*(x) \longrightarrow a^*(x)$ for any object x in $\mathcal{S}X$ satisfying two kinds of properties:

- the naturality square
- the following commutative diagram

$$\begin{tikzcd} a^* \circ F^*(y) \arrow[r, "(\eta' \bullet F^*)_y"] \arrow[d, "\alpha_y"'] & b^* \circ F^*(y) \arrow[d, "\beta_y"] \\ a^* \circ G^*(y) \arrow[r, "(\eta' \bullet G)_y"] & b^* \circ G^*(y) \end{tikzcd} \tag{4.44}$$

If we restrict x to being an object c' in $\mathbf{C}' \subseteq \mathcal{S}X$ then

$$H_{c'} := \eta'_{c'} : a^*(c') \longrightarrow b^*(c') \tag{4.45}$$

or by interpolating the identity $id_{\mathcal{S}X}$ as before, we have that

$$H_{c'} := \eta'_{c'} : a'^* \circ \mathbf{y}_{\mathbf{C}'}(c') \longrightarrow b'^* \circ \mathbf{y}_{\mathbf{C}'}(c') \tag{4.46}$$

where $a'^* := a^* \circ \Psi$ and $b'^* := b^* \circ \Psi$, are maps of objects in $\mathcal{M}^Z_{\mathcal{E}}$ to objects in $\mathcal{N}^Z_{\mathcal{E}}$. Indeed, the source of $\eta'_{c'}$ in (4.46) is the object in $\mathcal{M}^Z_{\mathcal{E}}$ that corresponds to the sort $\tilde{X}_{c'}$ that in turn corresponds to the object $X_{c'}$ in $\mathbf{C}'$. The target of $\eta'_{c'}$ is the object in $\mathcal{N}^Z_{\mathcal{E}}$ that corresponds to the (same) sort $\tilde{X}_{c'}$. This is obvious when we recall that $\mathbf{y}_{\mathbf{C}'}(c')$ is the object in the universal model of $\mathbb{T}_{X'}$ in $\mathcal{S}(\mathbf{C}', J')$ that corresponds to $\tilde{X}_{c'}$ (see also corollary (4.22)).

In order to prove that H is actually a homomorphism between the two models $\mathcal{M}^Z_{\mathcal{E}}, \mathcal{N}^Z_{\mathcal{E}}$, it suffices to show that it respects the interpretations of the function symbols of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{S}Z$. There are two types of functions in $\mathbb{T}_{\mathcal{E}}$. We will prove that H respects the interpretations of function symbols [F1] by using the naturality of η' and that it respects the interpretations of function symbols [F2] by using the commutativity of the diagram (4.44).

It is straightforward to verify that the naturality of η' yields respect of the function symbols [F1]. For any map $f : c'_1 \longrightarrow c'_2$ in $\mathbf{C}'$, the following is commutative

$$\begin{tikzcd} a'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_1) \arrow[r, "\eta'_{c'_1}"] \arrow[d, "a'^* \circ \mathbf{y}_{\mathbf{C}'}(f)"'] & b'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_1) \arrow[d, "b'^* \circ \mathbf{y}_{\mathbf{C}'}(f)"] \\ a'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_2) \arrow[r, "\eta'_{c'_2}"] & b'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_2) \end{tikzcd} \tag{4.47}$$

By the discussion following equation (4.46), $a'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_i)$ $(i = 1, 2)$ is the interpretation of the sort $\tilde{X}_{c'_i}$ in $\mathcal{M}^Z_{\mathcal{E}}$, so we can denote it $[c'_i]_1$. Similarly, $b'^* \circ \mathbf{y}_{\mathbf{C}'}(c'_i)$ is the interpretation of the sort $\tilde{X}_{c'_i}$ in $\mathcal{N}^Z_{\mathcal{E}}$ and we denote it $[c'_i]_2$. The same goes for the functions: $a'^* \circ \mathbf{y}_{\mathbf{C}'}(f)$ and $b'^* \circ \mathbf{y}_{\mathbf{C}'}(f)$ are the interpretations of the function symbols (of type [F1]) in the two models and hence denoted $[f]_1$ and $[f]_2$ respectively. Hence, by denoting $H_i := \eta'_{c'_i}$, the above square becomes

$$\begin{tikzcd} {[c'_1]_1} \arrow[r, "H^{(a,b)}_{X_1}"] \arrow[d, "{[f]_1}"'] & {[c'_1]_2} \arrow[d, "{[f]_2}"] \\ {[c'_2]_1} \arrow[r, "H^{(a,b)}_{X_2}"] & {[c'_2]_2} \end{tikzcd}$$

which proves that the components $H_{c'_i}$ respect the interpretations of function symbols of type [F1].

Now we turn to the square 4.44. For any object d in $\mathbf{D}$, we consider the components of η' of types

- $\eta'_{F^*(\mathbf{y}_{\mathbf{D}}(d))} = \eta'(F^\sigma(d)) = (\eta' \bullet F^*)_{\mathbf{y}_{\mathbf{D}}(d)}$
- $\eta'_{G^*(\mathbf{y}_{\mathbf{D}}(d))} = \eta'(G^\sigma(d)) = (\eta' \bullet G^*)_{\mathbf{y}_{\mathbf{D}}(d)}$

The commutative square (4.44) then reads

$$\begin{tikzcd} a'^{*} \circ \mathbf{y}_{\mathbf{C}'}(F^{\sigma}(d)) \arrow[r, "\eta'_{F^{*}(\mathbf{y}_{\mathbf{D}}(d))}"] \arrow[d, "\alpha_{\mathbf{y}_{\mathbf{D}}(d)}"'] & b'^{*} \circ \mathbf{y}_{\mathbf{C}'}(F^{\sigma}(d)) \arrow[d, "\beta_{\mathbf{y}_{\mathbf{D}}(d)}"] \\ a'^{*} \circ \mathbf{y}_{\mathbf{C}'}(G^{\sigma}(d)) \arrow[r, "\eta'_{G^{*}(\mathbf{y}_{\mathbf{D}}(d))}"] & b'^{*} \circ \mathbf{y}_{\mathbf{C}'}(G^{\sigma}(d)) \end{tikzcd} \tag{4.48}$$

The horizontal maps in the above diagram are $H_{F^{\sigma}(d)}$ and $H_{G^{\sigma}(d)}$ by virtue of the established notation of (4.46). Adhering to the nomenclature of the discussion that follows diagram (4.47), we denote $[F^{\sigma}(d)]_1 := a'^{*} \circ \mathbf{y}_{\mathbf{C}'}(F^{\sigma}(d))$, $[F^{\sigma}(d)]_2 := b'^{*} \circ \mathbf{y}_{\mathbf{C}'}(F^{\sigma}(d))$, $[G^{\sigma}(d)]_1 := a'^{*} \circ \mathbf{y}_{\mathbf{C}'}(G^{\sigma}(d))$ and $[G^{\sigma}(d)]_2 := b'^{*} \circ \mathbf{y}_{\mathbf{C}'}(G^{\sigma}(d))$. By also using the notation of (4.42), the above diagram becomes

$$\begin{tikzcd} {[F^{\sigma}(d)]_1} \arrow[r, "H_{F^{\sigma}(d)}"] \arrow[d, "{[k_d]_1}"'] & {[F^{\sigma}(d)]_2} \arrow[d, "{[k_d]_2}"] \\ {[G^{\sigma}(d)]_1} \arrow[r, "H_{G^{\sigma}(d)}"] & {[G^{\sigma}(d)]_2} \end{tikzcd} \tag{4.49}$$

This proves that the maps $H_{c'}$ for c' object in $\mathbf{C}'$ also respect the interpretations of function symbols of type [F2] in $\mathbb{T}_{\mathcal{E}}$. Hence, they constitute a well-defined homomorphism between the two models $\Sigma(a, \alpha)$ and $\Sigma(b, \beta)$.

We stipulate that the arrow part of the functor Σ is given by $\Sigma(\eta) = H_X^{(a,b)}$ as defined in equation (4.46).

The two constructions Λ and Σ are explicitly inverse to each other, so $\Sigma \circ \Lambda(\mathcal{M}_{\mathcal{E}}^{Z}) \cong \mathcal{M}_{\mathcal{E}}^{Z}$ for any model $\mathcal{M}_{\mathcal{E}}^{Z}$ and $\Lambda \circ \Sigma(a, \alpha)$ is isomorphic to (a, α) in $\mathbf{I}_Z$. Therefore in this section we have effectively proved the following

Theorem 4.28 *The Grothendieck topos $\mathcal{E}$ is the inserter in $\mathfrak{Top}$ of the diagram of the parallel functors F, G.*

Proof. We have demonstrated that the category $\mathbf{I}_Z$ is equivalent to the category to $\underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}})$ and hence it is equivalent to the category $\mathfrak{Top}(Z, \mathcal{E})$ due to the natural equivalence 4.26. More specifically, the functors Λ' and Σ' defined as the composites

$$\mathfrak{Top}(Z, \mathcal{E}) \xrightarrow{\sigma_Z} \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \xrightarrow{\Lambda} \mathbf{I}(Z)$$

$$\mathbf{I}(Z) \xrightarrow{\Sigma} \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \xrightarrow{\tau_Z} \mathfrak{T}op(Z, \mathcal{E})$$

are an equivalence of categories.

Now let $l : Z' \longrightarrow Z$ be a geometric morphism in $\mathfrak{T}op$. It induces a functor $\mathbf{I}(l) : \mathbf{I}(Z) \longrightarrow \mathbf{I}(Z')$ by the assignments

$$(a, \alpha) \mapsto (a \circ l, \alpha \bullet l) \quad \text{and} \quad \eta \mapsto \eta \bullet l$$

The fact that $\mathbf{I}(l)$ acts by precomposition means that the following square is commutative

$$\begin{tikzcd}
\mathbf{I}(Z) \arrow[r, "\Sigma_Z"] \arrow[d, "\mathbf{I}(l)"'] & \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \arrow[d, "l^*"] \\
\mathbf{I}(Z') \arrow[r, "\Sigma_{Z'}"] & \underline{Mod}(\mathcal{S}Z', \mathbb{T}_{\mathcal{E}})
\end{tikzcd}$$

is clearly commutative and therefore the equivalence $\mathbf{I}(Z) \simeq \mathfrak{T}op(Z, \mathcal{E})$ is natural with respect to Z. So $\mathcal{E}$ is the inserter in $\mathfrak{T}op$ of the diagram of the parallel functors F, G in the sense of section 4.6. ■

After the preceding lengthy analysis, it is not difficult to observe the following.

Remark 4.29 *Let $A : Z \longrightarrow \mathcal{E}$ a geometric morphism. Then the object part of the functor Λ' essentially assigns*

$$A \mapsto (i_{\mathcal{E}} \circ A \ , \ \epsilon \bullet)$$

Let $A_1, A_2 : Z \longrightarrow \mathcal{E}$ two geometric morphisms and $\eta : A_1 \Rightarrow A_2$ a natural transformation. Then the arrow part of the functor Λ' essentially assigns

$$\eta \mapsto \eta \bullet A$$

$i_{\mathcal{E}}$ and ϵ are the universal 1-cell and universal 2-cell of the geometric inserter $\mathcal{E}$ (section 4.5).

The following 2-categorical universal property of $\mathcal{E}$ is then an immediate consequence of the natural equivalence $\mathbf{I}(Z) \simeq \mathfrak{T}op(Z, \mathcal{E})$.

Lemma 4.30 *The triple $(\mathcal{E}, i_{\mathcal{E}}, \epsilon)$ satisfies the following two universal properties.*

(i) Given a geometric morphism $a : Z \longrightarrow X$ and a natural transformation $\alpha : F \circ a \Rightarrow G \circ a$, there is a unique geometric morphism $A : Z \longrightarrow \mathcal{E}$ such that $a = A \circ i_{\mathcal{E}}$ and $\alpha = A \bullet \epsilon$.

(ii) Given two geometric morphisms $a_1, a_2 : Z \longrightarrow X$ and two natural transformations $\alpha_1 : F \circ a_1 \Rightarrow G \circ a_1$ and $\alpha_2 : F \circ a_2 \Rightarrow G \circ a_2$ together with a natural transformation $\eta : a_1 \Rightarrow a_2$ such that

$$(G \bullet \eta) \circ \alpha_1 = \alpha_2 \circ (F \bullet \eta)$$

there is a unique natural transformation $H : A_1 \Rightarrow A_2$ such that $i_{\mathcal{E}} \bullet H = \eta$.

4.8 A criterion for adjunctions in $\mathfrak{Top}$

Let us suppose that we have two pairs of parallel geometric morphisms $F_1, G_1 : X \longrightarrow Y_1$ and $F_2, G_2 : X \longrightarrow Y_2$ with $\mathcal{S}X = \mathcal{S}(\mathbf{C}, J)$, $\mathcal{S}Y = \mathcal{S}(\mathbf{D_1}, K_1)$ and $\mathcal{S}Y_2 = \mathcal{S}(\mathbf{D_2}, K_2)$, with K_1 and K_2 being subcanonical topologies on $\mathbf{D_1}$ and $\mathbf{D_2}$ as before. We construct a site $(\mathbf{C_1}, J_1)$ as in definition 4.18 taking the geometric morphisms F_1, G_1 into account $(\mathbf{C_1}, J_1)$ and an other site $(\mathbf{C_2}, J_2)$ taking F_2, G_2 into account. It holds

$$\mathcal{S}X \simeq \mathcal{S}(\mathbf{C_1}, J_1) \simeq \mathcal{S}(\mathbf{C_2}, J_2)$$

Let $(\mathcal{E}, i_{\mathcal{E}})$ be the inserter corresponding to the pair F_1, G_1 and $(\mathcal{F}, i_{\mathcal{F}})$ the inserter corresponding to the pair F_2, G_2. Let $\mathbf{I}_{\mathcal{E}}(Z)$ and $\mathbf{I}_{\mathcal{F}}(Z)$ be the respective categories of lemma 4.25 for the two pairs. We obtain a pair of functors $\Lambda_1 \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \rightleftharpoons \mathbf{I}_{\mathcal{E}}(Z) : \Sigma_1$ and $\Lambda_2 \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{F}}) \rightleftharpoons \mathbf{I}_{\mathcal{F}}(Z) : \Sigma_2$ as in section 4.7 that are equivalences of categories. We define the functors Λ_1' and Σ_1' as the composites

$$\mathfrak{Top}(Z, \mathcal{E}) \xrightarrow{\sigma} \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \xrightarrow{\Lambda_1} \mathbf{I}_{\mathcal{E}}(Z)$$

$$\mathbf{I}_{\mathcal{E}}(Z) \xrightarrow{\Sigma_1} \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{E}}) \xrightarrow{\tau} \mathfrak{Top}(Z, \mathcal{E})$$

where σ and τ are the functors of the equivalence 4.26 and correspondingly the functors ya_2' and Σ_2'.

We assume now that there are geometric morphisms $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ and $\phi^{-1} : \mathcal{F} \longrightarrow \mathcal{E}$ such that the following conditions are both satisfied

- The pair ϕ, ϕ^{-1} is an equivalence of categories.

- The following two triangular diagrams commute.

$$\begin{tikzcd} \mathcal{E} \arrow[rr, shift left, "\phi"] \arrow[rr, phantom, "\simeq"] \arrow[dr, "i_{\mathcal{E}}"'] & & \mathcal{F} \arrow[ll, shift left, "\phi^{-1}"] \arrow[dl, "i_{\mathcal{F}}"] \\ & X & \end{tikzcd} \tag{4.50}$$

  in the sense that there are 2-cells $i_{\mathcal{F}} \circ \phi \Rightarrow i_{\mathcal{E}}$ and $i_{\mathcal{E}} \circ \phi^{-1} \Rightarrow i_{\mathcal{F}}$ that are isomorphisms. In other words ϕ and ϕ^{-1} are maps between $i_{\mathcal{E}}$ and $i_{\mathcal{F}}$ in the slice topos $\mathfrak{T}op/X$.

Let

$$\mathfrak{T}op(Z, \phi) : \mathfrak{T}op(Z, \mathcal{E}) \longrightarrow \mathfrak{T}op(Z, \mathcal{F}) \quad \text{and} \quad \mathfrak{T}op(Z, \phi^{-1}) : \mathfrak{T}op(Z, \mathcal{F}) \longrightarrow \mathfrak{T}op(Z, \mathcal{F})$$

be the induced functors whose action is defined by postcomposition with ϕ and ϕ^{-1} respectively. This pair of functors is obviously also an equivalence of categories. Furthermore, $\mathfrak{T}op(Z, \phi)$ and $\mathfrak{T}op(Z, \phi^{-1})$ induce the functors $\hat{\phi} : I(Z) \longrightarrow I'(Z)$ and $\hat{\phi}^{-1}$ defined as the composites

$$\hat{\phi} : \qquad \mathbf{I}_{\mathcal{E}}(Z) \xrightarrow{\Sigma_1'} \mathfrak{T}op(Z, \mathcal{E}) \xrightarrow{\mathfrak{T}op(Z, \phi)} \mathfrak{T}op(Z, \mathcal{F}) \xrightarrow{\Lambda_2'} \mathbf{I}_{\mathcal{F}}(Z)$$

$$\hat{\phi}^{-1} : \qquad \mathbf{I}_{\mathcal{F}}(Z) \xrightarrow{\Sigma_2'} \mathfrak{T}op(Z, \mathcal{F}) \xrightarrow{\mathfrak{T}op(Z, \phi^{-1})} \mathfrak{T}op(Z, \mathcal{E}) \xrightarrow{\Lambda_1'} \mathbf{I}_{\mathcal{E}}(Z)$$

By construction, the pair $\hat{\phi}, \hat{\phi}^{-1}$ is an equivalence of categories, i.e., $\mathbf{I}_{\mathcal{E}}(Z) \simeq \mathbf{I}_{\mathcal{F}}(Z)$. We pause and study the action of the functor $\hat{\phi}$ on the objects and arrows of $\mathbf{I}_{\mathcal{E}}(Z)$.

An arbitrary object of $I(Z)$ is a pair (a, α), where $a : Z \longrightarrow X$ is a geometric morphism and α is a 2-cell $F \circ a \Rightarrow G \circ a$. The geometric morphism Λ^{-1} takes this pair to the unique $A : Z \longrightarrow \mathcal{E}$ such that $a = i_{\mathcal{E}} \circ A$ (c.f. remark 4.29 and lemma 4.30). Subsequently, A is mapped to the object $\phi \circ A$ in $\mathfrak{T}op(Z, \mathcal{F})$. Finally, the geometric morphism Σ takes $\phi \circ A$ to the pair consisting of the geometric morphism $i_{\mathcal{F}} \circ \phi \circ A$ and a 2-cell $\hat{\phi}(\alpha) : F' \circ i_{\mathcal{F}} \circ \phi \circ A \Rightarrow G' \circ i_{\mathcal{F}} \circ \phi \circ A$. Taking into account the commutativity of the diagram 4.50, this pair can be written

$$\hat{\phi}(a, \alpha) = (a, \hat{\phi}(\alpha))$$

In a similar way we can demonstrate that for any arrow η in $I(Z)$,

$$\hat{\phi}(\eta) = \eta$$

The action of the functor $\hat{\phi}^{-1}$ is the same as above, assuming that (a, α) and η are an object and an arrow of the category $\mathbf{I}_{\mathcal{F}}(Z)$.

It will be useful to formulate and prove the following two lemmas.

Lemma 4.31 *Let $x_1, x_2 : Z \longrightarrow X$ be two points of X (at stage Z) and $\eta : x_1 \longrightarrow x_2$ a point transformation, i.e., a 2-cell between the geometric morphisms x_1, x_2. Let also f, g be natural transformations $f : F \circ x_1 \Rightarrow G \circ x_1$ and $g : F \circ x_2 \Rightarrow G \circ x_2$, such that the diagram below commutes*

$$\begin{tikzcd} Fx_1 \arrow[r, "f"] \arrow[d, "F(\eta)"'] & Gx_1 \arrow[d, "G(\eta)"] \\ Fx_2 \arrow[r, "g"] & Gx_2 \end{tikzcd} \tag{4.51}$$

then the following diagram is also commutative

$$\begin{tikzcd} F'x_1 \arrow[r, "\hat{\phi}(f)"] \arrow[d, "F'(\eta)"'] & G'x_1 \arrow[d, "G'(\eta)"] \\ F'x_2 \arrow[r, "\hat{\phi}(g)"] & G'x_2 \end{tikzcd} \tag{4.52}$$

Proof. The assumptions of the lemma can be rephrased as "(x_1, f) and (x_2, g) are two objects of $I(Z)$ and $\eta : (x_1, f) \longrightarrow (x_2, g)$ is an arrow between them in $I(Z)$". Indeed, the commutative square (4.51) is just the property expressed by the commutative diagram (4.25) in lemma 4.25). Applying the functor $\hat{\phi}$ yields that

$$\hat{\phi}(\eta) : \hat{\phi}(x_1, f) \longrightarrow \hat{\phi}(x_2, g)$$

is an arrow in $I'(Z)$. But in the discussion above we showed that $\hat{\phi}(\eta) = \eta$, $\hat{\phi}(x_1, f) = (x_1, \hat{\phi}(f))$ and $\hat{\phi}(x_2, g) = (x_2, \hat{\phi}(g))$ (up to equivalence). The fact that η is an arrow between $(x_1, \hat{\phi}(f))$ and $(x_2, \hat{\phi}(g))$ by definition implies that the second diagram of the lemma commutes. ■

Moreover, the symmetric of the lemma 4.31 can also be proved in exactly the same way (using $\hat{\phi}^{-1}$ instead of $\hat{\phi}$).

Lemma 4.32 *With the x_1, x_2, η as in lemma 4.31, if f, g are natural transformations $f : F'x_1 \Rightarrow G'x_1$ and $g : F'x_2 \Rightarrow G'x_2$ that make the L.H.S. square below commutative* *then the R.H.S square is also commutative.*

$$
\begin{tikzcd}
F'x_1 \arrow[r, "f"] \arrow[d, "F'(\eta)"'] & G'x_1 \arrow[d, "G'(\eta)"] \\
F'x_2 \arrow[r, "g"] & G'x_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Fx_1 \arrow[r, "\hat{\phi}^{-1}(f)"] \arrow[d, "F(\eta)"'] & Gx_1 \arrow[d, "G(\eta)"] \\
Fx_2 \arrow[r, "\hat{\phi}^{-1}(g)"] & Gx_2
\end{tikzcd}
$$

The importance of lemmas 4.31 and 4.32 become evident after their application in a special case. Suppose now that F, G are two geometric morphism in opposite directions.

$$
\begin{tikzcd}
X \arrow[r, shift left, "F"] & Y \arrow[l, shift left, "G"]
\end{tikzcd}
\tag{4.53}
$$

We consider the following two diagrams in $\mathfrak{T}op$.

$$
\begin{tikzcd}
X \times Y \arrow[r, shift left, "F \circ p_1"] \arrow[r, shift right, "p_2"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \times Y \arrow[r, shift left, "p_1"] \arrow[r, shift right, "G \circ p_2"'] & X
\end{tikzcd}
\tag{4.54}
$$

where p_1 and p_2 are the first and second projections of the product topos $X \times Y$. We denote $\mathcal{E}$ and $\mathcal{F}$ the inserters of these two diagrams respectively. Lemmas 4.25 and 4.28 essentially tell us that the points (at stage say Z) of $\mathcal{E}$ are triples $(x, y; f)$, where x, y are points of X and Y respectively and f is a map $f : Fx \longrightarrow y$. Similarly, the points of $\mathcal{F}$ (at the same stage Z) are triples $(x, y; g)$, where now g is a map $g : x \longrightarrow Gy$. Therefore, by construction, the $X \times Y$-topos $i_{\mathcal{E}} : \mathcal{E} \longrightarrow X \times Y$ is the classifying topos of the geometric theory (say $\mathbb{T}_{(Fx,y)}$) whose models in any $\mathcal{S}Z$ are homomorphisms (of models of $\mathbb{T}_Y$) $Fx \longrightarrow y$ *given* two models x and y in $\mathcal{S}Z$ of the geometric theories $\mathbb{T}_X$ and $\mathbb{T}_Y$ respectively. Similarly the $X \times Y$-topos $i_{\mathcal{F}} : \mathcal{F} \longrightarrow X \times Y$ classifies the geometric theory (say $\mathbb{T}_{(x,Gy)}$) whose models in $\mathcal{S}Z$ are homomorphisms (of models of $\mathbb{T}_X$) $x \longrightarrow Gy$ *given* two models x and y in $\mathcal{S}Z$ of the geometric theories $\mathbb{T}_X$ and $\mathbb{T}_Y$ respectively.

The points of the topoi $i_{\mathcal{E}} : \mathcal{E} \longrightarrow X \times Y$ and $i_{\mathcal{F}} : \mathcal{E} \longrightarrow X \times Y$ can be contemplated as the "geometric analogue" of the Hom-sets $\mathbf{C_2}(\mathbf{F}(c_1), c_2)$ and $\mathbf{C_1}(c_1, G(c_2))$ with $\mathbf{F}, \mathbf{G}$ being two functors $\mathbf{F} : \mathbf{C_1} \longrightarrow \mathbf{C_2}$ and $\mathbf{G} : \mathbf{C_2} \longrightarrow \mathbf{C_1}$ and $\mathbf{C_1}, \mathbf{C_2}$ being small categories.

In category theory, the existence of an isomorphism between the two Hom-sets, which is natural in both c_1 and c_2, is usually adopted (see [Mac71]) as the definition of adjoint situation between $\mathbf{F}$ and $\mathbf{G}$ ($\mathbf{F} \dashv \mathbf{G}$). The next theorem and its corollary will demonstrate that the equivalence of the topoi $\mathcal{E}$ and $\mathcal{F}$ over $X \times Y$ implies adjoint situation of the geometric morphisms F and G of figure 4.53.

Before we start, we draw attention on the fact that, among the equivalent definitions of an adjoint situation (e.g. [Mac71],IV, theorem 2), the one involving the validity of the triangle identities is "robust enough" to be used in our 2-categorical sense.

Theorem 4.33 *Let $F : X \longrightarrow Y$ and $G : Y \longrightarrow X$ be geometric morphisms between two Grothendieck topoi. Let also $\mathcal{E}$ and $\mathcal{F}$ be the inserters of the diagrams 4.54. Suppose that there are geometric morphisms $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ and $\phi^{-1} : \mathcal{F} \longrightarrow \mathcal{E}$ such that (ϕ, ϕ^{-1}) is an equivalence of categories and the following diagrams commute up to isomorphism*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, shift left, "\phi"] \arrow[rr, shift right, "\phi^{-1}"'] \arrow[dr, "i_{\mathcal{E}}"'] & & \mathcal{F} \arrow[dl, "i_{\mathcal{F}}"] \\ & X \times Y & \end{tikzcd} \tag{4.55}$$

or in other words $\mathcal{E}$ and $\mathcal{F}$ are equivalent topoi over $X \times Y$. Then, for any point $x : Z \longrightarrow X$ and any point $y : Z \longrightarrow Y$ there are natural transformations (2-cells in $\mathfrak{T}op$)

$$\eta_x : x \longrightarrow G \circ F \circ x \quad \varepsilon_y : F \circ G \circ y \longrightarrow y$$

such that the composite natural transformations

$$G \circ y \xrightarrow{\eta_{Gy}} G \circ F \circ G \circ y \xrightarrow{G \bullet \varepsilon_y} G \circ y \tag{4.56}$$

$$F \circ x \xrightarrow{F \bullet \eta_x} F \circ G \circ F \circ x \xrightarrow{\varepsilon_{Fx}} F \circ x \tag{4.57}$$

are isomorphic to the identity 2-cells id_{Gy} and id_{Fx} respectively.

Proof.

We denote the inserter of the left diagram in (4.54) as $\mathbf{I}_{\mathcal{E}}(Z)$ and the inserter of the right diagram as $\mathbf{I}_{\mathcal{F}}(Z)$. Let Z be any Grothendieck topos and $x : Z$s.t.X and $y : Z \longrightarrow Y$ be two arbitrary geometric morphisms that are fixed throughout this proof. Let also a and b be two geometric morphisms $Z \longrightarrow X \times Y$ given by

$$a =< G \circ y, F \circ G \circ y > \quad b =< G \circ y, y > \tag{4.58}$$

meaning that, e.g., a is given as the geometric morphism that has the product universal property

$$
\begin{tikzcd}
Z \arrow[r, "y"] & Y \arrow[r, "G"] \arrow[dl, "G"'] \arrow[d, "a"] & X \arrow[d, "F"] \\
X & X \times Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y
\end{tikzcd}
\tag{4.59}
$$

We consider the identity 2-cell $f = id_{FGy} : F \circ G \circ y \Rightarrow F \circ G \circ y$. f is also trivially a 2-cell

$$f := id_{FGy} : (F \circ p_1) \circ a \Rightarrow p_2 \circ a \tag{4.60}$$

which implies that (a, f) is an object of $\mathbf{I}_{\mathcal{E}}(Z)$. If furthermore we look at the identity 2-cell $id_{Gy} : G \circ y \Rightarrow G \circ y$, we observe that it is also a 2-cell

$$id_{Gy} : p_1 \circ b \Rightarrow (G \circ p_2) \circ b \tag{4.61}$$

i.e., the pair (b, id_{Gy}) is an object of the inserter $\mathbf{I}_{\mathcal{F}}(Z)$ (the inserter of the R.H.S. diagram (4.54)). By the analysis that precedes lemma (4.31), $\hat{\phi}^{-1}(b, id_{Gy}) = (b, \hat{\phi}^{-1}(id_{Gy}))$ and this is an object in $\mathbf{I}_{\mathcal{E}}(Z)$. By definition this says that $\hat{\phi}^{-1}(id_{Gy})$ is a 2-cell with $\hat{\phi}^{-1}(id_{Gy}) : (F \circ p_1) \circ b \Rightarrow p_2 \circ b$ and hence $\hat{\phi}^{-1}(id_{Gy}) : F \circ G \circ y \Rightarrow y$. We denote

$$g := \hat{\phi}^{-1}(id_{Gy}) \tag{4.62}$$

Finally, the fact that id_{Gy} is a 2-cell between $G \circ y \Rightarrow G \circ y$ and $\hat{\phi}^{-1}(id_{Gy})$ is a 2-cell between $F \circ G \circ y \Rightarrow y$ renders

$$\eta := < id_{Gy}, \hat{\phi}^{-1}(id_{Gy}) > \tag{4.63}$$

a 2-cell between $a \Rightarrow b$. Using f, g, η, $x_1 = a$ and $x_2 = b$, the square (4.51) of the lemma (4.31) becomes

$$
\begin{tikzcd}
F \circ G \circ y \arrow[r, "id_{FGy}"] \arrow[d, "F \bullet id_{Gy}"'] & F \circ G \circ y \arrow[d, "\hat{\phi}^{-1}(id_{Gy})"] \\
F \circ G \circ y \arrow[r, "\hat{\phi}^{-1}(id_{Gy})"] & G \circ y
\end{tikzcd}
\tag{4.64}
$$

which trivially commutes because $F \bullet id_{Gy} = id_{FGy}$! Therefore, by lemma (4.31), the diagram below also commutes

$$\begin{tikzcd} G \circ y \arrow[r, "\phi(id_{FGy})"] \arrow[d, "id_{Gy}"'] & G \circ F \circ G \circ y \arrow[d, "G \bullet \hat{\phi}^{-1}(id_{Gy})"] \\ G \circ y \arrow[r, "id_{Gy}"] & G \circ y \end{tikzcd} \tag{4.65}$$

where for the bottom horizontal map, we used the fact that $\hat{\phi} \circ \hat{\phi}^{-1}(id_{Gy})$ is isomorphic to id_{Gy} dictated by the assumption that $(\hat{\phi}, \hat{\phi}^{-1})$ is an equivalence of categories. So more precisely, the diagram (4.65) commutes up to isomorphism.

For the second identity, we turn to lemma 4.32. In the same fashion we can choose $a =< x, F \circ x >$ and $b =< G \circ F \circ x, F \circ x >$. Also, $f = \hat{\phi}(id_{Fx})$, which can also be checked that it is a natural transformation $p_1 \circ a \Rightarrow (G \circ p_2) \circ a$, and $g = id_{GFx}$. Finally, we choose $\eta =< \hat{\phi}(id_{Fx}), id_F x >$. With these choices the L.H.S. diagram of lemma 4.32 becomes

$$\begin{tikzcd} x \arrow[r, "\hat{\phi}(id_{Fx})"] \arrow[d, "\hat{\phi}(id_{Fx})"'] & G \circ F \circ x \arrow[d, "G \bullet id_{Fx}"] \\ G \circ F \circ x \arrow[r, "id_{GFx}"] & G \circ F \circ \end{tikzcd} \tag{4.66}$$

which again trivially commutes. Hence, by lemma 4.32, the following square also commutes

$$\begin{tikzcd} F \circ x \arrow[r, "id_{Fx}"] \arrow[d, "F \bullet \hat{\phi}(id_{Fx})"'] & Fx \arrow[d, "id_{Fx}"] \\ F \circ G \circ F \circ x \arrow[r, "\hat{\phi}^{-1}(id_{GFx})"] & F \circ x \end{tikzcd} \tag{4.67}$$

■

Corollary 4.34 *Let the assumptions and notation of theorem 4.33 hold. Then G is the right adjoint of F.*

Proof. We consider $Z := X \times Y$ and $x = pr_1$, $y = pr_2$ the first and second projections in $\mathfrak{T}op$ respectively. Then the triangle identities 4.56 and 4.57 can be expressed by the commutativity (up to isomorphism) of the following two diagrams

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "id_G"'] & G \circ F \circ G \arrow[d, "G \bullet \varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F \bullet \eta"] \arrow[dr, "id_F"'] & F \circ G \circ F \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \tag{4.68}$$

■

4.9 Conclusion

Let $F : X \longrightarrow Y$ and $G : Y \longrightarrow X$ be two geometric morphisms between two Grothendieck topoi. Suppose also that we know that X classifies a geometric theory $\mathbb{T}_X$ and Y a geometric theory $\mathbb{T}_Y$. Corollary 4.34 asserts that $F \dashv G$ when the inserters of the diagrams 4.54 are equivalent over $X \times Y$. Such an equivalence can be established *geometrically.* The recipe for that can be outlined as follows. Consider arbitrary models $\mathcal{M}_X^Z$ and $\mathcal{N}_Y^Z$ of $\mathbb{T}_X$ and $\mathbb{T}_Y$ respectively inside the sheaves of an arbitrary topos Z. Then consider the category, say $\mathbf{C}^Z_{(Fx,y)}$, whose objects are $\mathbb{T}_Y$-homomorphisms between models $F^*(\mathcal{M}_X^X) \longrightarrow \mathcal{N}_Y^Z$ and the category $\mathbf{C}^Z_{(x,Gy)}$ whose objects are homomorphisms $\mathcal{M}_X^Z \longrightarrow G^*(\mathcal{N}_Y^Z)$. Prove that there is a bijection between the objects of the two categories.

The specification of such a bijection need not be functorial. Indeed, an object assignment, e.g. $\mathbf{C}^Z_{(Fx,y)} \longrightarrow \mathbf{C}^Z_{(x,Gy)}$, determines *a fortiori* a functor between the two categories by geometricity (c.f. lemma 1.4). We finally point out that "inside the sheaves of an arbitrary topos Z" can be (mentally) substituted with "in **Sets**" as long as we restrict ourselves to the domain of geometric mathematics.

In chapter 6 we shall apply corollary 4.34 when proving the adjoint situation of a pair of functors between topoi that classify sheaves over strong proximity lattices ($\mathcal{B}$-sheaves).

Chapter 5

Sheaves Over Strong Proximity Lattices

5.1 Introduction

S. Vickers introduced the notion of a B-sheaf, i.e. a sheaf over a strong proximity lattice B in [Vic98b] (draft). This section is a reworking of the ideas and results of this paper.

Section 5.2 opens the chapter with a result about sheaves of sets: a presheaf over a locale is a sheaf iff it has binary (and hence finite) and directed pasting. This splitting of pasting prepares the ground for the definition of $\mathcal{B}$-sheaves. Distributive lattices can accommodate only the notion of finite pasting. The extra strong order of a strong proximity lattice is enough structure to support the notion of *continuity of approximation* which emulates directed pasting. All the basic definitions of *approximating presheaves* and $\mathcal{B}$-sheaves are given in section 5.3 where also some properties are studied, most notably the notion of the *interior* of an approximating presheaf.

Sheaves over a stably compact locale X and $\mathcal{B}$-sheaves over a strong proximity lattice B such that $X = \mathbf{RSpec}(B)$ are equivalent and this is proved in section 5.4. The theory of $\mathcal{B}$-sheaves is geometric, and this equivalence connects a non geometric construction with a geometric one. Geometricity of $\mathcal{B}$-sheaves is discussed in section 5.5. In particular it is proved that for a stably compact locale X, the exponential $[set]^X$ in $\mathfrak{Top}$ exists and classifies the geometric theory of $\mathcal{B}$-sheaves over the strong proximity lattice $\mathcal{B}X$.

Finally, section 5.6 introduces what effectively is sheavification for approximating presheaves.

The reader is prompted to compare the results in this chapter with the ones in [JJ82]. There, Johnstone and Joyal prove with different methods that a locale X is exponentiable as a topos iff it is *metastably locally compact*. The class of metastably locally compact locales comprises the stably locally compact locales (not necessarily compact).

5.2 A note on sheaves of sets over locales

This section is about sheaves over locales. Their defining property of pasting is studied developing separately its finite part (finite pasting) from the infinitary (directed pasting).

Let (F, r) be a presheaf over a locale X whose restriction maps are $r_{a_1}^{a_2}$ for $a_1 \leq a_2$ in ΩX. We state the standard condition for F to be a sheaf over X.

Let $\{a_i | i \in I\}$ be a family of opens ΩX. We say that a tuple $\{x(a_i) \in F(a_i) | i \in I\}$ is coherent iff for any two elements $x(a_1) \in F(a_1)$ and $x(a_2) \in F(a_2)$ in the tuple $r_{a_1 \wedge a_2}^{a_i}(x(a_1)) = r_{a_1 \wedge a_2}^{a_2}(x(a_2))$.

Definition 5.1 *A presheaf (F, r) over ΩX has arbitrary pasting iff for any family of opens $\{a_i \in \Omega X | i \in I\}$ with $\bigvee a_i = a$ and any coherent tuple $\{x(a_i) \in F(a_i) | i \in I\}$, there is a unique element $z \in F(a)$, such that $r_{a_i}^a(z) = x(a_i)$ for any $i \in I$.*

Definition 5.2 *A presheaf $F : \Omega X^{op} \longrightarrow$ **Sets** is a sheaf iff it has arbitrary pasting.*

We used the rather superfluous term "arbitrary pasting" in definition 5.1 in order to introduce three restricted cases of pasting: directed pasting, binary pasting and finite pasting.

Definition 5.3 *A presheaf (F, r) over X has directed pasting iff for any directed family of opens $\{a_i \in \Omega X | i \in I\}$ with $\bigvee^{\uparrow} a_i = a$ and any tuple $\{x(a_i) \in F(a_i) | i \in I\}$, there is a unique element $z \in F(a)$, such that $r_{a_i}^a(z) = x(a_i)$ for any $i \in I$.*

Let $lim_{i \in I} F(a_i)$ be the limit of the diagram $F : L \longrightarrow$ **Sets**, where L is the full subcategory of $(\Omega X, \leq)$ that includes the opens of the family $\{a_i | i \in I\}$). In concrete terms, any element $x \in lim_{i \in I} F(a_i)$ is a tuple of elements $\{x(a_i) \in F(a_i) | i \in I\}$ with the property that if $a_i \leq a_j$ with $i, j \in I$, then $x(a_i) = r_{a_i}^{a_j}(x(a_j))$ stemming from the fact that $lim_{i \in I} F(a_i)$ together with its projections is a cone of the diagram $F : L \longrightarrow$ **Sets**. We denote by $\Gamma : F(a) \longrightarrow \prod_{i \in I} F(a_i)$ the map that takes elements $z \in F(a)$ to tuples $\{r_{a_i}^a(z) | i \in I\}$. Such tuples are actually elements of $lim_{i \in I} F(a_i)$ because if $a_i \leq a_j$ with $i, j \in I$ with $p_{a_i}(\Gamma(z)) = x(a_i)$ and $p_{a_j}(\Gamma(z)) = x(a_j)$ then

$$\begin{array}{rcl} r_{a_i}^{a_j}(x(a_j)) & = & r_{a_i}^{a_j} \circ p_{a_j} \circ \Gamma(z) = \\ r_{a_i}^{a_j} \circ r_{a_j}^a(z) & = & r_{a_i}^a(z) = \\ p_{a_i} \circ \Gamma(z) & = & x(a_i) \end{array}$$

In categorical terms, Γ is the unique map with the property that $p_{a_i} \circ \Gamma = r^a_{a_i}$ for any $i \in I$, where p_{a_i} is the limit projection $lim_{i\in I}F(a_i) \longrightarrow F(a_i)$.

Theorem 5.4 *Let $(F, r) : \Omega^{op} \longrightarrow$ **Sets** be a presheaf over the locale X. Then F has directed pasting iff for any directed family $\{a_i \in \Omega X | i \in I\}$ with $\bigvee^{\uparrow}_i a_i = a$ the map,*

$$\Gamma : F(\bigvee{}^{\uparrow} a_i) \longrightarrow lim_{i\in I}F(a_i)$$

is an isomorphism.

Proof.

First let F have directed pasting. We are going to show that the map Γ is an isomorphism. Let $x = \{x(a_i)|i \in I\} \in lim_{i\in I}F(a_i)$. We show that x is a coherent tuple of elements. By assumption, $\{a_i\}$ is directed, so for any two elements $x(a_i), x(a_j)$ in the tuple, there is an element a_k with $k \in I$ and $a_i \leq a_k$ and $a_j \leq a_k$. Furthermore, by the defining property of the limit, $x(a_i) = r^{a_k}_{a_i}(x(a_k))$ and $x(a_j) = r^{a_k}_{a_j}(x(a_k))$. Therefore, $r^{a_i}_{a_i\wedge a_j}(x(a_i)) = r^{a_i}_{a_i\wedge a_j} \circ r^{a_k}_{a_i}(x(a_k)) = r^{a_k}_{a_i\wedge a_j}(x(a_k)) = r^{a_j}_{a_i\wedge a_j} \circ r^{a_k}_{a_j}(x(a_k)) = r^{a_j}_{a_i\wedge a_j}(x(a_j))$. So, by assumption, there is a unique $z \in F(a)$ such that $r^a_{a_i}(z)$ for any $i \in I$ which shows that Γ is an isomorphism.

For the opposite direction, let it be the case that $\Gamma : F(a) \longrightarrow lim_{i\in I}F(a_i)$ is an isomorphism and let $\{x(a_i) \in F(a_i)|i \in I\}$ be a coherent tuple of elements. It is easy to argue that this tuple is an element of $lim_{i\in I}F(a_i)$. Indeed, if $a_i \leq a_j$ with $i, j \in I$ then coherence forces $r^{a_j}_{a_i\wedge a_j}(x(a_j)) = r^{a_i}_{a_i\wedge a_j}(x(a_i))$ or $r^{a_j}_{a_i}(x(a_j)) = x(a_i)$. Since Γ is 1-1 and epi, there is a unique $z \in F(a)$ such that $\Gamma(z) = \{x(a_i)|i \in I\}$. But $\Gamma(z) = \{r^a_{a_i}(z)|i \in I\}$ which proves that F has directed pasting. ■

Next, we introduce the notions of binary and finite pasting.

Definition 5.5 *A presheaf (F, r) over X has binary pasting iff it possesses one of the following obviously equivalent properties.*

(I) For any pair of elements $x \in F(a_1)$ and $y \in F(a_2)$ with $r^{a_1}_{a_1\wedge a_2}(x) = r^{a_2}_{a_1\wedge a_2}(y)$, there is a unique element $z \in F(a_1 \vee a_2)$ with $r^{a_1\vee a_2}_{a_1}(z) = x$ and $r^{a_1\vee a_2}_{a_2}(z) = y$.

(II) F preserves pullbacks of the form

$$\begin{tikzcd} a_1 \vee a_2 \arrow[r] \arrow[d] & a_1 \arrow[d] \\ a_2 \arrow[r] & a_1 \wedge a_2 \end{tikzcd} \tag{5.1}$$

Definition 5.6 *A presheaf F over X has finite pasting iff for any finite tuple of elements $\{x_i \in F(a_i) | i = 1, ..., N\}$ that are coherent, there is a unique element $z \in F(\vee_{i=1}^{N} a_i)$, with $r_{a_i}^{\vee_{i=1}^{N} a_i}(z) = x_i$, for any $i = 1, ..., N$.*

Naturally we anticipate that binary pasting implies finite pasting. This is indeed true although there is some intricacy in how induction is used to prove it.

Lemma 5.7 *Let (F, r) be a presheaf over a locale X. Then F has finite pasting iff it has binary pasting.*

Proof. Let $\{x_i \in F(a_i) | i = 1, ..., N\}$ be a finite tuple of coherent elements. We introduce a "block" index $\mathbf{s}$ with $\emptyset \neq \mathbf{s} \subseteq \{1, ..., N\}$ and we write

$$a_{\mathbf{s}} := \wedge_{i \in \mathbf{s}} a_i$$

It is obvious that there is a unique $z_{\mathbf{s}} \in F(a_{\mathbf{s}})$ such that $i \in \mathbf{s} \Rightarrow z_{\mathbf{s}} = r_{a_{\mathbf{s}}}^{a_i}(x_i)$. Moreover, let $\{\mathbf{s}_r\}$ be *any* total ordering of the subsets $\mathbf{s} \subseteq \{1, ..., N\}$, i.e. $r = 1, ..., n$ where $n = 2^N - 1$. We are going to prove by induction on n that there is a unique $z \in F(\vee_{r=1}^{n} a_{\mathbf{s}_r})$ such that

$$r_{a_{\mathbf{s}_r}}^{\vee_{r=1}^{n} a_{\mathbf{s}_r}}(z) = z_{\mathbf{s}_r}$$

This suffices to prove finite pasting because $\vee_{r=1}^{n} a_{\mathbf{s}_r} = \vee_{i=1}^{N} a_i$. We observe that the claim is trivially true for $n = 1$. Suppose now that if $r = 1, ..., n-1$ then there is a unique $z' \in F(\vee_{r=1}^{n-1} a_{\mathbf{s}_r})$ such that

$$r_{a_{\mathbf{s}_r}}^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}(z') = z_{\mathbf{s}_r}$$

We are going to prove that

$$r_{(\vee_{r=1}^{n-1} a_{\mathbf{s}_r}) \wedge a_{\mathbf{s}_n}}^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}(z') = r_{\vee_{r=1}^{n-1} a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}^{a_{\mathbf{s}_n}}(z_{\mathbf{s}_n}) \tag{5.2}$$

This fact together with the binary pasting property of F imply that there is a unique element $z \in F((\vee_{r=1}^{n-1} a_{\mathbf{s}_r}) \vee a_{\mathbf{s}_n}) = F(\vee_{r=1}^{n} a_{\mathbf{s}_r})$, such that

$$r_{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}^{\vee_{r=1}^{n} a_{\mathbf{s}_r}}(z) = z' \quad \text{and} \quad r_{a_{\mathbf{s}_n}}^{\vee_{r=1}^{n} a_{\mathbf{s}_r}}(z) = a_{\mathbf{s}_n}$$

which will complete the induction.

Furthermore, to prove equation 5.2, we argue that it suffices to prove that

$$r_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}(z') = r_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}^{a_{\mathbf{s}_n}}(z_{\mathbf{s}_n}) \tag{5.3}$$

for all $r = 1, ..., n-1$. Indeed, the elements $r^{\vee_{r=1}^{n-1}(a_{\mathbf{s}_r})}_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}(z')$ are obviously coherent, so, by assumption, there is a unique $z'' \in F(\vee_{r=1}^{n-1}(a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}))$ such that

$$r^{\vee_{r=1}^{n-1}(a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n})}_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}(z'') = r^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}(z') \tag{5.4}$$

Now, the elements

$$z_1'' := r^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}_{\vee_{r=1}^{n-1}(a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n})}(z') \quad \text{and} \quad z_2'' = r^{a_{\mathbf{s}_n}}_{\vee_{r=1}^{n-1}(a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n})}(z_{\mathbf{s}_n}) \tag{5.5}$$

are clearly such elements because of equation 5.3. Therefore, uniqueness implies that $z_1'' = z_2''$ which is equation 5.2.

Finally, we prove equation 5.3 for $r = 1, ..., n$. The L.H.S. is the restriction of z' to $a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}$ which is an open (say) $a_{\mathbf{s}_{r'}}$, where the index $\mathbf{s}_{r'}$ is $\mathbf{s}_r \cup \mathbf{s}_n$ and hence $r' \neq r$. So, according to the assumption of the induction we have

$$r^{\vee_{r=1}^{n-1} a_{\mathbf{s}_r}}_{a_{\mathbf{s}_r} \wedge a_{\mathbf{s}_n}}(z') = z_{\mathbf{s}_{r'}}$$

The R.H.S. of 5.3 is also $z_{\mathbf{s}_{r'}}$ by the definition of the elements $z_{\mathbf{s}_r}$ and the fact that $a_{\mathbf{s}_{r'}} \leq a_{\mathbf{s}_n}$. This completes the proof. ■

So combining lemma 5.7 and theorem 5.4 we arrive to the main result of this section.

Theorem 5.8 *A presheaf over a locale $F : X \longrightarrow$ **Sets** is a sheaf iff it has binary pasting and transforms directed meets to codirected limits.*

Proof. Let (F, r) be a presheaf over X that fulfills the theorem's assumptions. From lemma 5.7 and theorem 5.4 we know that F has finite and directed pasting. It is easy to shoe that this amounts to having arbitrary pasting. Let $\{a_i \in \Omega X\}_{i \in I}$ be a family of opens in ΩX and $\{x(a_i) \in F(a_i) | i \in I\}$ a tuple of elements. We derive the family of opens

$$\{b_j \in \Omega X,\ j \in J | b_j = \vee_{i \in M} a_i,\ M \subseteq I \text{ finite}\}$$

From finite pasting, for each $b_j = a_{i_1} \vee ... \vee a_{i_n}$, there is a unique element $x(b_j)$ that restricts to $x(a_{i_k})$ for any $k = 1, ..., n$. The tuple $\{x(b_j) \in F(b_j) | j \in J\}$ is coherent and the family $\{b_j | j \in J\}$ is obviously directed. Therefore there is a unique element

$$z \in \bigvee_j \{b_j | j \in J\} = \bigvee_i \{a_i | i \in I\}$$

that restricts to the elements $\{x(b_j) | j \in J\}$ and hence to the elements $\{x(a_i) | i \in I\}$. ■

The following is an obvious generalisation.

Corollary 5.9 *Theorem 5.8 holds for F being $\mathbf{C}$-valued presheaf over a locale X, where $\mathbf{C}$ is any category with codirected limits. In particular it holds topos-valued presheaves (for a topos where sheaves are definable).*

5.3 $\mathcal{B}$-Sheaves and Their First Properties

If B is a strong proximity lattice, we are going to denote by $(B, \leq)$ the underlying poset and by $(B, \prec, =)$ the category with objects the elements of B and with arrows the relation $a \prec b$ plus the identities $a = a$.

Definition 5.10 *Let B be a strong proximity lattice. A presheaf with approximation over B is a triple (V, ϕ, θ) such that*

(i) (V, ϕ) *is a presheaf over the underlying poset* $(B, \leq)$, *i.e. for any* $a \leq b$ *in* B, *there are functions* $\phi_a^b : V(b) \longrightarrow V(a)$ *such that for any* $a \in B$, $\phi_a^a = id_{V(a)}$ *and if* $a \leq b \leq c$, *then* $\phi_b^c \circ \phi_a^b = \phi_a^c$. *We are going to refer to* $\phi := V(\leq)$ *as the weak restriction maps of* V.

(ii) (V, θ, id) *is a presheaf over the category* $(B, \prec, =)$, *i.e. for any* $a \prec b$ *there are functions* $\theta_a^b : V(b) \longrightarrow V(a)$ *such that if* $a \prec b \prec c$ *then* $\theta_b^c \circ \theta_a^b = \theta_a^c$. *Also,* $V(a = a) = id_{V(a)}$. *We are going to refer to* ϕ *as the strong restriction maps of* V.

(iii) The strong restriction maps absorb the weak: If $a \leq b \prec c \leq d$, *then* $\phi_c^d \circ \theta_b^c \circ \phi_a^b = \theta_a^d$.

Definition 5.11 *Let (V, ϕ, θ) and (W, β, δ) be two presheaves with approximation over a strong proximity lattice B. Then a morphism of presheaves with approximation or an approximating presheaf morphism is a transformation $f : V \longrightarrow W$ which is both natural with respect to the weak and strong restrictions. That means that f is defined by component functions $f_a : V(a) \longrightarrow W(a)$ for any $a \in B$, such that for $a_1 \leq a_2$ or $b_1 \prec b_2$, the respective naturality diagrams are commutative*

$$\begin{tikzcd}
V(a_2) \arrow[r, "f_{a_2}"] \arrow[d, "\phi_{a_1}^{a_2}"'] & W(a_2) \arrow[d, "\beta_{a_1}^{a_2}"] & V(b_2) \arrow[r, "f_{b_2}"] \arrow[d, "\theta_{b_1}^{b_2}"'] & W(b_2) \arrow[d, "\theta_{b_1}^{b_2}"] \\
V(a_1) \arrow[r, "f_{a_1}"] & W(a_2) & V(b_1) \arrow[r, "f_{b_1}"] & W(b_1)
\end{tikzcd} \tag{5.6}$$

Sometimes we will refer to the left square as the "weak naturality square" and to the left as the "strong naturality square". We denote by $\mathbf{Pre\mathcal{B}Sh}(B)$ *the category of presheaves with approximation over B and approximating presheaf morphisms.*

For any $a \in B$, consider the full subcategory J_a^s of $(B, \prec, =)$ that includes all the elements $a_i \in B$ with $a \prec a_i$. J_a^s is filtered because $\uparrow a$ is an ideal. Let $colim_{a \prec a_i} V(a_i)$ be the colimit of the diagram $V : J_a^s \longrightarrow \mathbf{Sets}$, where $(V, \phi, \theta))$ is a presheaf with approximation over B. Note that the superscript ^{s} signifies that the diagram is with respect to the strong restriction maps. There is an obvious map $\theta_a : colim_{a \prec a_i} V(a_i) \longrightarrow V(a)$ (unique in that it makes $\theta_a \circ \theta^{a_i} = \theta_a^{a_i}$, where θ^{a_i} are the colimit injections).

Definition 5.12 *Let (V, ϕ, θ) be a presheaf with approximation over B. V is continuous or equivalently V has continuous approximation iff the map $\theta_a : colim_{a \prec a_i} V(a_i) \longrightarrow V(a)$ is an isomorphism. We denote by* **ContPre$\mathcal{B}$Sh**(B) *the category of continuous presheaves with approximation over B and approximating presheaf morphisms, i.e.* **ContPre$\mathcal{B}$Sh**(B) *is a full subcategory of* **Pre$\mathcal{B}$Sh**(B).

In other words, if V is continuous $V(a)$ is approximated by the sets $\{V(a_i) | a \prec a_i\}$ 1 "strongly over" $V(a)$. This is in parallel with the notion of lattice continuity where $a = \bigvee^{\uparrow} \{a_i | a \succ a_i\}$. This has the following impact.

Lemma 5.13 *Let B be a strong proximity lattice and $(V, \phi, \theta) : B \longrightarrow$* **Sets** *a presheaf with continuous approximation. Then the strong restrictions ϕ completely determine the weak restrictions δ.*

In particular, if $b_1, b_2 \in B$ with $b_1 \leq b_2$ and $b_1 \prec b_2$, then for any $x \in W(b_2)$, $\phi_{b_1}^{b_2}(x) = \theta_{b_1}^{b_2}(x)$.

Proof. Let $b_1 \leq b_2$ in B. Then $V(b_2) \cong colim_{b_2 \prec b_i} V(b_i)$. Then, for any $b_i \succ b_2$, we have $b_1 \leq b_2 \prec b_i \Rightarrow b_1 \prec b_i$, so the functions $\phi_{b_1}^{b_2} \circ \theta_{b_2}^{b_i} = \theta_{b_1}^{b_i}$ together with $V(b_1)$ constitute a cone of the diagram $V : J_{b_2}^s \longrightarrow \mathbf{Sets}$. Therefore, there is a unique function $\alpha : V(b_2) \longrightarrow V(b_1)$ such that $\alpha \circ \theta_{b_2}^{b_i} = \theta_{b_1}^{b_i}$ for any $b_i \succ b_2$ which implies $\phi_{b_1}^{b_2} = \alpha$.

The second part of the lemma is more or less obvious now, but we are going to prove it formally in order to establish the notation henceforth used! Let $b_1 \leq b_2$ and $b_1 \prec b_2$. Continuity dictates that $V(b_2) \cong colim_{b_2 \prec b_i} V(b_i)$. Let $\theta^{b_i} : V(b_i) \longrightarrow colim_{b_2 \prec b_i} V(b_i)$ be the colimit injections and θ_{b_2} the obvious isomorphism $colim_{b_2 \prec b_i} V(b_i) \longrightarrow V(b_2)$. To prove that for any $x \in V(b_2)$, $\phi_{b_1}^{b_2}(x) = \theta_{b_1}^{b_2}(x)$, it suffices to prove that for any $b_i \succ b_2$ and for any $x \in V(b_i)$, $\phi_{b_1}^{b_2} \circ \theta_{a_2} \circ \theta^{b_i}(x) = \theta_{b_1}^{b_2} \circ \theta_{a_2} \circ \theta^{b_i}(x)$, which is the same as $\phi_{b_1}^{b_2} \circ \theta_{b_2}^{b_i}(x) = \theta_{b_1}^{b_2} \circ \theta_{b_2}^{b_i}(x)$. But (definition **??**) this last equality is the same as $\theta_{b_1}^{b_i}(x) = \theta_{b_1}^{b_i}(x)$ which is trivially true. ■

The following definition pertains to the lattice structure of the strong proximity lattice.

Definition 5.14 *Let (V, ϕ, θ) be a presheaf with approximation over B. Then V has pasting or equivalently V is a pasting presheaf with approximation iff $V(\perp) \cong \mathbf{1}$ and for* *any $a, b \in B$, the following is a pullback in* **Sets**

$$\begin{tikzcd} V(a \vee b) \arrow[r, "\phi_a^{a\vee b}"] \arrow[d, "\phi_b^{a\vee b}"'] & V(a) \arrow[d, "\phi_{a\wedge b}^a"] \\ V(b) \arrow[r, "\phi_{a\wedge b}^b"] & V(a \wedge b) \end{tikzcd} \tag{5.7}$$

We denote by $\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$ *the category of pasting presheaves with approximation over B and approximating presheaf morphisms;* $\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$ *is also a full subcategory of* $\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$.

Finally we have the following.

Definition 5.15 *A B-sheaf is a pasting, continuous presheaf with approximation over a strong proximity lattice B. We denote by $\mathcal{B}\mathbf{Sh}(B)$ the category of B-sheaves and approximating presheaf morphisms. Note that the category $\mathcal{B}\mathbf{Sh}(B)$ is a full subcategory of $\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$.*

Note: If B is a strong proximity lattice, we shall be referring to a "B-sheaf" or equivalently to a "$\mathcal{B}$-sheaf over B" (in the latter case $\mathcal{B}$- is calligraphic).

Next, we investigate the interplay between the continuity and pasting property of approximating presheaves. Given a presheaf with approximation V, there is an obvious continuous presheaf with approximation V' constructed out of V. What is less trivial is that V' has pasting if V has pasting.

Definition 5.16 *Let (V, ϕ, θ) be a presheaf with approximation over B. The interior of V, $(intV, \beta, \delta)$, is the presheaf with approximation defined by*

$$intV(a) = colim_{a \prec a_i} V(a_i)$$

and for $a \leq b$ ($a \prec b$), β_a^b (δ_a^b) are the obvious unique maps stemming from the fact that J_b^s is a full subcategory of J_a^s. We are going to denote the colimit injections as $\theta^{a_i,a} : V(a_i) \hookrightarrow colim_{a \prec a_i} V(a_i)$.

Lemma 5.17 *Let V be a pasting presheaf over a strong proximity lattice B. Then its interior $intV$ also has pasting.*

Proof. We are going to prove the property of definition 5.14. First we easily observe that all the injections into $colim_{\perp \prec a_i}$ factor through the map $V(\perp) \longrightarrow colim_{\perp \prec a_i}$ because $\perp \prec \perp$ and $V(\perp) \cong \mathbf{1}$ by assumption. So $intV(\perp) \cong \mathbf{1}$[?][expression]

Let $x \in V(a_i)$ and $y \in V(b_i)$ with $a_i \succ a$ and $b_i \succ b$, such that

$$\begin{aligned} \beta^a_{a\wedge b} \circ \theta^{a_i,a}(x) &= \beta^a_{a\wedge b} \circ \theta^{b_i,b}(y) \Leftrightarrow \\ \theta^{a_i,a\wedge b}(x) &= \theta^{b_i,a\wedge b}(y) \end{aligned}$$

Since the diagram $J^s_{a\wedge b}$ is filtered, the above equality is realised "before" the colimit, i.e. there is an element c with $a \wedge b \prec c \prec a_i \wedge b_i$ such that

$$\theta^{a_i}_c(x) = \theta^{b_i}_c(y) \tag{5.8}$$

B is a strong proximity lattice, so there are $a', b' \in B$ with $a \prec a'$ and $b \prec b'$ such that $a' \wedge b' \prec c$. Also, let c_1, c_2 be two interpolants with $a \prec c_1 \prec a_i$ and $b \prec c_2 \prec b_i$. We denote $a_0 := a' \wedge c_1$ and $b_0 := b' \wedge c_2$. The elements a_0 and b_0 have the following properties by construction.

$$\begin{array}{ll} a \prec a_0 \prec a_i & b \prec b_0 \prec b_i \\ a_0 \wedge b_0 \prec c & a \vee b \prec a_0 \vee b_0 \end{array}$$

Since $a_0 \wedge b_0 \prec c$, equation 5.8 becomes $\theta^{a_i}_{a_0\wedge b_0}(x) = \theta^{b_i}_{a_0\wedge b_0}(x)$ which is equivalent to

$$\phi^{a_0}_{a_0\wedge b_0} \circ \theta^{a_i}_{a_0}(x) = \phi^{b_0}_{a_0\wedge b_0} \circ \theta^{b_i}_{b_0}$$

V has pasting by assumption, so there is a unique $z \in V(a_0 \vee b_0)$such that

$$\phi^{a_0\vee b_0}_{a_0}(z) = \theta^{a_i}_{a_0} \text{ and } \phi^{a_0\vee b_0}_{b_0}(z) = \theta^{b_i}_{b_0}(x) \tag{5.9}$$

We need to prove that $\theta^{a_0\vee a_0,a\vee b}(z)$ is the required unique element. For any interpolant $a \prec a_1 \prec a_0$ we have the following

$$\begin{aligned} \beta^{a\vee b}_a \circ \theta^{a_0\vee a_0,a\vee b}(z) &= \theta^{a_0\vee b_0,a}(z) \\ &= \theta^{a_1,a} \circ \theta^{a_0\vee b_0}_{a_1}(z) \\ &= \theta^{a_1,a} \circ \theta^{a_0}_{a_1} \circ \phi^{a_0\vee b_0}_{a_0}(z) \\ &= \theta^{a_1,a} \circ \theta^{a_0}_{a_1} \circ \theta^{a_i}_{a_0}(x)(\text{equation 5.9}) \\ &= \theta^{a_1,a} \circ \theta^{a_i}_{a_1}(x) \\ &= \theta^{a_i,a}(x) \end{aligned}$$

Similarly we can prove $\beta_b^{a\vee b} \circ \theta^{a_0 \vee a_0, a\vee b}(z) = \theta^{a_i,a}(x)$. This proves existence.

To prove uniqueness, suppose there are two elements $z_1 \in V(a_{i_1})$ and $z_2 \in V(a_{i_2})$ with $a \vee b \prec a_{i_i}$ and $a \vee b \prec a_{i_2}$ such that

$$\begin{aligned} \beta_a^{a\vee b} \circ \theta^{a_{i_1}, a\vee b}(z_1) &= \beta_a^{a\vee b} \circ \theta^{a_{i_2}, a\vee b}(z_2) \\ \text{and } \beta_b^{a\vee b} \circ \theta^{a_{i_1}, a\vee b}(z_1) &= \beta_b^{a\vee b} \circ \theta^{a_{i_2}, a\vee b}(z_2) \end{aligned}$$

and with the property that the above expressions become equal when further weakly restricted to $intV(a \wedge b)$. Let d be an interpolant $a \vee b \prec d \prec a_{i_1} \wedge a_{i_2}$ and let $z_1' := \theta_d^{a_{i_1} \wedge a_{i_2}} \circ \phi_{a_{i_1} \wedge a_{i_2}}^{a_{i_1}}(z_1)$ and $z_2' := \theta_d^{a_{i_1} \wedge a_{i_2}} \circ \phi_{a_{i_1} \wedge a_{i_2}}^{a_{i_2}}(z_2)$. To prove that $\theta^{a_{i_1}, a\vee b}(z_1) = \theta^{a_{i_2}, a\vee b}(z_2)$, it suffices to prove that $\theta^{d, a\vee b}(z_1') = \theta^{d, a\vee b}(z_2')$ given that

$$\begin{aligned} \beta_a^{a\vee b} \circ \theta^{d, a\vee b}(z_1') &= \beta_a^{a\vee b} \circ \theta^{d, a\vee b}(z_2') := x \\ \text{and } \beta_b^{a\vee b} \circ \theta^{d, a\vee b}(z_1') &= \beta_b^{a\vee b} \circ \theta^{d, a\vee b}(z_2') := y \end{aligned}$$

and

$$\beta_{a\wedge b}^a(x) = \beta_{a\wedge b}^b(y) \tag{5.10}$$

Since we are dealing with filtered colimits, equation 5.10 implies that $\theta_c^d(z_1') = \theta_c^d(z_2')$ for some element c with $a \wedge b \prec c$. Also, there are elements c_1, c_2 with $a \prec c_1 \prec d$ and $b \prec c_2 \prec d$ such that $\theta_{c_1}^d(z_1') = \theta_{c_1}^d(z_2')$ and $\theta_{c_2}^d(z_1') = \theta_{c_2}^d(z_2')$. We call $c_1' := c \wedge c_1$ and $c_2' := c \wedge c_2$. It holds that $c_1' \vee c_2' \prec d$, therefore the last pair of equations yield

$$\begin{aligned} \phi_{c_1'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_1') &= \theta_{c_1'}^d(z_1') && (5.11) \\ = \theta_{c_1'}^d(z_2') &= \phi_{c_1'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_2') && (5.12) \end{aligned}$$

and similarly

$$\phi_{c_2'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_1') = \phi_{c_2'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_2') \tag{5.13}$$

Furthermore, it holds that $c_1' \wedge c_2' \leq c$, which means that if we post-compose $\phi_{c_1'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_1')$ and $\phi_{c_2'}^{c_1' \vee c_2'} \circ \theta_{c_1' \vee c_2'}^d(z_1')$ with $\phi_{c_1' \wedge c_2'}^{c_1'}$ and $\phi_{c_1' \wedge c_2'}^{c_2'}$, they become equal. Hence, by uniqueness of pasting

$$\theta_{c_1' \vee c_2'}^d(z_1') = \theta_{c_1' \vee c_2'}^d(z_2')$$

which in turn implies $\theta^{c_1'\vee c_2',a\vee b}\circ\theta^{d}_{c_1'\vee c_2'}(z_1') = \theta^{c_1'\vee c_2',a\vee b}\circ\theta^{d}_{c_1'\vee c_2'}(z_2')$, or $\theta^{d,a\vee b}(z_1') = \theta^{d,a\vee b}(z_2')$ as desired. ■

The choice of the word "interior" in definition 5.16 is justified by the following theorem.

Theorem 5.18 *Let V be a presheaf with approximation over B. Then its interior is a continuous presheaf with approximation. Moreover, the interior construction defines a functor* **int** *which is the right adjoint of the inclusion functor*

$$\mathbf{i_3} : \mathbf{ContPre\mathcal{B}Sh}(B) \hookrightarrow \mathbf{Pre\mathcal{B}Sh}(B)$$

This exhibits $\mathbf{ContPre\mathcal{B}Sh}(B)$ *as a co-reflective subcategory of* $\mathbf{Pre\mathcal{B}Sh}(B)$.

Proof. To define the functor **int**, we set $\mathbf{int}(V) = intV$, for any presheaf with approximation, as in definition 5.16. Let (V, ϕ, θ) and (W, β, δ) be two presheaves with approximation over B and $q : V \longrightarrow W$ an approximating presheaf morphism. We are going to define an approximating presheaf morphism between $\mathbf{int}V \longrightarrow \mathbf{int}W$.

Let $\theta^{a_i,a} : V(a_i) \hookrightarrow colim_{a\prec a_i}V(a_i)$ and $\delta^{a_i,a} : W(a_i) \hookrightarrow colim_{a\prec a_i}W(a_i)$ be the respective colimit injections as before. For $a \prec a_{i_1} \prec a_{i_2}$ in B, all the loops are commutative in the following diagram.

$$\begin{tikzcd}
V(a_{i_1}) \arrow[d, "q_{a_{i_1}}"'] & & V(a_{i_2}) \arrow[ll, "\theta^{a_{i_2}}_{a_{i_1}}"'] \arrow[d, "q_{a_{i_2}}"] \\
W(a_{i_1}) \arrow[dr, "\delta^{a_{i_1},a}"'] & & W(a_{i_2}) \arrow[ll, "\delta^{a_{i_2}}_{a_{i_1}}"'] \arrow[dl, "\delta^{a_{i_2},a}"] \\
& colim_{a\prec a_i}W(a_i) &
\end{tikzcd} \tag{5.14}$$

Indeed, the upper square commutes because of lemma 5.19 and the lower triangle because it is the universal cone of the diagram $W : J^s_a \longrightarrow \mathbf{Sets}$. So the functions $\delta^{a_i,a}\circ q_{a_i}$ with the set $colim_{a\prec a_i}W(a_i)$ constitute a cone of the diagram $V : J^s_a \longrightarrow \mathbf{Sets}$ and therefore there is a unique function $\mathbf{int}(q)_a : colim_{a\prec a_i}V(a_i) \longrightarrow colim_{a\prec a_i}W(a_i)$ with the property

$$\mathbf{int}(q)_a \circ \theta^{a_i,a} = \delta^{a_i,a} \circ q_{a_i} \tag{5.15}$$

for any $a_i \succ a$. Now we prove that $\mathbf{int}(q)$ is an approximating presheaf morphism between V and W. We start with the strong naturality property. We denote by $\mathbf{int}(\phi)$ and $\mathbf{int}(\theta)$ the weak and strong restriction maps of $\mathbf{int}(V)$ and by $\mathbf{int}(\beta)$ and $\mathbf{int}(\delta)$ the weak and strong restriction maps of $\mathbf{int}(W)$. We consider the following diagram for $a_1 \prec a_2 \prec a_i$

$$\begin{tikzcd}
V(a_i) \arrow[r, "q_{a_i}"] \arrow[d, "\theta^{a_i,a_2}"'] & W(a_i) \arrow[d, "\delta^{a_i,a_2}"] \\
\mathbf{int}(V)(a_2) \arrow[r, "\mathbf{int}(q)_{a_2}"] \arrow[d, "\mathbf{int}(\theta)^{a_2}_{a_1}"'] & \mathbf{int}(W)(a_2) \arrow[d, "\mathbf{int}(\delta)^{a_2}_{a_1}"] \\
\mathbf{int}(V)(a_1) \arrow[r, "\mathbf{int}(q)_{a_1}"] & \mathbf{int}(W)(a_1)
\end{tikzcd} \tag{5.16}$$

The left composite vertical map is just θ^{a_i,a_1} and the right composite vertical map is δ^{a_i,a_1}. Therefore, the outer diagram commutes as the defining property of $\mathbf{int}(q)$ (expression 5.15 above). For the same reason the upper square diagram also commutes. Therefore, the lower diagram, which is the strong naturality square of $\mathbf{int}(q)$, pre-composed with (any) colimit injection $\theta^{a_i,a}$ commutes which implies that it commutes.

In order to prove the weak naturality of $\mathbf{int}(q)$, we consider the corresponding diagram for $a_1 \leq a_2 \prec a_i$ with $\mathbf{int}(\phi)^{a_2}_{a_1}$ instead of $\mathbf{int}(\theta)^{a_2}_{a_1}$ and $\mathbf{int}(\beta)^{a_2}_{a_1}$ instead of $\mathbf{int}(\delta)^{a_2}_{a_1}$. The argument is the same as in the strong case because $\phi^{a_2}_{a_1} \circ \theta^{a_i,a_2} = \theta^{a_i,a_1}$ and $\beta^{a_2}_{a_1} \circ \delta^{a_i,a_2} = \delta^{a_i,a_1}$.

Finally, we prove that **int** is the right adjoint of the inclusion functor $\mathbf{i_3} : \mathbf{ContPre\mathcal{B}Sh}(B) \hookrightarrow \mathbf{Pre\mathcal{B}Sh}(B)$. Let (V, ϕ, θ) be a presheaf with approximation. It is easy to observe that the maps θ_a of definition 5.12 are the components of an approximating presheaf morphism $\mathbf{int}(V) \longrightarrow V$. Indeed, the strong naturality is demonstrated as follows for any pair $a_1 \prec a_2$ and any colimit injection θ^{a_i,a_2}:

$$\begin{aligned}
\theta_{a_1} \circ \mathbf{int}(\theta)^{a_2}_{a_1} \circ \theta^{a_i,a_2} &= \theta_{a_1} \circ \theta^{a_i,a_1} = \\
\theta^{a_i}_{a_1} &= \theta^{a_2}_{a_1} \circ \theta^{a_i}_{a_2} = \\
&= \theta^{a_2}_{a_1} \circ \theta_{a_2} \circ \theta^{a_i,a_2}
\end{aligned}$$

The strong naturality is demonstrated similarly. We denote by $\dot{\theta}$ the morphism whose components are the functions θ_a. Let (W, β, δ) be an other presheaf with approximation and $f : W \longrightarrow V$ an approximating morphism of presheaves. Then the following square commutes

$$\begin{tikzcd}
\mathbf{int}(W) \arrow[r, "\mathbf{int}(f)"] \arrow[d, "\dot{\delta}"'] & \mathbf{int}(V) \arrow[d, "\dot{\theta}"] \\
W \arrow[r, "f"] & V
\end{tikzcd} \tag{5.17}$$

To demonstrate this fact, we consider any $a \in B$ and any colimit injection $\delta^{a_i,a} : W(a_i) \hookrightarrow \mathbf{int}(W)(a)$. Then

$$\begin{aligned} f_a \circ \delta_a \circ \delta^{a_i,a} &= f_a \circ \delta_a^{a_i} \\ &= \theta_a^{a_i} \circ f_{a_i} \quad \text{(strong naturality of } f) \\ &= \theta_a \circ \theta^{a_i,a} \circ f_{a_i} \\ &= \theta_a \circ \mathbf{int}(f)_a \circ \delta^{a_i,a} \quad \text{(by the defining property of } \mathbf{int}(f)) \end{aligned}$$

Now, assume in addition that W is continuous, i.e. it has the property that $\dot{\delta} : \mathbf{int}(W) \longrightarrow W$ is an isomorphism (definition 5.12). We will prove that there is a unique morphism $f' : W \longrightarrow \mathbf{int}(V)$ such that $f = \dot{\theta} \circ f'$. Namely, let $f' := \dot{\delta}^{-1} \circ \mathbf{int}(f)$. The commutativity of the square 5.17 reads

$$\begin{aligned} \dot{\theta} \circ \mathbf{int}(f) = f \circ \dot{\delta} \quad &\Leftrightarrow \quad \dot{\theta} \circ \mathbf{int}(f) \circ \dot{\delta}^{-1} = f \\ &\Leftrightarrow \quad \dot{\theta} \circ f' = f \end{aligned}$$

We finally prove that f' is the unique such map. Let f'' be an approximating presheaf morphism such that for any $a \in B$, $\theta_a \circ f''_a = f_a$. We will show that for any $x \in W(a)$, $f''_a(x) = f'_(x)$. Since W is continuous, there is $x' \in W(a_i)$ for some $a_i \succ a$ such that $\delta_a^{a_i}(x') = x$. Let b be any interpolant $a \prec b \prec a_i$. We observe that since $\mathbf{int}(V)(b) = colim_{b \prec b_i} V(b_i)$, there is $y \in V(c)$ for some $c \succ b$ such that

$$f''_b \circ \delta_b^{a_i} = \theta^{c,b}(y) \tag{5.18}$$

We have the following sequence of equalities:

$$\begin{aligned} f''_a \circ \delta_a^{a_i}(x') &= f''_a \circ \delta_a^b \circ \delta_b^{a_i}(x') \\ &= \mathbf{int}(\theta)_a^b \circ f''_b \circ \delta_b^{a_i}(x') \quad \text{(by naturality of } f'') \\ &= \mathbf{int}(\theta)_a^b \circ \theta^{c,b}(y) \quad \text{(expression 5.18)} \\ &= \theta^{b,a} \circ \theta_b \circ \theta^{c,b}(y) \quad \text{(by definition of } \mathbf{int}(\theta)) \\ &= \theta^{b,a} \circ \theta_b \circ f''_b \circ \delta_b^{a_i}(x') \quad \text{(expression 5.18)} \\ &= \theta^{b,a} \theta f_b \circ \delta_b^{a_i}(x') \quad \text{(by assumption)} \\ &= \theta^{b,a} \circ \theta_b^{a_i} \circ f_{a_i}(x') \quad \text{(by naturality of } f) \\ &= \mathbf{int}(\theta)_a^{a_i} \circ f'_{a_i}(x') \\ &= f'_a \circ \delta_a^{a_i}(x') \quad \text{(by naturality of } f') \end{aligned}$$

This proves uniqueness of f'. ■

5.4 $\mathcal{B}$-sheaves are equivalent to sheaves

We recall from chapter 3 that if B is a strong proximity lattice, then there is a locale $X = \mathbf{RSpec}(B)$ generated by B whose frame is the rounded ideal completion of B. In the rest of the section we are going to prove that the category of B-sheaves and approximating presheaf morphisms is equivalent to the category $Sh(\mathbf{RSpec}(B))$ of sheaves over the locale $\mathbf{RSpec}(B)$ and sheaf morphisms. Before the main theorem, we are going to need some technical results.

Suppose that (V, ϕ, θ) is a presheaf with approximation over B. We consider the full subcategory L_a^s of $(B, \prec, =)$ that includes all the elements $a_i \in B$ with $a \succ a_i$. We denote $lim_{a \succ a_i} V(a_i)$ the limit of the diagram $V : L_a^s \longrightarrow \mathbf{Sets}$ and with $p_{a_i}^a$ the limit projections $p_{a_i}^a : lim_{a \succ a_i} V(a_i) \longrightarrow V(a)$. Furthermore, let $a_2 \geq a_1$ in B. For any $a_i \prec a_1$, $a_i \prec a_2$ and this implies that $lim_{a_2 \succ a_i} V(a_i)$ together with its projections $p_{a_i}^{a_2} : lim_{a_2 \succ a_i} V(a_i) \longrightarrow V(a_i)$, for any $a_i \prec a_1$, is a cone of the diagram $L_{a_1}^s$. Hence there is a unique function $\beta_{a_1}^{a_2} : lim_{a_2 \succ a_i} V(a_i) \longrightarrow lim_{a_1 \succ a_i} V(a_i)$ that satisfies $p_{a_i}^{a_1} \circ \beta_{a_1}^{a_2} = p_{a_i}^{a_2}$ for any $a_i \prec a_1$. Repeating the argument for $a_2 \succ a_1$ we define a function $\delta_{a_1}^{a_2}$.

Finally, for $a \succ a_i$ in B, the object $V(a)$ together with the restriction maps $\theta_{a_i}^a$ constitute a cone of the diagram L_a^s and therefore there is a unique function $q_a : V(a) \longrightarrow lim_{a \succ a_i} V(a_i)$ with the property $p_{a_i}^a \circ q_a = \theta_{a_i}^a$ for any $a_i \prec a$.

Lemma 5.19 *Let (V, ϕ, θ) be a presheaf with approximation over the strong proximity lattice B. Then the assignment $a \mapsto lim_{a \succ a_i} V(a_i) := W(a)$ for any $a \in B$, together with the maps $\beta_{a_1}^{a_2}$ and $\delta_{a_1}^{a_2}$ for $a_2 \geq a_1$ and $a_2 \succ a_1$ in B respectively define a presheaf with approximation over B.*

The maps q_a are the components of a natural transformation $q : (V, \theta, id) \longrightarrow (W, \delta, id)$. Furthermore, for any $a_1 \prec a_2$, $q_1 \circ p_{a_1}^{a_2} = \delta_{a_1}^{a_2}$.

Proof. It is straightforward to check that the maps β, δ satisfy the properties of the weak and strong restriction maps of definition 5.10. The second part of the lemma is slightly less trivial. Let $a_1 \prec a_2$ in B. Then for any $a_i \prec a_1$, we consider the following diagram in

Sets.

$$
\begin{tikzcd}
V(a_1) & & V(a_2) \arrow[ll, "\theta^{a_2}_{a_1}"'] \\
lim_{a_1 \succ a_i} V(a_i) \arrow[from=1-1, d, "q_{a_1}"'] \arrow[dr, "p^{a_1}_{a_i}"'] & & lim_{a_2 \succ a_i} V(a_i) \arrow[from=1-3, "q_{a_2}"] \arrow[ll, "\delta^{a_2}_{a_1}"'] \arrow[dl, "p^{a_2}_{a_i}"] \\
& V(a_i) &
\end{tikzcd}
\tag{5.19}
$$

By the definition of q, we have that $p^{a_1}_{a_i} \circ q_{a_1} = \theta^{a_1}_{a_i}$ and $p^{a_2}_{a_i} \circ q_{a_2} = \theta^{a_2}_{a_i}$ which entails that the total outlying diagram commutes. Also, by the definition of $\delta^{a_2}_{a_1}$, the triangular diagram commutes. Therefore, the upper square diagram commutes by virtue of the fact that the projections $p^{a_1}_{a_i}$ are epi. This is the strong naturality square of q.

To prove the last claim of the lemma, for any projection $P^{a_1}_{a_i}$ $(a_i \prec a_1)$, we have $p^{a_1}_{a_i} \circ q_{a_1} \circ p^{a_2}_{a_1} = \theta^{a_1}_{a_i} \circ p^{a_2}_{a_1}$ by the defining property of q. The last composite is equal to $p^{a_2}_{a_i} = p^{a_1}_{a_i} \circ \delta^{a_2}_{a_1}$ which proves the claim. ■

Corollary 5.20 *If in addition V is continuous then q is an approximating presheaf morphism.*

Proof. Let now $a_1 \leq a_2$. Then for any a_3 with $a_2 \prec a_3$, the map $\theta^{a_3}_{a_2} : V(a_3) \longrightarrow V(a_2)$ is the colimit injection $\theta^{a_3, a_2} : V(a_3) \hookrightarrow colim_{a_2 \prec a_i} V(a_i)$. The outer diagram below commutes because it is the strong naturality square of q (lemma 5.19)

$$
\begin{tikzcd}
V(a_1) \arrow[d, "q_{a_1}"'] & V(a_2) \arrow[l, "\phi^{a_2}_{a_1}"'] \arrow[d, "q_{a_2}"] & V(a_3) \arrow[l, "\theta^{a_3}_{a_2}"'] \arrow[d, "a_3"] \\
lim_{a_1 \succ a_i} V(a_i) & lim_{a_2 \succ a_i} V(a_i) \arrow[l, "\beta^{a_2}_{a_1}"'] & lim_{a_3 \prec a_i} V(a_i) \arrow[l, "\delta^{a_3}_{a_2}"']
\end{tikzcd}
\tag{5.20}
$$

For the same reason the right diagram also commutes. This means that $q_{a_1} \circ \phi^{a_2}_{a_1} \circ \theta^{a_3}_{a_2} = \beta^{a_2}_{a_1} \circ q_{a_2} \circ \theta^{a_3}_{a_2}$ for any injection $\theta^{a_3}_{a_2}$. Therefore, the left square diagram commutes. ■

Finally, we are going to rely on the following technical fact.

Lemma 5.21 *Let (V, ϕ, θ) be a continuous presheaf with approximation over B and q the approximating morphism of presheaves as in lemma 5.19 and corollary 5.20. Then* $\mathbf{int}(q)$ *is an isomorphism of B-sheaves.*

Proof. We have to prove that

$$colim_{a \prec a_i}\ lim_{a_i \succ a_j} V(a_j) \cong V(a)$$

In what follows, we denote $\gamma_a := \mathbf{int}(int)_a$.

$$\begin{tikzcd}
V(a_i) \arrow[r, "q_{a_i}"] \arrow[d, "\theta^{a_i}"'] & W(a_i) \arrow[d, "\delta^{a_i}"] \\
colim_{a \prec a_i} V(a_i) \arrow[r, "\gamma_a"] \arrow[dr, "\theta_a"'] & colim_{a \prec a_i} W(a_i) \arrow[d, dashed, "\alpha"] \\
& V(a)
\end{tikzcd} \tag{5.21}$$

For any $a_i \succ a$, the projections $p_a^{a_i} : W(a_i) := lim_{a_i \succ a_j} V(a_j) \longrightarrow V(a)$ together with the vertex $V(a)$ constitute a cone of the diagram $W : J_a^s \longrightarrow$ **Sets** because of the defining property of the restrictions δ. That means that there is a unique map $\alpha_a : colim_{a \prec a_i} W(a_i) \longrightarrow V(a)$ with the property

$$\alpha_a \circ \delta^{a_i} = p_a^{a_i} \tag{5.22}$$

where δ^{a_i} is the colimit injection $\delta^{a_i} : W(a_i) \longrightarrow colim_{a \prec a_i} W(a_i)$. Let θ_a be the obvious map as in the discussion preceding definition 5.12. By assumption V is continuous and hence θ_a is an isomorphism. We will prove that α_a has an inverse, namely the map

$$\alpha_a^{-1} := \gamma_a \circ \theta_a^{-1}$$

First, we recall that by the definition of q, for any $a_i \succ a$, $p_a^{a_i} \circ q_{a_i} = \theta_a^{a_i}$ (see right above lemma 5.19). Furthermore,

$$\begin{aligned}
p_a^{a_i} \circ q_{a_i} &= \theta_a^{a_i} \\
\Rightarrow \alpha_a \circ \delta^{a_i} \circ q_{a_i} &= \theta_a^{a_i} \quad \text{(by definition of } \alpha_a) \\
\Rightarrow \alpha_a \circ \gamma_a \circ \theta^{a_i} &= \theta_a^{a_i} \quad \text{(expression 5.15 in theorem 5.18)} \\
\Rightarrow \alpha_a \circ \gamma_a \circ \theta^{a_i} &= \theta_a \circ \theta^{a_i} \quad \text{(by definition of } \theta_a)
\end{aligned}$$

Since θ^{a_i} are the colimit injections, the last equality implies $\alpha_a \circ \gamma_a = \theta_a$ or that $\alpha_a \circ (\gamma_a \circ \theta_a^{-1}) = id_{V(a)}$. For the other direction, for any colimit injection $\delta^{a_i,a}$ and any interpolant a' with $a \prec a' \prec a_i$, we have

$$\gamma_a \circ \theta_a^{-1} \circ \alpha_a \circ \delta^{a_i,a} = \gamma_a \circ \theta_a^{-1} \circ p_a^{a_i} \quad \text{(by definition of } \alpha)$$

$$
\begin{aligned}
&= \gamma_a \circ \theta_a^{-1} \circ \theta_a^{a'} \circ p_a^{a_i} \\
&= \gamma_a \circ \theta^{a',a} \circ p_a^{a_i} \quad (\text{because } \theta_a^{a'} = \theta_a \circ \theta^{a',a}) \\
&= \delta^{a',a} \circ q_{a'} \circ p_a^{a_i} \quad (\text{by definition of } \gamma) \\
&= \delta^{a',a} \circ \delta_{a'}^{a_i} \quad (\text{lemma 5.19}) \\
&= \delta_a^{a_i}
\end{aligned}
$$

This proves that $(\gamma_a \circ \theta_a^{-1}) \circ \alpha_a = id_{colim_{a \prec a_i} lim_{a_i \succ a_j} V(a_j)}$. ■

Theorem 5.22 *Let B be a strong proximity lattice. Then the category $\mathcal{B}\mathbf{Sh}(B)$ is equivalent with the category $Sh(\mathbf{RSpec}(B))$ of ordinary sheaves over $\mathbf{RSpec}(B)$.*

Proof. First we are going to define a functor $\Psi : Sh(\mathbf{RSpec}(B)) \longrightarrow \mathcal{B}\mathbf{Sh}(B)$. Let $F : \mathbf{RSpec}(B) \longrightarrow \mathbf{Sets}$ be a sheaf. We denote by $r_{a_1}^{a_2} : F(a_2) \longrightarrow F(a_1)$ its restriction maps for $a_1 \leq a_2$.

As an intermediate step, let (V, ϕ, θ) be the triple where $V(a) := F(\downarrow a)$ for any $a \in B$, $\phi_{a_1}^{a_2} := r_{\downarrow a_1}^{\downarrow a_2}$ for $a_1 \leq a_2$ and $\theta_{a_1}^{a_2} := r_{\downarrow a_1}^{\downarrow a_2}$ for $a_1 \prec a_2$. (V, ϕ, θ) is obviously a presheaf with approximation over B. It is easy to demonstrate that V has pasting. Let $x \in V(a)$ and $x \in V(b)$ such that $\phi_{a\wedge}^{a}(x) = \phi_{a\wedge b}^{b}(y)$. This condition translates as $r_{\downarrow a \wedge \downarrow b}^{\downarrow a}(x) = r_{\downarrow a \wedge \downarrow b}^{\downarrow b}(y)$ or $r_{\downarrow (a\wedge b)}^{\downarrow a}(x) = r_{\downarrow (a\wedge b)}^{\downarrow b}(y)$. Because F has binary pasting, there is a unique $z \in F(\downarrow a \vee \downarrow b)$ such that $r_{\downarrow a}^{\downarrow a \vee \downarrow b}(z) = x$ and $r_{\downarrow b}^{\downarrow a \vee \downarrow b}(z) = y$. But lemma **??** says that $\downarrow a \vee \downarrow b = \downarrow (a \vee b)$ and therefore, the last statement translates that there is a unique $z \in V(a \vee b)$ such that $\phi_a^{a\vee b}(z) = x$ and $\phi_b^{a\vee b}(z) = y$.

Now we are ready to define the object part of the functor Ψ. We stipulate that

$$\Psi(F) = \mathbf{int}V = \mathbf{int}F(\downarrow \bullet) \tag{5.23}$$

where **int** is as in definition 5.16 and theorem 5.18. We denote the weak and strong restriction maps of $\Psi(F)$ as β and δ respectively. Lemma 5.17 guarantees that $(\Psi(F), \beta, \delta)$ has pasting and theorem 5.18 that it is continuous. For the arrow part of Ψ, let $f : F_1 \longrightarrow F_2$ be a natural transformation. The components $f_{\downarrow a} : F_1(\downarrow a) \longrightarrow F_2(\downarrow a)$ define a transformation $f' : V_1' \longrightarrow V_2'$, where $V_1'(a) = F(\downarrow a)$ and $V_2'(a) = F(\downarrow a)$, which is obviously weakly and strongly natural. We define $\Psi(F)(f) := \mathbf{int}(f')$.

For the opposite direction, we are going to construct a functor $\Phi : \mathcal{B}\mathbf{Sh}(B) \longrightarrow Sh(\mathbf{RSpec}(B))$. Let (U, ρ, σ) be a B-sheaf. For any rounded ideal $I \in \Omega(\mathbf{RSpec}(B))$, we define

$$\Phi(U)(I) = lim_{a \in I} U(a)$$

The limit is on the diagram $V : L^s_I \longrightarrow \mathbf{Sets}$ where L^s_I is the full subcategory of $(B, \prec, =)$ that includes all the elements of the ideal I. For $I_1 \subseteq I_2$, there is an obvious map $t^{I_2}_{I_1} : lim_{a \in I_2} U(a) \longrightarrow lim_{a \in I_1} U(a)$, unique in that it makes $p^{I_1}_a \circ t^{I_2}_{I_1} = p^{I_2}_a$ for any $a \in I_1$, where $p^{I_1}_a$ and $p^{I_2}_a$ are the respective projections of the two limits. We are going to prove that $(\Psi(U), t)$ is a sheaf over $\mathbf{RSpec}(B)$.

The arrow part of Φ is defined as follows. Let $g : U \longrightarrow U'$ be an approximating presheaf morphism between two B-sheaves (U, ρ, σ) and (U', ρ', σ') and let $p^I_{a_i} : lim_{a_i \in I} U(a_i) \longrightarrow U(a_i)$ and $p'^I_{a_i} : lim_{a_i \in I} U'(a_i) \longrightarrow U'(a_i)$ be the respective projections. Its strong naturality of g property reads that for any $a_1 \prec a_2$, $g_{a_1} \circ \sigma^{a_2}_{a_1} = \sigma'^{a_2}_{a_1} \circ g_{a_2}$. The composite maps $g_{a_i} \circ p^I_{a_i} : lim_{a_i \in I} U(a_i) \longrightarrow U'(a_i)$, for any $a_i \in I$, together with the vertex $lim_{a_i \in I} U(a_i)$ constitute a cone of the diagram whose limit is $lim_{a_i \in I} U'(a_i)$. That is because, for $a_1 \prec a_2$ in I, $g_{a_1} \circ p^I_{a_1} = g_{a_1} \circ \sigma^{a_2}_{a_1} \circ p^I_{a_2}$ which using the strong naturality of g becomes, $g_{a_1} \circ p^I_{a_1} = \sigma'^{a_2}_{a_1} \circ g_{a_2} \circ p^I_{a_2}$. Therefore, there is a unique map $\Phi(g)_I : lim_{a_i \in I} U(a_i) \longrightarrow lim_{a_i \in I} U'(a_i)$ such that $p'^I_{a_i} \circ \Phi(g)_I = g_{a_i} \circ p^I_{a_i}$, for any $a_i \in I$. The map $\Phi(g)$ is a natural transformation between $\Phi(U)$ and $\Phi(U')$ because for $I \subseteq J$ and any projection $p'^I_{a_i}$,

$$\begin{aligned} p'^I_{a_i} \circ t'^j_i \circ \Phi(g)_J &= p'^J_{a_i} \circ \Phi(g)_J \\ = g_{a_i} \circ p^J_{a_i} &= g_{a_i} \circ p^I_{a_i} \circ t^J_I \\ &= p'^I_{a_i} \circ \Phi(g)_I \circ t^J_I \end{aligned}$$

$(\Phi(U), t)$ is obviously a presheaf. To show that it has pasting we are going to rely on theorem 5.8 according to which, it suffices to prove that it has binary and directed pasting. First we show that it has binary pasting. Suppose that, for any pair of rounded ideals $I, J \in \mathbf{RSpec}(B)$, there are elements $x \in \Phi(U)(I)$ and $y \in \Phi(U)(J)$ that are coherent, i.e. they satisfy

$$t^I_{I \wedge J}(x) = t^J_{I \wedge J}(y) \tag{5.24}$$

We will prove that there is a unique element $z \in \Phi(U)(I \vee J)$ such that $t^{I \vee J}_I(z) = x$ and $t^{I \vee J}_j(z) = y$. It serves the task ahead to think of the elements x, y in concrete terms. Namely, x is an element of the limit $lim_{a \in I} U(a)$, so it can be represented as a tuple

$$\{x(a) | a \in I\} \in \prod \{U(a) | a \in I\} \tag{5.25}$$

subject to the restriction that if $x(a_1) \in U(a_1)$ and $x(a_2) \in U(a_2)$ are two entries of the tuple with $a_1 \prec a_2$, then the identity

$$\sigma^{a_2}_{a_1}(x(a_2)) = x(a_1) \tag{5.26}$$

must be satisfied. Note that we used the strong restriction map in equation 5.26 as the defining limits of $\Phi(U)$ are with respect to the strong restrictions. Nevertheless, if $x(a_1)$ and $x(a_2)$ are two entries of the tuple with $a_1 \leq a_2$ (weakly less) then we must also have

$$\rho^{a_2}_{a_1}(x(a_2)) = x(a_1) \tag{5.27}$$

For if $a_1 \leq a_2$ in I, then from roundedness there is $a_2' \in I$ such that $a_2 \prec a_2'$ and hence $a_1 \prec a_2'$. Then we have

$$\begin{aligned}
\rho^{a_2}_{a_1}(x(a_2)) &= \rho^{a_2}_{a_1} \circ \sigma^{a_2'}_{a_2}(x(a_2')) \text{ (because of 5.26)} \\
&= \sigma^{a_2'}_{a_1}(x(a_2')) \\
&= x(a_1) \text{ (because of 5.26)}
\end{aligned}$$

Similarly, y is a tuple $\{y(a) \in U(a) | a \in J\} \in \prod_{a \in J} U(a)$ with the property that if $a_1 \prec a_2$ (or $a_1 \leq a_2$) in J, then $\sigma^{a_2}_{a_1}(y(a_2)) = y(a_1)$ (or $\rho^{a_2}_{a_1}(y(a_2)) = y(a_1)$). By the definition of the restriction maps t, the restrictions $t^I_{I \wedge J}(x)$ and $t^J_{I \wedge J}(y)$ are the tuples $\{x(a) \in U(a) | a \in I \wedge J\}$ and $\{y(a) \in U(a) | a \in I \wedge J\}$. So, equation 5.24 can be read as

$$\{x(a) \in U(a) | a \in I \wedge J\} = \{y(a) \in U(a) | a \in I \wedge J\} \tag{5.28}$$

Now, for any $a_1 \in I$ and any $a_2 \in J$, the pair of elements $x(a_1)$ and $y(a_2)$ is coherent as we briefly demonstrate. Since I is rounded, there is $a_1' \in I$ with $a_1 \prec a_1'$. The identity 5.26 dictates that $x(a_1) = \sigma^{a_1'}_{a_1}(x(a_1'))$. Also $a_1 \wedge a_2 \leq a_1 \prec a_1'$ and so $a_1 \wedge a_2 \prec a_1'$. Therefore,

$$\begin{aligned}
\rho^{a_1}_{a_1 \wedge a_2}(x(a_1)) &= \rho^{a_1}_{a_1 \wedge a_2} \circ \sigma^{a_1'}_{a_1}(x(a_1')) \\
&= \sigma^{a_1'}_{a_1 \wedge a_2}(x(a_1')) \\
&= x(a_1 \wedge a_2) \text{ (because of 5.26 and } a_1 \wedge a_2 \in I)
\end{aligned}$$

By the same argument, also $\rho^{a_2}_{a_1 \wedge a_2}(y(a_2)) = y(a_1 \wedge a_2)$. The element $a_1 \wedge a_2$ obviously is an element of $I \cap J = I \wedge J$, therefore if we apply the coherence assumption for x and y, i.e. equation 5.28 we get $x(a_1 \wedge a_2) = y(a_1 \wedge a_2)$ and we just proved that this amounts to $\rho^{a_1}_{a_1 \wedge a_2}(x(a_1)) = \rho^{a_2}_{a_1 \wedge a_2}(y(a_2))$.

We exploit the coherence of pairs $x(a_1), y(a_2)$ as follows. U has pasting, so, for any

$a_1 \in I$ and $a_2 \in J$, there is a unique element $z(a_1 \vee a_2) \in U(a_1 \vee a_2)$ such that $\rho_{a_1}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = x(a_1)$ and $\rho_{a_2}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = y(a_2)$. We recall that the ideal $I \vee J$ is defined as in theorem 3.6(iii). Our goal is to show that there is a unique tuple of elements $z := \{z(a) \in U(a) | a \in I \vee J\}$ such that $t_I^{I \vee J}(z) = x$ and $t_J^{I \vee J}(z) = y$. We consider the following tuple

$$z := \{\sigma_a^{a_1 \vee a_2}(z(a_1 \vee a_2)) | a \in I \vee J \text{ and } a_1 \in I, a_2 \in J \text{ with } a \prec a_1 \vee a_2\} \tag{5.29}$$

Before we actually demonstrate why the above does the job, we have to prove that such a tuple is well defined and this will be rather painstaking. Let it be the case that there are $a_1, a_1' \in I$ and $a_2, a_2' \in J$ such that $a \prec a_1 \vee a_2$ and $a \prec a_1' \vee a_2'$. We have to prove that $\rho_a^{a_1 \vee a_2}(z(a_1 \vee a_2)) = \rho_a^{a_1' \vee a_2'}(z(a_1' \vee a_2'))$. First we show that $z(a_1 \vee a_2)$ and $z(a_1' \vee a_2')$ are coherent. We have that

$$\rho_{a_1 \wedge (a_1' \vee a_2')}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = x(a_1 \wedge (a_1' \vee a_2')) \text{ and}$$

$$\rho_{a_2 \wedge (a_1' \vee a_2')}^{a_1' \vee a_2'}(z(a_1' \vee a_2')) = y(a_2 \wedge (a_1' \vee a_2')) \tag{5.30}$$

because, e.g. $\rho_{a_1}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = x(a_1)$ and $a_1 \wedge (a_1' \vee a_2') \leq a_1$ and the entries of the tuple x have obey the property 5.27. Now, $a_1 \wedge (a_1' \vee a_2') \in I$ and $a_2 \wedge (a_1' \vee a_2') \in J$ so by a previous argument, $x(a_1 \wedge (a_1' \vee a_2'))$ and $y(a_2 \wedge (a_1' \vee a_2'))$ are coherent. Therefore, there is a unique element $z' \in U((a_1 \vee a_2) \wedge (a_1' \vee a_2'))$ such that

$$\rho_{a_1 \wedge (a_1' \vee a_2')}^{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}(z') = x(a_1 \wedge (a_1' \vee a_2')) \quad \text{and}$$

$$\rho_{a_2 \wedge (a_1' \vee a_2')}^{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}(z') = y(a_2 \wedge (a_1' \vee a_2')) \tag{5.31}$$

It holds that $a_1 \wedge (a_1' \vee a_2') \leq (a_1 \vee a_2) \wedge (a_1' \vee a_2') \leq a_1 \vee a_2$ and $a_2 \wedge (a_1' \vee a_2') \leq (a_1 \vee a_2) \wedge (a_1' \vee a_2') \leq a_1' \vee a_2'$, so the pair of equations 5.30 can be written

$$\rho_{a_1 \wedge (a_1' \vee a_2')}^{(a_1 \vee a_2) \wedge (a_1' \vee a_2')} \circ \rho_{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = x(a_1 \wedge (a_1' \vee a_2')) \text{ and}$$

$$\rho_{a_2 \wedge (a_1' \vee a_2')}^{(a_1 \vee a_2) \wedge (a_1' \vee a_2')} \circ \rho_{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}^{a_1' \vee a_2'}(z(a_1' \vee a_2')) = y(a_2 \wedge (a_1' \vee a_2')) \tag{5.32}$$

so that by uniqueness of z' (expressions 5.31) we can infer that

$$\rho_{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = \rho_{(a_1 \vee a_2) \wedge (a_1' \vee a_2')}^{a_1' \vee a_2'}(z(a_1' \vee a_2'))$$

which is the coherence property of the elements $z(a_1 \vee a_2)$ and $z(a_1' \vee a_2')$. So by the pasting property of U, there is a unique $z'' \in U(a_1 \vee a_2 \vee a_1' \vee a_2')$ such that

$$\rho_{a_1 \vee a_2}^{a_1 \vee a_2 \vee a_1' \vee a_2'}(z'') = z(a_1 \vee a_2) \quad \text{and} \quad \rho_{a_1' \vee a_2'}^{a_1 \vee a_2 \vee a_1' \vee a_2'}(z'') = z(a_1' \vee a_2') \tag{5.33}$$

which is enough to prove

$$\begin{array}{rcl} \sigma_a^{a_1 \vee a_2}(z(a_1 \vee a_2)) & = & \sigma_a^{a_1 \vee a_2} \circ \rho_{a_1 \vee a_2}^{a_1 \vee a_2 \vee a_1' \vee a_2'}(z'') \\ = \sigma_a^{a_1 \vee a_2 \vee a_1' \vee a_2'}(z'') & = & \sigma_a^{a_1' \vee a_2'} \circ \rho_{a_1' \vee a_2'}^{a_1 \vee a_2 \vee a_1' \vee a_2'}(z'') \\ & = & \sigma_a^{a_1' \vee a_2'}(z(a_1' \vee a_2')) \end{array}$$

Now we explain why the element z of expression 5.29 has the desired property $t_I^{I \vee J}(z) = x$ and $t_J^{I \vee J}(z) = y$. What we have to show is that, e.g. for any $a \in I$ with $a \in I \vee J$, $\sigma_a^{a_1 \vee a_2}(z(a_1 \vee a_2))$ for $a_1 \in I$ and $a_2 \in J$ with $a \prec a_1 \vee a_2$. Since $\sigma_a^{a_1 \vee a_2}(z(a_1 \vee a_2))$ is independent of the choice of a_1 and a_2, we chose an element a_1 such that $a \prec a_1$. Then $\sigma_a^{a_1 \vee a_2}(z(a_1 \vee a_2)) = \sigma_a^{a_1} \circ \rho_{a_1}^{a_1 \vee a_2}(z(a_1 \vee a_2)) = \sigma_a^{a_1}(x(a_1)) = x(a)$.

To conclude that $\Phi(U)$ has binary pasting, we must show uniqueness of the tuple of 5.29. Suppose that we have an other tuple $w := \{w(a) | a \in I \vee J\}$, such that $w(a) = x(a)$ if $a \in I$ and $w(a) = y(a)$ if $a \in J$. Let $w(a)$ with $a \prec a_1 \vee a_2$, $a_1 \in I, a_2 \in J$ be any entry of the tuple w. Then there are $a_1' \prec a_1$ and $a_2' \prec a_2$ such that $a \prec a_1' \vee a_2' \prec a_1 \vee a_2$. We have that $a_1' \in I$, $a_2' \in J$ and the element $a_1' \vee a_2'$ is then an element of $I \vee J$. The entries of w must observe the property of expression 5.26, so

$$w(a) = \sigma_a^{a_1' \vee a_2'}(w(a_1' \vee a_2')) \tag{5.34}$$

We have

$$\begin{array}{rcl} \rho_{a_1'}^{a_1' \vee a_2'}(w(a_1' \vee a_2')) & = & w(a_1') \text{ (as in 5.27)} \\ & = & x(a_1') \text{ (by assumption)} \end{array}$$

and similarly

$$\rho_{a_2'}^{a_1' \vee a_2'}(w(a_1' \vee a_2')) = y(a_1')$$

But we have shown that the elements $x(a_1')$ and $y(a_2')$ are coherent and the unique element that weakly restricts to them is $z(a_1' \vee a_2')$. Incorporating this information to equation 5.34, we get $\sigma_a^{a_1' \vee a_2'}(z(a_1' \vee a_2'))$ which completes the proof of uniqueness.

Now we prove that $\Phi \circ \Psi = id_{Sh(\mathbf{RSpec}(B))}$. Let (F, r) be a sheaf over $\mathbf{RSpec}(B)$.

With theorem 5.4 we showed that

$$F(I) = F(\bigvee{}^{\uparrow}\{\downarrow a_i | a_i \in I\}) \cong lim_{a_i \in I} F(\downarrow a_i)$$

Let $q_{a_i} : lim_{a_i \in I} F(\downarrow a_i) \longrightarrow F(\downarrow a_i)$ be the limit projections that coincide with the restriction maps $r^I_{\downarrow a_i}$. Also the maps $\theta_{a_i} : colim_{a_i \prec a_j} F(\downarrow a_j) \longrightarrow F(\downarrow a_j)$ are the components of an approximating morphism of presheaves $\theta : \mathbf{int}V \longrightarrow V$, where $V = F(\downarrow \bullet)$ as before (see proof of theorem 5.18). We denote by β the approximating morphism of presheaves $\Phi(\theta)$. The defining property of $\Phi(\theta)$ (see above in this proof) says that for any $a_i \in I$, the following square is commutative

$$\begin{tikzcd}
lim_{a_i \in I} colim_{a_i \prec a_j} F(\downarrow a_j) \arrow[r, "\beta_I"] \arrow[d, "p^I_{a_i}"'] & lim_{a_i \in I} F(\downarrow a_i) \arrow[d, "q^I_{a_i} = r^I_{\downarrow a_i}"] \\
colim_{a_i \prec a_j} F(\downarrow a_j) \arrow[r, "\theta_{a_i}"] & F(\downarrow a_1)
\end{tikzcd} \tag{5.35}$$

We will prove that the functions β_{a_i} are isomorphisms. To this end we define maps on the opposite direction. For any $a_i \in I$, we fix an element $a'_i \in I$ with $a_i \prec a'_i$ (such an element always exists because of roundedness of I). With notational consistency, we denote $\theta^{a'_i, a_i} : F(\downarrow a'_i) \hookrightarrow colim_{a_i \prec a_j} F(\downarrow a_j)$ the colimit injections. We will argue that the outer diagram below is commutative for any pair $a_1 \prec a_2$ in I.

$$\begin{tikzcd}
 & F(I) \arrow[dl, "r^I_{\downarrow a'_1}"'] \arrow[d, dashed, "\alpha_I"'] \arrow[dr, "r^I_{\downarrow a'_2}"] & \\
F(\downarrow a'_1) \arrow[d, "\theta^{a'_1, a_1}"'] & lim_{a_i \in I} colim_{a_i \prec a_j} F(\downarrow a_j) \arrow[dl, "p^I_{a_1}"'] \arrow[dr, "p^I_{a_2}"] & F(\downarrow a'_2) \arrow[d, "\theta^{a'_2, a_2}"] \\
colim_{a_1 \prec a_j} F(\downarrow a_j) & & colim_{a_2 \prec a_j} F(\downarrow a_j) \arrow[ll, "\delta^{a_2}_{a_1}"']
\end{tikzcd} \tag{5.36}$$

Indeed, if $a_1 \prec a'_1$ and $a_1 \prec a_2 \prec a'_2$, then $a_1 \prec a'_1 \wedge a'_2$ and so we have

$$\theta^{a'_1 \wedge a'_2, a_1} \circ r^{\downarrow a'_1}_{\downarrow (a'_1 \wedge a'_2)} \circ r^I_{\downarrow a'_1} \quad = \quad \theta^{a'_1 \wedge a'_2, a_1} \circ r^{\downarrow a'_2}_{\downarrow (a'_1 \wedge a'_2)} \circ r^I_{\downarrow a'_2} \Leftrightarrow$$

$$\begin{aligned}\theta^{a'_1,a_1} \circ r^I_{\downarrow a'_1} &= \theta^{a'_2}_{a_1} \circ r^I_{\downarrow a'_2} \\ &= \delta^{a_2}_{a_1} \circ \theta^{a'_2,a_2} \circ r^I_{\downarrow a'_2}\end{aligned}$$

Therefore, there exists a function α_I that makes the rest of the diagrams in 5.36 commutative. For any projection $q^I_{a_i} = r^I_{\downarrow a_i}$, the following (up to the isomorphism $F(I) \cong lim_{a_i \in I} F(\downarrow a_i)$)

$$\begin{aligned}q^I_{a_i} \circ \beta_I \circ \alpha_I &= \theta_{a_i} \circ p^I_{a_i} \circ \alpha_{a_i} \quad \text{(diagram 5.35)} \\ &= \theta_{a_i} \circ \theta^{a'_i,a_i} \circ r^I_{\downarrow a'_i} \quad \text{(diagram 5.36)} \\ &= \theta^{a'_i}_{a_i} \circ r^I_{\downarrow a'_i} \\ &= r^I_{\downarrow a_i} \\ &= q^I_{a_i}\end{aligned}$$

Therefore $\beta_I \circ \alpha_I = id_{F(I)}$. For the other direction we have for any projection $p^I_{a_i}$ (again up to isomorphism)

$$\begin{aligned}p^I_{a_i} \circ \alpha_I \circ \beta_I &= \theta^{a'_i,a_i} \circ r^I_{\downarrow a'_i} \circ \beta_I \quad \text{(diagram 5.36)} \\ &= \theta^{a'_i,a_i} \circ \theta_{a'_i} \circ p^I_{a'_i} \quad \text{(diagram 5.35)} \\ &= \theta^{a'_i}_{a_i} \circ p^I_{a'_i} \\ &= p^I_{a_i}\end{aligned}$$

This completes the proof of $\Phi \circ \Psi = id_{Sh(\mathbf{RSpec}(B))}$.

Finally, we prove that $\Psi \circ \Phi = id_{\mathcal{B}\mathbf{Sh}(B)}$. Let (U, ρ, σ) any B-sheaf. We recall that the map with components $q_a : U(a) \longrightarrow lim_{a \succ a_i} U(a_i)$ is an approximating presheaf morphism (corollary 5.20) and lemma 5.21 guarantees that $\mathbf{int}(q)$ is an isomorphism. We consider the map $\dot{\sigma} \circ \mathbf{int}(q)^{-1} : \Psi \circ \Phi(U) \longrightarrow U$ with components

$$colim_{a \prec a_i} lim_{a_i \succ a_j} U(a_j) \xrightarrow{\mathbf{int}(q)^{-1}_a} colim_{a \prec a_i} U(a_i) \xrightarrow{\sigma_a} U(a) \tag{5.37}$$

The composite $\dot{\sigma} \circ \mathbf{int}(q)^{-1}$ is an approximating morphism of presheaves because both $\mathbf{int}(q)$ and $\dot{\sigma}$ are (theorem 5.18) and in particular it is an isomorphism. This completes the proof of the theorem. ■

5.5 $\mathcal{B}$-sheaves as models of a geometric theory

Care has been taken so that all the definitions and results of the previous section are valid inside the sheaves of any Grothendieck topos and not just **Sets**. To make this point more concrete, we stress the following facts:

(i) The theory of approximating presheaves with pasting over a strong proximity lattice (definition 5.14) is *essentially algebraic* in the sense of Freyd [Fre72].

(ii) The theory of presheaves with continuous approximation over a strong proximity lattice (definition 5.12) is *geometric*.

(iii) Therefore, the theory of $\mathcal{B}$-sheaves over a strong proximity lattice (definition 5.15) is *geometric*.

Let us denote by $\mathbb{T}_{\mathcal{B}\mathbf{Sh}(B)}$ geometric theory of $\mathcal{B}$-sheaves over a strong proximity lattice B. Its language contains

sorts:		a sort $\tilde{V(a)}$ for each element $a \in B$.
functions:	1.	a function symbol $\tilde{\phi_a^{a'}}$ for each pair of elements $a, a' \in B$ with $a \leq a'$.
	2.	a function symbol $\tilde{\theta_a^{a'}}$ for each pair of elements $a, a' \in B$ with $a \prec a'$.

It is straightforward to translate into formal geometric axioms (in the above language) the essentially algebraic properties of definitions 5.10 and 5.14. The continuity axiom of definition 5.12 can be expressed as

1. $\forall x \in \tilde{V(a)}(\top \to \bigvee_{a \prec a'}\{\exists x' \in \tilde{V(a')}(x = \tilde{\theta_a^{a'}}(x'))\})$
2. $\forall x, y \in \tilde{V(a')}(\tilde{\theta_a^{a'}}(x) = \tilde{\theta_a^{a'}}(y) \to \bigvee_{a \prec a'' \prec a'}\{\tilde{\theta_{a''}^{a'}}(x) = \tilde{\theta_{a''}^{a'}}(y)\})$

This is the concrete property of the *filtered* colimit $colim_{a \prec a'} V(a')$. For the rest of the thesis, we use

$$\mathbf{Pre}\mathcal{B}\mathbf{Sh}_Z(B),\ \ \mathbf{ContPre}\mathcal{B}\mathbf{Sh}_Z(B),\ \ \mathbf{PastPre}\mathcal{B}\mathbf{Sh}_Z(B),\ \ \mathcal{B}\mathbf{Sh}_Z(B)$$

to denote the categories of approximating presheaves, continuous approximating presheaves, pasting approximating presheaves and $\mathcal{B}$-sheaves inside the sheaves of any Grothendieck topos Z. For example

$$\mathcal{B}\mathbf{Sh}_Z(B) := \underline{Mod}(\mathcal{S}Z, \mathbb{T}_{\mathcal{B}\mathbf{Sh}(B)})$$

where Z is any Grothendieck topos.

Now we revisit theorem 5.22. It It renders equivalent two notions; that of $\mathcal{B}$-sheaves which is geometric and that of ordinary sheaves which clearly is not. Unlike $\mathcal{B}$-sheaves, ordinary sheaves cannot be transferred freely between topoi. Nevertheless, the *fact* that they are equivalent holds inside the sheaves of any Grothendieck topos and not just **Sets**. We make the following upgrading of theorem 5.22 for future reference.

Remark 5.23 *Theorem 5.22 is valid inside the sheaves of any Grothendieck topos.*

Since the theory of $\mathcal{B}$-sheaves is geometric, it merits a classifying topos $[\mathcal{B} - \mathbf{sh}]$ and next we are going to uncover just that. Before we start we point out that, more accurately, we are in search of the classifying topos of $\mathcal{B}$-sheaves *given* a strong proximity lattice B where the latter is defined inside the sheaves of the our base topos, or more specifically in **Sets**. The theory of strong proximity lattices is algebraic, so after a model B is defined in **Sets**, we can pull it back along the inverse image functor of $Z \longrightarrow \mathbf{1}$ (the essentially unique map) to obtain the strong proximity lattice $!^*(B)$ in $\mathcal{S}Z$ (see section 3.3).

Theorem 5.24 *Let B be a strong proximity lattice and X be $\mathbf{RSpec}(B)$. Then the exponential $[set]^X$ (in $\mathfrak{Top}$) classifies the geometric theory of $\mathcal{B}$-sheaves over B.*

Proof. We must show that any point of $[set]^X$ at stage Z is equivalent to a $\mathcal{B}$-sheaf inside $\mathcal{S}Z$. This is clearly the case for $Z = \mathbf{1}$; a geometric morphism $\mathbf{1} \longrightarrow [set]^X$ corresponds to its exponential transpose under the exponentiation adjunction, i.e. to a geometric morphism $X \longrightarrow [set]$. The latter morphism corresponds to a sheaf of sets over X and theorem 5.22 says that such a sheaf is equivalent to a $\mathcal{B}$-sheaf over B.

To prove the claim for any topos Z, we argue as follows. Any geometric morphism $Z \longrightarrow [set]^X$ is equivalent to its exponential transpose $Z \times X \longrightarrow [set]$ and the latter is equivalent to geometric morphisms $Z \times X \longrightarrow Z \times [set]$ *over* the topos Z, i.e., such morphisms that make the following triangle commutative.

$$\begin{tikzcd} Z \times X \arrow[rr] \arrow[dr, "p_1"'] & & Z \times [set] \arrow[dl, "q_1"] \\ & Z & \end{tikzcd} \tag{5.38}$$

The map p_1 stands for first projection which is a localic map, so $p_1 : Z \times X \longrightarrow Z$ is a locale over Z. The first projection map $Z \times [set] \longrightarrow Z$ is the object classifier over Z, therefore the horizontal map in diagram 5.38 is equivalent to a sheaf of the locale $Z \times X$ over Z.

Now, $p_1 : Z \times X \longrightarrow Z$ is trivially the pullback

$$\begin{tikzcd} Z \times X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "!"] \\ Z \arrow[r, "!"] & \mathbf{1} \end{tikzcd} \tag{5.39}$$

So, by theorem 1.8, $p_1 : Z \times X \longrightarrow Z$ is just the locale $\mathbf{RSpec}(!^*(B))$, i.e., the locale X pulled back along the inverse image of $! : Z \longrightarrow \mathbf{1}$. This implies that a sheaf of the locale $Z \times X$ over Z is equivalent to a sheaf over X inside $\mathcal{S}Z$ and by theorem 5.22 and remark 5.23, this in turn is equivalent to $\mathcal{B}$-sheaf over $!^*(B)$. Therefore, in conclusion $Z \longrightarrow [set]^X$ is equivalent to a model of the theory of $\mathcal{B}$-sheaves inside $\mathcal{S}Z$. ■

The above theorem shows that the exponential $[set]^X$ exists in $\mathfrak{Top}$. By a Johnstone and Joyal result, this can be generalised for any topos instead of the object classifier $[set]$.

Theorem 5.25 *Let E be any topos. Then E is exponentiable in $\mathfrak{Top}$ iff the exponential $[set]^E$ exists in $\mathfrak{Top}$.*

Proof. [JJ82], theorem 4.5. ■

Therefore we have.

Corollary 5.26 *If X is a stably compact locale then X is exponentiable as a topos.*

Also, from the last part of the proof of theorem 5.24, we pick the following fact.

Corollary 5.27 *Let B be a strong proximity lattice in* **Sets**. *The object of $\mathfrak{Top}/Z$ that classifies the geometric theory of $!^*(B)$-sheaves over a Grothendieck topos Z is the first projection*

$$Z \times [set]^X \longrightarrow Z$$

5.6 The free pasting approximating presheaf

We are closing this chapter with a notion of *sheavification* for approximating presheaves. We seek a construction that, given an approximating presheaf over a strong proximity lattice B, it results to a pasting presheaf with approximation over B (not necessarily continuous).

As mentioned in the beginning of section 5.5, the theory of approximating presheaves with pasting is essentially algebraic. We adopt Freyd's version of essential algebraicity [Fre72]- see also [JV91] for a quick review. Without going into any detail, an essentially algebraic theory is the theory where some operations are only partial, with domain of definition stipulated equationally. In the case of approximating presheaves, the partially defined operations are the restrictions (weak and strong).

Most of the techniques of universal algebra can be extended to cover essentially algebraic theories. In particular, free constructions exist in the same sense as for algebraic theories and they are geometric. Owing to the applicability of universal algebra, we can readily accept the following.

Theorem 5.28 *Let* $\mathbf{i_1} : \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$

be the category inclusion (inside any topos). Then $\mathbf{i_1}$ *has a left adjoint* **past**. *In other words, if V is a presheaf with approximation, then* $\mathbf{past}(V)$ *is the free pasting presheaf with approximation over V.*

Remark 5.29 *The functor* **past** *provides a notion of sheavification for approximating presheaves. Sometimes it will be referred to as the sheavification functor.*

We have seen that the inclusion $\mathbf{i_3} : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$ has a right adjoint **int**. This together with lemma 5.17 has the following consequence.

Theorem 5.30 *If an approximating presheaf V over B is continuous then so is* $\mathbf{past}(V)$, *i.e.* $\mathbf{past}(V)$ *is a $\mathcal{B}$-sheaf.*

Proof. First we use the universal property of the counit of the adjunction $\mathbf{i_3} \dashv \mathbf{int}$. Let

$$\eta_V : V \longrightarrow \mathbf{i_1} \circ \mathbf{past}(V)$$

be the unit of the (other) adjunction $\mathbf{past} \dashv \mathbf{i_1}$ (i.e. the injection of generators). Then, omitting the category inclusions, there is a unique morphism $\zeta : V \longrightarrow \mathbf{int} \circ \mathbf{past}(V)$, such that the left diagram below commutes.

We continue to omit the symbols for the category inclusions. By lemma 5.17, $\mathbf{int} \circ \mathbf{past}(V)$ has pasting, so the universal property of the unit of the adjunction $\mathbf{past} \dashv \mathbf{i_1}$ says that given the morphism $\zeta : V \longrightarrow \mathbf{int} \circ \mathbf{past}(V)$, there is a unique morphism $\alpha : \mathbf{past}(V) \longrightarrow \mathbf{int} \circ \mathbf{past}(V)$ that makes the right diagram below commutative.

$$\begin{tikzcd}
\mathbf{int} \circ \mathbf{past}(V) \arrow[r, "\varepsilon_{\mathbf{past}(V)}"] & \mathbf{past}V \\
V \arrow[u, dashed, "\zeta"] \arrow[ur, "\eta_V"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{int} \circ \mathbf{past}(V) & \mathbf{past}(V) \arrow[l, dashed, "\alpha"'] \\
V \arrow[u, "\zeta"] \arrow[ur, "\eta_V"'] &
\end{tikzcd}$$

The morphism η_V is an injection, so

$$\varepsilon_{\mathbf{past}V} \circ \alpha \circ \eta_V \quad = \quad \zeta\eta_V$$

$$= \eta_V$$

implies that $\varepsilon_{\mathbf{past}V} \circ \alpha = id_{\mathbf{past}(V)}$. Similarly, the fact that $\varepsilon_{\mathbf{past}(V)}$ is epi, gives

$$\begin{aligned}
\varepsilon_{\mathbf{past}(V)} \circ \alpha &= id_{\mathbf{past}(V)} \Rightarrow \\
\varepsilon_{\mathbf{past}(V)} \circ \alpha \circ \varepsilon_{\mathbf{past}(V)} &= \varepsilon_{\mathbf{past}(V)} \Leftrightarrow \\
\alpha \circ \varepsilon_{\mathbf{past}(V)} &= id_{\mathbf{int} \circ \mathbf{past}(V)}
\end{aligned}$$

■

In view of the previous theorem, we denote as

$$\mathbf{past}' : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathcal{B}\mathbf{Sh}(B)$$

the functor that produces the free pasting presheaf over a continuous approximating presheaf. We denote by

$$\mathbf{i_4} : \mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B) \tag{5.40}$$

the inclusion functor. Then theorem 5.30 also implies that

$$\mathbf{past}' \dashv \mathbf{i_4}$$

Finally, the following are consequences of the fact that $\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$ is a full subcategory of $\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$ and that $\mathcal{B}\mathbf{Sh}(B)$ is a full subcategory of $\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B)$.

Lemma 5.31 *(i) Denote as* $\mathbf{i_1} : \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$ *the inclusion functor. Then* $\mathbf{past} \circ \mathbf{i_1} = id_{\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)}$*. This together with* $\mathbf{past} \dashv \mathbf{i_1}$ *imply that* $\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$ *is a reflective subcategory of* $\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$*.*

(ii) Denote as $\mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B)$ *the inclusion functor. Then* $\mathbf{past}' \circ \mathbf{i_4} = id_{\mathcal{B}\mathbf{Sh}(B)}$*. This together with* $\mathbf{past}' \dashv \mathbf{i_4}$ *imply that* $\mathcal{B}\mathbf{Sh}(B)$ *is a reflective subcategory of* $\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B)$*.*

Proof. (i) The claim basically says that if V is already a $\mathcal{B}$-sheaf over B then $\mathbf{sh}_B(V) = V$. We will prove that indeed V already has the universal property of the free presheaf above itself. Let $f : V \longrightarrow V'$ a presheaf with approximation morphism between V and any $\mathcal{B}$-sheaf V'. Then, by definition **??**, f is also a sheaf morphism between V and V'. So f

must be the unique extension of itself as in the diagram below.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "id_{\mathcal{B}Sh(B)}"'] & V' \\ V \arrow[ur, "f"'] & \end{tikzcd}$$

So V is the free sheaf above itself.

(ii) Trivially from (i). ■

Chapter 6

Functors Between Approximating Presheaves

6.1 Introduction

In the opening section 6.2 we introduce the geometric notion of the stalk of an approximating presheaf. The important result there is that sheavification does not alter the stalks in accordance with ordinary sheaf theory.

A strong homomorphism $\mu : B_2 \longrightarrow B_1$ induces two functors $\mathcal{B}\mathbf{Sh}(B_1) \rightleftharpoons \mathcal{B}\mathbf{Sh}(B_2)$ much the same way as a continuous map between two locales induces two functors between the respective categories of sheaves over the two locales. In this chapter we demonstrate just that- in two stages.

In sections 6.3 and 6.4 we construct two functors

$$\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1) \underset{\pi_\mu}{\overset{\rho_\mu}{\rightleftarrows}} \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2)$$

and in section 6.5 we prove that $\rho_\mu \dashv \pi_\mu$. We call ρ_μ, the *inverse image* functor and π_μ, the *direct image* functor; a brief justification of this choice is in the beginning of section 6.3.

The functor π_μ actually takes $\mathcal{B}$-sheaves to $\mathcal{B}$-sheaves but in order to obtain a functor $\mathcal{B}\mathbf{Sh}(B_2) \longrightarrow \mathcal{B}\mathbf{Sh}(B_1)$, we need to consider ρ_μ followed by the sheavification functor **past** of section 5.6. This is how we construct a pair of functors

$$\mathcal{B}\mathbf{Sh}(B_1) \underset{\mu_*}{\overset{\mu^*}{\rightleftarrows}} \mathcal{B}\mathbf{Sh}(B_2)$$

in section 6.6 and prove $\mu^* \dashv \mu_*$.

The functors μ^* and μ_* are functors between the (generalised) points of the classifying topoi $[set]^{\mathbf{RSpec}(B_1)}$ and $[set]^{\mathbf{RSpec}(B_2)}$ and have been constructed geometrically. Therefore, they determine two geometric morphisms

$$[set]^{\mathbf{RSpec}(B_1)} \underset{P_\mu}{\overset{R_\mu}{\rightleftarrows}} [set]^{\mathbf{RSpec}(B_2)}$$

Furthermore, by virtue of corollary 4.34, the adjunction $\mu^* \dashv \mu_*$ has a "sufficiently 2-categorical value" to assert the adjoint situation $R_\mu \dashv P_\mu$. All this is demonstrated in section 6.7. There it is also proved that R_μ is nothing else but the exponential geometric morphism $[set]^{\mathbf{RSpec}(\mu)}$ in $\mathfrak{Top}$. To this end, the insight of section 6.2 on stalks of $\mathcal{B}$-sheaves is used.

The last result has an immediate consequence. The functor **RSpec** is a retraction of the functor $\mathcal{B}$ (theorem 3.24), so if $f : X \longrightarrow Y$ is perfect and X, Y are stably compact locales, then the geometric morphism $[set]^f$ has a right adjoint. An important implication of this fact is demonstrated in section 6.8; the direct image functor f_* preserves filtered colimits that are externally indexed (i.e. in the base topos), or, using the terminology of the Moerdijk & Vermeulen monograph [MV97], f is relatively tidy.

All the results in this chapter are new.

6.2 The stalks of B-sheaves

We recall that, if B is a strong proximity lattice, a *point* of the locale $\mathbf{RSpec}(B)$ is a completely prime filter of rounded ideals of B. Under the isomorphism of lemma 3.25 completely prime filters of rounded ideals of B correspond to rounded prime filters of B. At the same time, we know how to calculate the stalks of sheaves over points of $\mathbf{RSpec}(B)$ but we do not have the notion yet of stalk for a $\mathcal{B}$-sheaf.

Theorem 6.1 *Let (V, ϕ, θ) be a $\mathcal{B}$-sheaf over a strong proximity lattice B. Let (F, r) be the sheaf over* **RSpec**(B) *that corresponds to V under the equivalence of theorem 5.22, i.e. $F = \Phi(V)$ in the notation of theorem 5.22. Finally, let H be a point of* **RSpec**(B), *i.e. a completely prime filter of rounded ideals of B. Then*

$$stalk_H \Phi(V) \cong colim_{a \in G} V(a) \tag{6.1}$$

where G is the rounded prime filter that corresponds to H under the isomorphism of lemma 3.25. The colimiting diagram on the R.H.S. of isomorphism 6.1 is with respect to the strong *restrictions.*

Proof. We know (theorem 5.22) that $stalk_H\Phi(V) \cong colim_{I\in H} lim_{a\in I} V(a)$. Also from lemma 3.25, $I \in H \Leftrightarrow I \cap G \neq$. So the L.H.S. of 6.1 can be written as

$$stalk_H\Phi(V) \cong colim_{I\cap G\neq} lim_{a_i\in I} V(a) \tag{6.2}$$

On the other hand, using lemma 5.21, the R.H.S. of 6.1 can be written as

$$colim_{a\in G} V(a) \cong colim_{a\in G} colim_{a\prec a'} lim_{a'\succ a_i} V(a_i) \cong colim_{a\in G} lim_{a\succ a_i} V(a_i) \tag{6.3}$$

The last isomorphism above holds because G is upper closed with respect to $\prec$.

First we define a function from the colimit 6.3 to the colimit 6.2. Let $r^I_J : F(J) \longrightarrow F(I)$ be the generic restriction map of $F = \Phi(V)$ as derived by the restrictions of the $\mathcal{B}$-sheaf V (theorem 5.22). We observe that, for any rounded filter G, $a \in G \Leftrightarrow \downarrow a \cap G \neq \emptyset$. Therefore, $colim_{a\in G} lim_{a\succ a_i} V(a_i)$ can be formally written as $colim_{\dot{I}\cap G\neq\emptyset} lim_{a_i\in \dot{I}} V(a_i)$ where $\dot{I}$ is any principal rounded ideal of B (i.e. $\dot{I} = \downarrow c$ for some $c \in B$). We fix the notation for the two types of colimit injections:

$$\begin{array}{c} lim_{a\succ a_i} V(a_i) \xhookrightarrow{r^{\downarrow a,G}} colim_{a\in G} lim_{a\succ a_i} V(a_i) \\ lim_{a_i\in I} V(a_i) \xhookrightarrow{r^{I,H}} colim_{I\cap G\neq\emptyset} lim_{a_i\in I} V(a_i) \end{array} \tag{6.4}$$

The set $colim_{I\cap G\neq\emptyset} lim_{a_i\in I} V(a_i)$ together with the injections $r^{\downarrow a,H}$, for $a \in G$, is a cone of the diagram whose colimit is $colim_{a\in G} lim_{a\succ a_i} V(a_i)$. So there is a unique function α_G that makes the following diagram commutative for any $a \in G$.

$$\begin{tikzcd} lim_{a\succ a_i} V(a_i) \arrow[dd, "r^{\downarrow a,H}"'] \arrow[dr, "r^{\downarrow a,G}"] & \\ & colim_{a\in G} lim_{a\succ a_i} V(a_i) \arrow[dl, "\alpha_G"] \\ colim_{I\cap G\neq\emptyset} lim_{a_i\in I} V(a_i) & \end{tikzcd} \tag{6.5}$$

Defining a function on the opposite direction needs more care. Given a rounded filter G, for all rounded ideals I with $I \cap G \neq \emptyset$, we fix an element $b \in I \cap G$. Suppose that $I_1 \leq I_2$

and the corresponding fixed elements are $b_1 \in I_1 \cap G$ and $b_2 \in I_2 \cap G$. We consider the following diagram.

$$\begin{tikzcd}
lim_{a_i \in I_1} V(a_i) \arrow[d, "r^{I_1}_{\downarrow b_1}"'] \arrow[ddr, "r^{I_1,H}"] & & lim_{a_i \in I_2} V(a_i) \arrow[ll, "r^{I_2}_{I_1}"'] \arrow[d, "r^{I_2}_{\downarrow b_2}"] \arrow[ddl, "r^{I_2,H}"'] \\
lim_{b_1 \succ a_i} V(a_i) \arrow[ddr, "r^{\downarrow b_1,G}"'] & & lim_{b_2 \succ a_i} V(a_i) \arrow[ddl, "r^{\downarrow b_2,G}"] \\
& colim_{I \cap G \neq \emptyset} lim_{a_i \in I} V(a_i) \arrow[d, dashed, "\beta_G"'] & \\
& colim_{a \in G} lim_{a \succ a_i} V(a_i) &
\end{tikzcd} \tag{6.6}$$

It can be verified by just composing the arrows that the outlying diagram commutes. Therefore, the set $colim_{a\in G} lim_{a \succ a_i} V(a_i)$ together with the composites $r^{\downarrow b,G} \circ r^I_{\downarrow b}$, for all I that meet G is a cone of the diagram with which has colimit $colim_{I \cap G \neq \emptyset} lim_{a_i \in I} V(a_i)$. Hence there is a unique function β_G such that

$$\beta_G \circ r^{I,H} = r^{\downarrow b,G} \circ r^I_{\downarrow b} \tag{6.7}$$

for any rounded ideal I that meets G. It remains to be shown that the composites $r^{\downarrow b,G} \circ r^I_{\downarrow b}$ are independent of the choice of the element b for any I. Let b_1, b_2 be two elements in the intersection $I \cap G$. Then the element $b_1 \vee b_2$ is an element of I because I is $\vee$-closed and it is also an element of G because G is upper closed. So we have

$$\begin{aligned}
r^{\downarrow b_2,G} \circ r^I_{\downarrow b_2} &= r^{\downarrow b_2,G} \circ r^{\downarrow (b_1 \vee b_2)}_{\downarrow b_2} \circ r^I_{\downarrow (b_1 \vee b_2)} = \\
r^{\downarrow b_1,G} \circ r^{\downarrow (b_1 \vee b_2)}_{\downarrow b_1} \circ r^I_{\downarrow (b_1 \vee b_2)} &= r^{\downarrow b_1,G} \circ r^I_{\downarrow b_1}
\end{aligned}$$

To prove that $\alpha_g \circ \beta_G = id$, it suffices to prove that for any colimit injection $r^{I,H}$ (i.e. for any I that meets G), $\alpha_g \circ \beta_G \circ r^{I,H} = r^{I,H}$. We have the following sequence of equalities

$$\begin{aligned}
\alpha_g \circ \beta_G \circ r^{I,H} &= \alpha_G \circ r^{\downarrow b,G} \circ r^I_{\downarrow b} \text{ (by equation 6.7)} \\
&= r^{\downarrow b,H} \circ r^I_{\downarrow b} \text{ (by diagram 6.5)} \\
&= r^{I,H}
\end{aligned}$$

Finally to prove that $\beta_G \circ \alpha_G = id$, it suffices to prove that for any colimit injection $r^{\downarrow a,F}$ (i.e. for any $a \in G$) it holds $\beta_G \circ \alpha_G \circ r^{\downarrow a,G} = r^{\downarrow a,F}$. In the following diagram, a

is any element in G and b is a fixed element in the intersection $\downarrow a \cap G$ (i.e. $b \prec a$ and $b \in G$).

$$\begin{tikzcd}
& lim_{a \succ a_i} V(a_i) \arrow[dl, "r^{\downarrow a}_{\downarrow b}"'] \arrow[d, "r^{\downarrow a,G}"] \arrow[dr, "r^{\downarrow a,H}"] & \\
lim_{b \succ a_i} V(a_i) \arrow[r, "r^{\downarrow b,G}"] & colim_{a \in G} lim_{a \succ a_i} V(a_i) \arrow[r, dashed, shift left, "\alpha_G"] & colim_{I \cap G \neq \emptyset} lim_{a_i \in I} V(a_i) \arrow[l, dashed, shift left, "\beta_G"]
\end{tikzcd} \tag{6.8}$$

For any $a \in G$, we have

$$\begin{aligned}
\beta_G \circ \alpha_G \circ r^{\downarrow a,G} &= \beta_G \circ r^{\downarrow a,H} \text{ (by diagram 6.5)} \\
&= r^{\downarrow b,G} \circ r^{\downarrow a}_{\downarrow b} \text{ (by equation 6.7)} \\
&= r^{\downarrow a,G}
\end{aligned}$$

This completes the proof of the theorem. ■

The above theorem in conjunction with theorem 3.24 have the following obvious consequence.

Corollary 6.2 *Let F be a sheaf over a stably compact locale X and x a point of X. Let also $V = \Psi(F)$ be the corresponding $\mathcal{B}$-sheaf over $\mathcal{B}X$ (vis-à-vis theorem 5.22) and F the rounded prime filter of $\mathcal{B}X$ that corresponds to x (given by the expression 3.28). Then we have*

$$stalk_x F \cong colim_{a \in F} V(a)$$

Theorem 6.1 dictates that if V is a $\mathcal{B}$-sheaf over a strong proximity lattice B, then its stalk over a rounded prime filter $F \subseteq B$ *has to be*

$$stalk_F V := colim_{a \in F} V(a) \tag{6.9}$$

In fact we are setting this as the definition of stalks for any presheaf with approximation.

Definition 6.3 *Let V be a presheaf with approximation over a strong proximity lattice B. Then its stalk is given by equation 6.9.*

When we write $stalk_x V$ or $colim_{x \vDash a} V(a)$ we shall be implying $colim_{a \in F} V(a)$, where F is the rounded prime filter of B that corresponds to the point x of **RSpec**(B) *(as in lemma 3.25).*

Remark 6.4 *The definition of stalks of presheaves with approximation is geometric (it involves a colimit) and so it makes sense inside the sheaves of any topos.*

Next, we are going to promote the stalk construction into a functor

$$\mathbf{Pre}\mathcal{B}\mathbf{Sh}_{\mathcal{S}Z}(B) \longrightarrow \mathcal{S}Z$$

and prove some of its properties. Here, $\mathbf{Pre}\mathcal{B}\mathbf{Sh}_{\mathcal{S}Z}(B)$ is the category of presheaves with approximation and approximating presheaf morphisms inside the sheaves of a topos Z. Once more, it suffices to argue about the case $Z = \mathbf{1}$ (i.e. $\mathcal{S}Z = \mathbf{Sets}$), as long as we argue geometrically and all the facts can be transferred inside $\mathcal{S}Z$ for any topos Z by means of the inverse image of the essentially unique functor $! : Z \longrightarrow \mathbf{1}$

Let $f : V_1 \longrightarrow V_2$ be an approximating presheaf map between two presheaves with approximation (V_1, ϕ, θ) and (V_2, β, δ) over B. Then a function $stalk_F(f)$ between the sets $stalk_F(V_1) \longrightarrow stalk_F(V_2)$ arises naturally. We denote as $\theta^{a_i,F} : V_1(a_i) \hookrightarrow colim_{a_i \in F} V_1(a_i)$ and $\delta^{a_i,F} : V_2(a_i) \hookrightarrow colim_{a_i \in F} V_2(a_i)$ the two generic colimit injections. As e.g. in the proof of theorem 5.18, we observe that the composites $\delta^{a_i,F} \circ f_{a_i}$ together with the vertex $colim_{a_i \in F} V_2(a_i)$ is a cone of the diagram $V_2 : J_F^s \longrightarrow \mathbf{Sets}$. By the universal property of $colim_{a_i \in F} V_1(a_i)$, there is a unique function (which we denote $stalk_F(f)_{a_i}$ that makes the following diagram commutative for any $a_i \in F$.

$$\begin{tikzcd}
V_1(a_i) \arrow[r, "f_{a_i}"] \arrow[d, "\theta^{a_i,F}"'] & V_2(a_i) \arrow[d, "\delta^{a_i,F}"] \\
stalk_F V_1 \arrow[r, dashed, "stalk_F(f)_{a_i}"] & stalk V_2
\end{tikzcd} \tag{6.10}$$

The assignments $V \mapsto stalk_F V$ and $f \mapsto stalk_F(f)$ define a functor

$$stalk_F : \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathbf{Sets}$$

We recall from ordinary sheaf theory that if X is a topological space, then, for each point of X, one can define a functor (usually referred to as the "skyscraper" functor) $sky_x : \mathbf{Sets} \longrightarrow \mathbf{Sh}(X)$ by stipulating that for any set Z and any open $a \in \Omega X$, $sky_x(Z)(a) = Z$ if $x \in a$ or $sky_x(Z)(a) = \mathbf{1}$ if $x \notin a$. Furthermore, it holds (e.g. Mac Lane & Moerdijk [MM92]) that if $\mathbf{in} : \mathbf{Sh}(X) \hookrightarrow \mathbf{PreSh}(X)$ is the inclusion functor, then sky_x is right adjoint to the functor $stalk_x \circ \mathbf{in}$. We are going to give a constructive version of the skyscraper functor in the case if $\mathcal{B}$-sheaves.

Let a be any element of B and F any prime rounded filter of B. By $[a \in F]$ we denote the set defined as

$$[a \in F] := \{* \in \mathbf{1} | a \in F\}$$

which is the constructive counterpart of the set which is the singleton if $a \in F$ and the empty set if $a \notin F$.

Definition 6.5 *Let* $\mathbf{sky_F} : \mathbf{Sets} \longrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$ *be the functor whose action on objects and arrows of* $\mathbf{Sets}$ *is as follows:*

- *If* Z *is a set and* $a \in B$*, then* $\mathbf{sky_F}(Z)(a)$ *is the exponential* $Z^{[a \in F]}$*.*
- *If* $f : Z_1 \longrightarrow Z_2$ *is a function, then* $\mathbf{sky_F}(f)$ *is the exponential transpose of the composite*
$$Z_1^{[a \in F]} \times [a \in F] \xrightarrow{\mathbf{ev}} Z_1 \xrightarrow{f} Z_2 \tag{6.11}$$
where $\mathbf{ev}$ *is the evaluation arrow.*

The above definition does not give an account of the weak and strong restriction maps of $\mathbf{sky_F}(Z)$ as they emerge naturally (and trivially) as we see next.

Indeed, for $a_1, a_2 \in B$ with $a_1 \leq a_2$ or $a_1 \prec a_2$, we have the following logical implications:

if $* \in [a \in F]$, then $a_1 \in F$ and hence $a_2 \in F$, i.e. $* \in [a_2 \in F]$ because F is upper closed.

By the definition of the skyscraper sheaf the above line can be read as:

if $* \in [a_1 \in F]$, then $\mathbf{sky_F}(Z)(a_1) = \mathbf{sky_F}(Z)(a_2) = Z$.

So, we define the natural weak and strong restriction maps to be $id : \mathbf{sky_F}(Z)(a_2) \longrightarrow \mathbf{sky_F}(Z)(a_1)$. We are going to use the simple notation $*|_{a_1}$ for both the weak and strong restriction $\mathbf{sky_f}(Z)(a_2) \longrightarrow \mathbf{sky_f}(Z)(a_1)$, for an element $* \in \mathbf{sky_f}(Z)(a_2)$ in the case where $a_1 \leq a_2$ or $a_1 \prec a_2$. The weak and strong restriction maps trivially compose in a way that make $\mathbf{sky_F}$ a presheaf with approximation over B. Moreover, the following lemma asserts that $\mathbf{sky_F}(Z)$ is a $\mathcal{B}$-sheaf over B.

Lemma 6.6 *For any set* Z*,* $\mathbf{sky_F}(Z)$ *is a B-sheaf over* B*. Therefore* $\mathbf{sky_f}$ *is a functor* $\mathbf{Sets} \longrightarrow \mathcal{B}\mathbf{Sh}(B)$*.*

Proof. The fact that $\mathbf{sky_F}(Z)$ is a pasting presheaf with continuous approximation amounts to the fact that F is a rounded prime filter.

First we prove that $\mathbf{sky_F}(Z)$ has pasting. Let $x \in \mathbf{sky_F}(Z)(a)$ and $y \in \mathbf{sky_F}(Z)(b)$ such that $x|_{a \wedge b} = y|_{a \wedge b}$. Suppose that $* \in [a \vee b \in F]$. Then either $a \in F$ or $b \in F$ because of primeness. This means that either $\mathbf{sky_F}(Z)(a) = Z$ or $\mathbf{sky_F}(Z)(b) = Z$. Suppose without loss of generality that $\mathbf{sky_F}(Z) = Z$. Then x as an element of $\mathbf{sky_F}(Z)(a \vee b)$ if restricted to $\mathbf{sky_F}(Z)(a)$ remains identical. This is well defined because if both $a, b \in F$, then $* \in [a \in F]$ and $* \in [b \in F]$ then $a \in F$ and $b \in F$ and because F is a filter, $a \wedge b \in F$ or $* \in [a \wedge b \in F]$ or $\mathbf{sky_F}(Z)(a \wedge b) = Z$ and all four restriction maps are identities.

Finally, it easy to prove that $\mathbf{sky_F}(Z)$ is continuous. Let $* \in [a \in F]$ or $a \in F$. Then $\mathbf{sky_F}(Z)(a) = Z$. The assumption also implies that for any $a' \succ a$, $a' \in F$ or $* \in [a' \in F]$. Hence, for any $a' \succ a$, $\mathbf{sky_F}(Z)(a') = Z$. Therefore, $colim_{a' \succ a}\mathbf{sky_F}(Z)(a') = Z$. ■

We also have the following result which is in accordance with the ordinary sheaf theory.

Theorem 6.7 *Let B a strong proximity lattice and let $\mathbf{i}$ be the inclusion of categories $\mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$. Then for any prime rounded filter F, $\mathbf{i} \circ \mathbf{sky_F}$ is the right adjoint $\mathbf{stalk_F}$.*

Proof. First we study the counit of the adjunction. We are seeking a natural transformation $\varepsilon : \mathbf{stalk_F} \circ \mathbf{i} \circ \mathbf{sky_f} \longrightarrow id_{\mathbf{Sets}}$. For any set Z, the components of ε are functions

$$\varepsilon_Z : colim_{a_i \in F} Z^{[a_i \in F]} \longrightarrow Z$$

The domain of ε is just Z and so we set $\varepsilon_Z = id_Z$. We need to show that for any function $g : \mathbf{stalk_F}(V) \longrightarrow Z$, where V is any presheaf and Z any set, there is a unique approximating presheaf morphism $h : V \longrightarrow \mathbf{i} \circ \mathbf{sky_F}(Z)$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{stalk_F} \circ \mathbf{i} \circ \mathbf{sky_F}(Z) \arrow[r, "\varepsilon_Z"] & Z \\
\mathbf{stalk_F}(V) \arrow[u, dashed, "\mathbf{stalk_F}(h)"] \arrow[ur, "g"'] &
\end{tikzcd} \tag{6.12}$$

We consider the following composite maps between $V(a) \times [a \in F] \longrightarrow Z$:

$$V(a) \xhookrightarrow{\theta^{a,F}} colim_{a \in F} V(a) \xrightarrow{g} Z \tag{6.13}$$

where $\theta^{a,F} : V(a) \hookrightarrow colim_{a \in F} V(a)$ are the colimit injections. We denote h the exponential transpose of the map whose components are given as the composites 6.13. To prove that it has the desired properties, we argue on the following diagram.

$$
\begin{tikzcd}
V(a) \arrow[rr, "h_a"] \arrow[d, hook, "\theta^{a,F}"'] & & Z^{[a\in F]} \arrow[d, hook, "in_a"] \\
\mathbf{stalk_F}(V) \arrow[rr, "\mathbf{stalk_F}(h)"] \arrow[d, "g"'] & & \mathbf{stalk_F} Z^{[a\in F]} \arrow[dll, "\varepsilon_Z"] \\
\mathbf{stalk_F}(V) & &
\end{tikzcd}
\tag{6.14}
$$

The arrow in_a is the colimit injection. The upper square commutes as it is the property, for any $a \in F$, of the map $\mathbf{stalk_F}(h)$ given a map h (see diagram 6.10). The lower triangular diagram is diagram 6.12.

To prove uniqueness of h, we observe that for any $a \in F$, $Z^{[a\in F]}$, the colimit injection in_a is the identity. We have established that $\mathbf{stalk_F} Z^{[a\in F]} \cong Z$ and defined $\varepsilon_Z = id_Z$. Therefore, if the lower diagram commutes, the outer diagram commutes and this yields $h_a = g \circ \theta^{a,F}$. This asserts that h thus defined is indeed the only possible map that makes the diagram 6.12 commutative.

To prove that h actually makes diagram 6.12 commutative, we observe that the outer diagram in 6.14 is commutative by the definition of h and the upper square is always commutative. This together with the fact that $\theta^{a,F}$ is 1-1 implies that the triangular diagram is commutative. ■

We recall (definition 5.15) that $\mathcal{B}\mathbf{Sh}(B)$ is a full subcategory of $\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$. That means that the inclusion functor

$$\mathbf{i} : \mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$$

is full and faithful. Similarly the following category inclusions are full and faithful

$$\mathbf{i_1} : \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)$$

$$\mathbf{i_2} : \mathcal{B}\mathbf{Sh}(B) \hookrightarrow \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$$

where $\mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)$ is as in definition 5.14. Using this insight, theorem 6.7 has the following corollary.

Corollary 6.8 *For any prime rounded filter F of a strong proximity lattice B, the following adjoint situations hold:*

(i) $\mathbf{stalk_F} \circ \mathbf{i} \dashv \mathbf{sky_F}$.

(ii) $\mathbf{stalk_F} \circ \mathbf{i_1} \dashv \mathbf{i_2} \circ \mathbf{sky_F}$.

Proof. (i) For any $\mathcal{B}$-sheaf V and any set Z, we have the following sequence of isomorphisms between hom-sets that are natural in both V and Z:

$$\begin{aligned} \mathbf{Sets}(\mathbf{stalk_F}(\mathbf{i}(V)), Z) &\cong \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)(\mathbf{i}(V), \mathbf{i} \circ \mathbf{sky_F}(Z)) \quad \text{(theorem 6.7)} \\ &\cong \mathcal{B}\mathbf{Sh}(B)(V, \mathbf{sky_F}(Z)) \ \ (\mathbf{i} \text{ is full and faithful}) \end{aligned}$$

(ii) In the same fashion, for any presheaf with pasting V and any set Z, we have

$$\begin{aligned} \mathbf{Sets}(\mathbf{stalk_F}(\mathbf{i_1}(V)), Z) &\cong \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)(\mathbf{i_1}(V), \mathbf{i} \circ \mathbf{sky_F}(Z)) \quad \text{(theorem 6.7)} \\ &\cong \mathbf{Pre}\mathcal{B}\mathbf{Sh}(B)(\mathbf{i_1}(V), \mathbf{i_1} \circ \mathbf{i_2} \circ \mathbf{sky_F}(Z)) \\ &\cong \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B)(V, \mathbf{i_2} \circ \mathbf{sky_F}(Z)) \ \ (\mathbf{i_1} \text{ is full and faithful}) \end{aligned}$$

■

Now we are in position to prove the main result of this section which is that sheavification leaves the stalks intact.

Theorem 6.9 *Let V be a presheaf over a strong proximity lattice B and let F be a rounded prime filter (point) of B. Then, the stalk of V over F is isomorphic to the stalk of the free pasting presheaf* $\mathbf{past}(V)$ *above V over F, or in other words*

$$\mathbf{stalk_F} \circ \mathbf{i_1}\mathbf{past}(V) \cong \mathbf{stalk_F}(V)$$

where **past** *is as in theorem 5.28.*

Proof. We have seen (theorem 5.28) that **past** is the left adjoint of $\mathbf{i_1}$

$$\mathbf{Pre}\mathcal{B}\mathbf{Sh}(B) \underset{\mathbf{i_1}}{\overset{\mathbf{past}}{\rightleftarrows}} \mathbf{PastPre}\mathcal{B}\mathbf{Sh}(B) \underset{\mathbf{i_2} \circ \mathbf{sky_f}}{\overset{\mathbf{stalk_F} \circ \mathbf{i_1}}{\rightleftarrows}} \mathbf{Sets}$$

The two adjoint pairs $\mathbf{stalk_F} \circ \mathbf{i_1} \dashv \mathbf{i_2} \circ \mathbf{sky_F}$ (corollary 6.8(ii)) and $\mathbf{past} \dashv \mathbf{i_1}$ compose nicely to yield that

$$\mathbf{stalk_F} \circ \mathbf{i_1} \circ \mathbf{past} \dashv \mathbf{i_1} \circ \mathbf{i_2} \circ \mathbf{sky_F} \quad \text{or} \quad \mathbf{stalk_F} \circ \mathbf{i_1} \circ \mathbf{past} \dashv \mathbf{i} \circ \mathbf{sky_F} \tag{6.15}$$

But by theorem 6.7 $\mathbf{stalk_F} \dashv \mathbf{i} \circ \mathbf{sky_F}$ and uniqueness of the left adjoint forces $\mathbf{stalk_F} \circ \mathbf{i_1} \circ \mathbf{past} = \mathbf{stalk_F}$ ■

If X is a locale then its points are morphisms $\mathbf{1} \longrightarrow X$ in **Loc**, i.e. continuous maps from the terminal object (of **Loc**) to X. The nomenclature of this thesis dictates that a global point of the *topos of sheaves* over X is still to be denoted as

$$x : \mathbf{1} \longrightarrow X \tag{6.16}$$

where now $\mathbf{1}$ is the terminal object and x an arrow, both in $\mathfrak{Top}$. The consistency of this notation is of course due to the full embedding of **Loc** in $\mathfrak{Top}$. By definition, an arrow in $\mathfrak{Top}$ is a geometric morphism $x : \mathbf{1} \longrightarrow X$ which amounts to a pair of functors $x^* : Sh(X) \longrightarrow \mathbf{Sets}$ and $x_* : \mathbf{Sets} \longrightarrow Sh(X)$ such that x_* is right adjoint to x^* and x^* is left exact. In standard textbooks (e.g. Mac Lane & Moerdijk) one can pin down the adjoint pair $x^* \dashv x_*$: if F is a sheaf over X and A a set then

$$x^* = stalk_x F \quad x_* = sky_x A$$

where $stalk_x F := colim_{x \vDash a} F(a)$ is the stalk of F over a point of the locale X and $sky_x A$ is the "skyscraper" sheaf with respect to the point x. Obviously, there are as many pairs $stalk_x \dashv sky_x$ as points of X.

Next, for X a stably compact locale, we are going to discuss the evaluation map

$$\mathbf{ev} : [set]^X \longrightarrow [set] \quad (\text{in } \mathfrak{Top})$$

We make two remarks before we start. First that the exponential $[set]^X$ exists in $\mathfrak{Top}$ (as shown in chapter 5 and also by Johnstone Joyal in [JJ82]) and hence merits an evaluation map. Second, that having given a geometric account of the pair $stalk_x \dashv sky_x$ in the case where X is stably compact, is going to facilitate the arguments.

We recall that $\mathbf{ev}$ is the map $\varepsilon_{[set]}$, where ε is the counit of the adjunction $(-)^X \dashv (- \times X)$ with $(-)^X, (-) \times X : \mathfrak{Top} \longrightarrow \mathfrak{Top}$. Note that this is an adjunction up to equivalence. The universal property of $\mathbf{ev}$ says that given a stably compact locale X, then for any Grothendieck topos Z and any geometric morphism $f : Z \times X \longrightarrow [set]$, there is an (up to equivalence) unique geometric morphism $f' : Z \longrightarrow [set]^X$ such that the following triangle commutes

$$\begin{tikzcd} Z \times X \arrow[d, dashed, "f' \times id"'] \arrow[dr, "f"] & \\ {[set]^X \times X} \arrow[r, "\mathbf{ev}"] & {[set]} \end{tikzcd} \tag{6.17}$$

Theorem 5.24 proves that $[set]^X \simeq [\mathcal{B}X\text{-sheaves}]$. therefore the (global) points of the product $[set]^X \times X$ are pairs (V, x), where V is a $\mathcal{B}$-sheaf over the strong proximity lattice

$\mathcal{B}X$ and x a point of the locale X. We claim that the geometric morphism $\mathbf{ev}$ acts on *global points* as

$$\mathbf{ev} : (V, x) \mapsto \mathbf{stalk}_x V := colim_{x \vDash a} V(a) \tag{6.18}$$

Although it is with respect of global points only, the above description is geometric since it involves a colimit and so it is sufficient to specify the map $\mathbf{ev}$ up to equivalence. We have to show that the map that, on global points, acts as in expression 6.18 has the universal property 6.17. Let us call it s. We first consider the special case where Z is $\mathbf{1}$. Then the triangle 6.17 pre-composed with a global point map $\mathbf{1} \longrightarrow$ becomes

$$\begin{tikzcd} \mathbf{1} \arrow[r, "x"] & \mathbf{1} \times X \arrow[d, dashed, "f' \times id"'] \arrow[dr, "f"] & \\ & {[set]^X \times X} \arrow[r, "s"] & {[set]} \end{tikzcd} \tag{6.19}$$

Fixing a geometric morphism f is the same as fixing a sheaf over X in **Sets** (which we still denote as f) and we know that this is the same up to equivalence with a $\mathcal{B}$-sheaf over $\mathcal{B}X$. Let us look at the composite

$$\mathbf{1} \xrightarrow{\quad x \quad} X \xrightarrow{\quad f \quad} [set] \tag{6.20}$$

Its inverse image functor $x^* \circ f^*$ first picks the sheaf f and then calculates its stalk at the point x. By theorem 6.1, this is the same (up to equivalence) as calculating the stalk of the corresponding $\mathcal{B}$-sheaf over $\mathcal{B}X$ and this description of $x^* \circ f^*$ has the advantage of being geometric. So, if we let $f' : \mathbf{1} \longrightarrow [set]^X \simeq [\mathcal{B}\text{-sheaves}]$ be the map that picks the same B-sheaf (i.e. the exponential transpose of f), the triangle 6.19 commutes and clearly f' is the unique map with such property (up to equivalence).

What the above tells us is that the functor

$$\Phi : \mathfrak{Top}(\mathbf{1}, [set]^X) \xrightarrow{\simeq} \mathfrak{Top}(X, [set]) \tag{6.21}$$

defined on objects by $f \mapsto s\circ < f, id >$ is an equivalence of categories. To generalise the argument for any topos Z, it suffices to consider the functor Φ and equivalence of 6.21 *over* any topos Z. The terminal object over Z is the identity map $id : Z \longrightarrow Z$. Also, $[set]^X$ is the classifying topos of $\mathcal{B}$-sheaves over $\mathcal{B}X$, therefore the L.H.S. of 6.21 becomes $\mathfrak{Top}/Z(Z, [!^*(\mathcal{B}X)\text{-sheaves}]_Z)$, where $[!^*(\mathcal{B}X)\text{-sheaves}]_Z$ is the classifying topos over Z of the geometric theory whose models are $!^*(\mathcal{B}X)$-sheaves and by corollary 5.27

this is the same as the Z-topos $p_2 : [\mathcal{B}X\text{-sheaves}] \times Z \longrightarrow Z$ (where p_2 is the second projection).[?] Furthermore, over Z, the locale X becomes the localic geometric morphism $p_2 : X \times Z \longrightarrow Z$ (the second projection) and the object classifier $[set]_Z$ over Z is $p_2 : [set] \times Z \longrightarrow Z$. Therefore, over Z, the equivalence 6.21 yields

$$\begin{aligned}
\mathfrak{Top}/Z(Z, [!^*(\mathcal{B}X)\text{-sheaves}]_Z) &\simeq \mathfrak{Top}/Z(X \times Z \longrightarrow Z, [set]_Z) \\
\Leftrightarrow \mathfrak{Top}/Z(Z, [\mathcal{B}X\text{-sheaf}] \times Z) &\simeq \mathfrak{Top}/Z(X \times Z, [set] \times Z) \\
\Leftrightarrow \mathfrak{Top}/Z(Z, [set]^X \times Z) &\simeq \mathfrak{Top}/Z(X \times Z, [set] \times Z) \text{ (by [?])} \\
\Leftrightarrow \mathfrak{Top}(Z, [set]^X) &\simeq \mathfrak{Top}(X \times Z, [set]) \text{ (by [?])}
\end{aligned}$$

This proves that the map that we denoted by s indeed satisfies the universal property reflected in diagram 6.17 and hence s is identical up to equivalence to $\mathbf{ev}$. This fact together with corollary 6.2 yield the following conclusion.

Theorem 6.10 *Let X be a stably compact locale. Then the evaluation map $\mathbf{ev} : [set]^X \times X \longrightarrow [set]$ acts on global points as*

$$(F, x) \mapsto stalk_x F \quad \textit{(non geometric version)}$$

where F is a sheaf over X and x a point of X, or equivalently as

$$(V, x) \mapsto \mathbf{stalk_F} V \quad \textit{(geometric version)}$$

where V is the corresponding $\mathcal{B}$-sheaf over $\mathcal{B}X$ and F the rounded prime filter that corresponds to x.

6.3 The inverse image functor

Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices. We show in this section that this induces a functor

$$\rho_\mu : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2) \longrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1)$$

between the categories of continuous presheaves with approximation over B_2 and B_1. The reason why we call this functor "inverse image" will be obvious in section 6.7 although it is not hard justify. In chapter 3 we demonstrated how the map μ gives rise to a perfect map $\mathbf{RSpec}(\mu) : \mathbf{RSpec}(B_1) \longrightarrow \mathbf{RSpec}(B_2)$, where $\mathbf{RSpec}(X), \mathbf{RSpec}(Y)$ are stably compact locales. We see, therefore, that $\mu : B_2 \longrightarrow B_1$ is pointing at the same direction as the defining frame homomorphism $(\mathbf{RSpec}\mu)^*$.

It is interesting that the object part construction $W \mapsto \rho_\mu(W)$, where W is an approximating presheaf over B_2, does not rely on the continuity of W (but it always yields a continuous presheaf with approximation over B_1). Nevertheless, we need to assume that W is continuous in order for ρ_μ to be functorial. Indeed, we show that given a morphism $g : W^{(1)} \longrightarrow W^{(2)}$, we can construct a map $f^w : \rho_\mu(W^{(1)}) \longrightarrow \rho_\mu(W^{(2)})$ which is natural with respect to the weak restrictions or, alternatively, a map $f^s : \rho_\mu(W^{(1)}) \longrightarrow \rho_\mu(W^{(2)})$ which is natural with respect to the strong restrictions. We find that the two maps coincide when $W^{(1)}$ and $W^{(2)}$ are continuous and so $f^w = f^s$ is then an approximating presheaf morphism.

All the constructions and proofs are geometric, so these categories may well be inside the sheaves of any topos. In particular the construction of ρ_μ involves colimits.

Definition 6.11 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices and $W : B_2 \longrightarrow$ **Sets** a continuous presheaf with approximation. By $V'(a)$ we denote the set*

$$V'(a) = \coprod_i \{W(b_i) \in B_1 X, \qquad a \prec \mu(b_i)\}$$

Now, let $x_1, x_2 \in V(a)$ and in particular $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$. We define the equivalence relation $\sim$ in $V(a)$, generated by

$$x_1 \sim x_2 \qquad \textit{if} \qquad b_2 \leq b_1 \textit{ and } a \prec \mu(b_2)$$

*So finally we define the assignment $V : B_1 \longrightarrow$ **Sets** that maps any $a \in B_1$ to $V'(a)/\sim$*

We make the observation that the equivalence relations on the sets $V'(a), a \in B_1$ are defined in relation only to the weak partial order of the strong proximity lattice. We are going to use the following lemma.

Lemma 6.12 [1] *In $V'(a), a \in B_1$, let $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$. Then $x_1 \sim' x_2$ iff there is $b_{12} \leq b_1 \wedge b_2 \in B_2$ with $a \prec \mu(a)$ such that $x_1{}_{|_{b_{12}}} = x_2{}_{|_{b_{12}}}$*

Proof. First we prove that $\sim'$ is an equivalence relation. It is obviously reflexive and symmetric. It takes a small proof to show that it is also transitive. Indeed, let $x_1, x_2, x_3 \in V(a)$ and in particular $x_1 \in W(b_1), x_2 \in W(b_2), x_3 \in W(b_3)$ such that $x_1 \sim' x_2$ and $x_2 \sim' x_3$. This means that there are $b_{12} \leq b_1 \wedge b_2$ and $b_{23} \leq b_2 \wedge b_3$ such that $x_1{}_{|_{b_{12}}} = x_2{}_{|_{b_{12}}}$

[1] I have an other version of this one, but I have to extract it from my laptop first! Please skip this lemma-we have gone through it on the board anyway.

and $x_2|_{b_{23}} = x_3|_{b_{23}}$. Since $b_{12} \wedge b_{23} \leq b_{12}$ and $b_{12} \wedge b_{23} \leq b_{23}$ we deduce that $x_1|_{b_{12}\wedge b_{23}} = x_2|_{b_{12}\wedge b_{23}}$ and $x_2|_{b_{12}\wedge b_{23}} = x_3|_{b_{12}\wedge b_{23}}$. Therefore, $x_1|_{b_{12}\wedge b_{23}} = x_3|_{b_{12}\wedge b_{23}}$. The restrictions of x_1 and x_3 coincide in $W(b_{12} \wedge b_{23})$, so they have to coincide in $W(b_{12} \wedge b_{23} \wedge b_1 \wedge b_3)$, because $b_{12} \wedge b_{23} \wedge b_1 \wedge b_3 \leq b_{12} \wedge b_{23}$. So we have proved that $x_1|_{b_{12}\wedge b_{23}\wedge b_1\wedge b_3} = x_3|_{b_{12}\wedge b_{23}\wedge b_1\wedge b_3}$ and $b_{12} \wedge b_{23} \wedge b_1 \wedge b_3 \leq b_1 \wedge b_3$ i.e. $x_1 \sim' x_3$ by the definition of $\sim'$.

Now we prove that in fact $\sim' = \sim$. Let $x_1, x_2 \in V(a)$ and $x_1 R x_2$. That implies that $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$ with $b_1 \leq b_2$ and $a \prec \Phi(b_1)$ such that $x_2|_{b_1} = x_1$. This equality can be rewritten as $x_1|_{b_1\wedge b_2} = x_2|_{b_1\wedge b_2}$ since $b_1 \wedge b_2 = b_1$. And this is the defining property of $x_1 \sim' x_2$. So $R \subseteq \sim'$. But because $\sim$ is the smallest equivalence relation being a superset of R, we deduce that $\sim \subseteq \sim'$.[!!!]

Working for the other direction, let $x_1, x_2 \in V(a)$ and in particular $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$ with $x_1|_{b_{12}} = x_2|_{b_{12}}$ for some $b_{12} \leq b_1 \wedge b_2$ in B_2. Then $x_1 R x_1|_{b_{12}}$ and $x_2 R x_2|_{b_{12}}$, so $x_1 \sim x_2$. Therefore $\sim' \subseteq \sim$.

This proves that the two equivalent relations are the same. Henceforth they are going to be denoted $\sim$. ■

We point out here that the equivalence relation $\sim$ is defined with respect to the strong order $\prec$ of the strong proximity lattice B_2; sometimes we shall denote as "$\sim^s$". In fact, we can also use the weak order $\leq$ of B_2 to define equivalence relations as follows. On the sets $V'(a)$ we introduce a relation R^w by stipulating that given that $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$ with $a \prec \mu(b_1)$ and $a \prec \mu(b_2)$, $x_1 R^w x_2$ iff $b_1 \leq b_2$ and $x_1 = \beta^{b_2}_{b_1}(x_2)$, where β are the weak restriction maps of W. We define $\sim^w$ to be the equivalence relation generated by R^w. In correspondence with the strong case, we also define a relation $\sim'^w$ on $V'(a)$ by stipulating that for $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$ with $a \prec \mu(b_1)$ and $a \prec \mu(b_2)$, $x_1 \sim'^w x_2$ iff there is $b_{12} \leq b_1 \wedge b_2$ with $a \prec \mu(b_{12})$ such that $\beta^{b_1}_{b_{12}}(x_2) = \beta^{b_2}_{b_{12}}(x_2)$. We have the following lemma.

Lemma 6.13 *Let $(W, \beta, \delta) : B_2 \longrightarrow$ **Sets** be a presheaf with approximation.*

(i) The relation $\sim'^w$ defined above is an equivalence relation and $\sim'^w = \sim^w$.

(ii) If in addition W is continuous then $\sim = \sim^w$.

Proof. This proof of (i) is a slight modification of the proof of lemma **??**. To prove that $\sim'^w$ is transitive and hence an equivalence relation, we first *weakly* restrict x_1 and x_2 to $b_{12} \wedge b_{23}$ where now $b_{12} \leq b_1 \wedge b_2$ and $b_{23} \leq b_2 \wedge b_3$. We next further weakly restrict to $b_{12} \wedge b_{23} \wedge b_1 \wedge b_3$. The restrictions of x_1 and x_3 coincide in $W(b_{12} \wedge b_{23})$, so they have to coincide in $W(b_{12}) \wedge b_{23} \wedge b_1 \wedge b_3)$, because $b_{12} \wedge b_{23} \wedge b_1 \wedge b_3 \leq b_{12} \wedge b_{23}$. So we have proved that $\beta^{b_1}_{b_{12}\wedge b_{23}\wedge b_1\wedge b_3}(x_1) = \beta^{b_2}_{b_{12}\wedge b_{23}\wedge b_1\wedge b_3}(x_2)$ and $b_{12} \wedge b_{23} \wedge b_1 \wedge b_3 \leq b_1 \wedge b_3$. Also

$a \prec \mu(b_{12} \wedge b_{23} \wedge b_1 \wedge b_3)$ because $a \prec \mu(b_{12})$, $a \prec \mu(b_{23})$, $a \prec \mu(b_1)$ and $a \prec \mu(b_3)$. Hence $x_1 \sim'^w x_3$ by the definition of $\sim'^w$.

Reflexivity of the weak order $\leq$ makes the proof of $\sim^w = \sim'^w$ trivial. For x_1, x_2 as in the proof of lemma **??**, $x_1 R^w x_2$ means that $b_1 \leq b_2$ and $x_1 = \beta^{b_2}_{b_1}(x_2)$. Choosing $b_{12} := b_1 \wedge b_2 = b_1 \leq b_1 \wedge b_2$ proves that $R^w \subseteq \sim'^w$. The other direction is obvious (and the same as lemma **??** with $b_{12} \leq b_1 \wedge b_2$ instead of $b_{12} \prec b_1 \wedge b_2$.)

Finally, suppose that (W, β, δ) is continuous. We prove that $\sim^w = \sim$. Let $x_1 \in W(b_1)$ and $x_2 \in W(b_2)$ with $a \prec \mu(b_i)$, $i = 1, 2$ and $x_1 \sim x_2$. Then there is $b_{12} \prec b_1 \wedge b_2$ with $a \prec \mu(b_{12})$ such that $\delta^{b_1}_{b_{12}}(x_1) = \delta^{b_2}_{b_{12}}(x_2)$. We repeat the simple trick of choosing $b'_{12} = b_{12} \wedge b_1 \wedge b_2$. Then $b'_{12} \leq b_1 \wedge b_2$, $a \prec \mu(b'_{12})$. We have also that $b'_{12} \leq b_{12} \prec b_1 \wedge b_2$, i.e., $b'_{12} \prec b_1 \wedge b_2$. x_1 and x_2 coincide when strongly restricted in $W(b_{12})$ so they have to coincide in $W(b'_{12})$ because $\delta^{b_i}_{b'_{12}} = \beta^{b_{12}}_{b'_{12}} \circ \delta^{b_i}_{b_{12}}(x_i)$ $(i = 1, 2)$. Lemma 5.13 guarantees that $\beta^{b_1}_{b_{12}}(x_1) = \delta^{b_1}_{b_{12}}(x_1)$ and $\beta^{b_2}_{b_{12}}(x_2) = \delta^{b_2}_{b_{12}}(x_2)$, therefore $\beta^{b_1}_{b'_{12}}(x_1) = \beta^{b_2}_{b'_{12}}(x_2)$ which means that $x_1 \sim^w x_2$.

For the other direction, let $x_1 \in W(b_1)$,$b_2 \in W(b_2)$ and $x_1 \sim^w x_2$, meaning that there is $b_{12} \leq b_1 \wedge b_2$ with $a \prec \mu(b_{12})$ such that $\beta^{b_1}_{b_{12}}(x_1) = \beta^{b_2}_{b_{12}}(x_2)$. Then, μ being a strong homomorphism, there is $b'_{12} \prec b_{12}$ with $a \prec \mu(b_{12})$. Then it holds that $\delta^{b_i}_{b'_{12}}(x_i) = \delta^{b_{12}}_{b'_{12}} \circ \beta^{b_i}_{b_{12}}(x_i)$ $(i = 1, 2)$ and so $\delta^{b_1}_{b'_{12}}(x_1) = \delta^{b_2}_{b'_{12}}(x_2)$. ■

The second part of the above lemma is a facet of the fact that if W is a presheaf with continuous approximation then its weak restriction maps are determined by its strong restriction maps.

Definition 6.14 *Assuming that (W, β, δ) is a presheaf with a (not necessarily continuous) approximation, for each $a \in B_1$, we denote*

$$V^s(a) := V'(a)/\sim \quad \textit{and} \quad V^w(a) := V'(a)/\sim^w$$

We now express them using categorical language. Consider the category $(B_2, \leq)$, i.e., the lattice B_2 qua poset with respect to its weak order. We denote by J^w_a the full subcategory of $(B_2, \leq)$ that includes all the elements b_i of B_2 such that $a \prec \mu(b_i)$. Then $W : J^w_a \longrightarrow$ **Sets** becomes a diagram in **Sets** with J^w_a the index category. When writing

$$colim^w_{a \prec \mu(b_i)} W(b_i)$$

we mean the colimit of the diagram $W : J^w_a \longrightarrow$ **Sets**. It is routine to check, by looking at definition **??**, that

$$colim^w_{a \prec \mu(b_i)} W(b_i) \cong V^w(a)/\sim^w$$

We can apply the same logic to the construction of $V^s(a)$. Now the structure $(B_2, \prec)$, i.e., the strong proximity lattice with only the strong order is not a category because of the absence of the identities. We therefore define $(B_2, \prec, id)$ to be the category whose objects are the elements of B_2 and whose arrows are those given by the relation $\prec$ together with the identity relation for all elements of B_2. As before, we define J_a^s to be the full subcategory of $(B_2, \prec, id)$ that includes all the elements b_i with $a \prec \mu(b_i)$. We denote by

$$colim^s_{a\prec\mu(b_i)} W(b_i)$$

the colimit of the diagram $W : J_a^s \longrightarrow \mathbf{Sets}$. Again it is easy to check that

$$colim^s_{a\prec\mu(b_i)} W(b_i) \cong V^s / \sim \tag{6.22}$$

(Adjoining the identities in the index category J_a^s is within our freedom because $x \sim^s x$). The following is a consequence of lemma 6.13.

Corollary 6.15 *Let $W : B_2 \longrightarrow$ **Sets** be a continuous presheaf. Then*

$$colim^w_{a\prec\mu(b_i)} W(b_i) \cong colim^s_{a\prec\mu(b_i)} W(b_i)$$

Using coproducts, the same isomorphisms are written as

$$\coprod_{a\prec\mu(b_i)} W(b_i)/\sim^w \cong \coprod_{a\prec\mu(b_i)} W(b_i)/\sim^s \tag{6.23}$$

The R.H.S. "strong" colimit (and its index category J_a^s) is henceforth going to be denoted without the superscript ^{s}.

The next step is to define weak and strong restriction maps ϕ, θ on the family of sets $V(a) := colim_{a\prec\mu(b_i)} W(b_i)$, $a \in B_1$.

Suppose that $a_1 \prec a_2$ in B_1. Then, with the above notation, $a_2 \prec \mu(b_i)$ implies $a_1 \prec \mu(b_i)$ for any $b_i \in B_2$, which renders J_{a_2} a full subcategory of J_{a_1}. The cone of $colim_{a_1\prec\mu(b_i)}$ is a cone of the diagram $W : J_{a_2} \longrightarrow$ **Sets**. So there must be a unique map

$$\theta^{a_2}_{a_1} : colim_{a_2\prec\mu(b_j)} W(b_j) \longrightarrow colim_{a_1\prec\mu(b_i)} W(b_i)$$

For $a_1 \leq a_2$ in B_1, the fact that $\leq \circ \prec = \prec$, allows us to repeat the argument of the above paragraph to determine a unique map

$$\phi^{a_2}_{a_1} : colim_{a_2 \prec \mu(b_j)} W(b_j) \longrightarrow colim_{a_1 \prec \mu(b_i)} W(b_i)$$

We have the following lemma.

Lemma 6.16 *Let (W, β, δ) be a presheaf with approximation over the strong proximity lattice B_2. Then (V, ϕ, θ), with $V(a) = colim_{a \prec \mu(b_i)} W(b_i)$ and ϕ, θ defined as above, is a presheaf with approximation over the strong proximity lattice X_1.*

Proof. We need to prove that the maps ϕ, θ behave as restriction maps, or more specifically that

(i) $\phi^a_a = id_{V(a)}$ and for $a_1 \leq a_2 \leq a_3$, $\phi^{a_2}_{a_1} \circ \phi^{a_3}_{a_2} = \phi^{a_3}_{a_1}$.

(ii) for $a_1 \prec a_2 \prec a_3$, $\theta^{a_2}_{a_1} \circ \theta^{a_3}_{a_2} = \theta^{a_3}_{a_1}$.

(iii) the strong restrictions absorb the weak ones, i.e., for $a_1 \leq a_2 \prec a_3 \leq a_4$, $\theta^{a_2}_{a_1} \circ \phi^{a_3}_{a_2} \circ \theta^{a_4}_{a_3} = \theta^{a_3}_{a_1}$.

All these properties can be easily verified by looking at the defining diagrams of ϕ and θ. ∎

Switching back to the concrete description of colimits in terms of coproducts (disjoint unions) we give the following equivalent definition of the weak (strong) restriction maps β (θ). For $a_1 \leq a_2$ $(a_1 \prec a_2)$, we define the obvious inclusion maps $\beta' : V'(a_2) \mapsto V'(a_1)$ $(\theta' : V'(a_2) \hookrightarrow V'(a_1))$. Then, denoting $[x]$ the equivalence class with respect to $\sim^w$ $(\sim^s)$ of an element $x \in V'(a_2)$, we have

$$[\beta'^{a_2}_{a_1}(x)] = \beta^{a_2}_{a_1}([x]) \quad ([\theta'^{a_2}_{a_1}(x)] = \theta^{a_2}_{a_1}([x]))$$

The following lemma demonstrates that the assignment of the lemma 6.16 yields a presheaf with continuous approximation.

Lemma 6.17 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices and W a presheaf with approximation over B_2. Then the presheaf with approximation (V, ϕ, θ) over B_1, defined as above, is continuous.*

Proof. We have to show that for every $a \in B_1$ the obvious map

$$\theta_a : colim_{a \prec a'} V(a_i) \longrightarrow V(a)$$

is an isomorphism. By evoking the freedom given to us by the isomorphism 6.22, we choose to construe V as the assignment

$$a \mapsto \coprod_{a \prec \mu(b_i)} W(b_i)/\sim^s$$

First we show that θ_a is an injection. We assume that for $z_1, z_2 \in colim_{a \prec a_i} V(a_i)$ we have $\theta_a(z_1) = \theta_a(z_2)$. We are going to prove that $z_1 = z_2$. We use θ with single superscript to denote the colimit injections and θ_a (single subscript) to denote the obvious map 6.3. We also denote $\theta_a(z_1) := x_1$ and $\theta_a(z_2) := x_2$.

Let x'_1 and x'_2 be any elements in $V'(a_1)$ and $V'(a_2)$ respectively whose image in the $colim_{a \prec a_i} V(a_i)$ is z_1 and z_2 respectively (i.e. $\theta'^{a_1}(x'_1) = z_1$ and $\theta'^{a_2}(x'_2) = z_2$). The universal property of θ_a says that $\theta'^{a_1}_a(x'_1) = \theta_a \circ \theta^{a_1}(x'_1) = x_1$ and $\theta'^{a_2}_a(x'_2) = \theta_a \circ \theta^{a_2}(x'_2) = x_2$. It suffices to prove that there is a' with $a \prec a' \prec a_1$ and $a \prec a' \prec a_2$ such that $\theta'^{a_1}_{a'}(x'_1) \sim^s \theta'^{a_2}_{a'}(x'_2)$, i.e., that x'_1 and x'_2 become equivalent before they "reach" the colimit.

Suppose in particular that $x'_1 \in W(b_1)$ and $x'_2 \in W(b_2)$ with $a \prec a_1 \prec \mu(b_1)$ and $a \prec a_2 \prec \mu(b_2)$. The assumption says that there is $b_{12} \prec b_1 \wedge b_2$ with $a \prec \mu(b_{12})$, such that $\delta^{b_1}_{b_{12}}(x'_1) = \delta^{b_2}_{b_{12}}(x'_2)$. The fact that $a_1 \prec \mu(b_1)$ implies $a_1 \wedge a_2 \wedge \mu(b_{12}) \leq a_1 \prec \mu(b_1)$ which yields

$$a_1 \wedge a_2 \wedge \mu(b_{12}) \prec \mu(b_1) \tag{6.24}$$

Similarly,

$$a_1 \wedge a_2 \wedge \mu(b_{12}) \prec \mu(b_2) \tag{6.25}$$

Also, by the definition of the strong proximity lattices, $a \prec a_1$, $a \prec a_2$ and $a \prec \mu(b_{12})$ implies that $a \prec a_1 \wedge a_2 \wedge \mu(b_{12})$. Since $\prec$ is an interpolative order, there is a' such that $a \prec a' \prec a_1 \wedge a_2 \wedge \mu(b_{12})$. Combining this with the expressions 6.24 and 6.25 we have that

$$a \prec a' \prec \mu(b_1) \text{ and } a \prec a' \prec \mu(b_2) \tag{6.26}$$

$$\begin{tikzcd}
V(a_1) \arrow[dr, "\theta^{a_1}_{a_1\wedge a_2}"'] & & V(a_2) \arrow[dl, "\theta^{a_2}_{a_1\wedge a_2}"] & \\
& V(a_1 \wedge a_2)) \arrow[dr, "\theta^{a_1\wedge a_2}_{a_1\wedge a_2\wedge\mu(b_{12})}"] & & V(\mu(b_{12})) \arrow[dl, "\theta^{\mu(b_{12})}_{a_1\wedge a_2\wedge\mu(b_{12})}"] \\
& & V(a_1 \wedge a_2 \wedge \mu(b_{12})) \arrow[d, "\theta^{a_1\wedge a_2\wedge\mu(b_{12})}_{a'}"'] & \\
& & V(a') \arrow[dd, "\theta^{a'}_a"'] \arrow[dr, "\theta^{a'}"] & \\
& & & colim_{a\prec a_i} V(a_i) \arrow[dl, "\theta_a"] \\
& & V(a) &
\end{tikzcd}$$

Expressions 6.26 guarantee that x'_1 and x'_2 can be strongly included in $V'(a')$, i.e.

$$\theta'^{a_1}_{a'}(x'_1), \theta'^{a_2}_{a'}(x'_2) \in V'(a') \quad \text{(as in proof of lemma 6.16)}$$

Furthermore, obviously

$$a' \prec a_1 \wedge a_2 \wedge \mu(b_{12})$$

which yields $a' \prec \mu(b_{12})$. Hence, by definition $\theta'^{a_1}_{a'}(x'_1) \sim^s \theta'^{a_2}_{a'}(x'_2)$ in $V'(a')$, or

$$\theta^{a_1}_{a'}([x'_1]) = \theta^{a_2}_{a'}([x'_2])$$

This in turn means that $z_1 = \theta^{a'}(x'_1) \sim^s \theta^{a'}(x'_2) = z_2$ which proves the 1-1 property.

Next we prove that the morphism θ_a is a surjection. Let $x \in V(a)$ and in particular $x \in W(b)$. This means that $a \prec \mu(b)$. By the interpolative property of $\prec$, there is $a' \in B_1$ such that $a \prec a' \prec \mu(b)$. So x can be written as $x = \theta^{a'}_a(z)$, with z an element of $V(a')$ with $a' \succ a$. Because of the universal property of θ_a, $\theta^{a'}_a$ has to factor through θ_a. So θ_a is epi. This completes the proof of $colim_{a\prec a_i} V(a_i) \cong V(a)$. $\blacksquare$

Corollary 6.18 *Given a strong homomorphism $\mu : B_2 \longrightarrow B_1$, we have determined an assignment (which we now denote ρ_μ) of presheaves with approximation W over B_2 to continuous presheaves V over B_1. This map is defined by*

$$\rho_\mu(W)(a) = colim_{a\prec\mu(b_i)} W(b_i)$$

and with weak and strong restrictions as in lemma 6.16

Next we are going to extend ρ_μ to a functor between the categories of presheaves with approximation over B_2 to the category of continuous presheaves over B_1. We have the following lemma.

Lemma 6.19 *Let $g : (W^{(1)}, \delta(1)) \longrightarrow (W^{(2)}, \delta(2)$ be a morphism of presheaves with approximation over B_2. We write*

$$\rho_\mu(W^{(1)}, \beta(1), \delta(1)) := (V^{(1)}, \phi(1), \theta(1)) \quad \textit{and} \quad \rho_\mu(W^{(2)}, \beta(2), \delta(2)) := (V^{(2)}, \phi(2), \theta(2))$$

We shall argue that for each $a \in B_1$, there is a unique map f_a as depicted in the diagram below.

$$\begin{tikzcd}
W^{(1)}(b_{j1}) \arrow[d, "g_{b_{j1}}"'] \arrow[ddr, "\delta^{b_{j1}}(1)"] & & W^{(1)}(b_{j2}) \arrow[ll, "\delta^{b_{j2}}_{b_{j1}}(1)"'] \arrow[d, "g_{b_{j2}}(1)"] \arrow[ddl, "\delta^{b_{j2}}"'] \\
W^{(2)}(b_{j1}) \arrow[ddr, "\delta^{b_{j1}}(2)"'] & & W^{(2)}(b_{j2}) \arrow[ll, "\delta^{b_{j2}}_{b_{j1}}(2)"'] \arrow[ddl, "\delta^{b_{j2}}(2)"] \\
 & colim_{a \prec \mu(b_i)} W^{(1)}(b_i) \arrow[d, dashed, "f_a"'] & \\
 & colim_{a \prec \mu(b_i)} (W^{(2)}(b_i)) &
\end{tikzcd} \tag{6.27}$$

Proof. Let us denote as $J^{(1)}_a$ and $J^{(2)}_a$ the two defining index diagrams of $colim_{a \prec \mathcal{B} f(b_i)} W^{(1)}(b_i)$ and $colim_{a \prec \mathcal{B} f(b_i)} (W^{(2)}(b_i))$ respectively, in the sense that, e.g. $colim_{a \prec \mathcal{B} f(b_i)} W^{(1)}(b_i)$ is the colimit of the diagram

$$W^{(1)} : J^{(1)} \longrightarrow \mathbf{Sets}$$

By assumption, g is a morphism of presheaves with approximation over B_2, i.e., a natural transformation between the functors $(W^{(1)}, \delta(1))$ and $(W^{(2)}, \delta(2))$ which makes the square diagram on top commutative. That means that the composites $\delta^{b_i}(2) \circ g_{b_i}$, for any b_i with $a \prec \mu(b_i)$, together with the vertex $colim_{a \prec \mu(b_i)} (W^{(2)}(b_i))$ constitute a cone of the diagram $J^{(1)}_a$. So there must be a unique map f_a such that $\delta^{b_i}(2) \circ g_{b_i} = f_a \circ \delta^{b_i}(1)$ for any index i. ■

We stress that this definition involved the *strong* restriction maps of $W^{(1)}$ and $W^{(2)}$. Next we prove that they commute with the strong restriction maps of $V^{(1)}$ and $V^{(2)}$.

Lemma 6.20 *The maps are the components of a natural transformation f between the functors*

$$(V^{(1)}, \theta(1)) \longrightarrow (V^{(2)}, \theta(2))$$

Proof.

$$\begin{tikzcd}
W^{(1)}(b_i) \arrow[r, "g_{b_i}"] \arrow[d, "\delta^{b_i}_{a_2}(1)"'] & W^{(2)}(b_i) \arrow[d, "\delta^{b_i}_{a_2}(2)"'] \\
colim_{a_2 \prec \mu(b_i)} W^{(1)}(b_i) \arrow[r, "f_{a_2}"] \arrow[d, "\theta^{a_2}_{a_1}(1)"'] & colim_{a_2 \prec \mu(b_i)} W^{(2)}(b_i) \arrow[d, "\theta^{a_2}_{a_1}(2)"'] \\
colim_{a_1 \prec \mu(b_i)} W^{(1)}(b_i) \arrow[r, "f_{a_1}"] & colim_{a_1 \prec \mu(b_i)} W^{(2)}(b_i)
\end{tikzcd}$$

We want to prove that the bottom square commutes for any $a_1 \prec a_2$. Since $\delta^{b_i}_{a_2}(1)$ is the generic colimit injection, it suffices to prove that, for any b_i with $a_2 \prec \mu(b_i)$,

$$\theta^{a_2}_{a_1}(2) \circ f_{a_2} \circ \delta^{b_i}_{a_2}(1) = f_{a_1} \circ \theta^{a_2}_{a_1}(1) \circ \delta^{b_i}_{a_2}(1) \tag{6.28}$$

But by the definition (6.16) of $\theta(1)$, $\theta^{a_2}_{a_1}(1) \circ \delta^{b_i}_{a_2}(1) = \delta^{b_i}_{a_1}$. Also, the top square is commutative being the defining diagram of f_{a_2}. So the L.H.S. of the expression 6.28 becomes $\theta^{a_2}_{a_1}(2) \circ \delta^{b_i}_{a_2}(2) \circ g_{b_i}$. But, by the definition of $\theta(2)$, $\theta^{a_2}_{a_1}(2) \circ \delta^{b_i}_{a_2}(2) = \delta^{b_i}_{a_1}(2)$, which means that expression 6.28 becomes

$$\delta^{b_i}_{a_1}(2) \circ g_{b_i} = f_{a_1} \circ \delta^{b_i}_{a_1}(1)$$

which obviously holds by the definition of f_{a_1}. This proves the claim of this lemma. ■

We saw f defined by the "strong" diagram 6.27 commutes with the strong restrictions. In general it does not commute with the weak restrictions though. On the other hand, if we make the additional assumption that $(W^{(1)}, \beta(1(, \delta(1))$ and $(W^{(2)}, \beta(2), \delta(2))$ have continuous approximations we have the following fact.

Lemma 6.21 *Let $W^{(1)}$ and $W^{(2)}$ be presheaves with continuous approximation over B_2. Let f be the natural transformation defined by the diagram 6.27. Then f is a natural transformation between the functors*

$$(V^{(1)}, \phi(1)) \longrightarrow (V^{(2)}, \phi(2))$$

i.e., they also commute with the weak restrictions.

Proof. For any $a \in B_1$, we can define a map

$$f_a^w : colim_{a \prec \mu(b_i)} W^{(1)}(b_i) \longrightarrow colim_{a \prec \mu(b_i)} W^{(2)}(b_i)$$

by repeating the argument of the diagram 6.27 but with weak restrictions β instead of the strong restrictions δ. This "weak" definition gives a unique map f_a^w between $colim^w_{a \prec \mu(b_i)} W^{(1)}(b_i) \longrightarrow colim^w_{a \prec \mu(b_i)} W^{(2)}(b_i)$ and lemma 6.15 guarantees that

$$f_a^w : colim_{a \prec \mu(b_i)} W^{(1)}(b_i) \longrightarrow colim_{a \prec \mu(b_i)} W^{(2)}(b_i)$$

We can also prove that these are the components of a natural transformation f^w between the functors $(V^{(1)}, \phi(1)) \longrightarrow (V^{(2)}, \phi(2))$ in the same way as in lemma 6.20. Therefore, to complete the claim, we have to prove that $f^s = f$. As before we denote by $\beta_a^{b_i}(1)$ and $\beta_a^{b_i}(2)$ the (weak) colimit injections

$$W^{(1)}(b_i) \hookrightarrow colim_{a \prec \mathcal{B}f(b_i)} W^{(1)}(b_i) \quad \text{and} \quad W^{(2)}(b_i) \hookrightarrow colim_{a \prec \mathcal{B}f(b_i)} W^{(2)}(b_i)$$

respectively. In order to prove $f_a^w = f_a$, it suffices to prove that for all b_i with $a \prec \mu(b_i)$,

$$f_a^w \circ \beta_a^{b_i}(1) = f_a \circ \beta_a^{b_i}(1) \tag{6.29}$$

Since $W^{(1)}$ and $W^{(2)}$ are continuous,

$$W^{(1)}(b_i) = colim_{b_i \prec b_{ij}} W^{(1)}(b_{ij}) \quad \text{and} \quad W^{(2)}(b_i) = colim_{b_i \prec b_{ij}} W^{(2)}(b_{ij})$$

and the maps $\delta_{b_i}^{b_{ij}}(1)$ and $\delta_{b_i}^{b_{ij}}(2)$ are (isomorphic to) the colimit injections. So to prove 6.29, it suffices to prove that

$$f_a^w \circ \beta_a^{b_i}(1) \circ \delta_{b_i}^{b_{ij}}(1) = f_a \circ \beta_a^{b_i}(1) \circ \delta_{b_i}^{b_{ij}}(1) \tag{6.30}$$

$$\begin{tikzcd}
colim_{a \prec \mu(b_i)} W^{(1)}(b_i) \arrow[d, dashed, "f_a^w"'] & W^{(1)}(b_i) \arrow[l, "\beta_a^{b_i}(1)"'] \arrow[d, "g_{b_i}"] & W^{(1)}(b_{ij}) \arrow[l, "\delta_{b_i}^{b_{ij}}(1)"'] \arrow[d, "g_{b_{ij}}"] \\
colim_{a \prec \mu(b_i)} W^{(2)}(b_i) & W^{(2)}(b_i) \arrow[l, "\beta_a^{b_i}(2)"'] & W^{(2)}(b_{ij}) \arrow[l, "\delta_{b_i}^{b_{ij}}(2)"']
\end{tikzcd}$$

Now, in the above figure, the right square diagram is the naturality square of g, so it commutes. The left square also commutes from the definition of the "weak" f_a^w which makes the outer diagram commutative. So 6.30 becomes

$$\beta_a^{b_i}(2) \circ \delta_{b_i}^{b_{ij}}(2) \circ g_{b_{ij}} = f_a \circ \beta_a^{b_i}(1) \circ \delta_{b_i}^{b_{ij}}(1) \tag{6.31}$$

A strong restriction followed by a weak colimit injection amount to a strong colimit injection, so the above expression becomes

$$\delta_a^{b_{ij}}(2) \circ g_{b_{ij}} = f_a \circ \delta_a^{b_{ij}}(1)$$

which is true for b_{ij} such that $a \prec \mu(b_{ij})$, being the defining property of the ("strong") map f_a. This proves that $f^w = f$. ■

The combination of lemmas 6.21 and 6.20 asserts that the assignment $g \mapsto f$ of lemma 6.19 produces a morphism of presheaves with approximation when $W^{(1)}$ and $W^{(2)}$ are continuous. So, by stipulating that

$$\rho_f(g) = f$$

the assignment ρ_f of corollary 6.18 extends to a functor

$$\rho_f : \mathbf{Cont}\mathcal{B}\mathbf{PreSh}(\mathbf{B_2}) \longrightarrow \mathbf{Cont}\mathcal{B}\mathbf{PreSh}(\mathbf{B_1})$$

6.4 The Direct Image Map

In this section we are going to define a functor

$$\mathcal{B}\mathbf{Sh}(B_1) \longrightarrow \mathcal{B}\mathbf{Sh}(B_2)$$

given a strong homomorphism $\mu : B_2 \longrightarrow B_1$ between two strong proximity lattices. This construction is more obvious and less technical than the inverse image functor construction of the previous section. The reason is that the direction of μ is convenient; a $\mathcal{B}$-sheaf over B_1 can be precomposed with μ to give a $\mathcal{B}$-sheaf over B_2. This is analogous to the direct image functor between sheaves over locales; a sheaf over a locale X_1 can be precomposed with f^* to give a sheaf over a locale X_2 given a continuous map $f : X_1 \longrightarrow X_2$. To elaborate the analogy we juxtapose the preservation of finite meets, finite joins and directed joins by f^* with the preservation of finite meets, preservation of finite joins by μ and its *strongness.*

Definition 6.22 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices. We define a functor π_μ between presheaves with approximation over B_1 and* *presheaves with approximation over B_2 as follows*

(i) *Let (V, ϕ, θ) be a presheaf with approximation over B_1 and $b \in B_2$. Then*

$$\pi_\mu(V)(b) := V \circ \mu(b)$$

The weak restrictions of $\pi_\mu(V)$ are given by $\beta^{b_2}_{b_1} := \phi^{\mu(b_2)}_{\mu(b_1)}$ (for $b_2 \leq b_1$) and the strong restrictions are similarly given by $\delta^{b_2}_{b_1} := \theta^{\mu(b_2)}_{b_1}$ (for $b_2 \prec b_1$) .

(ii) *For $f : V^{(1)} \longrightarrow V^{(2)}$ a morphism between presheaves with approximation over B_1*

$$(\pi_\mu(f))_b := f_{\mu(b)}$$

$$\begin{tikzcd}
\mathbf{Sets} & \mathbf{Sets} \\
\mathcal{B}X \arrow[u, "V"] & B_2 \arrow[l, "\mu"'] \arrow[u, "V \circ \mathcal{B}f"']
\end{tikzcd}$$

It is routine to check that the assignment π_μ indeed produces approximating presheaves over B_2. The maps β are weak restriction maps because μ is monotone and the maps δ are strong restrictions because μ preserves the strong order. For the same reasons $\pi_\mu(f)$ is a morphism of presheaves with approximation. Furthermore, we have the following.

Lemma 6.23 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between strong proximity lattices and (V, ϕ, θ) a presheaf with approximation over B_1. Then $\pi_\mu(V)$ is continuous if V is continuous.*

Proof. Suppose that V is a continuous presheaf with approximation. We will prove that $V \circ \mu$ is continuous, i.e. that $colim_{b \prec b_i} V \circ \mu(b_i) \cong V \circ \mu(b)$.

The fact that W is continuous implies that $V \circ \mu(b) = colim_{\mu(b_i) \prec a_i} V(a_i)$. We will prove that

$$colim_{b \prec b_i} V \circ \mu(b_i) \cong colim_{\mu(b) \prec a_i} V(a_i)$$

We denote δ^{b_i}, the generic injection of the L.H.S. colimit and θ^{a_i}, the generic injection of the R.H.S. colimit. We define a function

$$f : colim_{b \prec b_i} V \circ \mu(b_i) \longrightarrow colim_{\mu(b) \prec a_i} V(a_i)$$

in the following way:

Since μ is strong we have $b_i \succ b \Longrightarrow \mu(b_i) \succ \mu(b)$. Therefore the maps $\theta^{\mu(b_i)}$ together with the vertex $colim_{\mu(b)\prec a_i}V(a_i)$ constitute a cone of the defining diagram of $colim_{b\prec b_i}V \circ \mu(b_i)$. So there is a unique map

$$f : colim_{b\prec b_i}V \circ \mu(b_i) \longrightarrow colim_{\mu(b)\prec a_i}V(a_i)$$

that makes the left square diagram below commutative (for any $b_i \succ b$).

$$\begin{tikzcd}
 & & V(a_i) \arrow[dl, "\theta^{a_i}_{\mathcal{B}f(b_i)}"'] \arrow[dd, "\theta^{a_i}"] \\
V \circ \mathcal{B}f(b_i) \arrow[r, "\cong"] \arrow[d, "\delta^{b_i}"'] & V(\mathcal{B}f(b_i)) \arrow[dr, "\theta^{\mathcal{B}f(b_i)}"] & \\
colim_{b\prec b_i}V \circ \mathcal{B}f(b_i) \arrow[rr, dashed, "f"] & & colim_{\mathcal{B}f(b)\prec a_i}V(a_i)
\end{tikzcd}$$

We prove that f is epi. Let $y \in colim_{\mu(b)\prec a_i}V(a_i)$. Then there is $y' \in W(a_i)$ with $\mu(b) \prec a_i$, such that $\theta^{a_i}(y') = y$. Since μ is a strong homomorphism there is b_i, for an index i with $b \prec b_i$ such that $\mu(b_i) \prec a_i$. This implies that $\theta^{a_i}(y') = \theta^{\mu(b_i)} \circ \theta^{a_i}_{\mu(b_i)}(y')$ (because the R.H.S. diagram is a cone). Hence

$$y = \theta^{\mu(b_i)} \circ \theta^{a_i}_{\mu(b_i)}(y') = f \circ \delta^{b_i} \circ \theta^{a_i}_{\mu(b_i)}(y')$$

or, by calling $y'' = \delta^{b_i} \circ \theta^{a_i}_{\mu(b_i)}(y')$, we have $y = f(y'')$.

It is easy to demonstrate that f is into. For any colimit injection δ^{b_i}, the composite $f \circ \delta^{b_i}$ is an injection because it is equal to $\theta^{\mu(b_i)}$ (by commutativity of the right square diagram). So f must be an injection. ■

By virtue of lemma 6.23, we can stipulate that π_μ to be a functor

$$\pi_\mu : \mathcal{B}\mathbf{Sh}(B_1) \longrightarrow \mathcal{B}\mathbf{Sh}(B_2)$$

for any strong homomorphism $\mu : B_2 \longrightarrow b_1$ between two strong proximity lattices. Also we have the following.

Lemma 6.24 $\pi_\mu(V)$ *has pasting if* V *has pasting.*

Proof. This is fairly obvious. If V has pasting then, by definition $V(\perp) \cong \mathbf{1}$. But $\mu(\perp) = \perp$ because μ is a lattice homomorphism. So $V \circ \mu(\perp) = \mathbf{1}(= \{*\})$. Also

$$V : (B_1, \leq) \longrightarrow \mathbf{Sets}$$

preserves pullbacks of the form 5.7 that exist in $(B_1, \leq)$. Also,

$$\mu : (B_2, \leq) \longrightarrow (B_1, \leq)$$

preserves pullbacks of the same form in $(B_2, \leq)$ being a lattice homomorphism. Therefore, the composite $V \circ \mu$ preserves these diagrams in $(B_2, \leq)$. ■

6.5 Adjoint functors between continuous approximating presheaves

We begin this section by restating the main points about functors between presheaves with approximation over strong proximity lattices. Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices.

In section 6.3 we defined a functor

$$\rho_\mu : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2) \longrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1)$$

and in section 6.4 a functor

$$\pi_\mu : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1) \longrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2) \tag{6.32}$$

All the mathematics involved in their definitions and in proofs were geometric so the two categories $\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1)$ and $\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2)$ might as well be inside any topos Z. We reiterate that these two categories are designated to signify $\mathcal{S}Z$-valued continuous approximating presheaves, where Z is any Grothendieck topos.

Therefore, by virtue of geometricity, we have also defined a functor between the classifying topoi of the respective geometric theories of continuous presheaves with approximation over B_1 and over B_2.

$$[\mathbb{T}_{ContPreSh(B_1)}] \longrightarrow [\mathbb{T}_{ContPreSh(B_2)}] \tag{6.33}$$

whose induced functor between points (models) is 6.32.

In this section we are going to prove that π_μ is the right adjoint of ρ_μ and we are going to do that geometrically so that it will also be valid inside any topos. To that end, we rely on the theorem 4.33 and corollary 4.34. The recipe outlined in section 4.9 says that it sufficient to demonstrate a bijection between the "sets"

$$\mathbf{ContPre\mathcal{B}Sh}(B_2)(W, \pi_\mu(V)) \cong \mathbf{ContPre\mathcal{B}Sh}(B_1)(\rho_\mu(V), W) \tag{6.34}$$

Let ξ be a map in the L.H.S. of 6.34. It is a morphism between presheaves with (continuous) approximation over B_1, it amounts to a family of maps

$$\xi_b : W(b) \longrightarrow \pi_\mu(V)(a) \quad \text{or} \quad \xi_b : W(b) \longrightarrow V \circ \mu(b)$$

for any $b \in B_2$, subject to the naturality conditions with respect to the weak and strong restriction maps of W and $\pi_\mu(V)$. We are going to define a function

$$ContPre\mathcal{B}Sh(B_2)(W, \pi_\mu(V)) \cong ContPre\mathcal{B}Sh(B_1)(\rho_\mu(V), W)$$

with $\xi \mapsto \psi_\xi$ by using the argument of the following lemma.

Lemma 6.25 *Let $\xi : W \longrightarrow \pi_\mu(V)$ be a morphism of presheaves with approximation with W a continuous presheaf over B_2 and V a continuous presheaf over B_1. Let also, for each $a \in B_1$, J^s_a be the index category as in the discussion preceding corollary 6.15 and $colim_{a \prec \mu(b_i)} W(b_i) := \rho_\mu(W)(a)$ the colimit of the diagram $W : J^s_a \longrightarrow$* **Sets**. *Then there is a unique map $(\psi_\xi)_a$ that makes all the diagrams below commutative, i.e., $\theta^{\mu(b_i)}_a \circ \xi_{b_i} = (\psi_\xi)_a \circ \delta^{b_i}_a$, for any $b_i \in B_2$ with $a \prec \mu(b_i)$.*

$$\begin{tikzcd}
W(b_1) \arrow[d, "\xi_{b_1}"'] \arrow[ddr, "\delta^{b_1}_a"] & & W(b_2) \arrow[ll, "\delta^{b_2}_{b_1}"'] \arrow[d, "\xi_{b_2}"] \arrow[ddl, "\delta^{b_2}_a"'] \\
V \circ \mu(b_1) \arrow[ddr, "\theta^{\mu(b_1)}_a"'] & & V \circ \mu(b_2) \arrow[ll, "\pi_\mu(\theta)^{b_2}_{b_1}"'] \arrow[ddl, "\theta^{\mu(b_2)}_a"] \\
& colim_{a \prec \mu(b_i)} W(b_i) \arrow[d, dashed, "(\psi_\xi)_a"'] & \\
& V(a) &
\end{tikzcd} \tag{6.35}$$

Moreover, for $a_1 \prec a_2$, the following naturality property holds for ψ_ξ

$$\begin{tikzcd} colim_{a_2 \prec \mu(b_i)} W(b_i) \arrow[r, "(\psi_\xi)_{a_2}"] \arrow[d, "\rho_\mu(\delta)^{a_2}_{a_1}"'] & V(a_2) \arrow[d, "\theta^{a_2}_{a_1}"] \\ colim_{a_1 \prec \mu(b_i)} W(b_i) \arrow[r, "(\psi_\xi)_{a_1}"] & V(a_1) \end{tikzcd} \tag{6.36}$$

and also the corresponding naturality square for $a_2 \leq a_1$.

Proof. The composites $\theta_a^{\mu(b_i)} \circ \xi_{b_i}$ with $a \prec \mu(b_i)$ and the vertex $V(a)$ constitute a cone of the diagram $W : J_a^s \longrightarrow$ **Sets**. The reason is that the top square diagram always commutes for $b_1 \prec b_2$ because ξ is a morphism of presheaves with approximation and as such it is natural in $b \in B_2$. Therefore, for any $b_1 \prec b_2$

$$\begin{aligned} \theta_a^{\mu(b_i)} \circ \xi_{b_1} \circ \delta_{b_1}^{b_2} &= \delta_{b_1}^{b_2} \circ \pi_\mu(\theta)_{b_1}^{b_2} \circ \xi_{b_2} \\ \theta_a^{\mu(b_1)} \circ \theta_{\mu(b_1)}^{\mu(b_2)} &= \theta_a^{\mu(b_2)} \circ \xi_{b_2} \end{aligned}$$

where we used the fact that by definition, $\pi_\mu(\theta)_{b_1}^{b_2} = \theta_{\mu(b_1)}^{\mu(b_2)}$ and the composition law of the strong restriction maps θ for $a \prec \mu(b_1) \prec \mu(b_2)$. So there must be indeed a unique map

$$(\psi_\xi)_a : colim_{a_1 \prec \mu(b_i)} W(b_i) \longrightarrow V(a)$$

Now we prove the strong naturality of ψ_ξ. The layout of the proof is similar to that of lemma 6.20. To prove the commutativity of the diagram 6.36, it suffices to prove that for any colimit injection $\delta_{a_2}^{b_i} : W(b_i) \longrightarrow colim_{a_2 \prec \mu(b_i)} W(b_i)$, the following equality holds

$$(\psi_\xi)_{a_1} \circ \rho_\mu(\delta)_{a_1}^{a_2} \circ \delta_{a_2}^{b_i} = \theta_{a_1}^{a_2} \circ (\psi_\xi)_{a_2} \circ \delta_{a_2}^{b_i} \tag{6.37}$$

The defining property of ψ_ξ (see diagram 6.35) says that $(\psi_\xi)_{a_2} \circ \delta_{a_2}^{b_i} = \theta_{a_2}^{\mu(b_i)} \circ \xi_{b_i}$, it suffices to prove that

$$(\psi_\xi)_{a_1} \circ \rho_\mu(\delta)_{a_1}^{a_2} \circ \delta_{a_2}^{b_i} = \theta_{a_1}^{a_2} \circ \theta_{a_2}^{\mu(b_i)} \circ \xi_{b_i} \tag{6.38}$$

But $\theta_{a_1}^{a_2} \circ \theta_{a_2}^{\mu(b_i)} = \theta_{a_1}^{\mu(b_i)}$ and, by the definition of $\rho_\mu(\delta)$ (c.f. lemma 6.16), $\rho_\mu(\delta)_{a_1}^{a_2} \circ \delta_{a_2}^{b_i} = \delta_{a_1}^{b_i}$. So equation 6.38 becomes

$$\theta_{a_1}^{\mu(b_i)} \circ \xi_{b_i} = (\psi_\xi)_{a_1} \circ \delta_{a_1}^{b_i} \tag{6.39}$$

But again this is true as part of the definition of ψ_ξ (see diagram 6.35).

To prove the weak naturality of ψ_ξ amounts to proving the commutativity of the far right square in the diagram below

$$\begin{tikzcd} W(b_i) \arrow[r, "\delta^{b_i}_{a_2}"] \arrow[d, "\xi_{b_i}"'] & colim_{a_2 \prec \mu(b_i)} \arrow[r, "\rho_\mu(\beta)^{a_2}_{a_1}"] \arrow[d, "(\psi_\xi)_{a_2}"'] & colim_{a_1 \prec \mu(b_i)} \arrow[d, "(\psi_\xi)_{a_1}"] \\ V \circ \mu(b_i) \arrow[r, "\theta^{\mu(b_i)}_{a_2}"] & V(a_2) \arrow[r, "\phi^{a_2}_{a_1}"] & V(a_1) \end{tikzcd} \tag{6.40}$$

where now it involves the weak restriction maps $\rho_\mu(\beta)$ and ϕ. We work exactly as in the proof of strong naturality. We need to recall that the weak restriction maps $\rho_\mu(\beta)$ are also defined using the *strong* colimit diagram (see discussion preceding lemma 6.16). Therefore, the top horizontal maps compose to $\delta^{b_i}_{a_1}$ and the bottom horizontal maps to $\theta^{\mu(b_i)}_{a_1}$ (because the strong restriction maps absorb the weak ones). So the equality that needs to be proved is $\theta^{\mu(b_i)}_{a_1} \circ \xi_{b_i} = (\psi_\xi)_{a_1} \circ \delta^{b_i}_{a_1}$ which is exactly the equation 6.39.

So the conclusion is that ψ_ξ is a morphism of (continuous) presheaves with approximation over B_1. ■

This defines a functor

$$\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2)(W, \pi_\mu(V)) \longrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1)(\rho_\mu(V), W)$$

We are going to define a functor in the opposite direction.

Let $\psi : \rho_\mu(W) \longrightarrow V$ be a morphism of presheaves with approximation with W a continuous presheaf over B_2 and V a continuous presheaf over B_1. For any $b \in B_2$, we can consider the component $\psi_{\mu(b)}$ of the morphism ψ.

$$\psi_{\mu(b)} : colim_{\mu(b) \prec \mu(b_i)} W(b_i) \longrightarrow V(\mu(b)) \tag{6.41}$$

Now, continuity of W yields $W(b) \cong colim_{b \prec b_i} W(b_i)$ and perfectness of μ guarantees that $b \prec b_i \Rightarrow \mu(b) \prec \mu(b_i)$. So there is a unique colimit inclusion map $i_b : colim_{b \prec b_i} W(b_i) \longrightarrow colim_{\mu(b) \prec \mu(b_i)} W(b_i)$.

Definition 6.26 *Given a a morphism of presheaves with approximation $\psi : \rho_\mu(W) = colim_{a \prec \mu(b_i)} \longrightarrow V$, with the above notation and for any $b \in B_2$, we define maps the $(\xi_\psi)_b$ as the following composites*

$$W(b) \xrightarrow{\cong} colim_{b \prec b_i} W(b_j) \xrightarrow{i_b} colim_{\mu(b) \prec \mu(b_i)} \xrightarrow{\psi_{\mu(b)}} V \circ \mu(b) \tag{6.42}$$

Lemma 6.27 *The maps of definition 6.26 are the components of a morphism of presheaves with approximation over* B_2*.*

Proof. Once again we have to demonstrate both the strong and weak naturality of ξ_ψ. To prove the strong naturality, we first look at the diagram below for $b_1 \prec b_2 \prec b_i$ in B_2.

$$\begin{tikzcd} W(b_i) \arrow[r, "\delta^{b_i}_{b_2}"] \arrow[dr, "\delta^{b_i}_{b_1}"'] & colim_{b_2 \prec b_i} W(b_i) \arrow[r, "i_{b_2}"] \arrow[d, "\delta^{b_2}_{b_1}"'] & colim_{\mu(b_2)\prec\mu(b_i)} \arrow[d, "\rho_\mu(\delta)^{\mu(b_2)}_{\mu(b_1)}"] \\ & colim_{b_1 \prec b_i} W(b_i) \arrow[r, "i_{b_1}"] & colim_{\mu(b_1)\prec\mu(b_i)} \end{tikzcd} \tag{6.43}$$

The colimit injections $\delta^{b_i}_{b_2}$ and $\delta^{b_i}_{b_1}$ are indeed the strong restrictions (as their notation suggests) because $colim_{b_2\prec b_i} W(b_i) \cong W(b_2)$ and $colim_{b_1\prec b_i} W(b_i) \cong W(b_1)$. Also for the same reason, the left vertical map is the restriction $\delta^{b_2}_{b_1}$. By the definition of the maps i_{b_1} and i_{b_2}, the composites $i_{b_2} \circ \delta^{b_i}_{b_2}$ and $i_{b_1} \circ \delta^{b_i}_{b_1}$ are just the colimit injections $\delta^{b_i}_{\mu(b_2)}$ and $\delta^{b_i}_{\mu(b_1)}$ respectively. So the outer diagram commutes by the definition of $\rho_\mu(\delta)$. The triangular diagram on the left obviously also commutes as it is the composition rule of strong restriction maps δ. Hence, the square diagram on the right commutes. This is the strong naturality square of the map i in the definition 6.26. The map ψ is also strongly natural by assumption, therefore, the maps $(\xi_\psi)_b$ are the components of a strongly natural map ξ_ψ.

To prove that ξ_ψ is also weakly natural we work in exactly the same way taking into account the comments made in the corresponding part of the proof of lemma 6.25. So ξ_ψ is a morphism of presheaves with approximation over B_2. ■

Now we prove that the two translations $\xi \mapsto \psi_\xi$ and $\psi \mapsto \xi_\psi$ are inverse to each other.

Theorem 6.28 *For any continuous presheaf with approximation* V *over* B_1 *and any continuous presheaf with approximation* W *over* B_2*, we have the bijection*

$$\mathbf{ContPre\mathcal{B}Sh}(B_2)(W, \pi_\mu(V)) \cong \mathbf{ContPre\mathcal{B}Sh}(B_1)(\rho_\mu(V), W) \tag{6.44}$$

Proof.

$$
\begin{tikzcd}
W(b_1) \arrow[d, "\xi_{b_1}"'] \arrow[ddrr, "in^{b_1}"] & & & & W(b_2) \arrow[llll, "\delta^{b_2}_{b_1}"'] \arrow[d, "\xi_{b_2}"] \arrow[ddll, "in^{b_2}"'] \\
V \circ \mu(b_1) \arrow[dddrr, "\pi_\mu(\theta)^{b_1}_b"'] & & & & V \circ \mu(b_2) \arrow[llll, "\pi_\mu(\theta)^{b_2}_{b_1}"'] \arrow[dddll, "\pi_\mu(\theta)^{b_2}_b"] \\
& & colim_{b \prec b_i} W(b_i) \arrow[rr, shift left, "\alpha"] \arrow[d, dashed, "\mathbf{i_b}"'] & & W(b) \arrow[ll, shift left, "\gamma"] \\
& & colim_{\mu(b) \prec \mu(b_i)}(W(b_i)) \arrow[d, dashed, "(\psi_\xi)_{\mu(b)}"] & & \\
& & V \circ \mu(b) & &
\end{tikzcd}
\tag{6.45}
$$

By combining lemma 6.25 and definition 6.26, we verify that given a morphism of presheaves with approximation, $\xi : W \longrightarrow \pi_\mu(V)$, the map ξ_{ψ_ξ} which is the target of two successive assignments $\xi \mapsto \psi_\xi \mapsto \xi_{\psi_\xi}$ is defined as in the diagram 6.45. Its components $\left(\xi_{\psi_\xi}\right)_b$ are the composites

$$\left(\xi_{\psi_\xi}\right)_b = \left(\psi_\xi\right)_{\mu(b)} \circ i_b \circ \gamma$$

In diagram 6.45, $b \in B_2$ and $b \prec b_1 \prec b_2$. i_b is the colimit inclusion map as in definition 6.26 and $\alpha$and γ are the isomorphism maps ($\alpha \circ \gamma = id$ and $\gamma \circ \alpha = id$). Also, the maps in^{b_1} and in^{b_2} are the colimit injections and by virtue of the isomorphism $colim_{b \prec b_i} W(b_i) \cong W(b)$), we have

$$\alpha \circ in^{b_1} = \delta^{b_1}_b \quad \text{and} \quad \alpha \circ in^{b_2} = \delta^{b_2}_b \tag{6.46}$$

We are going to prove that $\left(\xi_{\psi_\xi}\right)_b = \xi_b$ which is by definition the same as $\left(\psi_\xi\right)_{\mu(b)} \circ i_b \circ \gamma = \xi_b$. This equality is the same (up to isomorphism) with $\left(\psi_\xi\right)_{\mu(b)} \circ i_b \circ \gamma\alpha = \xi_b \alpha$ or

$$\left(\psi_\xi\right)_{\mu(b)} \circ i_b = \xi_b \alpha \tag{6.47}$$

Since in^{b_1} for any $b_1 \succ b$ are the colimit injections, to prove equation 6.47 it suffices to prove it pre-composed with any such in^{b_1}, i.e.

$$\forall b_1 \succ b \quad \left(\psi_\xi\right)_{\mu(b)} \circ i_b \circ in^{b_1} = \xi_b \circ \alpha \circ in^{b_1} \tag{6.48}$$

But from the definition of i_b, $i_b \circ in^{b_1}$ is the same as the colimit injection $\delta^{b_1}_{\mu(b)} : W(b_1) \hookrightarrow colim_{\mu(b) \prec \mu(b_i)} W b_i$. So, equation 6.48 becomes $(\psi_\xi)_{\mu(b)} \circ \delta^{b_1}_{\mu(b)} = \xi_b \circ \alpha \circ in^{b_1}$. By virtue of the defining property of $(\psi_\xi)_{\mu(b)}$ (lemma 6.25) and equation 6.46, this last equation becomes $\pi_\mu(\theta)^{b_1}_b \circ \xi_{b_1} = \xi_b \circ \delta^{b_1}_b$. But this is just the strong naturality property of ξ, so it holds for any $b_1 \succ b$. Therefore, $\xi_{\psi_\xi} = \xi$.

Now we are going to prove the other direction, i.e. that the assignment $\psi \mapsto \xi_\psi \mapsto \psi_{\xi_\psi}$ produces a map identical to ψ.

$$
\begin{tikzcd}
W(b_k) \arrow[d, "\delta^{b_j}_{b_1}"] & & & & \\
W(b_1) \arrow[d, "i_{b_1}"'] & & & & W(b_2) \arrow[llll, "\delta^{b_2}_{b_1}"'] \arrow[d, "i_{b_2}"] \\
colim_{\mu(b_1) \prec \mu(b_j)} W(b_j) \arrow[d, "\psi_{\mu(b_1)}"'] \arrow[ddrr, "\rho_\mu(\delta)^{\mu(b_1)}_a"] & & & & colim_{\mu(b_2) \prec \mu(b_j)} W(b_j) \arrow[d, "\psi_{\mu(b_2)}"] \arrow[ddll, "\rho_\mu(\delta)^{\mu(b_2)}_a"'] \\
V \circ \mu(b_1) \arrow[ddrr, "\theta^{\mu(b_1)}_a"'] & & & & V \circ \mu(b_2) \arrow[llll, "\pi_\mu(\theta)^{b_2}_{b_1}"] \arrow[ddll, "\theta^{\mu(b_2)}_a"] \\
& & colim_{a \prec \mu(b_i)}(W(b_i)) \arrow[d, dashed, "(\psi_{\xi_\psi})_a"] & & \\
& & V(a) & &
\end{tikzcd}
$$

The above is part of the defining diagram of the component $(\psi_{\xi_\psi})_a$ for $a \prec \mu(b_1)$ and $b_1 \prec b_2$ (and hence $\mu(b_1) \prec \mu(b_2)$). The maps i_{b_1} and i_{b_2} are the colimit inclusions, e.g. $i_{b_1} : W(b_1) \cong colim_{b_1 \prec b_j} W(b_j) \longrightarrow colim_{\mu(b_1) \prec \mu(b_j)}$ (vis-à-vis definition 6.26). Hence the vertical composites $\psi_{\mu(b_1)} \circ i_{b_1}$ and $\psi_{\mu(b_2)} \circ i_{b_2}$ are the maps $(\xi_\psi)_{b_1}$ and $(\xi_\psi)_{b_2}$ of the definition 6.26 respectively. The map $(\psi_{\xi_\psi})_a$ is the unique map that makes all the diagrams commutative. To prove that $(\psi_{\xi_\psi})_a = \psi_a$, it suffices to prove this equality pre-composed with any colimit injection $\delta^{b_1}_a : W(b_1) \longrightarrow colim_{a \prec \mu(b_i)} W(b_i)$ for any $b_1 \in B_2$ with $a \prec \mu(b_1)$. Furthermore, to prove this last fact, it suffices to prove that for any $b_j \succ b_1$,

$$(\psi_{\xi_\psi})_a \circ \delta^{b_1}_a \circ in^{b_j} = \psi_a \circ \delta^{b_1}_a \circ \delta^{b_j}_{b_1} \tag{6.49}$$

where, due to continuity of W, $\delta_{b_1}^{b_j}$ are (up to isomorphism) the colimit injections

$$\delta_{b_1}^{b_j} : W(b_j) \hookrightarrow colim_{b_1 \prec b_j} W(b_1) \cong W(b_1)$$

Also, from the defining property of i_{b_1}, we can substitute $\delta_a^{b_1} \circ \delta_{b_1}^{b_j}$ with $\rho_\mu(\delta)_a^{\mu(b_1)} \circ \delta_{\mu(b_1)}^{b_j}$. Therefore, the equality we wish to prove becomes (for any $b_j \succ b_1$ and for any b_1 with $a \prec \mu(b_1)$)

$$\left(\psi_{\xi_\psi}\right)_a \circ \rho_\mu(\delta)_a^{\mu(b_1)} \circ \delta_{\mu(b_1)}^{b_j} = \psi_a \circ \rho_\mu(\delta)_a^{\mu(b_1)} \circ \delta_{\mu(b_1)}^{b_j} \tag{6.50}$$

But from the diagram 6.5, $\left(\psi_{\xi_\psi}\right)_a \circ \rho_\mu(\delta)_a^{\mu(b_1)} = \theta_a^{\mu(b_1)} \circ \psi_{\mu(b_1)}$, so 6.50 becomes

$$\theta_a^{\mu(b_1)} \circ \psi_{\mu(b_1)} \circ \delta_{\mu(b_1)}^{b_j} = \psi_a \circ \rho_\mu(\delta)_a^{\mu(b_1)} \circ \delta_{\mu(b_1)}^{b_j} \tag{6.51}$$

which is true because $\theta_a^{\mu(b_1)} \circ \psi_{\mu(b_1)} = \psi_a \circ \rho_\mu(\delta)_a^{\mu(b_1)}$ is just the strong naturality property of ψ for $a \prec \mu(b_1)$ which is implied in the assumption.

This completes the proof of the isomorphism 6.34. ■

Corollary 6.29 *The functor π_μ is the right adjoint of ρ_μ.*

6.6 Adjoint Functors between $\mathcal{B}$-sheaves

We saw in section 5.6 that for any strong proximity lattice B there is the sheavification functor

$$\mathbf{past}' : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathcal{B}\mathbf{Sh}(B) \tag{6.52}$$

that takes a presheaf V with approximation to its free pasting presheaf with approximation over V. This functor is the left adjoint of the inclusion functor

$$\mathbf{i_4} : \mathcal{B}\mathbf{Sh}(B) \longrightarrow \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B) \tag{6.53}$$

As briefly discussed in section 5.6, the application of **past** on a presheaf with approximation is a free construction of an essentially algebraic structure. Therefore, it is a geometric construction and the functors of the expressions 6.52 and 6.53 are considered inside the sheaves of any topos Z.

Also recall that if $\mu : B_2 \longrightarrow B_1$ is a strong homomorphism, ρ_μ and π_μ are the two functors

$$\pi_\mu : \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1) \rightleftharpoons \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2) : \rho_\mu$$

constructed in sections 6.3 and 6.4. In section 6.5 we proved that $\rho_\mu \dashv \pi_\mu$. Now we are going to "refine" this to an adjoint situation involving functors between the categories $\mathcal{B}\mathbf{Sh}(B_1)$ and $\mathbf{B_2}$.

Definition 6.30 *Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices.*

(i) We define $\mu^* : \mathcal{B}\mathbf{Sh}(B_2) \longrightarrow \mathcal{B}\mathbf{Sh}(B_1)$ to be the composite functor $\mathbf{past} \circ \rho_\mu \circ \mathbf{i_4}$.

(ii) We define $\mu_* : \mathcal{B}\mathbf{Sh}(B_1) \longrightarrow \mathcal{B}\mathbf{Sh}(B_2)$ to be the composite functor $\mathbf{past} \circ \pi_\mu \circ \mathbf{i_4}$.

$$\begin{tikzcd}
\mathcal{B}\mathbf{Sh}(B_1) \arrow[r, shift left, "\mu_*"] \arrow[d, shift right, "\mathbf{i_4}"'] & \mathcal{B}\mathbf{Sh}(B_2) \arrow[l, shift left, "\mu^*"] \arrow[d, shift right, "\mathbf{i_4}"'] \\
\mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1) \arrow[r, shift left, "\pi_\mu"] \arrow[u, shift right, "\mathbf{past}'"'] & \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2) \arrow[l, shift left, "\rho_\mu"] \arrow[u, shift right, "\mathbf{past}'"']
\end{tikzcd} \tag{6.54}$$

We need to point out the following two properties. The first is lemma 5.31(ii), i.e. that for any $\mathcal{B}$-sheaf V in $\mathcal{B}\mathbf{Sh}(B)$,

$$\mathbf{past}_B \circ \mathbf{i_4}(V) \cong V \tag{6.55}$$

The second property is given by the next lemma

Lemma 6.31 *For any $\mathcal{B}$-sheaf V in $\mathcal{B}\mathbf{Sh}(B)$, $\pi_\mu \circ \mathbf{i_4}(V) \cong \mathbf{i_4} \circ \mu_*(V)$.*

Proof. This is an immediate consequence of lemma 6.24; application of π_μ on a $\mathcal{B}$-sheaf yields a $\mathcal{B}$-sheaf. ■

The section culminates with the next theorem.

Theorem 6.32 *The functor μ_* is the right adjoint of μ^*.*

Proof. We have the following sequence of isomorphisms between categories of morphisms, for any $\mathcal{B}$-sheaf V over B_1 and any $\mathcal{B}$-sheaf W over B_2.

$$\begin{array}{lr}
\mathcal{B}\mathbf{Sh}(B_1)(\mu^*(W), V) & \\
\cong \mathcal{B}\mathbf{Sh}(B_1)(\mathbf{past} \circ \rho_\mu \circ \mathbf{i_4}(W), V) & \text{(by definition)} \\
\cong \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_1)(\rho_\mu \circ \mathbf{i_4}(W), \mathbf{i_4}(V)) & \text{(because } \mathbf{past} \dashv \mathbf{i_4} \\
\cong \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2)(\mathbf{i_4}(W), \pi_\mu \circ \mathbf{i_4}(V)) & \text{(because (corollary 6.29) } \rho_\mu \dashv \pi_\mu) \\
\cong \mathbf{ContPre}\mathcal{B}\mathbf{Sh}(B_2)(\mathbf{i_4}(W), \mathbf{i_4} \circ \mu_*(V)) & \text{(because of lemma 6.31)} \\
\cong \mathcal{B}\mathbf{Sh}(B_2)(\mathbf{past}' \circ \mathbf{i_4}(W), \mu_*(V)) & \text{(because } \mathbf{past}' \dashv \mathbf{i_4}) \\
\cong \mathcal{B}\mathbf{Sh}(B_2)(W, \mu_*(V)) & \text{(because of 6.55)}
\end{array}$$

So we have proved the isomorphism

$$\mathcal{B}\mathbf{Sh}(B_1)(\mu^*(W), V) \cong \mathcal{B}\mathbf{Sh}(B_2)(W, \mu_*(V)) \tag{6.56}$$

which by definition implies that μ_* is the right adjoint of μ^*. ■

6.7 Geometric morphisms between the exponentials $[set]^{\mathbf{RSpec}(B_1)}$ and $[set]^{\mathbf{RSpec}(B_2)}$

Here we use the 2-categorically sound criterion for adjunctions between Grothendieck topoi developed in chapter 4 to "lift" the adjunction of theorem 6.32 to an adjunction between the corresponding classifying topoi.

Let $\mu : B_2 \longrightarrow B_1$ be a strong homomorphism between two strong proximity lattices. In the previous three sections we defined an adjoint pair of functors $\mu^* \dashv \mu_*$

$$\mathcal{B}\mathbf{Sh}(B_1) \underset{\mu_*}{\overset{\mu^*}{\rightleftarrows}} \mathcal{B}\mathbf{Sh}(B_2) \tag{6.57}$$

All the work has been done geometrically and this means that the categories $\mathcal{B}\mathbf{Sh}(B_1)$ and $\mathcal{B}\mathbf{Sh}(B_2)$ can be construed in their internal sense, i.e. inside the sheaves of any Grothendieck topos. That means that the pair μ^*, μ_* determines a pair of *geometric morphisms* between the corresponding classifying topoi of the theories of $\mathcal{B}$-sheaves over B_1 and $\mathcal{B}$-sheaves over B_2 (see section 1.2). By theorem 5.24 we know that these classifying topoi are $[set]^{\mathbf{RSpec}(B_1)}$ and $[set]^{\mathbf{RSpec}(B_2)}$ respectively. Therefore, the pair of 6.57 uniquely determines a pair of functors

$$[set]^{\mathbf{RSpec}(B_1)} \underset{P_\mu}{\overset{R_\mu}{\rightleftarrows}} [set]^{\mathbf{RSpec}(B_2)} \tag{6.58}$$

Moreover, as proved in chapter 4, the fact that μ_* is the right adjoint of μ^* implies that P_μ is the right adjoint of R_μ. Indeed, in the notation of section 4.9, theorem 6.32 proves that there is a bijection

$$\mathbf{C}^Z_{(R_\mu V, W)} \cong \mathbf{C}^Z_{V, P_\mu W}$$

for any Z-point of $[set]^{\mathbf{RSpec}}(B_1)$, i.e. a $\mathcal{B}$-sheaf V over B_1 and any Z-point of $[set]^{\mathbf{RSpec}}(B_2)$, i.e. a $\mathcal{B}$-sheaf W over B_2.

We can easily demonstrate that we know the map $R_\mu : [set]^{\mathbf{RSpec}(B_2)} \longrightarrow [set]^{\mathbf{RSpec}(B_1)}$ from an other source. Let X and Y be two stably compact locales and $f : X \longrightarrow Y$ *any* *continuous* map between them. The topoi $[set]^X$ and $[set]^Y$ are exponential in the 2-category $\mathfrak{Top}$ of Grothendieck topoi. Category theory asserts that there is always an arrow in $\mathfrak{Top}$ (the evaluation arrow)

$$\mathbf{ev} : [set]^Y \times Y \longrightarrow [set]$$

Therefore we can define the following functor.

Definition 6.33 *Let $f : X \longrightarrow Y$ be a continuous map between two stably compact locales. We define $[set]^f : [set]^Y \longrightarrow [set]^X$ to be the exponential transpose of the composite*

$$[set]^Y \times X \xrightarrow{id_{[set]^Y} \times f} [set]^Y \times Y \xrightarrow{\mathbf{ev}} [set] \tag{6.59}$$

We have shown in chapter 3 that given a strong homomorphism $\mu : B_2 \longrightarrow B_1$, $\mathbf{RSpec}(\mu)$ is a perfect map between the locales $\mathbf{RSpec}(B_1) \longrightarrow \mathbf{RSpec}(B_2)$. Therefore, the following is a special case of the definition 6.33.

Definition 6.34 *Let $\mu : B_2 \longrightarrow B_1$. We define $[set]^{\mathbf{RSpec}(\mu)}$ to be the exponential transpose of the composite map*

$$\begin{tikzcd} {[set]^{\mathbf{RSpec}(B_2)} \times \mathbf{RSpec}(B_1)} \arrow[r, "id_{[set]^{\mathbf{RSpec}(B_2)}} \times \mathbf{RSpec}(\mu)"] & {[set]^{\mathbf{RSpec}(B_2)} \times \mathbf{RSpec}(B_2)} \arrow[d, "\mathbf{ev}"'] \\ & {[set]} \end{tikzcd} \tag{6.60}$$

We can actually describe the composite map of 6.60 concretely by its action on points. A (generalised) point of $[set]^{\mathbf{RSpec}(B_2)} \times \mathbf{RSpec}(B_1)$ is a pair consisting of a $\mathcal{B}$-sheaf W over B_2 (inside $\mathcal{S}Z$) and a point x of the locale $\mathbf{RSpec}(B_1)$ (theorem 5.24). Such a point x amounts to a completely prime filter of rounded ideals H of B_1 and lemma 3.25 says that this is the same as a rounded prime filter F of B_1. The map $id_{[set]^{\mathbf{RSpec}(B_2)}} \times \mathbf{RSpec}(\mu)$ takes such a pair (W, F) to a pair $(W, pt \circ \mathbf{RSpec}(\mu))(H)$, where $pt \circ \mathbf{RSpec}(\mu)$ is as in the expression 3.25 and by corollary 3.28 this is the same as $(W, \mathbf{RPFilt}(\mu)(F))$ where

$$\mathbf{RPFilt}(\mu)(F) = (\mu)^{-1}[F] := \{b \in B_2 | \exists a \in F : a = \mu(b)\}$$

Finally, the map $\mathbf{ev}$ calculates the stalk of W above the point $\mathbf{RPFilt}(\mu)(F)$. Hence, the map of the diagram 6.60 acts on points as

$$(W, F) \mapsto colim_{b \in (\mu)^{-1}[F]} W(b) := colim_{\mu(b) \in F} W(b) \tag{6.61}$$

After this insight we prove the following.

Theorem 6.35 *The functor R_μ is isomorphic to the functor $[set]^{\mathbf{RSpec}(\mu)}$.*

Proof. We know by category theory that R_μ is the exponential transpose of the composite $\mathbf{ev} \circ (R_\mu \times id_{\mathbf{RSpec}(B_1)})$, so effectively we need to demonstrate that the following square commutes

$$\begin{tikzcd}
{[set]^{\mathbf{RSpec}(B_2)} \times \mathbf{RSpec}(B_1)} \arrow[r, "id_{[set]^{\mathbf{RSpec}(B_2)}} \times \mathbf{RSpec}(\mu)"] \arrow[d, "R_\mu \times id_{\mathbf{RSpec}(B_1)}"'] & {[set]^{\mathbf{RSpec}(B_2)} \times \mathbf{RSpec}(B_2)} \arrow[d, "\mathbf{ev}"] \\
{[set]^{\mathbf{RSpec}(B_1)} \times \mathbf{RSpec}(B_1)} \arrow[r, "\mathbf{ev}"] & {[set]}
\end{tikzcd} \tag{6.62}$$

We also give an account of the action of $\mathbf{ev} \circ (\rho_\phi \times id_{\mathbf{RSpec}(B_1)})$ on points. The functor $R_\mu \times id_{\mathbf{RSpec}(B_1)}$ takes a pair (W, F) to a pair $(\mu^*(W), F)$, where μ^* is the functor of the definition 6.30. Then $\mathbf{ev}$ calculates the stalk of the $\mathcal{B}$-sheaf $\mu^*(W)$ above the point F. Also we know

$$\begin{aligned}
\mathbf{stalk}_F(\mu^*(W)) &\cong \mathbf{stalk}_F(\mathbf{past}(\rho_\mu(W))) \text{ (definition 6.30)} \\
&\cong \mathbf{stalk}_F(\rho_\mu(W)) \text{ (theorem 6.9)} \\
&\cong colim_{a \in F} \rho_\mu(W)(a) \text{ (definition 6.3)} \\
&\cong colim_{a \in F} colim_{a \prec \mu(b)} W(b) \text{ (section 6.3)} \\
&\cong colim_{\mu(b) \in F} W(b)
\end{aligned}$$

The last isomorphism holds because the sets $A := \{b \in B_2 | \mu(b) \in F\}$ and $\{b \in B_2 | \exists a \in F : a \prec \mu(b)\}$ are identical ($B \subseteq A$ because $\uparrow F = F$ and $A \subseteq B$ because F is rounded).

This shows that $\mathbf{ev} \circ (R_\mu \times id_{\mathbf{RSpec}(B_1)})$ has the same effect as $\mathbf{ev} \circ (id_{[set]^{\mathbf{RSpec}(B_2)}} \times \mathbf{RSpec}(\mu))$ on points of the topos $[set]^{\mathbf{RSpec}(B_2)}$ and this fact suffices to prove that the square 6.62 commutes (up to isomorphism). Therefore, the claim of the theorem is true because $[set]^{\mathbf{RSpec}(\mu)}$ and R_μ have isomorphic exponential transposes. ■

An immediate consequence of the above theorem is the following.

Corollary 6.36 *If $\mu : B_2 \longrightarrow B_1$ is a strong homomorphism between two strong proximity lattices then*

$$[set]^{\mathbf{RSpec}(\mu)} : [set]^{\mathbf{RSpec}(B_2)} \longrightarrow [set]^{\mathbf{RSpec}(B_1)}$$

has a right adjoint.

Proof. By theorem 6.35 and theorem 6.32. ■

In chapter 3 we saw that a perfect map between two stably compact locales always gives rise to a strong homomorphism between the corresponding strong proximity lattice. In this context, corollary 6.36 yields the following result.

Corollary 6.37 *(i) Let $f : X \longrightarrow Y$ be a perfect map between two stably compact locales. Then*

$$[set]^f : [set]^Y \longrightarrow [set]^X$$

has a right adjoint.

(ii) The functor $R_{\mathcal{B}f}$ acts on points as $f^* : Sh(Y) \longrightarrow Sh(X)$ and the functor $P_{\mathcal{B}f}$ as $f_* : Sh(X) \longrightarrow Sh(Y)$.

Proof. (i) We apply corollary 6.36 for the case $B_1 := \mathcal{B}\mathbf{X}$, $B_2 := \mathcal{B}\mathbf{Y}$ and $\mu = \mathcal{B}(f)$, where $\mathcal{B}$ is the functor of theorem 3.13. Thus, we establish that the map

$$[set]^{\mathbf{RSpec}\circ\mathcal{B}(f)} : [set]^{\mathbf{RSpec}\circ\mathcal{B}(Y)} \longrightarrow [set]^{\mathbf{RSpec}\circ\mathcal{B}(Y)} \tag{6.63}$$

has a right adjoint. But by theorem 3.24, $\mathbf{RSpec} \circ \mathcal{B} = id_{\mathbf{StKLoc}}$ and so the map in 6.63 reduces to the claimed one.

(ii) It is obvious that $R_{\mathcal{B}f}$ acts on points as f^* (up to equivalence) from the fact that $R_{\mathcal{B}f}$ is equivalent to $[set]^f$. That $P_{\mathcal{B}f}$ is equivalent to f_* follows from the uniqueness of the right adjoint. ■

6.8 Conclusion: perfect maps and filtered colimits

If $f : X \longrightarrow Y$ is a perfect map between two stably locales, then what is the characterisation of f as a geometric morphism $X \longrightarrow Y$ qua topoi? The answer given here will be that f is *relatively tidy* in the sense of Moerdijk & Vermeulen [MV97], i.e. that f preserves filtered colimits indexed by an external category. External category here means a category in **Sets** but **Sets** could be substituted with any Grothendieck topos with a natural number object.

We start by noting that for any Grothendieck topos X, the category of its points at stage (say) Z has all set-indexed filtered colimits (Johnstone [Joh77] Corollary 7.14). We are going to outline a demonstration of this fact in order to establish notation and context. First we recall the following theorem (see Johnstone [Joh77], section 2.2).

Theorem 6.38 *Let* $\mathbf{C}$ *be a category with finite limits and coequalisers and* $\mathfrak{I}$ *an internal category in* $\mathbf{C}$*. Then the functor* $\mathbf{\Delta} : \mathbf{C} \longrightarrow \mathbf{C}^{\mathfrak{I}}$ *that sends an object of* $\mathbf{C}$ *to the corresponding constant diagram has a left adjoint which we denote* $colim_{\mathfrak{I}}$*. If* $\mathbf{C}$ *is cartesian closed category, then* $\mathbf{\Delta}$ *has a right adjoint* $lim_{\mathfrak{I}}$*.*

By definition the colimit of an object $\mathcal{D}$ of $\mathbf{C}^{\mathfrak{I}}$, i.e. an internal $\mathfrak{I}$-diagram in $\mathbf{C}$ is the image of $\mathcal{D}$ along the functor $colim_{\mathfrak{I}}$.

Now we restrict to the case of interest where Z is a Grothendieck topos. Then we have that $Z\hat{\ }\mathfrak{I}$ [2] is also a topos and that

$$colim_{\mathfrak{I}} \dashv \Delta \quad \text{and} \quad \Delta \dashv lim_{\mathfrak{I}} \tag{6.64}$$

If furthermore $\mathfrak{I}$ is directed, then $colim_{\mathfrak{I}}$ preserves finite limits (Johnstone [Joh77], 2.58). The three functors of expression 6.64 can organise themselves as

$$\pi^* := \Delta, \;\; \pi_* := lim_{\mathfrak{I}}, \;\; \infty^* := colim_{\mathfrak{I}}, \;\; \infty_* := \Delta \tag{6.65}$$

to yield a pair of geometric morphisms

$$Z \underset{\pi}{\overset{\infty}{\rightleftarrows}} Z\hat{\ }\mathfrak{I} \tag{6.66}$$

Now let E be a Grothendieck topos classifying a geometric theory $\mathbb{T}_E$ and I a directed *small* category, i.e. a category in **Sets**. Then the exponential topos $X^{\hat{I}}$ (where $\hat{I}$ is the topos with $\mathcal{S}\hat{I} = [I, \mathbf{Sets}]$) classifies theories whose models (say) in $\mathcal{S}Z$ are I-diagrams of models of $\mathbb{T}_E$ in $\mathcal{S}Z$ (Johnstone & Joyal [JJ82], lemma 4.1).

Such models are equivalent to geometric morphisms $Z \longrightarrow E^{\hat{I}}$ by the classifying topos property and these are in turn equivalent to geometric morphisms $Z \times \hat{I} \longrightarrow E$ by the

[2] With this symbol we refer to the topos whose category of sheaves is $\mathcal{S}X^{\mathfrak{I}}$. See the discussion on topos notation in section 1.1.

exponentiation adjunction. The product topos $Z \times \hat{\mathcal{I}}$ is the trivial pullback in $\mathfrak{T}op$

$$\begin{tikzcd} Z \times \hat{\mathcal{I}} \arrow[r] \arrow[d] & \hat{\mathcal{I}} \arrow[d, "!"] \\ Z \arrow[r, "!"] & \mathbf{1} \end{tikzcd}$$

and so it is the topos $Z\hat{!}^*(I)$ whose sheaves are $\mathcal{S}Z^{!^*(I)}$, i.e. $!^*(I)$ diagrams in $\mathcal{S}Z$, where $!^*(I)$ is the $\mathcal{S}Z$-internalised version of I. This comes out of the Diaconescu's theorem (see [Joh77], Corollary 4.35).

We consider the definitions 6.65 and 6.66 for the case $\mathfrak{I} = !^*(I)$. The geometric morphisms ∞ and π induce, by pre-composition, a pair of functors in the point categories

$$\mathfrak{T}op(Z \times \hat{\mathcal{I}}, E) \underset{\mathfrak{T}op(\pi, E)}{\overset{\mathfrak{T}op(\infty, E)}{\rightleftarrows}} \mathfrak{T}op(Z, E) \tag{6.67}$$

or equivalently, a pair of functors between the categories

$$\underline{Mod}(\mathcal{S}(Z \times \hat{\mathcal{I}}), \mathbb{T}_E) \underset{\pi^\# := \Delta}{\overset{\infty^\# := colim_{!^*(I)}}{\rightleftarrows}} \underline{Mod}(\mathcal{S}Z, E) \tag{6.68}$$

The L.H.S. is the category of I-diagrams of models of $\mathbb{T}_E$ inside $\mathcal{S}Z$ and the functors $\infty^\#$ and $\pi^\#$ are the pullbacks along the inverse image functors ∞^* and π^*. It also holds $\infty^\# \dashv \pi^\#$, which means that calculating the colimit of a I-diagram of models of $\mathbb{T}_E$, amounts to obtaining its image along $\infty^\#$. Equivalently, a filtered diagram of points is a geometric morphism $Z \times \hat{\mathcal{I}} \longrightarrow E$ and its colimit is obtained by pre-composition with the geometric morphism ∞. This shows that the categories of points of Grothendieck topoi have filtered colimits just as the posets of points of locales have directed joins.

Moreover, it is trivial to demonstrate that the action of geometric morphisms on points preserves the filtered colimits.

$$\begin{tikzcd} Z \arrow[d, shift right, "\infty"'] \arrow[d, shift left, "\pi"] & & \\ Z \times \hat{I} \arrow[r, "\mathcal{D}"] & {[\mathbb{T}_E]} \arrow[r, "F"] & {[\mathbb{T}_H]} \end{tikzcd} \tag{6.69}$$

It is basically a manifestation of the associative property of the composition of arrows in $\mathfrak{T}op$! For let F be a geometric morphism $E \longrightarrow H$ and $\mathcal{D}$ a diagram of $\mathbb{T}_E$-models in $\mathcal{S}Z$, i.e. (equivalent to)an object of $\mathfrak{T}op(Z \times \hat{\mathcal{I}}, H)$. Then applying F on the diagram first and then taking the colimit corresponds to $(F \circ \mathcal{D}) \circ \infty$ whereas calculating the colimit first and then applying F corresponds to $F \circ (\mathcal{D} \circ \infty)$. Therefore we demonstrated the following.

Theorem 6.39 *The categories of points at any stage Z of Grothendieck topoi have all filtered colimits. Geometric morphisms acting on points preserve these filtered colimits.*

Let now $f : X \longrightarrow Y$ be a perfect map between two stably compact locales. In chapter 5 we showed that the exponentials $[set]^X$ and $[set]^Y$ classify $\mathcal{B}X$-sheaves and $\mathcal{B}Y$-sheaves respectively. In this chapter we showed that f induces a pair of geometric morphisms (with the notation of section 6.7)

$$[set]^X \underset{P_{\mathcal{B}f}}{\overset{R_{\mathcal{B}f}}{\rightleftarrows}} [set]^Y$$

such that $R_{\mathcal{B}f} \dashv P_{\mathcal{B}f}$ and $P_{\mathcal{B}f}$ acts on $\mathcal{B}X$-sheaves, or equivalently on $Sh(X)$, as $f_* : Sh(X) \longrightarrow Sh(Y)$. Let us consider diagram 6.69 with $[\mathbb{T}_E] \equiv [set]^X$, $[\mathbb{T}_H] \equiv [set]^Y$ and $F \equiv P_{\mathcal{B}f}$. The interpretation of

$$P_{\mathcal{B}f} \circ (\mathcal{D} \circ \infty) = (P_{\mathcal{B}f} \circ \mathcal{D}) \circ \infty$$

is that

$$\mu_* \text{ (colimits of diagrams of } !^*(\mathcal{B}X)\text{-sheaves)} = \text{colimits of } \mu_* \text{ (diagrams of } !^*(\mathcal{B}X)\text{-sheaves)} \tag{6.70}$$

We know that $!^*(\mathcal{B}X)$-sheaves are equivalent to sheaves over the locale $Z \times X \longrightarrow Z$, so the property 6.70 tells us that the geometric morphism

$$X \times Z \xrightarrow{f \times id} Y \times Z \tag{6.71}$$

in $\mathfrak{Top}/Z$ has a direct image functor that preserves filtered colimits that are indexed in $\mathcal{S}Z$. Therefore f is relatively tidy (relative to any base topos) (see definition 1.24 in introduction).

Conversely, if X, Y are any two stably compact locales and $f : Sh(X) \longrightarrow Sh(Y)$ a geometric morphism such that f_* preserves filtered colimits, then f_* it preserves filtered colimits of the representable sheaf $\mathbf{y}(X)$ which is equivalent with the fact that $f_* : \Omega X \longrightarrow \Omega Y$ preserves directed joins or that $f : X \longrightarrow Y$ is perfect. So we have established

Corollary 6.40 *A map between two stably compact locales is perfect iff it is relatively tidy.*

In chapter 2 we demonstrated that the Beck-Chevalley condition holds for *lax pullbacks* of perfect maps in **Loc**. Now we revisit the Beck-Chevalley condition. Combining Moerdijk and Vermeulen's result (see theorem 1.25 in the introduction) and corollary 6.40 we obtain the following.

Corollary 6.41 *We write the base topos as B, implying that $\mathcal{S}B \equiv$ **Sets**, the constructive version of sets we have been working inside throughout the thesis. Let $h : Y \longrightarrow Z$ be a perfect map between two stably compact locales in $\mathbf{Loc}/B$ and $f : X \longrightarrow Z$ any geometric morphism whose domain is any topos over B. Consider the lax pullback in $\mathfrak{Top}/B$*

$$\begin{tikzcd} X \Rightarrow_Z Y \arrow[r, "\vartheta_2"] \arrow[d, "\vartheta_1"'] \arrow[dr, phantom, "\nearrow"] & Y \arrow[d, "h"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{6.72}$$

(i) The geometric morphism ϑ_1 is proper.

(ii) The Beck-Chevalley condition holds for this lax pullback, i.e. the natural transformation

$$f^* \circ h_* \Rightarrow \vartheta_{1*} \circ \vartheta_2^*$$

is an isomorphism.

We finish with a characterisation of stable compactness in $\mathfrak{Top}$. We consider the terminal object $\mathbf{1}$ in **Loc**. We know that $\mathbf{1}$ is compact regular and hence stably compact. We also know that the map (in **Loc**)

$$! : X \longrightarrow \mathbf{1}$$

is proper iff X is compact. So assuming that X is stably compact entails that $! : X \longrightarrow \mathbf{1}$ is a proper and hence perfect (between two stably compact locales). Therefore, by corollary 6.40, the essentially unique geometric morphism $! : X \longrightarrow \mathbf{1}$ is relatively tidy in $\mathfrak{Top}$, i.e. the direct image functor $!_*$ which is the global sections functor

$$\Gamma : Sh(X) \longrightarrow \mathbf{Sets}$$

preserves filtered colimits. Moerdijk & Vermeulen call a topos that possesses this property *strongly compact*. We, thus, demonstrated the following.

Corollary 6.42 *If a locale is stably compact, then it is strongly compact as a topos.*

Chapter 7

Further Work

[1]

We do not know yet what a stably compact topos is. There are a number of approaches one can adopt in order to generalise stable compactness from locales to Grothendieck topoi. Such a possible generalisation is in the spirit of the work of Johnstone and Joyal [JJ82] on continuous categories. Define a *strong proximity category* to be a small category with all finite limits and finite colimits with also an extra class of "strong" arrows that obey a suitable generalisation of the properties of the strong order of a strong proximity lattice (see definition 3.1). A Grothendieck topos X should then be called stably compact, iff there is such a strong proximity category $\mathbf{B}$ such that $\mathcal{S}X$ is equivalent to the *ind-completion* $ind - \mathbf{B}$ of $\mathbf{B}$. Here by $Ind - \mathbf{B}$ we mean the category whose objects are the "rounded" filtered diagrams of $\mathbf{B}$.

In this section we wish to outline a different avenue based on the generalised Priestley duality (section 1.4 or end of section 2.1 for a short spatial account), according to which, the category of stably compact spaces and perfect maps is equivalent to the category of partially ordered compact Hausdorff spaces and continuous monotone functions. In other words, a stably compact space X is equivalent to a compact Hausdorff space $\mathbf{Patch}X$ together with a partial order on the points of $\mathbf{Patch}X$ stemming from the specialisation order of X.

A possible generalising direction of the above equivalence is the following. If a topos is stably compact then it is equivalent to a *locally ordered* compact Hausdorff space (or compact local pospace) as in definition 1.27. We elaborate this idea.

The set of points of a topological space are endowed with the specialisation order. Correspondingly (and more generally), the points of a topos are connected with *speciali-*

[1]Most of the arguments and intuitions in this chapter are non constructive. Lemmas 7.1 and 7.2 are constructive.

sation morphisms. We would like, in the special case where X is a stably compact topos, its category of points to be (equivalent to) a category whose objects are the points of a compact Hausdorff space and whose arrows amount to a local partial order.

An example could be a topos X whose category of (say) global points $\mathfrak{T}op(\mathbf{1}, \overleftarrow{S^1})$ (denote it $\overleftarrow{S^1}$) is equivalent to the 1-dimensional circle S^1 together with specialisation morphisms given as follows. If $e^{i\vartheta_1}$ and $e^{i\vartheta_2}$ are two points on S^1, then the set of arrows $e^{i\vartheta_1} \longrightarrow e^{i\vartheta_2}$ is isomorphic to the set of paths

$$n_{\vartheta_1,\vartheta_2} : [0,1] \longrightarrow S^1$$

given by

$$n_{\vartheta_1,\vartheta_2}(t) := e^{\vartheta_1 + (2n\pi + \vartheta_2 - \vartheta_1)t}$$

i.e. anticlockwise paths from $e^{i\vartheta_1}$ to $e^{i\vartheta_2}$ winding $0, 1, 2, ...$ times around the circle.

Intuitively at least, the category $\overleftarrow{S^1}$ is equivalent to the locally ordered circle of section 1.6.

We demonstrate the passage from topological spaces (specialisation order, global partial order) to topoi (specialisation morphisms, local partial order). Consider the unit interval $\overrightarrow{[0,1]}$ with the upper (Scott) topology (stably compact space). Suppose we attempt to "bend" it in such a way that its top and bottom points become identical. This can be done by means of the coequaliser of the diagram

$$\mathbf{1} \underset{\perp}{\overset{\top}{\rightrightarrows}} \overrightarrow{[0,1]}$$

The coequaliser of the above diagram in **Sp** is obviously the terminal local **1**. Identifying the top and bottom points of $\overrightarrow{[0,1]}$ causes all the points in between to become equal to $\top = \perp$ due to the specialisation order of $\overrightarrow{[0,1]}$.

On the other hand, such a coequaliser in $\mathfrak{T}op$ gives non trivial results. We concretely work out what happens in the case of the Sierpinski locale, trying to argue constructively when we can[2].

Lemma 7.1 *By the symbol* $\mathbb{N}$ *here we understand the monoid* $(\mathbb{N}, +, 0)$*, i.e. the category with a single object and whose set of arrows endowed with composition is isomorphic to the set of natural numbers endowed with addition. Then* $\hat{\mathbb{N}}$ *is the coequaliser in* $\mathfrak{T}op$ *of*

[2] This example is attributed to S.Vickers (presentation at the 74th Peripatetic Seminar on Sheaves and Logic, Cambridge, 2000

the diagram

$$\mathbf{1} \overset{\top}{\underset{\perp}{\rightrightarrows}} \$ \tag{7.1}$$

where \$ is the Sierpinski topos (see section 1.2).

Proof. Recall that for any small category $\mathbf{C}$, the representable presheaves $\mathbf{y_C}(c)$, for any object c of $\mathbf{C}$, are flat and therefore points (models) of $\hat{}\,\mathbf{C}$. It follows that any point of $\hat{}\,\mathbf{C}$ is a filtered colimit of the *principal* points $\mathbf{y_C}$. In our case where $\mathbf{C}$ is the monoid $\mathbb{N}$, we have one principal point corresponding to the single object of $\mathbb{N}$. This principal point obviously has to be a set (call it) N together with an endomorphism $s : N \longrightarrow N$. We define the obvious map $e : \$ \longrightarrow \hat{}\,\mathbb{N}$ on points $\perp \longrightarrow \top$ of \$ as below

$$\begin{array}{ccc} \top & & N \\ \Uparrow & \stackrel{e}{\longmapsto} & \Uparrow \; s \\ \perp & & N \end{array}$$

The map e trivially agrees on $\perp$ and $\top$.

Conversely, suppose that there is a map $e' : \$ \longrightarrow E$ to a topos E agreeing on $\perp$ and $\top$, i.e. points x, of E with $x \cong y$. Let $\mathbf{C}(E)$ be the category of diagrams of the form $x \Rightarrow y$, where x, are points of E with $x \cong y$. To prove that there is a (up to an isomorphism) unique $i : \hat{}\,\mathbb{N} \longrightarrow E$, such that $e' = i \circ e$, it suffices to show that there is an equivalence

$$\top(\hat{}\,\mathbb{N}, E) \simeq \mathbf{C}(E)$$

Define a functor $\top(\hat{}\,\mathbb{N}, E) \longrightarrow \mathbf{C}(E)$ by

$$\begin{array}{ccc} E & & F(N) \\ F \;\Uparrow & \stackrel{e}{\longmapsto} & \Uparrow \; F(s) \\ \hat{}\,\mathbb{N} & & F(N) \end{array}$$

This functor is trivially full, faithful and essentially surjective. This proves the claim of the lemma. ∎

Lemma 7.2 *The points of the functor topos* $\hat{}\,\mathbb{N}$ *are equivalent to sets* M *such that*

(i) M *is inhabited.*

(ii) If $x, y \in M$ *then there is* $n \in \mathbb{N}$ *with* $x = n \cdot y$ *or* $y = n \cdot x$.

(iii) If $n \cdot x = n \cdot y$ *for some* $n \in \mathbb{N}$ *and* $x, y \in M$, *then* $x = y$.

(iv) *Only the identity map* 0 *has fixpoints, i.e. if* $n \cdot x = x$ *for some* $n \in \mathbb{N}$ *and* $x \in M$*, then* $n = 0$.

Proof. We know by the Diaconescu theorem that points $\mathbf{1} \longrightarrow \hat{}\mathbb{N}$ are equivalent to flat presheaves $\mathbb{N} \longrightarrow \mathbf{Sets}$ and the latter are the same as the filtered presheaves of definition 4.12. We will check that the conditions of definition 4.12 are the same as the conditions of this lemma.

First assume that $M : \mathbb{N} \longrightarrow \mathbf{Sets}$ is a filtering presheaf. $\mathbb{N}$ has a single object, so M is a single set (with automorphisms) and therefore condition (i) of definition 4.12 coincides with condition (i) of this lemma in the case of $\hat{}\mathbb{N}$.

Applying condition (ii) of definition 4.12 here, we get that for any two elements $x, y \in M$, there is $z \in M$ and natural numbers m, n such that $x = m \cdot z$ and $y = n \cdot z$. If $m \geq n$, we get that $x = (m - n) \cdot n \cdot z = (m - n) \cdot y$ and similarly we get $y = (n - m) \cdot x$.

Now suppose that $n \cdot x = n \cdot y$. Then by condition (ii) of this lemma, there is k with $x = k \cdot y$ or $y = k \cdot x$. Without loss of generality we assume the latter. Then $n \cdot x = n \cdot k \cdot x$, i.e. $n \cdot x = (n + k) \cdot x$. By condition (iii) of definition 4.12, there is element z and natural number l such that $x = l \cdot z$ and $n \cdot l = (n + k) \cdot l$, i.e. $n + l = n + k + l$. This implies $k = 0$, i.e. $x = y$.

Finally condition (iv) follows immediately by writing $x = 0 \cdot x$. Then by evoking again condition (iii) of definition 4.12, if $n \cdot x = x = 0 \cdot x$, there is k with $n + k = 0 + k$, which gives $n = 0$.

Conversely, conditions (i) and (ii) of this lemma yield conditions (i) and (ii) of definition 4.12 immediately. Now assume that $n \cdot x = m \cdot x$. Without loss of generality suppose that $n \geq m$. Then the assumption can be written $m \cdot (n - m) \cdot x = m \cdot x$. Condition (iii) of this lemma guarantees that $(n - m) \cdot x = x$ and then condition (iv) forces $n - m = 0$ or $n = m$. ■

Corollary 7.3 *The topos* $\hat{}\mathbb{N}$ *is coherent and hence strongly compact.*

Proof. $\hat{}\mathbb{N}$ classifies a geometric whose models in **Sets** are given by lemma 7.2. We observe that this geometric theory does not have infinite disjunctions of formulae which implies that $\hat{}\mathbb{N}$ is coherent[3]. Also, any coherent topos is strongly compact (c.f. example 1.22 in section 1.5), therefore $\hat{}\mathbb{N}$ is strongly compact. ■

Lemma 7.4 *Classically,* $\hat{}\mathbb{N}$ *has only the following two points (up to isomorphism).*

- N*, i.e. the set of natural numbers acted on by addition.*

[3] There must be an even more obvious reason why $\hat{}\mathbb{N}$ is coherent! Are all topoi $\hat{}\mathbf{C}$ with $\mathbf{C}$ a category with finite objects coherent?

- *Z, i.e. the set of integers acted on by addition.*

Proof. First assume that there is an element $x_0 \in M$, such that for any $y \in M$, $x_0 \neq 1 \cdot y$. We show that in this case M is isomorphic to N.

The assumption implies that if $x_0 = n \cdot y$ then $n = 0$. For if $x_0 = n \cdot y$ and $n \neq 0$, then we get $x_0 = 1 \cdot (n-1) \cdot y$ which contradicts the assumption. Now define a function $f : N \longrightarrow M$, by $n \mapsto n \cdot x_0$. This is obviously a presheaf homomorphism and is also 1-1 by condition (iii) in lemma 7.2. To prove it also onto, consider $y \in M$. Then, by condition (ii) of lemma 7.2, there is n such that either $y = n \cdot x_0$ or $x_0 = n \cdot y$. The former case means that $y = f(n)$ and the latter case means that $n = 0$, by our assumption, and so $y = 0 = 0 \cdot x_0 = f(0)$.

Therefore we have proved that $M \cong N$.

Now assume that there is no element $x_0 \in M$, such that for any $y \in M$, $x_0 \neq 1 \cdot y$, i.e. for any x there is y such that $x = 1 \cdot y$. This by induction implies that, for a any n and any x, there is a unique y such that $x = n \cdot y$. We fix an arbitrary $x_0 \in M$. We define a map $g : Z \longrightarrow M$, by

$$z \mapsto \left\{ \begin{array}{ll} z \cdot x_0 & \text{if } z \geq 0 \\ \text{the unique } y \text{ s.t. } x_0 = (-z) \cdot y & \text{if } z \leq 0 \end{array} \right\}$$

First we show that g is a presheaf homomorphism, i.e. it is natural with respect to actions of $\mathbb{N}$, or more specifically $g(z+n) = n \cdot (g(z))$. If $z \geq 0$, then $g(n+z) = (n+z) \cdot x_0 = n \cdot (z \cdot x_0)$. If $z \leq 0$, we further distinguish two cases

(i) $n + z \geq 0$. Then $g(n+z) = (n+z) \cdot x_0 = (n+z) \cdot (-z) \cdot y = n \cdot y = n \cdot g(z)$.

(ii) $n + z \leq 0$. Then $(-n-z)n \cdot y = (-z) \cdot y = x_0$. So by the definition of g, $n \cdot y = g(n+z)$.

Next we prove that g is 1-1. Assume that $g(z) = g(z')$. We distinguish three cases.

(i) If $z, z' \geq 0$, then we get $z \cdot x_0 = z' \cdot x_0$ and so $z = z'$ by the fact that M is filtered.

(ii) If $z' \leq 0 \leq z$, then by the definition of g, $g(z') = y'$ with $x_0 = (-z') \cdot y'$. So $g(z') = g(z) \Rightarrow y' = z \cdot x_0 = z \cdot (-z) \cdot y' = (z - z') \cdot y'$. Therefore $z - z' = 0$ because of condition (iv) of lemma 7.2.

(iii) If $z, z' \leq 0$, then the assumption gives $g(z) = y = g(z')$ with $x_0 = (-z) \cdot y = (-z') \cdot y$. Therefore $z = z'$ again by the fact that M is filtered.

Finally we show that g is onto. Let $y \in M$. By condition (ii) of lemma 7.2, there is natural number n such that either $y = n \cdot x_0$ or $x_0 = n \cdot y$. In the former case $y = g(n)$ and in the latter case $y = g(-n)$.

Therefore we have proved that $M \cong Z$. ■

Remark 7.5 *The point Z can be regarded as colimit of the filtered diagram below in the category of monoids*

$$\begin{tikzcd} N \arrow[r, "+"] \arrow[drr, "0"'] & N \arrow[r, "+"] \arrow[dr, "-1"] & \dots \arrow[r, "+"] & N \arrow[r, "+"] \arrow[dl, "-n"'] & N \arrow[r, "+"] \arrow[dll, "-n-1"] & \dots \\ & & Z & & & \end{tikzcd}$$

where by, e.g. $-n$, above we denote the colimit injection that embeds N *into* Z *in such a way that* 0 *is sent to* $-n$.

In our context, the topos ˆℕ seems to be a good example of a stably compact topos. What we do next is write ˆℕ as a *localic groupoid*.

A *groupoid* is a category in which each arrow is an isomorphism. Most generally, it is given by a set G of arrows, a set X of objects and the structure maps below

$$\begin{tikzcd} G \times_X G \arrow[r, "c"] & G \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] \arrow[d, "in"'] & X \arrow[l, "id"'] \\ & G & \end{tikzcd} \tag{7.2}$$

Here s denotes the source, t the target, id the map that assigns the identity arrow to each $x \in X$, c the composition of two composable arrows and in the map that gives the inverse arrow. A groupoid is called *localic* when G and X are locales and all the structure maps are continuous functions. Given such a localic groupoid, a G-sheaf is a sheaf over the locale X which is also equipped with a continuous G-action. We denote by $\mathcal{S}_G X$ the category of G-sheaves over X. It is a topos and we follow Moerdijk in calling it the *classifying topos of the groupoid* $G \rightrightarrows X$. We refer primarily to two sources: [JT84] and [Moe88]. For further study, see [Moe90] and [BM98]. In [JT84], Joyal and Tierney prove the following result that places localic groupoids in the centre of topos theory.

Theorem 7.6 *For every topos E, there is a localic groupoid $G \rightrightarrows X$ such that there is an equivalence*

$$\mathcal{S}E \simeq \mathcal{S}_G X$$

We return now to the topos ˆ$\mathbb{N}$. It is not hard to demonstrate the following.

Lemma 7.7 ˆ$\mathbb{N}$ *is the classifying topos of the localic groupoid*

$$\$ + \mathbf{2} \rightrightarrows \$$$

where $\mathbf{2}$ *is the discrete space with two points. The* \$ *component of the arrows set contains the identities and the* $\mathbf{2}$ *component provides an isomorphism* $\bot \rightleftarrows \top$ *between the points* $\top$ *and* $\bot$ *of* \$.

Proof. We argue with the stalks of the G-sheaves over \$. Recall that taking stalks is functorial. A sheaf over \$ basically amounts to a function between two sets $s : M_\bot \longrightarrow M_\top$. An object of the topos $\mathcal{S}_{\$+\mathbf{2}}\$$ is a sheaf over $\mathcal{S}\$$ that also allows for the non trivial action of $\mathbf{2}$, it is therefore a function $s : M_\bot \longrightarrow M_\top$ with $M_\bot \cong M_\top := M$. So the category $\mathcal{S}_{\$+\mathbf{2}}\$$ is obviously isomorphic to the category $\mathbf{Sets}^{\mathbb{N}}$. ■ The next step is an attempt to write the topos $\overleftarrow{S^1}$ as the classifying topos of a localic groupoid. Consider the localic groupoid

$$G_0 + G_1 + G_{-1} \rightrightarrows X$$

Here X and G_0 are each the disjoint union of three copies of $\overrightarrow{[0,3]}$, the closed interval $[0,3]$ with the upper topology. G_1 and G_{-1} are each the disjoint union of three copies of $\overrightarrow{[0,1]}$. (Note that $X \cong G_0 \cong G_1 \cong_{-1}$, but we use different interval lengths for easier description.)

The component G_0 contains the identities for each point of X, i.e. the source and target maps $G_0 \longrightarrow X$ are both the identity maps. To describe the structure maps $s, t : G_1 \longrightarrow X$, we put indices to the connected components of X and G_1:

$$X := \overrightarrow{[0,3]_1} + \overrightarrow{[0,3]_2} + \overrightarrow{[0,3]_3}$$

and

$$G_1 := \overrightarrow{[0,1]_1} + \overrightarrow{[0,1]_2} + \overrightarrow{[0,1]_3}$$

We define the source and target maps by

$$\begin{array}{lll}
s : \overrightarrow{[0,1]_1} \longrightarrow \overrightarrow{[0,3]_1} & & x \mapsto x+2 \\
t : \overrightarrow{[0,1]_1} \longrightarrow \overrightarrow{[0,3]_2} & & x \mapsto x \\
s : \overrightarrow{[0,1]_2} \longrightarrow \overrightarrow{[0,3]_2} & & x \mapsto x+2 \\
t : \overrightarrow{[0,1]_2} \longrightarrow \overrightarrow{[0,3]_3} & & x \mapsto x \\
s : \overrightarrow{[0,1]_3} \longrightarrow \overrightarrow{[0,3]_3} & & x \mapsto x+2 \\
t : \overrightarrow{[0,1]_3} \longrightarrow \overrightarrow{[0,3]_1} & & x \mapsto x
\end{array}$$

Finally, the component G_{-1} contains the inverses of the arrows in G_1, i.e. the structure maps $G_{-1} \rightrightarrows X$ are as $G_1 \rightrightarrows X$ but with source and target maps swapped. Note that all the structure maps are perfect and e.g. $s, t : G_1 \longrightarrow X$ are not proper.

Although the details of such a construction are not worked out, we anticipate that points of the classifying topos $\mathcal{S}_G X$ of the above groupoid $G \rightrightarrows X$, together with their specialisation morphisms constitute a category equivalent to $\overrightarrow{S^1}$. Moreover, even classically and as in the case of $\hat{}\,\mathbb{N}$, there must be an extra point or points of $\mathcal{S}_G X$ that come into existence as colimits of filtered subcategories of $\overrightarrow{S^1}$.

Definition 7.8 *A localic groupoid $G \rightrightarrows X$ is stably compact iff G and X are both stably compact and all its structure maps are perfect.*

Now the general question is: what conditions must a localic groupoid $G \rightrightarrows X$ obey so that the category of points of its classifying topos is equivalent with a compact local pospace. The beginning of a conjecture is that $G \rightrightarrows X$ must be a stably compact localic groupoid. We expect that additional restrictions must also be imposed, probably of algebraic topological nature.

Let us iterate that by the term "stably compact topos" we understand a topos whose category of points and specialisation morphisms is equivalent to a compact local pospace. Suppose also that a "local compact regular poset" is the localic analogue of a compact local pospace. Then we have the following half-finished correspondence.

<table>
<tr><td colspan="2">Priestley duality</td></tr>
<tr><td>Ordered Stone locales
&
monotone continuous maps</td><td>Coherent locales
&
perfect maps</td></tr>
<tr><td colspan="2">Generalised Priestley duality</td></tr>
<tr><td>Compact regular posets
&
monotone continuous maps</td><td>Stably compact locales
&
perfect maps</td></tr>
<tr><td colspan="2">Topos generalised Priestley duality (conjectured)</td></tr>
<tr><td>Local compact regular posets
&
dimaps</td><td>Stably compact localic groupoids (plus further conditions?)
&
relatively tidy maps</td></tr>
</table>

A further question concerns the patch construction. We saw (theorem 2.3) that the functor **Patch** : **StKLoc** $\longrightarrow$ **KRegLoc** is a right adjoint and so it preserves limits. Therefore it preserves the localic groupoid structure.

$$\begin{tikzcd}
\mathbf{Patch}(G) \arrow[r, shift left, "\mathbf{Patch}(s)"] \arrow[r, shift right, "\mathbf{Patch}(t)"'] \arrow[d, "\varepsilon_G"'] & \mathbf{Patch}(X) \arrow[d, "\varepsilon_X"] \\
G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X
\end{tikzcd}$$

If the bottom localic groupoid is classified by a stably compact topos, we would like the top localic groupoid to be classified by a compact regular locale. For example, if $G \rightrightarrows X$ is the groupoid in the example with the locally ordered circle above, is the classifying topos $\mathcal{S}_{\mathbf{Patch}G}\mathbf{Patch}X$ equivalent to $\mathcal{S}S^1$?

Bibliography

[BB88] B.Banaschewski and G.C.L.Brümmer. Stably continuous frames. *Mathematical Proceedings of the Cambridge Philosophical Society***104**(1988), 7-19.

[BBH83] B.Banaschewski, G.C.L.Brümmer and K.A.Hardie. Biframes and bispaces. *Quaestiones Mathematicae***6**(1983), 13-25.

[BM98] C.Butz and I.Moerdijk. Representing topoi by topological groupoids. *Journal of Pure and Applied Algebra*, **130**, (1998), 223-235.

[Bor91(1)] F.Borceux. *Handbook of Categorical Algebra 1.* Cambridge University Press, 1994.

[Bor92(2)] F.Borceux. *Handbook of Categorical Algebra 2.* Cambridge University Press, 1994.

[Esc01] M.H. Escardó. The regular-locally-compact coreflection of a stably locally compact locale. *Journal of Pure and Applied Algebra*, **157** 1, 41-55, 2001.

[Esc01b] M.H. Escardó. On the compact-regular coreflection of a stably compact locale. *Proceedings of the 15th conference on Mathematical Foundations of Programming Semantics (MFPS XV)*, Elsevier, Volume **20** of Electronic Notes in Theoretical Computer Science.

[FGR98] L.Fajstrup, E.Goubault and M.Raussen. Detecting deadlocks in concurrent systems. *Proceedings of the 9th International Conference on Concurrency Theory*, 1998.

[FGR99] L.Fajstrup, E.Goubault and M.Raussen. Algebraic topology and concurrency. Submitted to *Theoretical Computer Science*, also *Technical Report*, Aalborg University, 1999.

[Fre72] P.Freyd. Aspects of Topoi. *Bulletin of Australian Mathematical Society***7**, (1972), 1-76.

[FŠ90] P.Freyd and A.Ščedrov. Categories, Allegories. *Vol. 39 North-Holland Mathematical Library.* North-Holland 1990.

[GHKLM80] G.K.Gierz, K.H.Hofmann, K.Keimel, J.D.Lawson, M.Mislove and D.S.Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, Berlin, 1980.

[JJ82] P.T.Johnstone and A.Joyal. Continuous categories and exponential toposes. *Journal of Pure and Applied Algebra* **25** (1982), 255-296.

[Joh77] P.T.Johnstone. *Topos Theory.* L.M.S. Mathematical Monographs **10**, Academic Press, New York, 1977.

[Joh79] P.T. Johnstone. Factorization and pullback theorems for localic geometric morphisms. *Université Catholique de Louvain, Séminaire de Mathemathique Pure.* Rapport no. **79** (1979).

[Joh82] P.T.Johnstone. *Stone Spaces.* Cambridge Studies in Advanced Mathematics **3**, 1982.

[Joh02] P.T.Johnstone. *Sketches of an elephant : a topos theory compendium Vol 1& 2.* Oxford logic guides ; 43, 2002.

[JPRSWW78] P.Johnstone, R.Par, R.Roseburgh, S.Schumacher, R.Wood, G.Wraith. Indexed categories and their applications. *Lecture Notes in Mathematics*, Springer-Verlag 1978.

[JS96] A.Jung and P.Sünderhauf. On the duality of compact vs. open. In S. Andima, R. C. Flagg, G. Itzkowitz, P. Misra, Y. Kong and R. Kopperman (eds.), *Papers on General Topology and Applications: Eleventh Summer Conference at University of Southern Maine.* Annals of the New York Academy of Sciences, **806**, 1996, 214-230.

[JT84] A.Joyal and M.Tierney. An extension of the Galois theory of Grothendieck. *Memoirs of the American Mathematical Society***309**, 1984.

[JV91] P.T.Johnstone and S.J. Vickers. Preframe presentations present. In A.Carboni, M.C.Pedicchio and G.Rosolini (eds.), *Category Theory-Proceedings, Como, 1990.* Springer Lecture Notes in Mathematics **1488**, Springer Verlag 1991, 193-212.

[Kel89] G.M.Kelly. Elementary observations on 2-categorical limits. *Bulletin of Australian Mathematical Society (2)*39(1989), 301-317.

[Lin] T.W. Lindgren. *Proper Morphisms of Topoi.* Thesis, Rutgers University.

[Mac71] S.Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag, 1971.

[Moe88] I.Moerdijk. The classifying topos of a continuous groupoid I. *Transactions of the American Mathematical Society* **310**, (1988), 629-668.

[Moe90] I.Moerdijk. The classifying topos of a continuous groupoid II. *Cahiers de Topologie et Géométrie Différentielle Catégoriques.* **31** (1990), no. 2, 137-168.

[MM92] S.Mac Lane and I.Moerdijk. *Sheaves in Geometry and Logic.* Springer Verlag, Berlin, 1992.

[MV97] I.Moerdijk and J.Vermeulen. Proper maps of toposes. *Utrecht preprint* **1039**, 1997.

[MV99] I.Moerdijk and J.Vermeulen. Proof of a conjecture of A.Pitts. *Journal of Pure and Applied Algebra* **143** (1999),329-338.

[Nac50] L.Nachbin. *Topologia e ordem.* University of Chicago Press, 1950. In Portugese. English translation: *Topology and Order.* Van Nostrand, Princeton, 1965.

[Pit85] A.Pitts. On product and change of base for toposes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques.* **XXVI-1** (1985), 43-61.

[PR91] J.Powers and E.Robinson. A characterization of pie limits. *Mathematical Proceedings of the Cambridge Philosophical Society***110**(1991), 33-47.

[Smy77] M.Smyth. Effectively given domains. *Theoretical Computer Science* **5** (1977), 257-274.

[Tow96] C.F.Townsend. *Preframe Techniques in Constructive Locale Theory.* PhD thesis, Department of Computing, Imperial College, London, 1996.

[Tow97] C.F.Townsend. Localic Priestley duality. *Journal of Pure and Applied Algebra* **116** (1997), 323-335.

[Ver94] J.J.C. Vermeulen. Proper maps of locales. *Journal of Pure and Applied Algebra* **92** (1994),79-107.

[Vic97] S.J.Vickers. Localic completion of quasimetric spaces. *Research Report DoC97/2*, Department of Computing, Imperial College, London, 1997.

[Vic98a] S.J.Vickers. Strongly Algebraic =SFP(Topically). *Mathematical Structures in Computer Science* **11** (2002), 717-742.

[Vic98b] S.J.Vickers. Sheaves and frame presentations. *Draft*

[Vic99a] S.J.Vickers. Topical categories of domains. *Mathematical Structures in Computer Science* **9** (1999), 569-616.

[Vic01] S.J.Vickers. The double powerlocale and exponentiation: a case study in geometric logic. Submitted to *Theory and Applications of Categories*.